Thermal noise in advanced gravitational wave interferometric antennas: A comparison between arbitrary order Hermite and Laguerre Gaussian modes

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We study the thermal noise caused by mechanical or thermomechanical dissipation in mirrors of interferometric gravitational wave antennas. We give relative figures of merit for arbitrary Hermite-Gauss or Laguerre-Gauss optical beams regarding the Brownian and thermoelastic noises (substrate and coating) in the infinite mirror approximation.

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I. INTRODUCTION

One of the main limits to the sensitivity of interferometric gravitational wave antennas existing, like Virgo or LIGO, is the thermal noise caused by thermodynamical processes, essentially in the material of the mirrors of long Fabry-Perot cavities. For the next generation of instruments (''advanced'' detectors), several strategies have been proposed to reduce this kind of noise.

One possible way [1] is to use optical modes of transverse quantum numbers higher than the fundamental's, expecting a large cancellation of surface fluctuations by averaging on a widely spread phase surface. The analyses were restricted up to now to axially symmetrical modes, excluding Hermite-Gauss modes and nonaxisymmetrical Laguerre-Gauss modes. Axial symmetry allows a simple calculation, but is not a fundamental restriction. The question of the efficiency of more general modes, not necessarily axisymmetric, remained thus open. We show here that a general calculation is possible at least in the limit of a mirror seen as a half space limited by an infinite plane. This is a first step towards a general assessment using a more accurate theory with finite mirrors. We think however that even in this restricted framework, our results may give some ideas about the respective efficiencies of symmetrical and nonsymmetrical modes of any kind. We focus on relative merit factors of the various readout beams, merit factors that are invariant with respect to the material's elastic and thermoelastic parameters.

II. THEORETICAL BASIS

A. Fluctuation-dissipation

1. Brownian noise

We use the fluctuation-dissipation theorem (FDT) and the resulting principle derived by Levin [2]. Recall briefly that the FDT states that the power spectral density (PSD) of Brownian noise in an elementary dynamical system described by a degree of freedom x and a driving force F is given by

$$S_x(f) = \frac{4k_B T}{(2\pi f)^2}\mathcal{R}e[Z]$$

where T is the temperature of the system, k_B the Boltzmann constant, and Z the mechanical impedance, i.e. $Z \equiv \tilde{v}/\tilde{F}$, if $v \equiv \dot{x}$. At this point, Levin has shown that at low frequency (far from mechanical resonance of the mirror's substrate) the real part of the impedance is given by

$$\mathcal{R}\,e[Z] = 4\pi f \Phi U$$

where U is the elastic energy stored in the mirror by a static pressure distribution having the transverse profile of the intensity of the readout beam, and normalized to 1N, so that U has dimension J/N^2. Φ is a loss angle characterizing internal dissipation (the inverse of the mechanical Q factor). Obviously, at frequencies near resonances, the theory becomes far more intricate (see [3,4]). The question of the spectral density of noise nevertheless amounts therefore to the calculation of U for all possible beam profiles, the low-frequency tail (a part of major interest) of the PSD being simply:

$$S_x(f) = \frac{4k_B T}{\pi f}\Phi U. \tag{1}$$

2. Thermoelastic noise

Another source of noise is related to the coupling of temperature fluctuations with the linear thermal expansion of the material. The Levin formula again holds, and we have [5]

$$S_x(f) = \frac{4k_B T}{\omega^2}W \tag{2}$$

where W is the averaged dissipated power. The pressure distribution being normalized to 1N, W has dimension W/N^2. How to compute U and W is the scope of the next paragraph.

B. Basics of static linear elasticity

The aim of the present paragraph is to define notations to be employed in the following. Recall briefly that under applied internal and/or external forces, a stressed solid is deformed so that the position of atoms inside moves according to a displacement vector field $\vec{u}(\vec{r})$. The aim of elasticity theory [6] is to determine $\vec{u}$ from the applied forces and boundary conditions. We consider an infinite solid limited only by an infinite plane $z = 0$ and a pressure distribution $p(x, y)$ applied on that plane. (This is a picture of a light beam hitting a mirror large with respect to the beam transverse extension.)

1. Static strain energy

The elastic energy stored in the solid may be evaluated in the case of only one interface (the plane $z = 0$) as

$$U = -\frac{1}{2}\int_{\mathbb{R}^2} p(x, y)u_z(x, y, z = 0). \tag{3}$$

Our problem is now reduced to compute u_z from $p(x, y)$. The strain tensor E_{ij} is related to $\vec{u}$ by

$$E_{ij}(\vec{r}) = \frac{1}{2}[\partial_i u_j(\vec{r}) + \partial_j u_i(\vec{r})].$$

In an isotropic medium and in the linear regime, the stress tensor $\Theta_{ij}(\vec{r})$ is related in turn to the strain tensor by a relation analogous to Hooke's law:

$$\Theta_{ij} = \lambda E\delta_{ij} + 2\mu E_{ij}$$

where $E \equiv E_{ii}$ is the trace of the strain tensor, and where λ and μ are the Lamé coefficients. Recall that the Lamé coefficients are related to the Young modulus Y and to the Poisson ratio σ by

$$\lambda = \frac{\sigma Y}{(1+\sigma)(1-2\sigma)}, \qquad \mu = \frac{Y}{2(1+\sigma)}.$$

The displacement vector must obey the elastodynamics equation (ρ being the density of the bulk material):

$$\partial_j \Theta_{ij}(\vec{r}) = \rho\frac{\partial^2 u_i}{\partial t^2}(\vec{r}).$$

In the static case, this reduces to the Navier-Cauchy equation:

$$\partial_j \Theta_{ij}(\vec{r}) = 0. \tag{4}$$

Moreover, the boundary conditions on the limiting plane are

$$\Theta_{zz}(x, y, z = 0) = p(x, y), \qquad \Theta_{xz}(x, y, z = 0) = 0, \qquad \Theta_{yz}(x, y, z = 0) = 0. \tag{5}$$

Another way of evaluating the strain energy is to compute the energy density given by

$$\epsilon(x, y, z) = \frac{1}{2}E_{ij}\Theta_{ij} = \frac{1}{2}[\lambda E^2 + 2\mu(E_{ij}E_{ij})].$$

Then

$$U = \int_V \epsilon(x, y, z)dxdydz \tag{6}$$

where V is the space region of interest. Equation (3) may be employed for the infinite substrate whereas Eq. (6) is mandatory in the case of the coatings (limited slab).

2. Thermoelastic dissipated energy

The averaged dissipated energy (see Eq. (2)) due to temperature fluctuations is related to the trace of the strain tensor by [5,7,8]:

$$W = KT\left[\frac{\alpha Y}{(1-2\sigma)\rho C}\right]^2 \int(\vec{\nabla}E)^2 dV \tag{7}$$

where α is the linear thermal expansion coefficient, ρ the density and C the specific heat of the material.

C. Optical beams in the paraxial approximation of diffraction

It is well known that long cavities having weakly curved mirrors have eigen modes accurately described by the paraxial theory of diffraction (PTD). In the PTD, a main propagation direction (optical axis) is assumed, and we take it in the z direction. The coordinates in the transverse plane are either Cartesian (x, y) or polar (r, ϕ). If the set of all possible light amplitude distributions (i.e. of integrable squared modulus) in some plane $z = z_0$ is given the structure of a Hilbert space, there exist at least two well-known complete bases, namely, the Hermite-Gauss and the Laguerre-Gauss basis. Each basis is a family of wave functions assumed to propagate along the z direction, described at abscissa z by their complex amplitude and labeled by 2 quantum numbers (n, m). For Hermite-Gauss modes, the expression relevant for our purpose (ignoring phase factors) is

$$\mathcal{H}_{m,n}(x, y) = \sqrt{\frac{2}{\pi w^2 2^{m+n} m!n!}} H_m\left(\sqrt{2}\frac{x}{w}\right)H_n\left(\sqrt{2}\frac{y}{w}\right) \times \exp\left(-\frac{r^2}{w^2}\right) \tag{8}$$

where w is a width parameter, and where the $H_n(x)$ are the Hermite polynomials. The corresponding intensity distribution, needed according to the preceding subsection, is thus

$$I_{m,n}(x, y) = I_m(x) \times I_n(y) \tag{9}$$

with (for any integer N and real t):

$$I_N(t) = \sqrt{\frac{2}{\pi}}\frac{1}{2^N N! w} H_N(\sqrt{2}t/w)^2 \mathrm{e}^{-2t^2/w^2}.$$

For the Laguerre-Gauss modes, the transverse plane is described in polar coordinates, and we have for the modulus $\mathcal{L}_m^{(n)}$ of the wave function:

$$\mathcal{L}_m^{(n)}(r,\phi) = \sqrt{\frac{4}{(1+\delta_{n0})\pi w^2}\frac{m!}{(m+n)!}}\left(\frac{2r^2}{w^2}\right)^{n/2} L_m^{(n)}\left(\frac{2r^2}{w^2}\right) \times \exp\left(-\frac{r^2}{w^2}\right)\cos n\phi \tag{10}$$

where the $L_m^{(n)}(x)$ are the generalized Laguerre polynomials. The intensity distribution is consequently:

$$I_m^{(n)}(r,\phi) = \frac{2}{(1+\delta_{n0})\pi w^2}\frac{m!}{(m+n)!}\left(\frac{2r^2}{w^2}\right)^{n} L_m^{(n)}\left(\frac{2r^2}{w^2}\right)^2 \times \exp\left(-\frac{2r^2}{w^2}\right)(1+\cos 2n\phi). \tag{11}$$

In all following calculations, the intensities are viewed (as said in Sec. II A) as pressure distributions normalized to 1N, so that $I(r,\phi)$ or $I(x,y)$ have dimension m^{-2}.

III. NOISE OF HERMITE-GAUSS MODES

A. Brownian noise: Substrate

It is possible to find a displacement vector $\vec{u}$ satisfying the Navier-Cauchy equations and all but one boundary conditions by taking the following displacement vector field:

$$\begin{aligned} u_x(x,y,z) &= \frac{i}{4\pi^2}\int_{\mathbb{R}^2}\frac{p}{k}\frac{\mu - kz(\lambda+\mu)}{\lambda+2\mu}A(p,q)\mathrm{e}^{-kz}\mathrm{e}^{i(px+qy)}dpdq \\ u_y(x,y,z) &= \frac{i}{4\pi^2}\int_{\mathbb{R}^2}\frac{q}{k}\frac{\mu - kz(\lambda+\mu)}{\lambda+2\mu}A(p,q)\mathrm{e}^{-kz}\mathrm{e}^{i(px+qy)}dpdq \\ u_z(x,y,z) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left(1+kz\frac{\lambda+\mu}{\lambda+2\mu}\right)A(p,q)\mathrm{e}^{-kz}\mathrm{e}^{i(px+qy)}dpdq \end{aligned} \tag{12}$$

where $k \equiv \sqrt{p^2+q^2}$ and where $A(p,q)$ is an arbitrary function of (p,q). If we consider a Hermite-Gauss mode, the remaining boundary condition is thus:

$$\Theta_{zz}(x,y,z=0) = I_{m,n}(x,y)$$

where $I_{m,n}(x,y)$ is given by (9). On the other hand, after some calculations, we find:

$$\begin{aligned} \Theta_{zz}(x,y,z=0) &= -\frac{1}{4\pi^2}\int_{\mathbb{R}^2}kA(p,q)\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \times \mathrm{e}^{i(px+qy)}dpdq \\ &= -\frac{Y}{2(1-\sigma^2)}\frac{1}{4\pi^2}\int_{\mathbb{R}^2}kA(p,q) \times \mathrm{e}^{i(px+qy)}dpdq. \end{aligned}$$

So that $kA(p,q)$ appears as the Fourier transform of the intensity distribution. We get

$$kA(p,q) = -\frac{2(1-\sigma^2)}{Y}\tilde{I}_m(p)\tilde{I}_n(q)$$

with

$$\begin{aligned} \tilde{I}_N(u) &= \sqrt{\frac{2}{\pi}}\frac{1}{2^N N!}\int_{\mathbb{R}}\mathrm{e}^{-2x^2/w^2}H_N(\sqrt{2}x/w)^2\mathrm{e}^{iux}dx \\ &= \frac{2}{\sqrt{\pi}}\frac{1}{2^N N!}\int_0^\infty \mathrm{e}^{-t^2}H_N(t)^2\cos\left(\frac{uw}{\sqrt{2}}t\right)dt \end{aligned} \tag{13}$$

which is simply [9]:

$$\tilde{I}_N(u) = \mathrm{e}^{-u^2w^2/8}L_N(u^2w^2/4) \tag{14}$$

where the $L_N(x) \equiv L_N^{(0)}(x)$ are the ordinary Laguerre polynomials. The Fourier transform of the intensity is thus

$$\tilde{I}_{m,n}(p,q) = \mathrm{e}^{-k^2w^2/8}L_m(p^2w^2/4)L_n(q^2w^2/4). \tag{15}$$

Owing to (12), we can consider $A(p,q)$ as the Fourier transform of $u_z(x,y,z=0)$. We have the following expression for the elastical energy:

$$U_{m,n} = -\frac{1}{2}\int_{\mathbb{R}^2}dxdy u_z(x,y,z=0)I_{m,n}(x,y).$$

After the Parseval-Plancherel theorem (PPT), we have as well in the Fourier space:

$$U_{m,n} = -\frac{1}{8\pi^2}\int_{\mathbb{R}^2}dpdq\tilde{u}_z(p,q,z=0)\tilde{I}_{m,n}(p,q)$$

or as well

$$\begin{aligned} U_{m,n} &= \frac{1-\sigma^2}{4\pi^2 Y}\int_{\mathbb{R}^2}dpdq\frac{1}{k}\tilde{I}_{m,n}(p,q)^2 \\ &= \frac{1-\sigma^2}{4\pi^2 Y}\int_0^\infty dk\mathrm{e}^{-k^2w^2/4}\int_0^{2\pi}d\alpha\tilde{I}_m(k\cos\alpha)^2 \times \tilde{I}_n(k\sin\alpha)^2 \end{aligned}$$

and finally

$$U_{m,n} = \frac{1-\sigma^2}{2\sqrt{\pi}Yw}\pi^{-3/2}\int_0^\infty dt\mathrm{e}^{-t^2}\int_0^{2\pi}d\alpha L_m(t^2\cos^2\alpha)^2 \times L_n(t^2\sin^2\alpha)^2.$$

In the case $m = n = 0$ (fundamental mode), we get

$$U_{0,0} = \frac{1-\sigma^2}{2\sqrt{\pi}Yw}$$

which is the result first given in [4]. In all what follows, we shall give results as relative to $U_{0,0}$ under the form of a merit factor g expressing the PSD reduction obtained by increasing the order of the mode. We have here, for HG modes:

$$g_{0,m,n} \equiv \frac{U_{m,n}}{U_{0,0}} = \pi^{-3/2}\int_0^\infty dt e^{-t^2}\int_0^{2\pi} d\alpha L_m(t^2\cos^2\alpha)^2 L_n(t^2\sin^2\alpha)^2.$$

We give on Table I the values of the first $g_{0,m,n}$'s. Note that the global PSD scales as $1/w$. A general remark: All merit factors g, here and in the following, result from infinite integrals involving a Gaussian function times a more or less complicated polynomial. These integrals are therefore analytically and exactly computable using any symbolic calculation software, giving a rational number (e.g. $g_{0,1,1} = 1275/2048$). Owing to the approximative nature of the theory, we give the result as a decimal number (which is not the result of a numerical integration).

B. Brownian noise: Coatings

The coating, i.e. the thin reflective layer at the surface of the mirror, has its own elastic parameters and loss angle. The total strain energy of the mirror is therefore the sum of the substrate energy, plus the coating contribution. In Eq. (1), we may replace ϕU by $\phi_S U_S + \phi_C U_C$, where subscript S refers to the substrate, and subscript C to the coating. For evaluating U_C, it is no more possible to use Eq. (3); one has to integrate the energy density. The volume integral reduces in fact, assuming almost constant stresses in the layer, to a surface integral times the thickness δ_C of the coating. This leads [8] to

$$U_C = \delta_C \frac{(1+\sigma)(1-2\sigma)}{Y}\Omega_1\varpi_1$$

with

$$\varpi_1 = \int_{\mathbb{R}^2} dpdq\tilde{I}_{m,n}(p,q)^2 \tag{16}$$

and where Ω_1 is a factor depending on the elastic constants of the coating's material [8]. In practical cases, Ω_1 is of the order of unity, and exactly 1 if the constants are the same as the substrate's. We note that the integral (16) is nothing but the $\mathcal{L}^2$ norm of the intensity profile, so that it can be expressed either in the Fourier or the direct space (PPT). We get

$$\varpi_1 = \frac{\pi}{w^2} g_{1,n,m}.$$

The relevant merit factor is

$$g_{1,n,m} = G_m \times G_n \tag{17}$$

with

$$G_N \equiv \frac{2}{\sqrt{\pi}}\int_0^\infty dt e^{-t^2} L_N(t^2)^2.$$

Table II gives the results for the first $g_{1,m,n}$'s. Let us note that the corresponding PSD scales as $1/w^2$.

C. Thermolastic noise: Substrates

It is easy to derive from (12) the trace of the strain tensor. We have:

$$E(x,y,z) = -\frac{2(1-2\sigma)(1+\sigma)}{4\pi^2 Y}\times\int_{\mathbb{R}^2} dpdq\tilde{I}_{mn,}(p,q)e^{i(px+qy)}e^{-kz}$$

from what we obtain, in particular,

$$\int_{\mathbb{R}^2} dxdy\left[\left(\frac{\partial E}{\partial x}\right)^2 + \left(\frac{\partial E}{\partial y}\right)^2 + \left(\frac{\partial E}{\partial z}\right)^2\right] = \frac{2(1-2\sigma)^2(1+\sigma)^2}{\pi^2 Y^2}\int_{\mathbb{R}^2} k^2 dpdq\tilde{I}_{m,n}(p,q)^2 e^{-2kz} \tag{18}$$

and finally

$$\int(\vec{\nabla}E)^2 dV = \frac{(1-2\sigma)^2(1+\sigma)^2}{\pi^2 Y^2}\int_{\mathbb{R}^2} kdpdq\tilde{I}_{m,n}(p,q)^2$$

or as well

$$\int(\vec{\nabla}E)^2 dV = \frac{4(1-2\sigma)^2(1+\sigma)^2}{\sqrt{\pi}Y^2 w^3} g_{2,m,n}$$

and finally, for the dissipated power

TABLE I. Some numerical values of $g_{0,m,n}$.

m n	0	1	2	3	4	5
0	1	.781	.683	.622	.579	.546
1	.781	.623	.550	.504	.471	.446
2	.683	.550	.488	.449	.421	.400
3	.622	.504	.449	.414	.389	.370
4	.579	.471	.421	.389	.366	.348
5	.546	.446	.400	.370	.348	.332

TABLE II. Some numerical values of $g_{1,m,n}$.

m n	0	1	2	3	4	5
0	1	.75	.641	.574	.528	.493
1	.75	.562	.480	.431	.396	.370
2	.641	.480	.410	.368	.338	.316
3	.574	.431	.368	.330	.303	.283
4	.528	.396	.338	.303	.279	.260
5	.493	.370	.316	.283	.260	.243

TABLE III. Some numerical values of the $g_{2,m,n}$.

m / n	0	1	2	3	4	5
0	1	.906	.866	.841	.824	.811
1	.906	.791	.741	.712	.691	.676
2	.866	.741	.688	.656	.634	.618
3	.841	.712	.656	.623	.600	.583
4	.824	.691	.634	.600	.577	.559
5	.811	.676	.618	.583	.559	.542

$$W = \frac{4KT\alpha^2(1+\sigma)^2}{\sqrt{\pi}\rho^2C^2w^3} g_{2,m,n}$$

with

$$g_{2,m,n} = \frac{2}{\pi\sqrt{\pi}} \times \int_0^\infty dt t^2 e^{-t^2} \int_0^{2\pi} d\alpha L_n(t^2\cos^2\alpha)^2 L_m(t^2\sin^2\alpha)^2.$$

The PSD of equivalent displacement is given by Eq. (2). Table III gives the first values of the $g_{2,m,n}$. Let us emphasize that the PSD scales as $1/w^3$.

d. Thermoelastic noise: Coatings

Thermoelastic noise in coatings has been discussed by several authors (we do not repeat the details, see [10,11]); the theory foresees a scaling law in w analogous to that of coating Brownian noise, so that the relative merit factors are identical and already given by Table II.

IV. NOISE OF LAGUERRE-GAUSS MODES

A. Brownian noise: Substrate

The case of Laguerre-Gauss modes is more difficult. We have an intensity profile which is (see Eq. (11)) the sum of two terms, one is axially symmetrical, the second has angular parity $\cos 2n\phi$. Each of the two terms causes its own displacement vector. The theory being linear, the global displacement is thus the sum of two displacement vectors: $\vec{u}_a(r,z)$ for the axisymmetrical part, and $\vec{u}_n(r,\phi,z)$ for the ϕ-dependent term. An axisymmetrical displacement giving null tangential stresses on the plane $z=0$ and satisfying the Navier-Cauchy equations has the form of a Hankel transform:

$$\vec{u}_a(r,z) = \begin{cases} u_r(r,z) = -\int_0^\infty A_a(k)\frac{\mu - kz(\lambda+\mu)}{\lambda+2\mu} e^{-kz} J_1(kr)kdk \\ u_\phi(r,z) = 0 \\ u_z(r,z) = \int_0^\infty A_a(k)\left(1+\frac{\lambda+\mu}{\lambda+2\mu}kz\right) e^{-kz} J_0(kr)kdk \end{cases}. \tag{19}$$

$A_a(k)$ is an arbitrary function to be determined. In all the following, the $J_n(z)$ are the Bessel functions of the first kind. The stress component normal to the surface $z=0$ is

$$\Theta_{zz}(r, z=0) = -2\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\int_0^\infty A_a(k)J_0(kr)k^2dk = -\frac{Y}{2(1-\sigma^2)}\int_0^\infty A_a(k)J_0(kr)k^2dk.$$

The condition $\Theta_{zz}(r,z=0) = I_m^{(n)}(r)$ gives, after inverting the Hankel transform of kernel J_0:

$$A_a(k) = -2\frac{1-\sigma^2}{Y}\frac{1}{k}\tilde{I}_{a,m}^{(n)}(k) \tag{20}$$

with the following definition:

$$\tilde{I}_{a,m}^{(n)}(k) \equiv \int_0^\infty J_0(kr)I_m^{(n)}(r)rdr. \tag{21}$$

This has been already calculated in [1], and the result is

$$\tilde{I}_{a,m}^{(n)}(k) = \frac{1}{2\pi}e^{-y}L_m(y)L_{m+n}(y) \qquad (y \equiv k^2w^2/8). \tag{22}$$

The z component of the displacement is (at $z=0$)

$$u_{a,z} = -2\frac{1-\sigma^2}{Y}\int_0^\infty \tilde{I}_{a,m}^{(n)}(k)J_0(kr)dk.$$

Let us consider now the ϕ-dependent term. The corresponding displacement vector giving null tangential stresses on the plane $z=0$, satisfying the Navier-Cauchy equations and having the right angular parity, is

$$u_r(r,\phi,z) = \int_0^\infty A(k)\partial_r J_{2n}(kr)e^{-kz}\frac{\mu-kz(\lambda+\mu)}{\lambda+2\mu} \times dk\cos 2n\phi$$

$$u_\phi(r,\phi,z) = -\int_0^\infty \frac{1}{r}A(k)J_{2n}(kr)e^{-kz}\frac{\mu-kz(\lambda+\mu)}{\lambda+2\mu} \times dk\sin 2n\phi$$

$$u_z(r,\phi,z) = \int_0^\infty A(k)J_{2n}(kr)e^{-kz}\left(1+kz\frac{\lambda+\mu}{\lambda+2\mu}\right) \times kdk\cos 2n\phi. \tag{23}$$

The boundary condition reads

$$\int_0^\infty A(k)J_{2n}(kr)k^2dk\cos 2n\phi = I_m^{(n)}(r)\cos 2n\phi$$

from what we get, after inverting the Hankel transform of kernel J_{2n},

$$A(k) = -2\frac{1-\sigma^2}{Y}\frac{1}{k}\tilde{I}_m^{(n)}(k)$$

with the definition

$$\tilde{I}_m^{(n)}(k) \equiv \int_0^\infty J_{2n}(kr) I_m^{(n)}(r) r dr. \quad (24)$$

The result is (see [12])

$$\tilde{I}_m^{(n)}(k) = \frac{1}{2\pi}\frac{m!}{(m+n)!}e^{-y}y^n L_m^{(n)}(y)^2 \qquad (y \equiv k^2w^2/8). \quad (25)$$

At this point, the global pressure distribution is

$$p(r,\phi) = I_m^{(n)}(r) + I_m^{(n)}(r)\cos 2n\phi$$

whereas the global z component of the displacement at $z = 0$ is

$$u_z(r,\phi) = -2\frac{1-\sigma^2}{Y}\left(\int_0^\infty dk J_0(kr)\tilde{I}_{a,m}^{(n)}(k)^2 + \int_0^\infty dk J_{2n}(kr)\tilde{I}_m^{(n)}(k)^2\cos 2n\phi\right)$$

and consequently

$$U_m^{(n)} = -\frac{1}{2}\int_0^\infty dkrdr\int_0^{2\pi} d\phi u_z(r,\phi)p(r,\phi) = \frac{2\pi(1-\sigma^2)}{Y}\left(\varpi_{0,a,m}^{(n)} + \frac{1}{2}\varpi_{0,m}^{(n)}\right)$$

with the definitions

$$\varpi_{0,a,m}^{(n)} = \int_0^\infty dk\tilde{I}_{a,m}^{(n)}(k)^2 = \frac{1}{4\pi^{3/2}w}G_{a,m}^{(n)}$$

$$\varpi_{0,m}^{(n)} = \int_0^\infty dk\tilde{I}_m^{(n)}(k)^2 = \frac{1}{4\pi^{3/2}w}G_m^{(n)}$$

with

$$G_{a,m}^{(n)} \equiv 2\sqrt{\frac{2}{\pi}}\int_0^\infty dx e^{-2x^2}L_m(x^2)^2L_{m+n}(x^2)^2$$

and

$$G_m^{(n)} \equiv 2\sqrt{\frac{2}{\pi}}\left(\frac{m!}{(m+n)!}\right)^2\int_0^\infty dx e^{-2x^2}x^{4n}L_m^{(n)}(x^2)^4$$

and at the end

$$U_m^{(n)} = U_{0,0}g_{0,m}^{(n)} \quad (26)$$

with

$$g_{0,m}^{(0)} = G_{a,m}^{(0)} \qquad g_{0,m}^{(n)} = G_{a,m}^{(n)} + \frac{1}{2}G_m^{(n)} (n \neq 0).$$

See Tables IV and V.

TABLE IV. Some numerical values of the $g_{0,m}^{(n)}$.

m / n	0	1	2	3	4	5
0	1	.598	.462	.390	.343	.310
1	.781	.558	.458	.398	.357	.326
2	.623	.475	.402	.356	.323	.297
3	.540	.427	.368	.330	.302	.280
4	.486	.393	.344	.310	.286	.267
5	.448	.367	.324	.295	.273	.256

B. Brownian noise: Coating

The fact already mentioned that the Brownian noise of the coating depends on the $\mathcal{L}^2$ norm of the intensity, which can be evaluated in the direct or Fourier space, makes the result very simple. Indeed, due again to the PPT, the $\mathcal{L}^2$ norms of both radial distributions of intensity (corresponding to the axisymmetrical term and to the term in $\cos 2n\phi$ respectively) for the Laguerre-Gauss modes are given by the two equivalent integrals:

$$\mathcal{J}_m^{(n)} \equiv 2\left(\frac{m!}{(m+n)!}\right)^2\int_0^\infty e^{-2x}x^{2n}L_m^{(n)}(x)^4 dx = 2\int_0^\infty e^{-2x}L_m(x)^2L_{m+n}(x)^2 dx$$

each of these gives the merit figure for an axisymmetrical $\mathrm{LG}_m^{(n)}$ mode. Now, considering the factor of $1/2$ arising from the angular integration (of $\cos^2 2n\phi$), the global merit figure corresponding to the sum of the two contributions is

$$g_{1,m}^{(n)} = \left(1+\frac{1}{2}\right)\mathcal{J}_m^{(n)}(n \neq 0) \qquad g_{1,m}^{(0)} = \mathcal{J}_m^{(0)}. \quad (27)$$

See Table VI for the merit figures (keeping in mind that the PSD scales in $1/w^2$). We recall the results (Table VII) for axisymmetrical LG modes (modes in $\exp(in\phi)$).

C. Thermoelastic noise: Substrate

Here the trace of the strain tensor is again shared between the axisymmetrical part and the angle dependent part. For the axisymmetrical part E_a we have [8]

TABLE V. Recall some numerical values of the $G_{a,m}^{(n)}$ corresponding to axial symmetry (modes in $\exp(in\phi)$).

m / n	0	1	2	3	4	5
0	1	.598	.462	.390	.343	.310
1	.687	.496	.409	.356	.319	.292
2	.571	.440	.374	.331	.301	.278
3	.505	.402	.348	.312	.286	.265
4	.459	.374	.328	.297	.274	.255
5	.426	.352	.311	.284	.263	.246

TABLE VI. Some numerical values of the $g_{1,m}^{(n)}$.

m / n	0	1	2	3	4	5
0	1	.5	.344	.266	.218	.186
1	.75	.469	.351	.285	.241	.210
2	.562	.375	.292	.242	.208	.183
3	.469	.322	.256	.215	.187	.166
4	.410	.287	.231	.196	.172	.153
5	.369	.261	.212	.181	.160	.144

TABLE VII. Some numerical values of the $\mathcal{J}_m^{(n)}$.

m / n	0	1	2	3	4	5
0	1	.5	.344	.266	.218	.186
1	.5	.312	.234	.190	.161	.140
2	.375	.250	.194	.161	.139	.122
3	.312	.215	.170	.143	.125	.111
4	.273	.191	.154	.131	.114	.102
5	.246	.174	.141	.121	.107	.096

$$E_a(r) = \frac{2(1-2\sigma)(1+\sigma)}{Y}\int_0^\infty kdk\tilde{I}_{a,m}^{(n)}(k)J_0(kr)\mathrm{e}^{-kz}.$$

The angularly dependent part is

$$E_n(r,\phi) = \frac{2(1-2\sigma)(1+\sigma)}{Y} \times \int_0^\infty kdk\tilde{I}_m^{(n)}(k)J_{2n}(kr)\mathrm{e}^{-kz}\cos 2n\phi.$$

Now the gradient of E_n has the following components:

$$\partial_r E_n(r,z,\phi) = \mathcal{E}_1\cos 2n\phi$$
$$\frac{1}{r}\partial_\phi E_n(r,z,\phi) = \mathcal{E}_2\sin 2n\phi$$
$$\partial_z E_n(r,z,\phi) = \mathcal{E}_3\cos 2n\phi$$

with, respectively,

$$\mathcal{E}_1(r,z) = \frac{2(1-2\sigma)(1+\sigma)}{Y}\int_0^\infty k^2dk\tilde{I}_m^{(n)}(k)J'_{2n}(kr)\mathrm{e}^{-kz}$$
$$\mathcal{E}_2(r,z) = -\frac{2(1-2\sigma)(1+\sigma)}{Y}\int_0^\infty k^2dk\tilde{I}_m^{(n)}(k)\frac{2n}{kr}J_{2n}(kr) \times \mathrm{e}^{-kz}$$
$$\mathcal{E}_3(r,z) = -\frac{2(1-2\sigma)(1+\sigma)}{Y}\int_0^\infty k^2dk\tilde{I}_m^{(n)}(k)J_{2n}(kr) \times \mathrm{e}^{-kz}$$

so that

$$\int_0^{2\pi}(\vec{\nabla}E_n)^2d\phi = \pi(\mathcal{E}_1^2+\mathcal{E}_2^2+\mathcal{E}_3^2).$$

Now, we have on one hand:

$$\mathcal{E}_1+\mathcal{E}_2 = \frac{2(1-2\sigma)(1+\sigma)}{Y} \times \int_0^\infty k^2dk\tilde{I}_m^{(n)}(k)J_{2n-1}(kr)\mathrm{e}^{-kz}$$

and on the other hand the product $\mathcal{E}_1\mathcal{E}_2$ vanishes in a radial integration because for $n\neq 0$, the function J_{2n} is zero both at $r=0$ and at $r=\infty$. We get, therefore,

$$\int_0^{2\pi}d\phi\int_0^\infty rdr(\vec{\nabla}E_n)^2 = \pi\int_0^\infty rdr[(\mathcal{E}_1+\mathcal{E}_2)^2+\mathcal{E}_3^2]$$

and at the end, using the closure relation for Bessel functions,

$$\int(\vec{\nabla}E_n)^2dV = \frac{4\pi(1-2\sigma)^2(1+\sigma)^2}{Y^2}\varpi_2$$

with

$$\varpi_2 = \int_0^\infty k^2dk\tilde{I}_m^{(n)}(k)^2.$$

An entirely analogous calculation can be carried out for the axisymmetrical part E_a, except that a factor of 2π appears in the azimuthal integration instead of π, so that

$$\int(\vec{\nabla}E_n)^2dV = \frac{8\pi(1-2\sigma)^2(1+\sigma)^2}{Y^2}\varpi_{2,a}$$

with

$$\varpi_{2,a} = \int_0^\infty k^2dk\tilde{I}_{a,m}^{(n)}(k)^2.$$

Now we have after some straightforward calculation

$$\varpi_2 = \frac{1}{2\pi\sqrt{\pi}w^3}G_{2,m}^{(n)}, \qquad \varpi_{2,a} = \frac{1}{2\pi\sqrt{\pi}w^3}G_{2,a,m}^{(n)}$$

with:

$$G_{2,m}^{(n)} = 8\sqrt{\frac{2}{\pi}}\left(\frac{m!}{(m+n)!}\right)^2\int_0^\infty t^{4n+2}L_m^{(n)}(t^2)^4\mathrm{e}^{-2t^2}dt$$

and

$$G_{2,a,m}^{(n)} = 8\sqrt{\frac{2}{\pi}}\int_0^\infty t^2L_m(t^2)^2L_{m+n}(t^2)^2\mathrm{e}^{-2t^2}dt.$$

By adding the two contributions to W, we get at the end:

$$W = \frac{4(1-2\sigma)^2(1+\sigma)^2}{\sqrt{\pi}Y^2w^3}g_{2,m}^{(n)} \qquad (28)$$

with

$$g_{2,m}^{(n)} = G_{2,a,m}^{(n)} + \frac{1}{2}G_{2,m}^{(n)} \qquad (n\neq 0), \quad \text{and} \quad g_{2,m}^{(0)} = G_{2,m}^{(0)}.$$

See in Table VIII the first values of the $g_{2,m}^{(n)}$, keeping in mind that the PSD scales as w^3.

TABLE VIII. Some numerical values of the $g_{2,m}^{(n)}$.

m / n	0	1	2	3	4	5
0	1	.754	.642	.573	.526	.490
1	.906	.749	.667	.612	.571	.539
2	.791	.658	.591	.546	.513	.487
3	.734	.610	.549	.509	.479	.456
4	.698	.578	.521	.484	.456	.434
5	.673	.556	.500	.465	.439	.418

TABLE IX. Some numerical values of the $G_{2,a,m}^{(n)}$: case of axisymmetrical LG modes (modes in $\exp(in\phi)$).

m / n	0	1	2	3	4	5
0	1	.754	.642	.573	.526	.490
1	.437	.390	.357	.333	.314	.299
2	.329	.306	.287	.272	.260	.250
3	.275	.261	.248	.238	.229	.221
4	.242	.231	.222	.214	.207	.201
5	.218	.210	.203	.197	.191	.186

We recall hereafter (Table IX) the values for axisymmetrical LG modes. The thermoelastic noise being directly related to strain gradient, we are not surprised to see that modes having intensity nodes give worse merit factors than the axisymmetric modes of same quantum numbers.

D. Thermoelastic noise: Coatings

As in the preceding sections, the scaling law being analogous to the case of Brownian noise, the relative merit factors are already given by Tables VI and VII.

V. CLIPPING

The preceding calculations were dealing with infinite mirrors. This makes sense for beams of cross section which are reasonably small compared to the radius a of the mirror. It also makes sense to compare HG and LG modes in the same conditions. But we keep in mind that in real world, mirrors have finite radii. If the w parameter is imposed from geometrical considerations, the preceding numerical results are sufficient. In the case where w can be freely chosen, one has to choose it for obtaining the largest cross section consistent with negligible clipping losses (i.e. the fraction of light power escaping the mirror). It is well known that the effective cross section is an increasing function of the order (n, m). The ratio $\rho_{n,m} = a/w_{n,m}$ giving (arbitrarily) 1 ppm loss for the $\mathcal{H}_{m,n}$ (or the ratio $\rho_m^{(n)} = a/w_m^{(n)}$ for $\mathcal{L}_m^{(n)}$) is thus an increasing function of (m, n), and this should be included in the comparison between modes. To be specific, the results of the two preceding tables should be corrected, giving a new factor:

TABLE X. Some numerical values of the ratio $\rho \equiv a/w$ for 1 ppm clipping losses (Hermite-Gauss modes).

m / n	0	1	2	3	4	5
0	2.628	2.889	3.123	3.333	3.525	3.703
1	2.889	3.093	3.295	3.485	3.662	3.829
2	3.123	3.295	3.475	3.647	3.812	3.968
3	3.333	3.485	3.647	3.807	3.961	4.108
4	3.525	3.662	3.812	3.961	4.106	4.245
5	3.703	3.829	3.968	4.108	4.245	4.379

TABLE XI. Some numerical values of the ratio $\rho \equiv a/w$ for 1 ppm clipping losses (Laguerre-Gauss modes).

m / n	0	1	2	3	4	5
0	2.628	3.145	3.548	3.891	4.197	4.475
1	2.889	3.347	3.720	4.044	4.336	4.603
2	3.093	3.520	3.873	4.184	4.465	4.723
3	3.267	3.675	4.014	4.313	4.586	4.830
4	3.423	3.815	4.143	4.434	4.699	4.916
5	3.565	3.946	4.265	4.549	4.803	4.968

$$\hat{g}_{n,m} = g_{n,m} \times \left(\frac{\rho_{n,m}}{\rho_{0,0}}\right)^{s+1}$$

where s is the scaling factor ($s = 0$ for the Brownian thermal noise in the substrate, $s = 1$ for the Brownian or thermoelastic thermal noise in the coating, $s = 2$ for the thermoelastic noise in the substrate, see [11]). We therefore give in the two following Tables X and XI the values of the ratio, obtained by a numerical integration.

VI. CONCLUSION

The calculation of relative figures of merit for arbitrary Hermite or Laguerre Gaussian beams shows that axial symmetry is always preferable. Regarding Brownian noise, it shows that the efficiency of increasing the order of the modes is a general property, whatever the symmetry properties of the intensity distribution of the beam are. At comparable order, the efficiency is however the best for axially symmetrical LG modes, then for nonaxially symmetrical LG, and eventually for HG modes. Regarding thermoelastic noise, it is clear that it may increase with the order of the readout beam if the w parameter is taken into account (in particular in the substrate case).

The global conclusion is that there is a benefit in using high-order modes. These assessments were made in the approximation of an infinite mirror surface, and we are trying to extend the analysis to finite mirrors.

The gain in the various kinds of thermal noises is obviously not the only criterion in the design of a real optical configuration. Numerical simulations and laboratory experiments are necessary to assess the compatibility of high-order modes operation with the locking systems and the stability of the mirrors steering control loops in an actual interferometer.

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