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Singularity problem and phase-space noncanonical noncommutativity

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The Wheeler-DeWitt equation arising from a Kantowski-Sachs model is considered for a Schwarzschild black hole under the assumption that the scale factors and the associated momenta satisfy a noncanonical noncommutative extension of the Heisenberg-Weyl algebra. An integral of motion is used to factorize the wave function into an oscillatory part and a function of a configuration space variable. The latter is shown to be normalizable using asymptotic arguments. It is then shown that on the hypersurfaces of constant value of the argument of the wave function's oscillatory piece, the probability vanishes in the vicinity of the black hole singularity.

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I. INTRODUCTION

Pioneering work in the context of quantum gravity and string theory [1] has led to great interest in noncommutativity aspects of quantum gravity, quantum field theory [2,3] and quantum mechanics [4]. It is generally believed that some sort of noncommutativity may arise as one approaches the Planck scale. Noncommutativity is also shown to be relevant in the context of black holes (BHs) physics [5–8]. To avoid having to deal with an infinite number of degrees of freedom, one usually resorts to minisuperspace models. Starting from a suitable metric, one obtains via the Arnowitt-Deser-Misner procedure the Wheeler-DeWitt (WDW) equation. Most often the solutions of the latter are not square integrable and thus one faces the problem of determining a "time" variable and a measure, such that on constant time hypersurfaces, the wave function is normalizable and the square of its modulus is a bona fide probability density. A square integrable wave function would allow us, for instance, to address the problem of the BH singularity by computing the probability near the singularity.

Some steps in that direction were taken in Refs. [5–7]. The Kantowski-Sachs (KS) metric was chosen, as it is integrable and it involves two scale factors, which is the minimum number of degrees of freedom for incorporating noncommutativity. Moreover, by a surjective mapping, one can describe the interior of the Schwarzschild BH by the KS model. In Ref. [7] only configuration space noncommutativity was considered: $[\Omega, \beta] = i\theta$, where Ω, β are the scale factors and θ is a real constant. However, it is found in the cosmological and BH context that this type of noncommutativity does not lead to any new qualitative features with respect to the commutative problem. On the other hand, it was shown in Refs. [5,6] that by imposing noncommutativity in the momentum sector as well, i.e., $[P_\Omega, P_\beta] = i\eta$, where η is a real constant and P_Ω, P_β are momenta conjugate to Ω, β, respectively, two interesting features arise: (i) For the Schwarzschild BH, it is found that the potential for the Schrödinger-like problem has a stable minimum. In the neighborhood of this minimum, one is able to perform a saddle point evaluation of the partition function and compute the relevant BH temperature and entropy. (ii) The solution of the WDW equation was shown to factorize into the product of an oscillatory function and a function which displays a conspicuous damping behavior. This damping does not lead to a full-fledged square integrable function, but the wave function does vanish at infinity, where the BH singularity is located. This leads one to conjecture whether another type of noncommutativity might yield a truly normalizable wave function. In this article it is shown that this is indeed possible. The crux of the argument lies on the choice of a noncanonical noncommutative algebra, which is nevertheless isomorphic to the Heisenberg-Weyl (HW) algebra. By noncanonical, it is meant that the commutation relations are not constants. Algebras of this kind have been thoroughly investigated in noncommutative quantum field theory (see, for instance, [9] and references therein) and in mathematical physics. For instance, the algebraic structure of the reduced phase-space formulation of systems with second class constraints is typically a noncanonical noncommutative one [10].

In what follows, the KS metric formulation of the Schwarzschild BH is reviewed, and the noncommutative algebra and the isomorphic mapping to the HW algebra are

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defined. From this map, a quantum representation of the noncanonical algebra is determined, which is then used to obtain the noncommutative WDW equation. Through a suitable constant of motion, the WDW equation is reduced into an ordinary differential equation. By resorting to asymptotic arguments and some results from the spectral theory of operators, it is shown that the solutions of this ordinary differential equation are square integrable.

II. PHASE-SPACE NONCOMMUTATIVE QUANTUM COSMOLOGY

In here the following system of units is used: $\hbar = c = G = 1$. The Schwarzschild BH is described by the following metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (1)$$

where r is the radial coordinate and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. For $r < 2M$ the time and radial coordinates are interchanged ($r \leftrightarrow t$) so that space-time is described by the metric [6]

$$ds^2 = -\left(\frac{2M}{t} - 1\right)^{-1} dt^2 + \left(\frac{2M}{t} - 1\right)dr^2 + t^2 d\Omega^2. \quad (2)$$

This is an anisotropic metric; thus for $r < 2M$ a Schwarzschild BH can be described as an anisotropic cosmological space-time. Indeed, the metric (2) can be mapped by the KS metric [6,7], which, in the Misner parametrization, can be written as

$$ds^2 = -N^2 dt^2 + e^{2\sqrt{3}\beta} dr^2 + e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega} d\Omega^2, \quad (3)$$

where Ω and β are scale factors, and N is the lapse function. For $r < 2M$, the identification

$$N^2 = \left(\frac{2M}{t} - 1\right)^{-1}, \qquad e^{2\sqrt{3}\beta} = \left(\frac{2M}{t} - 1\right), \quad e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega} = t^2 \quad (4)$$

allows for turning the metric Eq. (3) into the metric Eq. (2). The black hole singularity lies at $t = 0$. Inverting the last two equations in (4), one finds $e^{2\sqrt{3}\beta} = (2M - t)/t$ and $e^{-2\sqrt{3}\Omega} = t(2M - t)$. Thus, the singularity corresponds to $\beta, \Omega \rightarrow +\infty$. Notice that the lapse function vanishes at the singularity.

In the present work, one considers the following noncanonical extension of the HW algebra:

$$\begin{aligned} [\hat{\Omega}, \hat{\beta}] &= i\theta\left(1 + \epsilon\theta\hat{\Omega} + \frac{\epsilon\theta^2}{1 + \sqrt{1 - \xi}}\hat{P}_\beta\right), \\ [\hat{P}_\Omega, \hat{P}_\beta] &= i(\eta + \epsilon(1 + \sqrt{1 - \xi})^2\hat{\Omega} + \epsilon\theta(1 + \sqrt{1 - \xi})\hat{P}_\beta), \\ [\hat{\Omega}, \hat{P}_\Omega] &= [\hat{\beta}, \hat{P}_\beta] = i(1 + \epsilon\theta(1 + \sqrt{1 - \xi})\hat{\Omega} + \epsilon\theta^2\hat{P}_\beta), \end{aligned} \quad (5)$$

where θ, η and ϵ are positive constants and $\xi = \theta\eta < 1$. The remaining commutation relations vanish. For $\epsilon \neq 0$ it implies that the noncommutative commutation and uncertainty relations are themselves position and momentum dependent. Also notice that $\epsilon = 0$ corresponds to the canonical phase-space noncommutativity [5,6,11]. One could add that the well known Darboux's theorem ensures that one can always find a system of coordinates, where locally the algebra looks like the HW algebra. For this algebra, this statement also holds globally, so that the algebra is isormorphic to the HW algebra. Actually, the algebra of the equations in (5) is the simplest noncanonical extension that is globally isomorphic to the HW algebra. It is an open issue whether this algebraic structure can be physically motivated or derived from a more fundamental theory. The present study shows, on the other hand, that this noncanonical extension has a direct impact on the singularity problem, as discussed below.

The isomorphism between the algebra of the equations in (5) and the HW algebra is called a Darboux (D) transformation. D transformations are not unique. Indeed, the composition of a D transformation with a unitary transformation yields another D transformation. Suppose that $(\hat{\Omega}_c, \hat{P}_{\Omega_c}, \hat{\beta}_c, \hat{P}_{\beta_c})$ obey the HW algebra:

$$[\hat{\Omega}_c, \hat{P}_{\Omega_c}] = [\hat{\beta}_c, \hat{P}_{\beta_c}] = i. \quad (6)$$

The remaining commutation relations vanish. Then a suitable transformation would be

$$\begin{aligned} \hat{\Omega} &= \lambda\hat{\Omega}_c - \frac{\theta}{2\lambda}\hat{P}_{\beta_c} + E\hat{\Omega}_c^2, \\ \hat{\beta} &= \lambda\hat{\beta}_c + \frac{\theta}{2\lambda}\hat{P}_{\Omega_c} \\ \hat{P}_\Omega &= \mu\hat{P}_{\Omega_c} + \frac{\eta}{2\mu}\hat{\beta}_c, \\ \hat{P}_\beta &= \mu\hat{P}_{\beta_c} - \frac{\eta}{2\mu}\hat{\Omega}_c + F\hat{\Omega}_c^2. \end{aligned} \quad (7)$$

Here, μ, λ are real parameters such that $(\lambda\mu)^2 - \lambda\mu + \frac{\xi}{4} = 0 \Leftrightarrow 2\lambda\mu = 1 \pm \sqrt{1 - \xi}$, and one chooses the positive solution given the invariance of the physics on the D map [11], and

$$\begin{aligned} E &= -\frac{\theta}{1 + \sqrt{1 - \xi}}F, \\ F &= -\frac{\lambda}{\mu}\epsilon\sqrt{1 - \xi}(1 + \sqrt{1 - \xi}). \end{aligned} \quad (8)$$

The inverse D is easily computed:

$$\begin{aligned}\hat{\Omega}_c &= \frac{1}{\sqrt{1-\xi}}\left(\mu\hat{\Omega} + \frac{\theta}{2\lambda}\hat{P}_\beta\right),\\ \hat{P}_{\Omega_c} &= \frac{1}{\sqrt{1-\xi}}\left(\lambda\hat{P}_\Omega - \frac{\eta}{2\mu}\hat{\beta}\right),\\ \hat{\beta}_c &= \frac{1}{\sqrt{1-\xi}}\left(\mu\hat{\beta} - \frac{\theta}{2\lambda}\hat{P}_\Omega\right),\\ \hat{P}_{\beta_c} &= \frac{1}{\sqrt{1-\xi}}\Bigg[\lambda\hat{P}_\beta + \frac{\eta}{2\mu}\hat{\Omega}\\ &\quad - \frac{F\mu}{\sqrt{1-\xi}}\left(\hat{\Omega} + \frac{\theta}{1+\sqrt{1-\xi}}\hat{P}_\beta\right)^2\Bigg].\end{aligned} \tag{9}$$

It can be shown using Eqs. (7) and (9) that the algebra of Eq. (5) implies the HW algebra, Eq. (6), and vice versa. Of course, as already pointed out, one could consider other quadratic transformations relating the two algebras. These other possibilities are obtained from D by composition with a unitary transformation. They all lead to the same physical predictions.

One considers now the Hamiltonian associated to the WDW equation for the KS metric and a particular factor ordering [5],

$$\hat{H} = -\hat{P}_\Omega^2 + \hat{P}_\beta^2 - 48e^{-2\sqrt{3}\hat{\Omega}}. \tag{10}$$

Upon substitution of the equations in (7), one obtains

$$\begin{aligned}\hat{H} &= -\mu^2\hat{P}_{\Omega_c}^2 - \frac{\eta^2}{4\mu^2}\hat{\beta}_c^2 - \eta\hat{\beta}_c\hat{P}_{\Omega_c} + \mu^2\hat{P}_{\beta_c}^2 + \frac{\eta^2}{4\mu^2}\hat{\Omega}_c^2\\ &\quad + F^2\hat{\Omega}_c^4 - \eta\hat{\Omega}_c\hat{P}_{\beta_c} + 2\mu F\hat{\Omega}_c^2\hat{P}_{\beta_c} - \frac{\eta F}{\mu}\hat{\Omega}_c^3\\ &\quad - 48\exp\left(-2\sqrt{3}\lambda\hat{\Omega}_c + \frac{\sqrt{3}\theta}{\lambda}\hat{P}_{\beta_c} - 2\sqrt{3}E\hat{\Omega}_c^2\right).\end{aligned} \tag{11}$$

In the previous expression, the HW operators have the usual representation: $\hat{\Omega}_c = \Omega_c$, $\hat{\beta}_c = \beta_c$, $\hat{P}_{\Omega_c} = -i\frac{\partial}{\partial\Omega_c}$ and $\hat{P}_{\beta_c} = -i\frac{\partial}{\partial\beta_c}$. Let us now define the operator $\hat{A}$, which is analogous to the constant of motion that was defined for the problem discussed in Ref. [5] (which considers the same space-time setup, but assumes a canonical phase-space noncommutative algebra):

$$\hat{A} = \mu\hat{P}_{\beta_c} + \frac{\eta}{2\mu}\hat{\Omega}_c. \tag{12}$$

A simple calculation reveals that for the noncommutative algebra of Eq. (5), $\hat{A}$ also commutes with the Hamiltonian $\hat{H}$, Eq. (10), $[\hat{H}, \hat{A}] = 0$. This quantity corresponds to the momentum P_β shifted by $(\eta/(2\mu^2))\Omega$, which can be seen as analogous to the canonical conjugate momentum in the presence of a gauge field, where $\eta/2\mu^2$ corresponds to the electric charge and Ω to the gauge field. Thus, one seeks for solutions $\psi(\Omega_c, \beta_c)$ of the WDW equation,

$$\hat{H}\psi = 0, \tag{13}$$

which are simultaneous eigenstates of $\hat{A}$. The most general solution of the eigenvalue equation $\hat{A}\psi = a\psi$ with a real is

$$\psi_a(\Omega_c, \beta_c) = \mathcal{R}(\Omega_c)\exp\left[\frac{i\beta_c}{\mu}\left(a - \frac{\eta}{2\mu}\Omega_c\right)\right], \tag{14}$$

where $\mathcal{R}(\Omega_c)$ is an arbitrary C^2 function of Ω_c, which is assumed to be real.

From Eqs. (11), (13), and (14), one gets after some algebraic manipulation

$$\begin{aligned}&\mu^2\mathcal{R}'' - 48\exp\left(-\frac{2\sqrt{3}}{\mu}\Omega_c - 2\sqrt{3}E\Omega_c^2 + \frac{\sqrt{3}\theta a}{\mu\lambda}\right)\mathcal{R}\\ &\quad - \frac{2\eta}{\mu}(\Omega_c + F\Omega_c^3)\mathcal{R} + F^2\Omega_c^4\mathcal{R} + a^2\mathcal{R}\\ &\quad + \left(\frac{\eta^2}{\mu^2} + 2aF\right)\Omega_c^2\mathcal{R} = 0.\end{aligned} \tag{15}$$

The dependence on β_c has completely disappeared and one is effectively left with an ordinary differential equation for $\mathcal{R}(\Omega_c)$. Through the substitution, $\Omega_c = \mu z$, $d^2/d\Omega_c^2 = \frac{1}{\mu^2}\frac{d^2}{dz^2}$ and $\mathcal{R}(\Omega_c(z)) := \phi_a(z)$, one obtains a second-order linear differential equation, which is a Schrödinger-like equation

$$-\phi_a''(z) + V(z)\phi_a(z) = 0, \tag{16}$$

where the potential function, $V(z)$, reads

$$\begin{aligned}V(z) &= -(\eta z - a)^2 - F^2\mu^4 z^4 - 2F\mu^2(\eta z - a)z^2\\ &\quad + 48\exp\left(-2\sqrt{3}z - 2\sqrt{3}\mu^2 E z^2 + \frac{\sqrt{3}\theta a}{\mu\lambda}\right).\end{aligned} \tag{17}$$

Equation (16) depends explicitly on the noncommutative parameters θ, η, ϵ and the eigenvalue a.

Notice that $E > 0$. One concludes from Eq. (17) that asymptotically ($z \rightarrow \infty$), the potential function is dominated by the term

$$V(z) \sim -F^2\mu^4 z^4, \tag{18}$$

which leads to $L^2(\mathrm{IR})$ solutions of the Schrödinger equation (16). Indeed, the Hamiltonian $H = -\frac{\partial^2}{\partial z^2} - F^2\mu^4 z^4$ has a continuous and real spectrum [12] in L^2, and the eigenfunction corresponding to the eigenvalue zero and eigenvalue a of $\hat{A}$ has the asymptotic form

$$\phi_a(z) \sim \frac{1}{z}\exp\left[\pm i\frac{F\mu^2}{3}z^3\right]. \tag{19}$$

Consequently, a generic solution of the WDW equation can be written as

$$\psi(\Omega_c, \beta_c) = \int C(a)\psi_a(\Omega_c, \beta_c)da, \tag{20}$$

where $C(a)$ are arbitrary complex constants and $\psi_a(\Omega_c, \beta_c)$ is of the form

$$\psi_a(\Omega_c, \beta_c) = B\phi_a\left(\frac{\Omega_c}{\mu}\right)\exp\left[\frac{i\beta_c}{\mu}\left(a - \frac{\eta}{2\mu}\Omega_c\right)\right]. \quad (21)$$

Here $\phi_a(\Omega/\mu)$ is a the solution of Eq. (16) and B is a normalization constant such that

$$\int |\psi_a(\Omega_c, \beta_c)|^2 d\Omega_c = B^2 \int \left|\phi_a\left(\frac{\Omega_c}{\mu}\right)\right|^2 d\Omega_c = 1. \quad (22)$$

Given that the wave function is oscillatory in β_c, it is natural to fix a constant β_c hypersurface. This suggests that a measure $d\xi = \delta(\beta - \beta_c)d\beta d\Omega_c$ should be considered for the definition of the inner product in $L^2(\mathrm{IR}^2)$ and for the computation of probabilities. For the wave function (20) one then finds

$$\|\psi\|_{L^2(\mathrm{IR}^2,d\xi)} = \left(\int \psi(\Omega_c, \beta)\psi^*(\Omega_c, \beta)d\xi\right)^{1/2} = \left(\int C(a)C^*(a')\left(\int \psi_a(\Omega_c, \beta_c)\psi^*_{a'}(\Omega_c, \beta_c)d\Omega_c\right)dada'\right)^{1/2}, \quad (23)$$

and since the wave functions $\psi_a(\Omega_c, \beta_c)$ are solutions of a hyperbolic-type equation, in general, they are not orthogonal to each other, and hence one uses the Cauchy-Schwartz inequality to compute the $L^2(\mathrm{IR}^2, d\xi)$ inner product between $\psi_a(\Omega_c, \beta_c)$ and $\psi^*_{a'}(\Omega_c, \beta_c)$. It follows that

$$\|\psi\|_{L^2(\mathrm{IR}^2,d\xi)} \leq \int |C(a)| \|\psi_a(\Omega_c, \beta_c)\|_{L^2(\mathrm{IR}^2,d\xi)} da = \int |C(a)| da, \quad (24)$$

where, in the last step, the relation (22) was used. One concludes that for $C(a) \in L^1(\mathrm{IR})$ the wave function (20) belongs to $L^2(\mathrm{IR}^2, d\xi)$; i.e. it is squared integrable on constant β_c hypersurfaces. Hence, the BH probability $P(r = 0, t = 0)$ at the singularity can now be calculated:

$$P(r = 0, t = 0) = \lim_{\tilde{\Omega}_c, \beta_c \to +\infty} \int_{\tilde{\Omega}_c}^{+\infty} \int_{-\infty}^{+\infty} |\psi(\Omega_c, \beta)|^2 d\xi = \lim_{\tilde{\Omega}_c, \beta_c \to +\infty} \int_{\tilde{\Omega}_c}^{+\infty} |\psi(\Omega_c, \beta_c)|^2 d\Omega_c = 0, \quad (25)$$

where, in the last step, $\psi(\Omega_c, \beta_c) \in L^2(\mathrm{IR}^2, d\xi)$ has been used and so the integral in Ω_c vanishes, independently of the value of β_c.

III. CONCLUSIONS

In this work a KS cosmological model for the interior of a Schwarzschild BH is considered. It is shown that a noncanonical form of the phase-space noncommutativity leads to a natural factorization of the solutions of the WDW equation into an oscillatory function of the scale factor β_c and a square integrable function of Ω_c. This is the key property of the model. The fact that the general solution of the WDW equation is square integrable on the constant β_c hypersurfaces makes it possible to use the standard rules of quantum mechanics to compute the probability of finding the BH at a specific configuration (this computation is, of course, meaningless for the case of nonsquare integrable wave functions, like the ones obtained in the model studied in Ref. [6]). Using asymptotic arguments, the probability of finding the BH in the vicinity of the Schwarzschild singularity was then shown to vanish. This result seems to be quite robust, as the asymptotic property upon which it is based is shared by all the solutions of the WDW equation, and is thus valid independently of the particular wave function.

This is the most one can expect from the quantum approach to the Schwarzschild BH. One does not have a metric, but only a wave function and the associated probability distribution for a set of possible metrics. In the standard approach, the metric is obtained from the wave function $\psi(\Omega_c, \beta_c)$ through semiclassical arguments (see [13] and references therein). This yields semiclassical corrections to the KS metric (3). But from the above discussion, one can conclude that, independently of the particular wave function $\psi(\Omega_c, \beta_c)$ that solves the WDW equation, the semiclassical KS metric will not display the Schwarzschild singularity.

The encountered regularization of the BH singularity is a relevant novel result and relies on several steps. One first approaches the Schwarzschild BH through a map to the KS cosmological model. Noncanonical phase-space noncommutative relations are then imposed for the KS scale factors and the respective canonical conjugate momenta. The identification of a constant of motion of the associated classical problem leads to the operator $\hat{A}$, which commutes with the Hamiltonian of the system. Through the eigenvalue equation of $\hat{A}$ it is possible to reduce the problem of determining the solutions of the WDW equation to the one of solving a Schrödinger-like ordinary second-order differential equation, whose potential is asymptotically dominated by a quartic term. The corresponding solutions of the Hamiltonian operator are L^2 and their asymptotic behavior admits a suitable probability definition at the singularity, which is shown to be vanishing.

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