Quantum stress tensor fluctuations of a conformal field and inflationary cosmology

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We discuss the additional perturbation introduced during inflation by quantum stress tensor fluctuations of a conformally invariant field such as the photon. We consider both a kinematical model, which deals only with the expansion fluctuations of geodesics, and a dynamical model which treats the coupling of the stress tensor fluctuations to a scalar inflaton. In neither model do we find any growth at late times, in accordance with a theorem due to Weinberg. What we find instead is a correction which becomes larger the earlier one starts inflation. This correction is non-Gaussian and highly scale dependent, so the absence of such effects from the observed power spectra may imply a constraint on the total duration of inflation. We discuss different views about the validity of perturbation theory at very early times during which currently observable modes are trans-Planckian.

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I. INTRODUCTION

The inflationary paradigm has been remarkably successful in predicting observed features of the cosmic microwave background (CMB) radiation and the large scale structure of the Universe. If inflation is driven by a nearly free, massless quantum field, then a generic prediction is a spectrum of primordial fluctuations which is Gaussian and almost scale invariant [1–5]. For a recent review, see for example [6]. The best test of these predictions comes from CMB observations by the Wilkinson Microwave Anisotropy Probe satellite, which has found a spectrum of temperature fluctuations consistent with Gaussian, nearly scale-invariant primordial fluctuations [7].

However, in addition to the dominant effect coming from tree order fluctuations of the scalar inflaton and of the graviton, there should also be some effects from loop corrections of these fields with themselves and with other fields. The latter will be the topic of this paper, particularly a one loop effect which can be interpreted in terms of quantum stress tensor fluctuations. The fluctuations of quantum stress tensors and their physical effects have been discussed by several authors in recent years [8–13]. For a recent review with further references, see Ref. [14]. Quantum stress tensor fluctuations necessarily have a skewed, highly non-Gaussian, probability distribution, although the explicit form of this distribution has only been found in two-dimensional spacetime models [15].

The effect with which we are concerned is quite different from the effect of adding a noise term in the inflaton equation of motion. The latter effect has been discussed by several authors, including Calzetta and Gonorazky [16], who also considered several types of couplings to the inflaton field, Lombardo and Nacir [17], and by Wu, *et al.* [18]. A review of this line of work is given in the recent book by Calzetta and Hu [19]. The latter authors are working in a fixed background spacetime and generally assume a Gaussian spectrum of noise. We are working in a model in which quantum stress tensor fluctuations induce fluctuations of the spacetime geometry, and are dealing with non-Gaussian fluctuations, as noted above. As we will show, the physical predictions of our model are distinct from those of the models in the papers just cited.

Reference [20] studied the possible contributions of quantum stress tensor fluctuations of a conformally invariant field to primordial density perturbations in inflationary models. It was found that these contributions can be proportional to a power of the scale factor change during inflation, and hence potentially large enough to observe. Because they are associated with a non-scale invariant and non-Gaussian contribution, they can at best be a subdominant part of the primordial density perturbations. This fact was used in Ref. [20] to infer upper bounds on the duration of inflation. These bounds are compatible with adequate inflation to solve the horizon and flatness problems, but raise the possibility that the total duration of inflation might be observable. This possibility goes against a commonly held view that inflation erases the memory of anything which occurred previously, and hence increasing its duration beyond the minimum needed to solve the horizon and flatness problems can produce no observable effect. However, contrary indications to this view had previously

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been published in the form of arguments that inflation cannot be eternal to the past [21,22], although these arguments were based on general considerations which do not make specific predictions of observable effects. Winitzki [23] has recently suggested a model in which inflaton field fluctuations can produce violations of the null energy condition and possible effects of the total inflationary expansion.

The purpose of the present paper is to re-examine and improve the analysis in Ref. [20]. In Sec. II, we discuss a kinematic model which makes no explicit reference to the inflaton field, but examines the gravitational effects of the stress tensor fluctuation upon timelike geodesics. In Sec. III, we give a detailed treatment of a dynamical model in which the stress tensor fluctuations alter the dynamics of a scalar inflaton field. In both models, a correction to the power spectrum of density fluctuations is computed. Section IV discusses the implications of our results and some associated conceptual issues, especially the role of trans-Planckian modes. Our analysis is summarized in Sec. V.

Before concluding this section we should mention some conventions. A hat is used to denote the spatial Fourier transform of any field $A(t, \mathbf{x})$

$$\hat{A}(t, \mathbf{k}) \equiv \frac{1}{(2\pi)^3} \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} A(t, \mathbf{x}). \tag{1}$$

We represent the *power spectrum* of A by the symbol $\mathcal{P}_A(k, t)$, which is defined as follows from the correlator of two $\hat{A}$ fields:

$$\langle \hat{A}(t, \mathbf{k}) \hat{A}(t, \mathbf{k}') \rangle \equiv \mathcal{P}_A(k, t) \times \frac{\delta(\mathbf{k} + \mathbf{k}')}{4\pi k^3}. \tag{2}$$

In (the usual) cases for which $\hat{A}(t, \mathbf{k})$ is time independent we drop time from the argument list of the power spectrum, as in $\mathcal{P}_A(k)$. We consider the loop counting parameter of quantum gravity to be $\kappa^2 \equiv 16\pi G$. Our curvature tensors follow the Landau-Lifshitz spacelike convention, which is also the Misner-Thorne-Wheeler (+ + +) convention,

$$R^\rho{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\alpha}\Gamma^\alpha{}_{\mu\sigma} \quad \text{and} \quad R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}. \tag{3}$$

A very important point for understanding our analysis and results is that we normalize the Friedmann-Robertson-Walker scale factor to unity at the end of inflation, rather than at the current time. Also note that we use the subscript "0" sometimes to signify "background" and sometimes to denote that the subscripted quantity is evaluated at the beginning of inflation. So t_0 is the time at which inflation begins, rather than the current time as in much of the literature on cosmology. We indicate the current time by the subscript "now," so the wave number $k = 2\pi/\lambda$ is measured in units of the comoving distance at the end of inflation, and it can be expressed in terms of the current wave number $k_{\rm now} = 2\pi/\lambda_{\rm now}$ through the relation $k = a_{\rm now} k_{\rm now}$.

II. THE KINEMATIC MODEL REVISITED

Here we will review and modify a model first presented in Ref. [20]. The point is to give a simple computation of the extra part of the power spectrum of energy density fluctuations due to a conformally invariant quantum field. (See Fig. 1 for the relation between our contribution and the usual tree order result.) A rigorous derivation involves solving the coupled, linearized inflaton-graviton equations with the conformal stress tensor as a source. We will do that in Sec. III. Here we avoid any mention of the inflaton field, and we require only the background metric

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \\ &= a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j), \end{aligned} \tag{4}$$

where t is the comoving time, η is the conformal time, and a is the scale factor.

What we do instead is to assume that the stress energy consists of a perfect fluid with energy density $\rho(t, \mathbf{x})$, pressure $p(t, \mathbf{x}) = w\rho(t, \mathbf{x})$ (with constant equation of state parameter w) and 4-velocity $u^\mu(t, \mathbf{x})$, in comoving coordinates such that $u^\mu \partial_\mu = \partial/\partial t$. Then we use energy conservation,

$$\dot{\rho} + (\rho + p)\theta = 0, \tag{5}$$

with $\dot{\rho} \equiv \partial\rho/\partial t$, to infer the perturbed energy density $\delta\rho(t, \mathbf{x})$ by perturbing the expansion $\theta(t, \mathbf{x}) \equiv u^\mu{}_{;\mu}$

$$\frac{\partial}{\partial t}\left(\frac{\delta\rho(t, \mathbf{x})}{\rho_0(t)}\right) = -(1 + w)\delta\theta(t, \mathbf{x}). \tag{6}$$

The key to simplifying the computation is deriving the perturbed expansion $\delta\theta(t, \mathbf{x})$ from the Raychaudhuri equation

$$\begin{aligned} u^\mu \partial_\mu \theta = -R_{\mu\nu}u^\mu u^\nu - \tfrac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega^{\mu\nu}\omega^{\mu\nu} \\ + (u^\mu{}_{;\nu}u^\nu)_{;\mu}. \end{aligned} \tag{7}$$

We shall drop the shear $\sigma^{\mu\nu}$, the vorticity $\omega^{\mu\nu}$, and the acceleration $(u^\mu{}_{;\nu}u^\nu)_{;\mu}$, at which point one can obtain $\delta\theta(t, \mathbf{x})$ from $\delta R_{\mu\nu}(t, \mathbf{x}) = \frac{1}{2}\kappa^2 T^{\rm conf}_{\mu\nu}(t, \mathbf{x})$, for the particular part of the total perturbation that concerns us. (See Fig. 1.) Here $T^{\rm conf}_{\mu\nu}$ denotes the stress tensor of the conformal field.

This makes for a wonderfully simple analysis in which we need never consider the perturbed inflaton field or components of the perturbed metric. Unfortunately, it is not correct, as we will see in Sec. III. Ignoring $\sigma^{\mu\nu}$ and $\omega^{\mu\nu}$ is valid at linearized order for single-scalar inflation, but the acceleration term contributes at linearized order and that spoils the simple relation between $\delta\theta(t, \mathbf{x})$ and the conformal stress tensor. So the result we shall derive in this

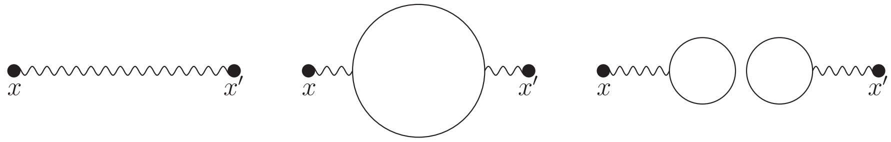

FIG. 1. Various contributions to the power spectrum of primordial perturbations. Wavy lines stand for graviton-inflaton fields and solid lines denote conformal fields. The left-most diagram represents the tree order contribution which is usually reported. The center diagram gives the one loop contribution from conformal matter which is the subject of our analysis. The right-most diagram represents the (unobserved) term which is neglected by subtracting off the expectation value of the conformal stress tensor.

section is off by an important factor of $(k/H)^4$, but it does depend correctly on the initial time.

Conformal invariance allows the stress tensor correlation function

$$C_{\mu\nu\alpha\beta}(x, x') = \langle T_{\mu\nu}(x)T_{\alpha\beta}(x')\rangle - \langle T_{\mu\nu}(x)\rangle\langle T_{\alpha\beta}(x')\rangle \quad (8)$$

to be written in terms of the flat space stress tensor correlation function

$$C^{\rm RW}_{\mu\nu\alpha\beta}(x, x') = a^{-2}(\eta)a^{-2}(\eta')C^{\rm flat}_{\mu\nu\alpha\beta}(x, x'). \quad (9)$$

Here the components of $C^{\rm RW}_{\mu\nu\alpha\beta}(x, x')$ are understood to be in the second set of coordinates in Eq. (4). Although the conformal anomaly term in $\langle T_{\mu\nu}(x)\rangle$ breaks conformal symmetry, this term cancels out of the correlation function, Eq. (8). The construction of the stress tensor correlation function for nonconformal fields is considerably more complex. It has recently been done in de Sitter spacetime for minimally coupled and massive scalar fields by Perez-Nadal, Roura, and Verdaguer [13]. Their results might be used to extend our analysis to nonconformal fields.

The flat spacetime energy density correlation function of the conformal field is $\mathcal{E}(\Delta\eta, r)$, where $\Delta\eta = \eta - \eta'$ and $r = |\mathbf{x} - \mathbf{x}'|$. The expansion correlation function can be expressed in terms of $\mathcal{E}(\Delta\eta, r)$ as

$$\langle\delta\theta(\eta_1, \mathbf{x})\delta\theta(\eta_2, \mathbf{x}')\rangle = \frac{1}{4}\kappa^4 a^{-2}(\eta_1)a^{-2}(\eta_2)\int_{\eta_0}^{\eta_1}\frac{d\eta}{a(\eta)} \times \int_{\eta_0}^{\eta_2}\frac{d\eta'}{a(\eta')}\mathcal{E}(\Delta\eta, r). \quad (10)$$

For the case of the electromagnetic field,

$$\mathcal{E}_{\rm em}(\Delta\eta, r) = {\rm Re}\left\{\frac{(\Delta\eta^2 + 3r^2)(r^2 + 3\Delta\eta^2)}{\pi^4[r^2 - (\Delta\eta + i\varepsilon)^2]^6}\right\}, \quad (11)$$

and the expression for the conformal scalar field case is identical except for an additional factor of 1/12. [Note that this expression corrects an error in Eq. (39) of Ref. [20].] This expression is ultraviolet finite in spite of being one loop [11].

After reheating, the expansion fluctuations cause differential redshifting and consequent density fluctuations, in accordance with the conservation law Eq. (5) for a perfect fluid. The fluid flow approach to density perturbations has been discussed by several authors [24–27]. Let $\delta_\rho = \delta\rho/\rho$ be the fractional density fluctuation at conformal time $\eta = \eta_s$, the last scattering surface. Its spatial correlation function is given in terms of the expansion correlation function by

$$\langle\delta_\rho(\eta_s, \mathbf{x})\delta_\rho(\eta_s, \mathbf{x}')\rangle = (1 + w)^2\int_{\eta_r}^{\eta_s}\frac{d\eta_1}{a(\eta_1)}\int_{\eta_r}^{\eta_s}\frac{d\eta_2}{a(\eta_2)} \times \langle\delta\theta(\eta_1, \mathbf{x})\delta\theta(\eta_2, \mathbf{x}')\rangle. \quad (12)$$

Here reheating occurs at $\eta = \eta_r$ and the equation of state after reheating is $p = w\rho$.

Let

$$F_0(r) = \langle\delta\theta(\eta_r, \mathbf{x})\delta\theta(\eta_r, \mathbf{x}')\rangle \quad (13)$$

be the variance of the expansion at the end of inflation. In many cases, the dominant contribution to the density fluctuations arises from effects occurring during inflation. Then contributions to the expansion correlation function coming from stress tensor fluctuations after reheating may be neglected. However, we still need to account for the evolution of $\delta\theta$ after reheating. If we ignore the effects of classical density perturbations and pressure gradients, as well as the quantum stress tensor, then $\delta\theta$ satisfies [See Eq. (13) in Ref. [20]]

$$\frac{d\delta\theta}{dt} = -\frac{2}{3}\theta_0\delta\theta, \quad (14)$$

where $\theta_0 = 3\dot{a}/a$ is the unperturbed Robertson-Walker expansion. The solution of this equation can be written as

$$\delta\theta(\eta) = \frac{\delta\theta(\eta_r)}{a^2(\eta)}, \qquad \eta \geq \eta_r, \quad (15)$$

where we have set the scale factor at reheating to unity, $a(\eta_r) = 1$. In the approximation where we consider only stress tensor fluctuations during inflation, after reheating we have

$$\langle\delta\theta(\eta_1, \mathbf{x})\delta\theta(\eta_2, \mathbf{x}')\rangle = a^{-2}(\eta_1)a^{-2}(\eta_2)F_0(r). \quad (16)$$

The density perturbation correlation function now becomes

$$\langle\delta_\rho(\eta_s, \mathbf{x})\delta_\rho(\eta_s, \mathbf{x}')\rangle \approx (1 + w)^2 F_0(r)\left[\int_{\eta_r}^{\eta_s}\frac{d\eta_1}{a^3(\eta_1)}\right]^2. \quad (17)$$

The time integral now depends only upon the form of the scale factor between reheating and last scattering. We consider a model in which the inflation is described by a de Sitter metric,

$$a(\eta) = -\frac{1}{H\eta}, \qquad \eta \le \eta_r, \tag{18}$$

and the subsequent period is radiation dominated ($w = 1/3$),

$$a(\eta) = H\eta + 2. \tag{19}$$

Here H is the Hubble parameter of de Sitter space, and both $a(\eta)$ and $da/d\eta$ are continuous at $\eta = \eta_r = -1/H$. Then

$$\int_{\eta_r}^{\eta_s} \frac{d\eta_1}{a^3(\eta_1)} = \int_{-1/H}^{\eta_s} \frac{d\eta_1}{(H\eta_1 + 2)^3} \approx \frac{1}{2H}. \tag{20}$$

The last step follows because $a(\eta_s) \gg 1$. Now we obtain

$$\langle \delta_\rho(\eta_s, \mathbf{x})\delta_\rho(\eta_s, \mathbf{x}')\rangle = \frac{4}{9H^2} F_0(r). \tag{21}$$

We will next take spatial Fourier transforms and write, for example,

$$\hat{\mathcal{E}}_{\rm em}(\Delta\eta, k) = \frac{1}{(2\pi)^3}\int d^3x e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \mathcal{E}_{\rm em}(\Delta\eta, r). \tag{22}$$

The explicit form of $\hat{\mathcal{E}}_{\rm em}(\Delta\eta, k)$ is

$$\hat{\mathcal{E}}_{\rm em}(\Delta\eta, k) = -\frac{k^4\sin(k\Delta\eta)}{960\pi^5\Delta\eta} = -\frac{k^5}{960\pi^5}\int_0^1 du\cos(ku\Delta\eta). \tag{23}$$

The Fourier transform of $F_0(r)$ is

$$\hat{F}_0(k) = \frac{1}{4}\kappa^4 \int_{\eta_0}^{\eta_r} \frac{d\eta}{a(\eta)} \int_{\eta_0}^{\eta_r} \frac{d\eta'}{a(\eta')} \mathcal{E}(\Delta\eta, k). \tag{24}$$

We next evaluate the integrals in this expression, using the second form in Eq. (23), to find in the limit that $k|\eta_0| \gg 1$,

$$\hat{F}_0(k) \sim \frac{\kappa^4 k^4 H^2}{11\,520\pi^4}\left(-|\eta_0|^3 + \frac{3}{\pi k}|\eta_0|^2 + \cdots\right). \tag{25}$$

Note that the magnitude of $\hat{F}_0(k)$ grows as $|\eta_0| \to \infty$ for fixed k. The growth was found in Ref. [20], but there only the $|\eta_0|^2$ term appeared. The reason for this is that in Ref. [20], the calculations were done in coordinate space until the last step, where a term in $F_0(r) \propto 1/r^6$ was Fourier transformed into a term in $\hat{F}_0(k) \propto k^3$. However, this procedure is not sensitive to the possibility of delta-function terms in $F_0(r)$. The leading term in Eq. (25) arises from just such a term, one proportional to $\nabla^4\delta(\mathbf{x} - \mathbf{x}')$.

Let $P_{\delta_\rho}(k)$ denote the spatial Fourier transform of $\langle \delta_\rho(\eta_s, \mathbf{x})\delta_\rho(\eta_s, \mathbf{x}')\rangle$, which is related to the power spectrum by $\mathcal{P}_{\delta_\rho}(k) = 4\pi k^3 P_{\delta_\rho}(k)$. From Eqs. (21) and (25), we find

$$P_{\delta\rho}(k) \equiv \frac{1}{(2\pi)^3}\int d^3x e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\langle \delta_\rho(\eta_s, \mathbf{x})\delta_\rho(\eta_s, \mathbf{x}')\rangle \approx \frac{\kappa^4 k}{25\,920\pi^4}\left(-|k\eta_0|^3 + \frac{3}{\pi}|k\eta_0|^2 + \cdots\right). \tag{26}$$

Note that if we take the $k^4|\eta_0|^3$ term seriously for all k, then its spatial Fourier transform contributes a term proportional to $\nabla^4\delta(\mathbf{x} - \mathbf{x}')$ in $\langle \delta_\rho(\eta_s, \mathbf{x})\delta_\rho(\eta_s, \mathbf{x}')\rangle$. To the extent that measurements are made in position space by comparing proxies for $\delta\rho(t, \mathbf{x})$ at $\mathbf{x} \neq \mathbf{x}'$, this term will not contribute. Here we assume that we may ignore this term and retain only the nonlocal η_0^2 effect. Recalling the definition Eq. (2), our result for the one loop contribution to the $\delta\rho$ power spectrum from conformal matter is then

$$[\mathcal{P}_{\delta_\rho}(k)]_{\rm conf} \approx \frac{\kappa^4 k^4}{2160\pi^4}\left(\frac{k}{Ha(t_0)}\right)^2. \tag{27}$$

Apart from numerical factors, this is equivalent to Eq. (48) in Ref. [20]. We will see in the next section that the prefactor of k^4 should really be H^4; however, that still leaves the result very strongly biased toward short wavelength perturbations and far from scale invariant. The possible implications will be discussed in Sec. IV.

III. A DYNAMICAL MODEL

The computation of $[\mathcal{P}_{\delta_\rho}(k)]_{\rm conf}$ we have just completed involved three basic steps:

(i) Inferring the post-inflationary density contrast from the perturbed expansion

$$\delta_\rho(t, \mathbf{x}) = -(1 + w)\int_{t_r}^{t} dt'\delta\theta(t, \mathbf{x}). \tag{28}$$

(ii) Approximating the post-inflationary Raychaudhuri equation as (14), so that the density contrast during radiation domination becomes

$$\delta_\rho(t, \mathbf{x}) \approx -\frac{2}{3H(t_r)}\delta\theta(t_r, \mathbf{x}). \tag{29}$$

(iii) Computing the perturbed expansion which is accumulated during inflation by using the Raychaudhuri Eq. (7), under the assumption that the acceleration term $(u^\mu{}_{;\nu}u^\nu)_{;\mu}$ makes no contribution at linearized order.

Equation (28) is exact. While Eq. (29) is certainly not exact, it does represent a reasonable approximation when Fourier transformed and restricted to superhorizon modes. The problematic step is ignoring the acceleration term to compute the $\delta\theta(t_r, \mathbf{x})$ induced by conformal matter fluctuations during inflation. It turns out that the acceleration term depends linearly upon the inflaton perturbation, and we must study the coupled gravity-inflaton system to get a reliable result for $\delta\theta(t_r, \mathbf{x})$. Having done this, we use Eqs. (28) and (29) as before to compute $[\mathcal{P}_{\delta_\rho}(k)]_{\rm conf}$.

In this section we study the coupling of a single-scalar inflaton field with the gravitational perturbations of a spatially flat Robertson-Walker spacetime. These perturbations are in turn driven by the fluctuations of the stress tensor of a conformal quantum field. The unperturbed metric is of the form in Eq. (4). During inflation, this metric will be approximately that of de Sitter spacetime, although with a slowly varying Hubble parameter H. We assume that the unperturbed metric satisfies Einstein's equations with the stress tensor of a spatially homogeneous inflaton field $\varphi_0(t)$ as the source. Let the inflaton be self-coupled by a potential $V(\varphi)$. Then the Einstein equations become

$$3H^2 = \tfrac{1}{2}\kappa^2(\tfrac{1}{2}\dot\varphi_0^2 + V_0), \tag{30}$$

and

$$-2\dot H - 3H^2 = \tfrac{1}{2}\kappa^2(\tfrac{1}{2}\dot\varphi_0^2 - V_0). \tag{31}$$

Here dots again denote derivatives with respect to t, $H = \dot a/a$ and $V_0 = V(\varphi_0)$. The scalar field equation is

$$-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) + V'(\varphi) = 0, \tag{32}$$

which becomes

$$\ddot\varphi_0 + 3H\delta\dot\varphi_0 + V_0' = 0. \tag{33}$$

To the extent that V_0 is not constant, the unperturbed spacetime will not be exactly de Sitter space.

A. Coupled equations for inflaton and metric perturbations

We next wish to consider linear perturbations of this spacetime in a gauge in which

$$g_{tt} = -1, \tag{34}$$

so the perturbed metric may be written as

$$ds^2 = -dt^2 + 2a(t)h_{ti}(t,\mathbf{x})dtdx^i + a^2(t)[\delta_{ij} + h_{ij}(t,\mathbf{x})]dx^i dx^j. \tag{35}$$

It is convenient to define the conformally transformed, spatial metric,

$$\tilde g_{ij} \equiv \delta_{ij} + h_{ij}. \tag{36}$$

The determinant of the full metric can be broken up into three factors,

$$-g = a^6 \times \det(\tilde g) \times [1 + h_{ti}h_{tj}\tilde g^{ij}] = a^6[1 + h + O(h^2)], \tag{37}$$

where $h = \delta^{ij}h_{ij}$ is the trace of the metric perturbation.

In addition to the metric perturbation, the inflaton field will have inhomogeneous perturbations:

$$\varphi(t,\mathbf{x}) = \varphi_0(t) + \delta\varphi(t,\mathbf{x}). \tag{38}$$

The scalar field perturbations will satisfy

$$\delta\ddot\varphi + 3H\delta\dot\varphi + \tfrac{1}{2}\dot\varphi_0\dot h + V_0''\delta\varphi = 0, \tag{39}$$

which follows from the expansion of Eq. (32) to first order both in $\delta\varphi$ and in h.

The Einstein equations may be written as

$$R_{\mu\nu} = \tfrac{1}{2}\kappa^2(T^{\rm total}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}T^{\rm total}_{\rho\sigma}), \tag{40}$$

where the total stress tensor is the sum of contributions from the inflaton field and the conformal quantum field:

$$T^{\rm total}_{\mu\nu} = T^{\rm infl}_{\mu\nu} + T^{\rm conf}_{\mu\nu}. \tag{41}$$

We focus here on the time-time component of the Einstein equation. The first-order expansion of R_{tt} is

$$R_{tt} = -3\dot H - 3H^2 - \frac{1}{2}(\ddot h + 2H\dot h) + \frac{1}{a}(\dot h_{ti,i} + Hh_{ti,i}) + O(h^2). \tag{42}$$

The inflaton stress tensor satisfies

$$T^{\rm infl}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}T^{\rm infl}_{\rho\sigma} = \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}V(\varphi). \tag{43}$$

The first-order expansion of the tt component of this expression is

$$\partial_t\varphi\partial_t\varphi - V(\varphi) = \dot\varphi_0^2 - V(\varphi_0) + 2\dot\varphi_0\delta\dot\varphi - V'(\varphi_0)\delta\varphi. \tag{44}$$

Thus, the equation for the first-order metric perturbation can be written as

$$-\frac{1}{2}(\ddot h + 2H\dot h) + \frac{1}{a}(\dot h_{ti,i} + Hh_{ti,i}) = \frac{\kappa^2}{2}[2\dot\varphi_0\delta\dot\varphi - V'(\varphi_0)\delta\varphi + U], \tag{45}$$

where we define

$$U = T^{\rm conf}_{tt}, \tag{46}$$

the energy density of the conformal field in the comoving frame.

Define the normal vector to the surfaces of constant φ by

$$u^\mu = -\frac{g^{\mu\nu}\partial_\nu\varphi}{\sqrt{-g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi}}. \tag{47}$$

The first-order expansion of the spatial components of this vector is

$$u^i = \frac{h_{ti}}{a} + \frac{\partial_i\delta\varphi}{a^2\dot\varphi_0}. \tag{48}$$

We can impose the gauge condition

$$u^\mu = \delta^\mu_t, \tag{49}$$

from which the condition $g_{tt} = -1$ follows. In addition, this leads to a relation between $\delta\varphi$ and h_{ti} to first order:

$$h_{ti}(t, \mathbf{x}) = -\partial_i\left[\frac{\delta\varphi(t, \mathbf{x})}{a(t)\dot{\varphi}_0(t)}\right]. \tag{50}$$

This relation allows us to eliminate the h_{ti} terms in Eq. (45) and write

$$-\kappa^2\dot{\varphi}_0\delta\dot{\varphi} + \frac{1}{2}\kappa^2 V_0'\delta\varphi - \frac{\nabla^2}{a^2}\frac{\partial}{\partial t}\left(\frac{\delta\varphi}{\dot{\varphi}_0}\right) - \frac{1}{2a^2}\frac{\partial}{\partial t}(a^2\dot{h}) = \frac{1}{2}\kappa^2 U. \tag{51}$$

Equations (39) and (51) form a pair of coupled second-order equations for the metric and scalar field perturbations. These equations may be rewritten by expressing φ_0, V_0, and their derivatives in terms of the Hubble parameter $H(t)$ and its derivatives. The sum of Eqs. (30) and (31) leads to

$$\kappa\dot{\varphi}_0 = 2\sqrt{-\dot{H}}, \tag{52}$$

and their difference leads to

$$\kappa^2 V_0 = 2\dot{H} + 6H^2. \tag{53}$$

From these relations, we find

$$\kappa\ddot{\varphi}_0 = -\frac{\ddot{H}}{\sqrt{-\dot{H}}}, \qquad \kappa\dddot{\varphi}_0 = -\frac{\dddot{H}}{\sqrt{-\dot{H}}} - \frac{\ddot{H}^2}{2(-\dot{H})^{3/2}}, \tag{54}$$

and

$$\kappa V_0' = \frac{\ddot{H}}{\sqrt{-\dot{H}}} - 6H\sqrt{-\dot{H}}, \qquad V_0'' = -\frac{\dddot{H}}{2\dot{H}} + \frac{\ddot{H}^2}{4\dot{H}^2} - \frac{3H\ddot{H}}{2\dot{H}} - 3\dot{H}. \tag{55}$$

Note that the homogeneous form (setting $U = 0$) of Eqs. (39) and (51) has a solution when $\delta\varphi = \dot{\varphi}_0$ and $\dot{h} = 6\dot{H}$. We can reduce the order of the system by scaling out this solution and defining

$$B(x) = \frac{\delta\varphi(x)}{\dot{\varphi}_0(t)}. \tag{56}$$

We can now rewrite Eq. (39) as

$$\dot{h} = 6\dot{H}B - 2\left(2\frac{\ddot{\varphi}_0}{\dot{\varphi}_0} + 3H\right)\dot{B} - 2\ddot{B}. \tag{57}$$

This allows us to eliminate h from Eq. (51), and write

$$\mathcal{O}\dot{B} = \tfrac{1}{2}\kappa^2 U, \tag{58}$$

where $\mathcal{O}$ is the operator defined by

$$\mathcal{O} = \left[\partial_t^2 + \left(5H + 2\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\right)\partial_t + 4\dot{H} + 6H^2 + 4H\frac{\ddot{\varphi}_0}{\dot{\varphi}_0} - 2\frac{\ddot{\varphi}_0^2}{\dot{\varphi}_0^2} + 2\frac{\dddot{\varphi}_0}{\dot{\varphi}_0} - \frac{\nabla^2}{a^2}\right]. \tag{59}$$

Equation (58) is a third-order equation which we will solve using a retarded Green's function.

It will be convenient to take a spatial Fourier transform, and define the operator

$$\mathcal{O}_k = \partial_t^2 + \left(5H + 2\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\right)\partial_t + 4\dot{H} + 6H^2 + 4H\frac{\ddot{\varphi}_0}{\dot{\varphi}_0} - 2\frac{\ddot{\varphi}_0^2}{\dot{\varphi}_0^2} + 2\frac{\dddot{\varphi}_0}{\dot{\varphi}_0} + \frac{k^2}{a^2}. \tag{60}$$

Let $G(t, t', k)$ be the retarded Green's function of this operator, which satisfies the equation

$$\mathcal{O}_k G(t, t', k) = \delta(t - t'), \tag{61}$$

with the boundary condition

$$G(t, t', k) = 0 \quad \text{if } t < t'. \tag{62}$$

Let Ψ_1 and Ψ_2 be two linearly independent solutions of the homogeneous equation

$$\mathcal{O}_k\Psi(t, k) = 0. \tag{63}$$

The Green's function may be expressed as

$$G(t, t', k) = \frac{1}{W(t', k)}[\Psi_1(t_<, k)\Psi_2(t_>, k) - \Psi_1(t, k)\Psi_2(t', k)], \tag{64}$$

where $t_<$ and $t_>$ are the lesser and the greater, respectively, of t and t', and

$$W(t, k) = \Psi_1(t, k)\dot{\Psi}_2(t, k) - \dot{\Psi}_1(t, k)\Psi_2(t, k) \tag{65}$$

is the Wronskian.

The homogeneous solutions Ψ_i are difficult to obtain in general. However, if we make a "slow roll" approximation in which time derivatives of H, and hence of φ_0 and of V_0, are assumed to be small, then we have approximately

$$\mathcal{O}_k \approx \partial_t^2 + 5H\partial_t + 6H^2 + \frac{k^2}{a^2}. \tag{66}$$

In this approximation, the solutions of Eq. (63) are

$$\Psi_1(t, k) = a^{-2}(t)e^{ik\int_{t_0}^{t} dt_1 a^{-1}(t_1)}, \tag{67}$$

and $\Psi_2(t, k) = \Psi_1^*(t, k)$. Here t_0 is an arbitrary constant. Now the Wronskian is

$$W(t, k) = -\frac{2ik}{a^5(t)}, \tag{68}$$

and the retarded Green's function may be written as

$$G(t, t', k) = -\frac{a^3(t')}{ka^2(t)} \sin\left[k \int_{t'}^{t} \frac{dt_1}{a(t_1)}\right], \tag{69}$$

for $t \geq t'$.

B. Inflaton field fluctuations

In this subsection, we wish to calculate the fluctuations in φ which are driven by the stress tensor fluctuations of the conformal field. Let $G(x, x')$ be a coordinate space Green's function for the operator $\mathcal{O}$, which satisfies

$$\mathcal{O}\, G(x, x') = \delta(x - x') = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}'). \tag{70}$$

A particular solution of Eq. (58) can be written as

$$\dot{B}(t', \mathbf{x}') = \frac{1}{2}\kappa^2 \int d^4x_1 G(x, x_1)U(x_1). \tag{71}$$

We now treat U and hence B as fluctuating fields, and write the correlation function for $\dot{B}$ as

$$\langle \dot{B}(t, \mathbf{x})\dot{B}(t, \mathbf{x}')\rangle = \frac{1}{4}\kappa^4 \int d^4x_1 d^4x_2 G(x, x_1)G(x', x_2) \times \langle U(x_1)U(x_2)\rangle. \tag{72}$$

Next we convert from comoving to conformal time, using $d\eta = dt/a(t)$, and use the relation between the (comoving) energy density in Robertson-Walker spacetime to that in flat spacetime,

$$\langle U(x_1)U(x_2)\rangle = a^{-4}(\eta_1)a^{-4}(\eta_2)\mathcal{E}(\Delta\eta, r), \tag{73}$$

where $\mathcal{E}(\Delta\eta, r)$ is the flat spacetime energy density correlation function, with $r = |\mathbf{x}_1 - \mathbf{x}_2|$ and $\Delta\eta = \eta_1 - \eta_2$. This leads to

$$\begin{aligned}\langle \partial_\eta B(\eta, \mathbf{x})\partial_\eta B(\eta', \mathbf{x}')\rangle &= \frac{1}{4}\kappa^4 a(\eta)a(\eta') \int_{\eta_0}^{\eta} \frac{d\eta_1}{a^3(\eta_1)} \int_{\eta_0}^{\eta'} \frac{d\eta_2}{a^3(\eta_2)} \\ &\times \int d^3x_1 d^3x_2 [G(\eta, \eta_1, \mathbf{x} - \mathbf{x}_1)G(\eta', \eta_2, \mathbf{x}' - \mathbf{x}_2) \\ &\times \mathcal{E}(\Delta\eta, r)]. \end{aligned}\tag{74}$$

Here the boundary condition $\partial_\eta B(\eta, \mathbf{x}) = 0$ at $\eta = \eta_0$ has been imposed.

Next we take spatial Fourier transforms, and define

$$\langle \partial_\eta B(\eta, \mathbf{x})\partial_{\eta'} B(\eta', \mathbf{x}')\rangle = \int d^3k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \langle \partial_\eta B \partial_{\eta'} B\rangle_k, \tag{75}$$

and analogous transforms of $G(\eta, \eta_1, \mathbf{x} - \mathbf{x}_1)$ and of $\mathcal{E}(\Delta\eta, r)$. Then we may write

$$\langle \partial_\eta B \partial_{\eta'} B\rangle_k = \frac{1}{4}\kappa^4 a(\eta)a(\eta') \int_{\eta_0}^{\eta} \frac{d\eta_1}{a^3(\eta_1)} \int_{\eta_0}^{\eta'} \frac{d\eta_2}{a^3(\eta_2)} \times [G(\eta, \eta_1, k)G(\eta', \eta_2, k)\hat{\mathcal{E}}(\Delta\eta, k)]. \tag{76}$$

The correlation function for $\partial_\eta B$ may be integrated to yield the mean squared inflaton fluctuation at the end of inflation, $\eta = \eta_r$. If $B = 0$ at $\eta = \eta_0$, then

$$\langle B^2(\eta_r)\rangle_k = \int_{\eta_0}^{\eta_r} d\eta \int_{\eta_0}^{\eta_r} d\eta' \langle \partial_\eta B \partial_{\eta'} B\rangle_k. \tag{77}$$

The form for the Green's function from the slow roll approximation, Eq. (69), may be expressed as

$$G(\eta, \eta', k) = -\frac{a^3(\eta')}{ka^2(\eta)} \sin[k(\eta - \eta')]. \tag{78}$$

With this form, and the de Sitter space scale factor, Eq. (18) we find

$$\begin{aligned}\langle B^2(\eta_r)\rangle_k = \frac{\kappa^4 H^2}{4k^2} &\int_{\eta_0}^{\eta_r} d\eta\eta \int_{\eta_0}^{\eta_r} d\eta'\eta' \int_{\eta_0}^{\eta} d\eta_1 \\ &\times \sin[k(\eta - \eta_1)] \int_{\eta_0}^{\eta'} d\eta_2 \\ &\times \sin[k(\eta' - \eta_2)]\hat{\mathcal{E}}(\Delta\eta, k). \end{aligned}\tag{79}$$

It is interesting to compare this with the result Eq. (10) of the kinematical model. After first horizon crossing, $\delta\hat{\theta}(\eta, \mathbf{k}) \sim H^2\hat{B}(\eta, \mathbf{k})$ [See Eq. (86) below], so the big difference between Eqs. (79) and (10) is the extra factors of

$$\frac{H^2}{k} \int_{\eta_0}^{\eta} d\eta_1 \sin[k(\eta - \eta_1)] \frac{H^2}{k} \int_{\eta_0}^{\eta'} d\eta_2 \sin[k(\eta' - \eta_2)]. \tag{80}$$

These terms describe how stress tensor fluctuations from very early times are communicated by the inflaton field to the late time geometry, and they effectively introduce a factor of $(H/k)^2 \times (H/k)^2 = (H/k)^4$ to the result of the kinematical model.

Finally, we may use the form of $\hat{\mathcal{E}}(\Delta\eta, k)$ given in Eq. (23), and perform the integrations in Eq. (79) using the algebraic computer program MATHEMATICA. The result is rather complicated, but in the limit of large $|\eta_0|$, it becomes

$$\langle B^2(\eta_r)\rangle_k \sim -\frac{\kappa^4 H^2 |\eta_0|^3}{122\,880\pi^4} + \frac{\kappa^4 H^2 \eta_0^2}{153\,600\pi^3 k} + \ldots. \tag{81}$$

As with our result Eq. (25) for $\hat{F}_0(k)$, the leading contribution for large $|\eta_0|$ corresponds to a term which is ultralocal in position space, in this case proportional to $\delta(\mathbf{x} - \mathbf{x}')$.

In the subsequent analysis, we will also encounter expectation values of quadratic forms involving time derivatives of B, such as $\langle B\dot{B}\rangle$ and $\langle \dot{B}^2\rangle$. However, one may check that all of these terms are at most proportional to $|\eta_0|$ and hence subdominant compared to $\langle B^2\rangle$ in the limit of large $|\eta_0|$.

1. Density fluctuations from the conservation law

As discussed at the beginning of this section, we can compute the density contrast using Eqs. (28) and (29) once we have $\delta\theta(t_r, \mathbf{x})$, the perturbed expansion at the end of inflation. The expansion θ can be obtained from its definition

$$\theta \equiv u^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}u^\mu). \tag{82}$$

In our gauge, Eq. (49), this becomes

$$\theta(t, \mathbf{x}) = \frac{\partial}{\partial t}\ln(\sqrt{-g}). \tag{83}$$

We may use Eq. (37) to write

$$\begin{aligned}\theta &= 3H + \frac{1}{2}\frac{\partial}{\partial t}\ln(\tilde{g}) + \frac{1}{2}\frac{\partial}{\partial t}\ln[1 + h_{ti}h_{tj}\tilde{g}^{ij}] \\ &= 3H + \frac{1}{2}\dot{h} + O(h^2).\end{aligned} \tag{84}$$

Recall that $\theta_0 = 3H$ is the expansion of the comoving geodesics in Robertson-Walker spacetime, so the first-order perturbation of the expansion is

$$\delta\theta(t, \mathbf{x}) = \tfrac{1}{2}\dot{h}(t, \mathbf{x}). \tag{85}$$

One infers $\dot{h}(t, \mathbf{x})$ from $B(t, \mathbf{x})$ using Eq. (57). Because we only need it at the end of inflation the $\dot{B}$ and $\ddot{B}$ terms can be dropped, and we can use the radiation domination result $\dot{H} = -2H^2$ to conclude

$$\delta\theta(t_r, \mathbf{x}) \approx -6H^2 B(t_r, \mathbf{x}). \tag{86}$$

Hence, Eqs. (28) and (29) give the following expression for the density contrast during radiation domination:

$$\delta_\rho(t, \mathbf{x}) \approx 4HB(t_r, \mathbf{x}). \tag{87}$$

This should be valid when Fourier transformed and restricted to superhorizon modes.

To find the power spectrum we first compute the spatial Fourier transform of the δ_ρ correlator using Eqs. (87) and (81)

$$\begin{aligned}P_{\delta\rho}(k) &\equiv \frac{1}{(2\pi)^3}\int d^3x e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\langle\delta_\rho(\eta_s, \mathbf{x})\delta_\rho(\eta_s, \mathbf{x}')\rangle \\ &\approx \frac{\kappa^4 H^4 k^{-3}}{7680\pi^4}\left(-|k\eta_0|^3 + \frac{4\pi}{5}|k\eta_0|^2 + \cdots\right).\end{aligned} \tag{88}$$

Multiplying by $4\pi k^3$ gives the power spectrum. As for the kinematic model (26), we assume that we may drop the $|\eta_0|^3$ term which is ultralocal and presumably not part of the observed power spectrum. This leaves us with

$$[\mathcal{P}_{\delta_\rho}(k)]_{\text{conf}} \approx \frac{\kappa^4 H^4}{2400\pi^2}\left(\frac{k}{Ha(t_0)}\right)^2. \tag{89}$$

Thus, the dynamical model also produces a non-scale-invariant spectrum biased toward the blue end of the spectrum, although less so than in the case of the kinematic model.

Models with noise terms in the inflaton equation of motion on a fixed background spacetime [16–18] can also produce a blue spectrum, although the basic physics of these models is quite different from our model, and the details of the spectrum depend on the details of the noise term and differ from that of Eq. (89). None of the models in the papers just cited exhibit the dependence upon the physical momentum which we find.

2. Density fluctuations from the Sachs-Wolfe effect

An alternative approach to calculate density or temperature fluctuations is to study the effects of metric perturbations on the redshifts of photons, as was first done by Sachs and Wolfe [28]. Equation (39) of their paper may be expressed as

$$\frac{\Delta T}{T} = \int_{t_r}^{t_s} dt\left[\hat{e}^i h_{ti,t}(x) - \frac{1}{2}\hat{e}^i\hat{e}^j h_{ij,t}(x)\right]. \tag{90}$$

This formula gives the differential redshift, and hence temperature fluctuation, of a photon propagating from $t = t_r$ to $t = t_s$ along a null geodesic in the direction of the unit vector $\hat{e}^i$. The integrand is understood to be evaluated along the unperturbed null geodesic. In contrast to the previous discussion, we now need expressions for the individual components of the spatial metric perturbation, h_{ij}. For this purpose, it is convenient to express the scalar part of the perturbed metric, Eq. (35) as

$$ds^2 = -dt^2 - 2B_{,i}dx^i dt + [a^2(1-2\psi)\delta_{ij} - 2E_{,ij}]dx^i dx^j. \tag{91}$$

Here we follow the notation of Mukhanov [6], as modified in Ref. [29]. As before, h_{ti} is given by Eq. (50). The spatial components of scalar perturbations are

$$h_{ij}(t, \mathbf{x}) = -2\delta_{ij}\psi(t, \mathbf{x}) - \frac{2}{a^2(t)}\partial_i\partial_j E(t, \mathbf{x}). \tag{92}$$

The quantities which appear in the integrand of Eq. (90) may be written in terms of B, ψ and E as

$$\hat{e}^i h_{ti,t}(x) = -\left(\frac{\hat{\mathbf{e}}\cdot\nabla}{a}\right)(\dot{B} - HB) \tag{93}$$

and

$$-\frac{1}{2}\hat{e}^i\hat{e}^j h_{ij,t}(x) = \dot{\psi} + \left(\frac{\hat{\mathbf{e}}\cdot\nabla}{a}\right)^2(\dot{E} - 2HE). \tag{94}$$

Expressions for ψ and for $\dot{E} - 2HE$, Eqs. (A23) and (A25), respectively, are derived in the Appendix. First we note that $\dot{E} - 2HE$ contains two types of terms, those which involve $\dot{B}$ and $\ddot{B}$, and those which depend upon U or $\dot{U}$ evaluated at the same time as $\dot{E} - 2HE$. Both of these types of terms will give a subdominant contribution, which is either independent of $|\eta_0|$ or small compared to the $|\eta_0|^3$

and $|\eta_0|^2$ terms. The same comment applies to all terms in ψ, except for the HB term. Note that in coordinate space, Eqs. (A23) and (A25) contain the nonlocal operator $1/\nabla^2$, which is nonlocal in space only, not in time. In any case, calculations are best done in Fourier space, where $1/\nabla^2$ is replaced by $-1/k^2$.

Thus, we may take

$$\psi \approx -HB. \tag{95}$$

If we drop the $\dot{B}$ term in Eq. (93), and use the fact that

$$\frac{d}{dt} = \frac{\partial}{\partial t} - \left(\frac{\hat{\mathbf{e}} \cdot \nabla}{a}\right) \tag{96}$$

is the total derivative along our null geodesic, we may write

$$\frac{\Delta T}{T} = -\int_{t_r}^{t_s} dt \frac{d(HB)}{dt} \approx (HB)_{t_r}. \tag{97}$$

In the last step, we used the fact that the dominant contribution will come from the lower limit of the integral. If we recall that here

$$\frac{\Delta\rho}{\rho} = 4\frac{\Delta T}{T}, \tag{98}$$

we again obtain Eq. (87).

Density perturbations are often treated using the gauge-invariant potentials, which is yet another possible approach. However, both the fluid flow approach and the Sachs-Wolfe formula, Eq. (90), are themselves gauge invariant and somewhat simpler for our purposes than the gauge-invariant potentials.

IV. DENSITY PERTURBATIONS FROM QUANTUM STRESS TENSOR FLUCTUATIONS

A. Possible constraints on the duration of inflation

Let us now discuss the possible physical implications of the conformal matter contribution to the power spectrum. Recalling that $\kappa^2 = 16\pi G$, and that current wave numbers $k_{\rm now}$ correspond to $k = (ak)_{\rm now}$, our result (89) can be expressed as

$$[\mathcal{P}_{\delta_\rho}]_{\rm conf} \approx \frac{8G^2H^4}{75}\left(\frac{(ak)_{\rm now}}{a_0 H}\right)^2. \tag{99}$$

This is not scale invariant, and it is associated with highly non-Gaussian fluctuations. In contrast, observations of large scale structure and the cosmic microwave background radiation are consistent with the primordial perturbation spectrum being approximately scale invariant and Gaussian [7]

$$\mathcal{P}_{\mathcal{R}}(k_{\rm now}) \approx (2.441^{+0.088}_{-0.092}) \times 10^{-9}\left(\frac{k_{\rm now}}{0.002\ {\rm Mpc}^{-1}}\right)^{-0.037\pm 0.012}. \tag{100}$$

(The primordial curvature and density contrast power spectra are related by $\mathcal{P}_{\mathcal{R}} = \frac{9}{16}\mathcal{P}_{\delta_\rho}$ [6].) Note that the weak scale dependence of the observed power spectrum (100) is actually in the opposite (red) sense to the massive blue tilt we predict from conformal matter. Hence the contribution from conformal matter can only represent a tiny part of the total power spectrum. Because our result Eq. (99) grows like $1/a^2(t_0)$ as the start of inflation is pushed back to earlier and earlier times, one can derive a bound on the duration of inflation by requiring that Eq. (99) is small enough to not affect the measured result, Eq. (100).

It will facilitate the discussion to recall some reasonably generic predictions of single-scalar inflation in the slow roll approximation. The tree order results for the scalar and tensor power spectra are [27]

$$[\mathcal{P}_{\mathcal{R}}(k_{\rm now})]_{\rm tree} \approx \frac{GH^2(t_k)}{\pi\epsilon(t_k)}, \qquad [\mathcal{P}_h(k_{\rm now})]_{\rm tree} \approx \frac{16}{\pi}GH^2(t_k), \tag{101}$$

where $\epsilon(t) \equiv -\dot{H}/H^2$, and t_k is the time of first horizon crossing,

$$a_{\rm now}k_{\rm now} = a(t_k)H(t_k). \tag{102}$$

The absence of much scale dependence in the observed result, Eq. (100), is explained by $H(t)$ being approximately constant during inflation. (This is why our de Sitter approximation of Sec. III was well motivated.) Of course a nearly constant $H(t)$ makes the slow roll parameter $\epsilon(t) = -\dot{H}/H^2$ close to zero. The enhancement of the scalar power spectrum by $1/\epsilon(t_k)$ explains why it has been observed, while the tensor contribution has so far not been resolved. At 95% confidence the bound on their ratio is [7]

$$r \equiv \frac{\mathcal{P}_h(0.002\ {\rm Mpc}^{-1})}{\mathcal{P}_{\mathcal{R}}(0.002\ {\rm Mpc}^{-1})} < 0.22. \tag{103}$$

With the theoretical results Eq. (101) and the scalar observation Eq. (100), this implies a bound on the inflationary Hubble parameter

$$GH^2 = \frac{\pi}{16} \times r \times \mathcal{P}_{\mathcal{R}}(0.002\ {\rm Mpc}^{-1}) \lesssim 10^{-10}. \tag{104}$$

Note that our one loop contribution Eq. (99) is suppressed by $GH^2\epsilon$ relative to the tree effect Eq. (101). It can only become observable when inflation begins at such an early time that these factors are canceled by the square of the physical wave number in Hubble units, $(k/a_0H)^2$.

The bound we get on t_0 derives from requiring the predicted contribution from conformal matter Eq. (99) to be smaller than the observed result Eq. (100) for the largest wave number $k_{\rm now}$ for which data exists. We take $k_{\rm now} \approx 10^{-24}\ {\rm cm}^{-1} \approx 2 \times 10^{-38}$ GeV, which corresponds to structures of about 2 Mpc in physical size, or about 5 arcminutes of angular scale [30]. Let T_R stand for the reheat temperature, and let us assume efficient reheating so that

$$H^2 \approx 8\pi G T_R^4 \approx 8\pi \times 10^{10}\ \mathrm{GeV}^2 \left(\frac{T_R}{10^{12}\ \mathrm{GeV}}\right)^4. \quad (105)$$

The Universe has expanded by about a factor of 10^3 since the last scattering time t_s (when the temperature was $T_S \approx 1$ eV) and recall that we normalize the scale factor to unity at the end of inflation, hence

$$a_{\mathrm{now}} \approx 10^3 a(t_s) \approx 10^3 \frac{T_R}{T_S} \approx 10^{24} \left(\frac{T_R}{10^{12}\ \mathrm{GeV}}\right). \quad (106)$$

For these parameters our result Eq. (99) implies

$$[\mathcal{P}_{\mathcal{R}}(k_{\mathrm{now}})]_{\mathrm{conf}} \approx \frac{3G^2H^4}{50}\left(\frac{a_{\mathrm{now}}k_{\mathrm{now}}}{a_0 H}\right)^2 \approx \frac{5\times 10^{-94}}{a_0^2}\left(\frac{T_R}{10^{12}\ \mathrm{GeV}}\right)^6. \quad (107)$$

Requiring this conformal contribution to be less (by a factor of 10, say) than the observed spectrum Eq. (100) gives

$$\frac{1}{a_0} \lesssim 10^{42}\left(\frac{10^{12}\ \mathrm{GeV}}{T_R}\right)^3. \quad (108)$$

Recall that sufficient inflation to solve the horizon and flatness problems requires $1/a_0 \gtrsim 10^{23}$, so Eq. (108) allows more than enough inflation for this purpose. Note that this bound is, apart from being improved by a factor of 10^3, equivalent to Eq. (92) in Ref. [20]. However, the latter result was derived using an overly simplified dynamical model which did not fully account for the coupling between the inflaton field and the perturbations of the spacetime geometry.

B. The trans-Planckian issue

We now turn to some of the conceptual issues which are raised by the calculations described in the previous sections. One of these concerns the use of trans-Planckian modes, that is, modes whose physical wavelengths are less than the Planck length of $\ell_p \equiv \sqrt{G} \approx 10^{-33}$ cm, as measured by an observer at the start of inflation. The estimates in the previous subsection dealt with perturbations with a present wavelength on the order of $\lambda_{\mathrm{now}} \approx 10^{25}$ cm $\approx 10^{58}\ell_p$. The physical wavelength of this mode at the beginning of inflation is,

$$\lambda_0 = a_0\lambda = a_0 \times \frac{\lambda_{\mathrm{now}}}{a_{\mathrm{now}}}. \quad (109)$$

If we assume that a_0 is at the bound (108)—which means conformal matter contributes 10% of the measured power spectrum at the smallest observed scales—then the initial wavelength is

$$\lambda_0 \approx 10^{-8}\ell_p\left(\frac{T_R}{10^{12}\ \mathrm{GeV}}\right)^2. \quad (110)$$

For most values of T_R, this is at/or below the Planck length. This raises two questions:

(i) Is it valid to extrapolate low energy dynamics such as electromagnetism to trans-Planckian scales?

(ii) Is it valid to apply perturbation theory for trans-Planckian modes?

No one knows what dynamical principles might apply at Planck scales, but it is of course acceptable to carry out a study, as we have done, based on the explicitly stated assumption that they are unchanged. What does not seem alright is employing perturbation theory. One must not be mislead by the fact that the tree order effect $\sim\kappa^2H^2/\epsilon$ is small; the series is an expansion in powers of the large parameter $(\kappa k/a_0)^2$,

$$\mathcal{P}_{\delta_\rho}(k) \sim \kappa^2H^2\left\{\frac{\alpha_0}{\epsilon} + \alpha_1\left(\frac{\kappa k}{a_0}\right)^2 + \alpha_2\left(\frac{\kappa k}{a_0}\right)^4 + \ldots\right\}. \quad (111)$$

If one makes the usual assumption that the pure numbers α_ℓ are of order one then the only way of making the one loop term comparable to the tree order result must also make the two loop and higher terms comparable. The conclusion seems unavoidable that perturbation theory must break down, for mode k, as the initial time is pushed back to the point for which $\kappa k/a_0 \sim 1$. We do not possess a nonperturbative computational technique so what actually happens at earlier times is a matter of conjecture and lively debate within the community [31–33].

One view is based on the observation that the far ultraviolet contains so many modes that even very small deviations from quiescence in each of them must produce enormous fluctuations that would invalidate semiclassical inflation. Hence it must be, the argument goes, that a nonperturbative resummation of loop corrections such as Eq. (111) exhibits no large effect, even for very early initial times. For each wave number k there would be a time T_k such that $\kappa k/a(T_k) \ll 1$, after which our perturbative treatment is valid. As long as t_0 comes after T_k, making t_0 smaller causes the one loop effect to grow as we predict, with higher loop contributions still negligible. But if t_0 is pushed before T_k then the higher loop corrections become important and the whole series approaches a constant. If this view is correct then, for $t_0 < T_k$, one could only employ the perturbative treatment of this paper by starting the evolution of $\hat{B}(t,\mathbf{k})$ at $t = T_k$, not at $t = t_0$. And the correct initial condition would be something close to quiescence at $t = T_k$. This is a nonlocal initial condition, but then quantum effects typically are nonlocal.

A different view is motivated by the similarity of these issues to those which arise in black hole physics. The original derivation of the Hawking effect [34] assumes free quantum field theory on a fixed background spacetime and requires trans-Planckian modes. This derivation is analogous to our treatment in the previous sections. It is possible to reproduce the Hawking effect without the use of trans-Planckian modes [35,36], but only by postulating a nonlinear dispersion relation, which breaks local Lorentz symmetry.

If there is a new physical principle which avoids trans-Planckian modes, then economy of thought would suggest that it should be the same principle for both black hole physics and for cosmology. Ideally, one might hope for an experimental or observational test of trans-Planckian physics. The power spectrum which we have derived using trans-Planckian modes has the potential to provide such a test. If inflation lasted just slightly less than the amount given by Eq. (108), then the model described above predicts a non-scale invariant and non-Gaussian component in the cosmic microwave background which might be detectable.

C. Relation to Weinberg's theorem

Neither the comoving wave number k nor scale factor $a(t)$ are physical, only their ratio, $k/a(t)$. Of course ratios of the scale factor at different times are also physical. Because we normalize the scale factor to one at the end of inflation, the various factors of k in our results must really be interpreted as the physical wave number at the end of inflation, k/a_r. Therefore, the kinematic model estimate of $[\mathcal{P}_{\delta_\rho}(k)]_{\rm conf} \sim \kappa^4 k^6/[H^2 a_0^2]$ seems to suggest a one loop correction to the power spectrum which not only violates scale invariance by the factor $(k/a_r)^6$ but also grows at late times like $(a_r/a_0)^2$. Such growth would contradict a bound of at most logarithmic growth established by Weinberg [37]. (Weinberg's result was derived for minimally coupled scalars but it can easily be extended to conformally coupled particles [38].) In fact one can see from the dynamical model of Sec. III that there is no growth at late times; what happens instead is that the principal effect arises from fluctuations near the time t_0 when the interaction is turned on, after which it rapidly approaches a constant. By comparing our one loop correction with the usual tree order result

$$[\mathcal{P}_{\delta_\rho}(k)]_{\rm tree} \sim \frac{\kappa^2 H^2}{\epsilon} \quad \text{versus} \qquad [\mathcal{P}_{\delta_\rho}(k)]_{\rm conf} \sim \kappa^2 H^2 \times \frac{\kappa^2 k^2}{a_0^2}, \tag{112}$$

it will be seen that our contribution consists of the tree result (without the inverse of $\epsilon \equiv -\dot{H}/H^2$), multiplied by a typical one loop correction of the square of κ times the mode's physical energy at the initial time. Later times contribute far less because the mode's physical energy redshifts so rapidly. There is no mystery about why the effect can be large at very early times because the mode is trans-Planckian then and should induce large gravitational effects. Of course this again raises concerns about using perturbation theory and low energy dynamics. What Weinberg considered was quantum loop effects from the "safe" regime of late times during which perturbative general relativity must be valid. Our results are in perfect agreement with his bound; indeed, they fail to show even the logarithmic growth allowed for by the bound and achieved by nonconformal matter.

V. SUMMARY AND DISCUSSION

In this paper we have evaluated the extra contribution to inflationary density perturbations from the quantum stress tensor fluctuations of a conformal field such as the photon. This was done in a simple, kinematical model and then in a more accurate, but much more complicated, dynamical model. Our main result is that the power spectrum of the energy density at wave number k goes like the tree order result (without enhancement by $1/\epsilon$) times $(E(t_0)/M_{\rm Pl})^2$, where $E(t_0) = k/a(t_0)$ is the mode's physical energy at the beginning of inflation and $M_{\rm Pl}$ is the Planck mass. If a perturbative computation such as this could be trusted to arbitrarily early times, the absence of such a massive blue tilt in the observed power spectrum would seem to imply a bound on the total duration of inflation. This constraint allows enough inflation to solve the horizon and flatness problems.

Our result derives from very early times and rapidly approaches a constant, so it does not contradict Weinberg's bound [37,38], on the maximum possible growth of quantum corrections at late times. However, our result does involve a problematic extrapolation of known physical laws to the trans-Planckian regime, and the even more problematic assumption that perturbation theory can be used at times and on modes for which the physical energy density is trans-Planckian. Opinion on these issues is divided [31–33] and we have tried to present both sides. It is worth pointing out that stress tensor fluctuations from very early times would also induce significant non-Gaussianities if one were to compute them perturbatively, using known physical laws, as we have done for the 2-point correlator.

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APPENDIX: SOME RELATIONS INVOLVING SPATIAL PERTURBATIONS

In this appendix, we will derive some relations relating to the spatial parts of the metric perturbations which are used in Sec. III B 2. The Einstein equations, Eq. (40), may be expressed as

$$R_{\mu\nu} - \tfrac{1}{2}\kappa^2[\partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}V(\varphi)] = \tfrac{1}{2}\kappa^2 T^{\rm conf}_{\mu\nu}. \qquad (A1)$$

It is convenient to remove the scale factors from the conformal stress tensor by defining

$$U = T^{\rm conf}_{tt} = \hat{T}_{tt}, \qquad T^{\rm conf}_{ti} = a\hat{T}_{ti}, \qquad T^{\rm conf}_{ij} = a^2\hat{T}_{ij}. \qquad (A2)$$

Note that $\hat{T}_{\mu\nu}$ are the components of the conformal stress tensor in a local orthonormal frame defined by $d\hat{t} = dt$ and $d\hat{x}^i = adx^i$. The h_{ti} and h_{ij} defined in Eq. (35) are the metric perturbations in this frame.

Because the conformal stress tensor is itself a first-order perturbation we can express its tracelessness using only the zeroth order metric,

$$g^{\rho\sigma}T^{\rm conf}_{\rho\sigma} = 0 \Rightarrow \hat{T}_{kk} = \hat{T}_{tt} + O(h^2). \qquad (A3)$$

Similar simplifications can be made to the relations implied by stress-energy conservation,

$$g^{\rho\sigma}T^{\rm conf}_{\mu\rho;\sigma} = 0 \Rightarrow \frac{1}{a^3}\partial_t(a^3\hat{T}_{tt}) = \frac{1}{a}\hat{T}_{tk,k} - H\hat{T}_{kk} + O(h^2), \qquad (A4)$$

$$\frac{1}{a^3}\partial_t(a^4\hat{T}_{ti}) = \hat{T}_{ik,k} + O(h^2). \qquad (A5)$$

The expansion of the time-time component of Eq. (A1) was performed in Sec. III A so here we focus on the remaining components. The first-order expansions of the required components of the Ricci tensor are, using the metric of Eq. (35),

$$R_{ti} = \dot{h}_{k[i,k]} + \frac{1}{a}h_{t[k,i]k} + (3H^2 + \dot{H})ah_{ti} + O(h^2), \qquad (A6)$$

$$R_{ij} = (3H^2 + \dot{H})a^2\delta_{ij} + (3H^2 + \dot{H})a^2 h_{ij} + \frac{1}{2a}\partial_t(a^3\dot{h}_{ij}) - \frac{1}{a}\partial_t(a^2 h_{t(i,j)}) + h_{k(i,j)k} - \frac{1}{2}h_{ij,kk} - \frac{1}{2}h_{,ij} - Ha\delta_{ij}h_{tk,k} + \frac{1}{2}Ha^2\delta_{ij}\dot{h} + O(h^2). \qquad (A7)$$

The remaining first-order Einstein equations become

$$\dot{h}_{k[i,k]} + 2\dot{H}\partial_i\Phi = \tfrac{1}{2}\kappa^2 a\hat{T}_{ti}, \qquad (A8)$$

and

$$\Big[\partial_t^2 + 3H\partial_t - \frac{\nabla^2}{a^2}\Big]h_{ij} + \frac{1}{a^2}[h_{ik,kj} + h_{jk,ki} - h_{,ij}] + H\delta_{ij}\dot{h} - 2\delta_{ij}[\ddot{H} + 6H\dot{H}]B + \frac{2}{a^2}\dot{B}_{,ij} + \frac{2H}{a^2}[\delta_{ij}\nabla^2 + \partial_i\partial_j]B = \kappa^2\hat{T}_{ij}. \qquad (A9)$$

To understand what Eqs. (A8) and (A9) imply, it is useful to make a decomposition of h_{ij} into irreducible representations of the rotation group,

$$h_{ij} \equiv h^{\rm TT}_{ij} + h^{\rm T}_{i,j} + h^{\rm T}_{j,i} - \frac{1}{2}\Big(\delta_{ij} - 3\frac{\partial_i\partial_j}{\nabla^2}\Big)h^{\rm L} + \frac{1}{2}\Big(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\Big)h. \qquad (A10)$$

This is a decomposition into transverse-tracefree (TT), transverse (T), longitudinal (L) and trace parts. Here $h^{\rm TT}_{ii} = 0 = h^{\rm TT}_{ij,j}$ and $h^{\rm T}_{i,i} = 0$ as usual. We can make similar decompositions of the conformal stress tensor,

$$\hat{T}_{ti} \equiv T^{\rm T}_{ti} + \partial_i T^{\rm L}_t, \qquad (A11)$$

$$\hat{T}_{ij} \equiv T^{\rm TT}_{ij} + T^{\rm T}_{i,j} + T^{\rm T}_{j,i} - \frac{1}{2}\Big(\delta_{ij} - 3\frac{\partial_i\partial_j}{\nabla^2}\Big)T^{\rm L} + \frac{1}{2}\Big(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\Big)T. \qquad (A12)$$

The following identities facilitate extraction of the longitudinal and trace parts,

$$\delta_{ij} = -\frac{1}{2}\Big(\delta_{ij} - 3\frac{\partial_i\partial_j}{\nabla^2}\Big) + \frac{3}{2}\Big(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\Big), \qquad (A13)$$

$$\partial_i\partial_j = -\frac{1}{2}\Big(\delta_{ij} - 3\frac{\partial_i\partial_j}{\nabla^2}\Big)\nabla^2 + \frac{1}{2}\Big(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\Big)\nabla^2. \qquad (A14)$$

Equivalently, the longitudinal part is obtained by the action of the projection operator

$$L_{ij} = \frac{\partial_i\partial_j}{\nabla^2}, \qquad (A15)$$

so that $h^{\rm L} = L_{ij}h_{ij}$. The longitudinal part of (A9) is,

$$\Big(\partial_t^2 + 3H\partial_t + \frac{\nabla^2}{a^2}\Big)h^{\rm L} + \Big(H\partial_t - \frac{\nabla^2}{a^2}\Big)h - (2\ddot{H} + 12H\dot{H})B + \frac{\nabla^2}{a^2}(2\partial_t + 4H)B = \kappa^2 T^{\rm L}. \qquad (A16)$$

Note that stress-energy conservation (A5) implies,

$$(\partial_t + 3H)(aT^{\rm L}_t) = T^{\rm L}, \qquad (A17)$$

and Eq. (A8) implies

$$\dot{h}^{\rm L} = \dot{h} - 4\dot{H}B + \kappa^2 a T_t^{\rm L}. \quad \text{(A18)}$$

$$\frac{\nabla^2}{a^2}h^{\rm L} + \left(\partial_t^2 + 4H\partial_t - \frac{\nabla^2}{a^2}\right)h - 4\dot{H}\dot{B} - (6\ddot{H} + 24H\dot{H})B + \frac{\nabla^2}{a^2}(2\partial_t + 4H)B = 0. \quad \text{(A19)}$$

Now eliminate the $\dot{h}$ terms using Eq. (57) and eliminate the resulting $\partial_t^3 B$ term using Eq. (58). The resulting simplification of (A19) is,

$$\frac{\nabla^2}{a^2}(h^{\rm L} - h + 4HB) + \left(-8H\frac{\ddot{\varphi}_0}{\dot{\varphi}_0} - 12H^2 + 4\dot{H}\right)\dot{B} - 4H\ddot{B} = \kappa^2 U. \quad \text{(A20)}$$

At this point, it is convenient to switch to the variables ψ and E defined in Eq. (91), in terms of which

$$h^{\rm L} = -2\psi - 2\frac{\nabla^2}{a^2}E, \quad \text{(A21)}$$

and

$$h = -6\psi - 2\frac{\nabla^2}{a^2}E. \quad \text{(A22)}$$

Now Eq. (A20) may be written as

$$\psi + HB - \frac{a^2}{\nabla^2}\left[H\ddot{B} + \left(2H\frac{\ddot{\varphi}_0}{\dot{\varphi}_0} + 3H^2 - \dot{H}\right)\dot{B} + \frac{\kappa^2}{4}U\right] = 0. \quad \text{(A23)}$$

The time derivative of Eq. (A22) is

$$\dot{h} = -6\dot{\psi} - 2\frac{\nabla^2}{a^2}(\dot{E} - 2HE). \quad \text{(A24)}$$

Next substitute this relation and the time derivative of Eq. (A23) into Eq. (57), and eliminate the $\partial_t^3 B$ term using Eq. (58). The result is

$$\frac{\nabla^2}{a^2}(\dot{E} - 2HE) = \ddot{B} + \left(\frac{\ddot{H}}{\dot{H}} + 3H\right)\dot{B} - \kappa^2\frac{a^2}{\nabla^2}\left(3HU + \frac{3}{4}\dot{U}\right). \quad \text{(A25)}$$

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