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Leading effect of *CP* violation with four generations

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In the standard model with a fourth generation of quarks, we study the relation between the Jarlskog invariants and the triangle areas in the 4×4 Cabibbo-Kobayashi-Maskawa matrix. To identify the leading effects that may probe the *CP* violation in processes involving quarks, we invoke small mass and small angle expansions, and show that these leading effects are enhanced considerably compared to the three-generation case by the large masses of fourth-generation quarks. We discuss the leading effect in several cases, in particular, the possibility of large *CP* violation in $b \to s$ processes, which echoes the heightened recent interest because of experimental hints.

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I. INTRODUCTION AND MOTIVATION

The experimental discovery [1] of *CP* violation (CPV) in 1964 came as a surprise, but it provided the clue that led Sakharov to suggest [2] the three conditions that need to be satisfied to explain a long standing and profound puzzle: the disappearance of antimatter from the very early Universe. It was Kobayashi and Maskawa (KM) who proposed [3], in 1973, that CPV can arise from charged current weak interactions, if there is a third generation of quarks. This proposal became part of the standard model (SM). The 3×3 quark mixing matrix can describe all flavor physics measurements to date, the crowning glory being the measurement of the fundamental CPV phase in modes such as $B^0 \to J/\psi K_S$ by the B factory experiments [4]. However, even the Nobel committee noted [5] that the KM phase is insufficient for the Sakharov conditions, typically by a factor of 10^{-10} or worse.

The numerics for the 10^{-10} factor can be most easily seen by a dimensional analysis of the so-called Jarlskog invariant CPV measure [6,7] of the three-generation standard model (SM3),

$$J = (m_t^2 - m_u^2)(m_t^2 - m_c^2)(m_c^2 - m_u^2)(m_b^2 - m_d^2)(m_b^2 - m_s^2) \times (m_s^2 - m_d^2)A, \quad (1)$$

and comparing with the baryon-to-photon ratio n_B/n_γ of our Universe, which is of order 10^{-9}. In Eq. (1), $A \simeq 3 \times 10^{-5}$ [4] is the area of, e.g. the triangle formed by the three sides of the unitarity relation $V_{ud}^* V_{ub} + V_{cd}^* V_{cb} + V_{td}^* V_{tb} = 0$ of SM3. Note that all possible triangle areas have the same value in SM3. Since J has 12 mass dimensions, normalizing by say v, the vacuum expectation value of electroweak symmetry breaking, or more trivially the electroweak phase transition temperature $T_{\rm EW} \simeq 100$ GeV, one finds $J/T_{\rm EW}^{12} \lesssim 10^{-20}$ falls short of n_B/n_γ by at least 10^{-10}.

The main suppression factor of J comes mainly from the small masses, $m_s^2 m_c^2 m_b^4 (/T_{\rm EW}^8)$, rather than from A. Noting this, one of us had suggested [8] that, *if one had an extra generation of quarks*, i.e. a four-generation standard model (SM4), then an analog to J in SM3,

$$J_{(2,3,4)}^{sb} = (m_{t'}^2 - m_c^2)(m_{t'}^2 - m_t^2)(m_t^2 - m_c^2)(m_{b'}^2 - m_s^2) \times (m_{b'}^2 - m_b^2)(m_b^2 - m_s^2)A_{234}^{sb}, \quad (2)$$

would be enhanced by ~15 orders of magnitude with respect to J. The staggering enhancement is brought about by the heavy t', b' quark masses, which are taken to be in the range of 300 to 600 GeV. A phenomenological analysis [9] for $A_{234}^{sb} = {\rm Im}[(V_{ts}V_{tb}^*)^* V_{t's}V_{t'b}^*]$ has been taken into account. It was further argued [8] that the proximity to the $m_d \cong m_s \cong 0$ degeneracy limit [7] on the scale v implies an almost three-generation world involving 2-3-4 generation quarks, in which Eq. (2) is indeed the leading term. Unlike the case for SM3, this large enhancement likely allows SM4 to provide sufficient CPV for the matter or baryon asymmetry of the Universe, which provides strong support for the possible existence of a fourth generation. The scenario can be directly searched for quite definitely [10] at the LHC. In fact, because of recent experimental activities at the Tevatron, be it direct search for the t' [11] or b' [12] quarks, or CPV studies in $B_s \to J/\psi \phi$ [13,14] or the recent hint of dimuon asymmetry [15], interest in the fourth generation has been steadily growing [16,17]. Our interest here, however, is more fundamental.

It may still be questioned whether the analogy of Eq. (2) and (1) between SM4 and SM3 covers the whole truth. Indeed, the expression of invariants probing the *CP* nonconservation becomes much more complicated when one adds a fourth generation, and one should inspect all the invariant quantities in SM4 more carefully. In the following section, we first review the various discussions of necessary and sufficient conditions for CPV. We then invoke the "natural ordering"—the apparent hierarchy of mass-mixing parameters in the quark sector—to make a small mass expansion. The fact that mixing angles involving the fourth generation cannot be large is less useful, since the pattern of mixing angles (e.g. $|V_{t's}|$ vs. $|V_{t'b}|$) is less clear at the moment, precisely because of the recent hints for possibly large CPV effects in $b \to s$ transitions. However, we are able to identify, from the phenomenological indication that $b \to d$ transitions appear consistent with SM3, the condition that simplifies the Jarlskog invariants further, and confirm that, in our world, Eq. (2) is indeed (close to) the leading effect for CPV. In the above process, we are also able to find the next-to-leading terms. We offer some discussion on the approximations made, before giving our conclusion. More tedious algebra and a discussion on the relations between triangle areas are given in the Appendices.

II. CONDITIONS FOR *CP* CONSERVATION

Many studies have been made on the necessary and sufficient conditions for *CP* conservation with three and four (or more) generations. In SM3, which is a very special case, we have only one condition for *CP* conservation:

$$\begin{aligned}\tfrac{1}{6}\,\mathrm{Im}\,\mathrm{tr}[S, S']^3 &= -\mathrm{Im}\,\mathrm{tr}(S^2 S' S S'^2) = J(1,2,3)\\ &= v(1,2,3)v'(1,2,3)A = 0,\end{aligned} \tag{3}$$

where $S^{(\prime)}$ is the up(down)-type Hermitian squared mass matrix, defined as $S = MM^\dagger$ with M the quark mass matrix. All the primed symbols hereafter denote down-type quantities. In Eq. (3), v is the Vandermonde determinant of squared masses,

$$v(\alpha, \beta, \gamma) = (m_\alpha^2 - m_\beta^2)(m_\beta^2 - m_\gamma^2)(m_\gamma^2 - m_\alpha^2), \tag{4}$$

and $J(1,2,3)$ [which is identical to J in Eq. (1)] is the Jarlskog invariant,

$$J(\alpha, \beta, \gamma) = v(\alpha, \beta, \gamma)\mathrm{Im}\,\mathrm{tr}(P_\alpha S' P_\beta S' P_\gamma S'), \tag{5}$$

where P_α is the projection operator for the indicated flavor, $SP_\alpha = m_\alpha^2 P_\alpha$.

That the number of conditions for *CP* conservation in SM3 is exactly one reflects the unique CPV phase in the quark mixing matrix. With fourth generations, we have two more Cabibbo-Kobayashi-Maskawa (CKM) phases, hence more conditions are needed for *CP* conservation. The number of conditions may, however, be larger than three due to the complexity of, and interdependency between, invariants. For instance, Botella and Chau [18] showed that there are nine independent triangle areas in SM4, rather than the single area in SM3, and *CP* is conserved *if and only if* all nine areas vanish. Note that these triangles are not "unitary triangles," since in SM4 the unitarity relations give quadrangles. However, every two sides of these quadrangles still form triangles, and we refer to these triangles as "CKM triangles." Namely,

$$A^{u_1 u_2}_{d_1 d_2} \equiv \mathrm{Im}[(V_{u_1 d_1} V^*_{u_1 d_2})^* V_{u_2 d_1} V^*_{u_2 d_2}] = 0, \quad \forall\ u_1 \neq u_2,\quad d_1 \neq d_2 \tag{6}$$

is defined quite in the same way as the conventions in SM3. One can see the number of total possible triangles is $(C_2^4)^2 = (4!/2!2!)^2 = 36$, but the unitarity conditions reduce this number to $(C_2^3)^2 = (3!/2!)^2 = 9$ (the corresponding numbers for SM3 are therefore 9 and 1, respectively). These CKM triangles, though rephasing invariant, may not be fully independent of each other (see Appendix B for a discussion). Furthermore, quark masses do not appear explicitly, although we know that CPV would vanish under certain mass degeneracy conditions.

Equation (3), which gives the Jarlskog invariant CPV measure for SM3 as in Eq. (1), is of course invariant under any change of flavor basis. Extending to SM4, Jarlskog showed [7] that it is the sum over four Jarlskog invariants of the form in Eq. (5), or three-cycles, that is

$$\begin{aligned}-\mathrm{Im}\,\mathrm{tr}(S^2 S' S S'^2) = J(2,3,4) + J(1,3,4) + J(1,2,4)\\ + J(1,2,3).\end{aligned} \tag{7}$$

The Jarlskog proposal is that one would have *CP* conservation *if and only if* all four invariants vanish. This proposal shows a transparent analogy between SM3 and SM4.

There are other basis-independent approaches, however, to the conditions for *CP* conservation. Gronau, Kfir, and Loewy (GKL) introduced [19] five more invariants in addition to Eq. (3), and proposed that *CP* is conserved in SM4 *if and only if* all six invariants vanish,

$$\begin{aligned}\mathrm{Im}\,\mathrm{tr}(S^2 S' S S'^2) &= \mathrm{Im}\,\mathrm{tr}(S^2 S' S S'^3) = \mathrm{Im}\,\mathrm{tr}(S^2 S'^2 S S'^3)\\ &= \mathrm{Im}\,\mathrm{tr}(S' S S'^2 S S'^3) = \mathrm{Im}\,\mathrm{tr}(S^3 S' S S'^2)\\ &= \mathrm{Im}\,\mathrm{tr}(S^3 S' S S'^3) = 0.\end{aligned} \tag{8}$$

Whether these two sets of conditions are really sufficient for *CP* conservation has been debated [7,20,21]. What is certain is that, if *CP* is violated, some of these quantities would be nonzero. Both sides do agree that *the two sets of conditions are equivalent if there is no vanishing element in the quark mixing matrix V in SM4.*

The pragmatic question is how these quantities appear in process amplitudes that give rise to *observable* measures of *CP* violation. In SM3, we know that the Jarlskog invariant enters various CPV measures. It further encodes the notion that, if any two like-charge quarks are degenerate in mass, or if the CKM triangle area vanishes, there would be no *CP* violation. Taking this as a hint, it is clear that the CKM triangle areas should enter various CPV measures, together with some mass difference factors, so the GKL and Jarlskog invariants should play a role in these measures. The generation labels of the CKM triangles in the measure should tell us what the related processes are for the search of *CP* violation. Note that the GKL and Jarlskog invariants are basis independent, and thus more likely to appear in physical quantities.

The invariants in Eqs. (5) and (8) are, however, rather complicated, and it is not apparent how the fourth generation effect on *CP* violation emerges. In contrast, the suggested leading effect of Eq. (2) is much more intuitive, and rather similar to the SM3 result of Eq. (1), but with the emphasis placed clearly on $b \to s$ transitions. We should try to express the invariants, whether of the GKL [19] or Jarlskog [7] kind, in terms of mass difference factors and CKM triangle areas, just like in SM3, and then identify the leading terms by considering the physical limits, such as physical quark masses. An extensive work that expands the GKL invariants into the nine CKM triangles [18] has been done in Ref. [22], but no comparison between different terms were made. For example, one has,

$$\mathrm{Im}\,\mathrm{tr}(S^2 S' S S'^2) = \sum v(1, i, j) v'(1, a, b) A^{ij}_{ab}, \qquad (9)$$

where the summation is over (i, j) and $(a, b) \in \{(2, 3), (3, 4), (4, 2)\}$, with similar expansions for the other invariants in Eq. (8). Comparing different GKL invariants would be less meaningful, however, as there are several different mass dimensions. Even if they are shown to enter some CPV measure, it would be difficult to tell whether the fourth generation enhances the *CP* violation or not. On the other hand, all the Jarlskog invariants have the same dimension, and the general form of Eq. (5) is maintained as one extends from SM3 to SM4.

In the following, we will express the Jarlskog invariants in terms of the nine CKM triangles, then identify the leading terms in these invariants, by adopting proper physical limits. We find that the suggestion of Eq. (2) is indeed the leading term, but various next-to-leading terms are only smaller by roughly a factor of 10.

III. SMALL MASS AND ANGLE EXPANSION OF JARLSKOG INVARIANTS

A. Jarlskog invariants and CKM triangles

Let us express the Jarlskog invariants of Eq. (5) in terms of the nine CKM triangles of Eq. (6). For convenience, we choose to decompose the down-type Jarlskog invariants into CKM triangle areas. The up-type relations can be obtained analogously. Since the invariants can be evaluated in any basis, we are allowed to choose S' to be diagonal and write $\hat{S} = V^\dagger M V$, where V is the familiar quark mixing matrix. The Jarlskog invariants become

$$\begin{aligned} J'(a, b, c) &= -v'(a, b, c)\mathrm{Im}\,\mathrm{tr}(P'_a S P'_b S P'_c S) \\ &= -v'(a, b, c)\mathrm{Im}(\hat{S}_{ab}\hat{S}_{bc}\hat{S}_{ca}) \\ &= v'(a, b, c)\sum_{i,j,k=1}^{4} m_i^2 m_j^2 m_k^2 \\ &\quad \times \mathrm{Im}(V_{ia}V^*_{ib}V_{jb}V^*_{jc}V_{kc}V^*_{ka}), \end{aligned} \qquad (10)$$

where the indices a, b, c are chosen all differently within $\{1, 2, 3, 4\}$, but not summed. With an eye toward Eq. (2), we use the unitarity relation

$$V_{1a}V^*_{1b} = -V_{2a}V^*_{2b} - V_{3a}V^*_{3b} - V_{4a}V^*_{4b} + \delta_{ab}, \qquad (11)$$

to eliminate all 1's for the indices that are summed over. We factor out any real factor that appears in $\mathrm{Im}(\cdots)$ of Eq. (10), and the remaining four CKM matrix elements then form a CKM triangle area.

After some algebra, which is rendered to Appendix A, the four down-type Jarlskog invariants are reduced to

$$\begin{aligned} J'(2, 3, 4) &= -v'(2, 3, 4)\Lambda_{234}, \\ J'(1, 3, 4) &= v'(1, 3, 4)[\Lambda_{234} - v(1, 3, 4)A^{34}_{34} \\ &\quad - v(1, 2, 3)A^{23}_{34} - v(1, 4, 2)A^{42}_{34}], \\ J'(1, 2, 3) &= v'(1, 2, 3)[\Lambda_{234} - v(1, 3, 4)A^{34}_{23} \\ &\quad - v(1, 2, 3)A^{23}_{23} - v(1, 4, 2)A^{42}_{23}], \\ J'(1, 4, 2) &= v'(1, 4, 2)[\Lambda_{234} - v(1, 3, 4)A^{34}_{42} \\ &\quad - v(1, 2, 3)A^{23}_{42} - v(1, 4, 2)A^{42}_{42}], \end{aligned} \qquad (12)$$

where

$$\begin{aligned} \Lambda_{234} = {}& (m_2^2 - m_1^2)(m_3^2 - m_1^2)[(m_4^2 - m_2^2)(|V_{22}|^2 A^{23}_{34} + |V_{23}|^2 A^{23}_{42} + |V_{24}|^2 A^{23}_{23}) \\ & - (m_4^2 - m_3^2)(|V_{32}|^2 A^{23}_{34} + |V_{33}|^2 A^{23}_{42} + |V_{34}|^2 A^{23}_{23})] \\ & + (m_3^2 - m_1^2)(m_4^2 - m_1^2)[(m_2^2 - m_3^2)(|V_{32}|^2 A^{34}_{34} + |V_{33}|^2 A^{34}_{42} + |V_{34}|^2 A^{34}_{23}) \\ & - (m_2^2 - m_4^2)(|V_{42}|^2 A^{34}_{34} + |V_{43}|^2 A^{34}_{42} + |V_{44}|^2 A^{34}_{23})] \\ & + (m_4^2 - m_1^2)(m_2^2 - m_1^2)[(m_3^2 - m_4^2)(|V_{42}|^2 A^{42}_{34} + |V_{43}|^2 A^{42}_{42} + |V_{44}|^2 A^{42}_{23}) \\ & - (m_3^2 - m_2^2)(|V_{22}|^2 A^{42}_{34} + |V_{23}|^2 A^{42}_{42} + |V_{24}|^2 A^{42}_{23})]. \end{aligned} \qquad (13)$$

This rather compact form, though still quite complicated, should be compared with the many terms of Ref. [22], obtained by expanding the six GKL invariants [19] of Eq. (8) in terms of the nine CKM triangle areas [18] of Eq. (6). We note again that the GKL invariants are of several different mass dimensions, and it is not easy to compare the relative importance of the numerous possible terms. In contrast, all four Jarlskog invariants have the same mass dimension, hence are more readily compared with one another. As remarked already, for SM4 with no vanishing CKM mixing matrix elements, GKL and Jarslkog approaches are equivalent, but the latter is clearly more convenient for our purpose.

B. Small mass and angle expansions

One of our goals is to identify the leading effects in the Jarlskog invariants. We now depart from generality by noting the fact of a clear hierarchy of physical quark masses in nature, namely $m_{b'}^2 \gg m_b^2 \gg m_s^2 \gg m_d^2$. This implies that the last two factors in Eq. (12) are much more suppressed than the first two. Likewise, the up-type hierarchy $m_{t'}^2 > m_t^2 \gg m_c^2 \gg m_u^2$ further suppresses the first and third terms in Λ_{234}, as well as the terms with factors $\upsilon(1, 2, 3)$ and $\upsilon(1, 2, 4)$ in $J'(1, 3, 4)$. Dropping these m_s^2 and m_c^2 suppressed terms, Eqs. (12) and (13) become

$$J'(2,3,4) \simeq -(m_{b'}^2 - m_s^2)(m_{b'}^2 - m_b^2)(m_b^2 - m_s^2)\Lambda_{234},$$

$$J'(1,3,4) \simeq (m_{b'}^2 - m_d^2)(m_{b'}^2 - m_b^2)(m_b^2 - m_d^2) \times [\Lambda_{234} - (m_{t'}^2 - m_u^2)(m_{t'}^2 - m_t^2) \times (m_t^2 - m_u^2)A_{bb'}^{tt'}], \tag{14}$$

$$\Lambda_{234} \simeq (m_t^2 - m_u^2)(m_{t'}^2 - m_u^2)[(m_t^2 - m_c^2)(-|V_{ts}|^2 A_{bb'}^{tt'} + |V_{tb}|^2 A_{sb'}^{tt'} - |V_{tb'}|^2 A_{sb}^{tt'}) + (m_{t'}^2 - m_c^2) \times (|V_{t's}|^2 A_{bb'}^{tt'} - |V_{t'b}|^2 A_{sb'}^{tt'} + |V_{t'b'}|^2 A_{sb}^{tt'})], \tag{15}$$

where we have returned all indices to physical labels, and kept the small subtracted masses in the explicit mass differences in these remainder terms, for sake of correspondence with Eq. (2).

As there are still quite a few terms in Eq. (15), we expand further in the strength of $|V_{ij}|^2$. Phenomenologically, we now know [17] that the rotation angles in the CKM matrix are small, i.e. $|V_{ij}|^2 \ll |V_{kk}|^2 \sim 1$, which holds not only in SM3, but seems to extend into SM4 as well. The Cabibbo angle appears to be the largest rotation angle, while $|V_{ts}|$ cannot be much different from the SM3 value of $\simeq 0.04$, and $|V_{t's}|^2$ and $|V_{t'b}|^2$ should be of order 10^{-2} or less [23]. Assuming small rotation angles, we drop the off-diagonal $|V_{ij}|^2$ terms in Λ_{234}, and get

$$J'(2,3,4) \sim -(m_{t'}^2 - m_u^2)(m_t^2 - m_u^2)(m_{b'}^2 - m_s^2) \times (m_{b'}^2 - m_b^2)(m_b^2 - m_s^2)[(m_t^2 - m_c^2)|V_{tb}|^2 A_{sb'}^{tt'} + (m_{t'}^2 - m_c^2)|V_{t'b'}|^2 A_{sb}^{tt'}], \tag{16}$$

$$J'(1,3,4) \sim (m_{t'}^2 - m_u^2)(m_t^2 - m_u^2)(m_{b'}^2 - m_d^2)(m_{b'}^2 - m_b^2) \times (m_b^2 - m_d^2)[(m_t^2 - m_c^2)|V_{tb}|^2 A_{sb'}^{tt'} + (m_{t'}^2 - m_c^2)|V_{t'b'}|^2 A_{sb}^{tt'} - (m_{t'}^2 - m_t^2)A_{bb'}^{tt'}]. \tag{17}$$

If one looks at the order of magnitude of quark masses, Eq. (16) is similar to Eq. (2). But they are not exactly the same: more than one CKM area carry the heaviest mass factor $m_{t'}^4 m_t^2 m_{b'}^4 m_b^2$, and the masses of u, d quarks also enter the expression.

In Eqs. (16) and (17), we have used "$\sim$" rather than "$\simeq$," because we have treated the CKM triangle areas as "free parameters" while dropping the off-diagonal $|V_{ij}|^2$ terms. These areas are characterized not only by the strength of CKM matrix elements, but also their relative phases, which makes clear that these two equations are for illustrative purposes only. Note that we have not assumed further hierarchical structure in the mixing elements. This is because of the possible indication of a large CPV effect involving $b \to s$ transitions; hence, we do not know whether the hierarchy structure of $|V_{ub}|^2 \ll |V_{cb}|^2 \ll |V_{us}|^2 \ll 1$ would extend to elements involving the fourth generation. We remark that, without assuming further structure in the CKM elements, unlike the application of mass hierarchies that lead to Eqs. (14) and (15), had we applied $|V_{ij}|^2 \ll |V_{kk}|^2 \sim 1$ (where $i \neq j$) first to Eqs. (12) and (13), not much simplification would be gained, and there would still be four Jarlskog invariants.

What CKM angle pattern could be noteworthy for further simplifications?

C. Leading effect of Jarlskog invariants

For our purpose of finding the leading effect of CPV, what pattern of small off-diagonal elements in the CKM matrix could provide additional approximate relations between triangle areas as defined in Eq. (6)? Taking note of the three specific triangle areas that enter Eq. (15), we note that the unitarity quadrangle

$$V_{td}V_{t'd}^* + V_{ts}V_{t's}^* + V_{tb}V_{t'b}^* + V_{tb'}V_{t'b'}^* = 0, \tag{18}$$

a special case of our starting Eq. (11), would approach a triangle, if $|V_{td}V_{t'd}^*|$ is small compared to the other terms. If this is the case, then any two of the other three sides would form the same triangle area, i.e.

$$A_{sb}^{tt'} \simeq -A_{sb'}^{tt'} \simeq A_{bb'}^{tt'}, \quad \text{for } |V_{td}V_{t'd}^*| \ll 1, \tag{19}$$

which relates the three CKM triangles appearing in Λ_{234} of Eq. (15). Applying Eq. (19) to Eq. (15), we get

$$\Lambda_{234} \simeq (m_t^2 - m_u^2)(m_{t'}^2 - m_u^2)[-(m_t^2 - m_c^2)(1 - |V_{td}|^2)A_{sb}^{tt'} + (m_{t'}^2 - m_c^2)(1 - |V_{t'd}|^2)A_{sb}^{tt'}] \simeq (m_t^2 - m_u^2)(m_{t'}^2 - m_u^2)(m_{t'}^2 - m_t^2)A_{sb}^{tt'}, \tag{20}$$

where we have assumed the smallness of both $|V_{t'd}|^2$ and $|V_{td}|^2$ in the second step, which is somewhat stronger than what is needed for Eq. (19) to hold. Substituting Eq. (20) into Eq. (14), we obtain

$$J'(2,3,4) \simeq -(m_{b'}^2 - m_s^2)(m_{b'}^2 - m_b^2)(m_b^2 - m_s^2)(m_t^2 - m_u^2) \times (m_{t'}^2 - m_u^2)(m_{t'}^2 - m_t^2)A_{sb}^{tt'}, \tag{21}$$

$$J'(1,3,4) \simeq 0 + \text{subleading terms}. \tag{22}$$

With $A_{234}^{sb} = A_{sb}^{tt'}$, $J'(2,3,4)$ in Eq. (21) is indeed the same as $J_{(2,3,4)}^{sb}$ of Eq. (2), except that m_c^2 is replaced by m_u^2, which makes little difference since m_u^2 and m_c^2 are negligible compared with m_t^2 and $m_{t'}^2$. What seems a little curious is that the starting point of Eq. (18) is more relevant for $t' \to t$ transitions, but it leads to the result in Eq. (21) [through Eq. (19)], which seems more relevant to $b \to s$ transitions.

In Ref. [23], the authors also showed that $A_{sb'}^{tt'}$ and $A_{sb}^{tt'}$ have similar area through a phenomenological study. Such a study started [9] from the hint of new physics CPV in $b \to s$ transitions, while $b \to d$ transitions (including B_d mixing-dependent CPV) mimic SM3, as first noted by Ref. [24]. If we take the approximation in Eq. (19), it implies that $|V_{ts}|$ and $|V_{t's}|$ are stronger than the counterparts involving d, which implies interesting CPV effects in $b \to s$ processes, including in $B_s \to J/\psi\phi$ [9,17,25]. This is precisely where we are finding several experimental hints [13–15]! We have thus clarified the phenomenological link and reasoning behind Eq. (21), which echoes quite well those given in Ref. [8] for Eq. (2), but in a more hand-waving way.

One may question whether our choice to eliminate the index "1" via Eq. (11) can keep its generality, once we introduce the hierarchy of quark masses. To check this, we note that one could apply small mass expansion to Eq. (10) directly and obtain the same results without passing any algebra, or even using Eq. (11). Taking Eq. (2) as a guide, by collecting terms with factor $m_{t'}^4 m_t^2$ in the summation in $J'(2,3,4)$ in Eq. (10), one has

$$\begin{aligned} J'(2,3,4) &\sim m_{t'}^4 m_t^2 m_{b'}^4 m_b^2 [|V_{t'b'}|^2 A_{sb}^{tt'} - |V_{t'b}|^2 A_{sb'}^{tt'} \\ &\quad + |V_{t's}|^2 A_{bb'}^{tt'}], \\ J'(1,3,4) &\sim -m_{t'}^4 m_t^2 m_{b'}^4 m_b^2 [|V_{t'b'}|^2 A_{sb}^{tt'} - |V_{t'b}|^2 A_{sb'}^{tt'} \\ &\quad + (|V_{t's}|^2 - 1) A_{bb'}^{tt'}], \end{aligned} \tag{23}$$

which is exactly what we have in Eqs. (14) and (15) when neglecting any terms of order equal to or smaller than $m_t^2/m_{t'}^2$. If we invoke $|V_{t'd}|$ to be much smaller than the other three elements, and Eq. (19) is satisfied, then we again get a formula for $J'(2,3,4)$ that is in line with Eq. (21), while $J'(1,3,4)$ cancels away as in Eq. (22).

One may then question the utility of the algebra, which constitutes the bulk of the paper. Note that Eq. (23) does not satisfy the requirements for the $t - t'$ degeneracy limit. This can be remedied by collecting the $m_{t'}^2 m_t^4$ terms as well. But if one wishes to explore other subleading effects, then Eq. (23) offers no guidance, and to explore these, one might as well resort to Eqs. (12) and (13). As we will show in the discussion below, it is possible, through the structure of the CKM matrix, that Eq. (21) [hence Eq. (23)] in fact is absent. Thus, Eqs. (12) and (13) offer the general starting point, independent of the quark mass hierarchy. It provided an easy way to evaluate the leading effects with the hierarchy taken into account, and can always be used in discussing special CKM structures. Our formulas show the complete mass factors in front of each CKM triangle area, which all have the form of difference of mass squares. This feature allows us to explore some more general cases, as we will discuss in the next section.

IV. DISCUSSION

At the end of the previous section, we have seen the implications for very small $|V_{td}|$ and $|V_{t'd}|$ and quark mass hierarchy. Only one of the four Jarlskog invariants, $J'(2,3,4)$ of Eq. (21), remains nonzero, while the other three are subleading, and hence could in principle vanish if we have exact $|V_{td}| = |V_{t'd}| = 0$ and $m_u = m_c$. In this case, it would be an effective three-generation world, where the first generation decouples from the other three heavier generations, when taking the extra freedom in $|V_{cd}|$ provided by $u - c$ degeneracy. This seems to be in contrast to the assertion by Jarlskog in Ref. [7] that when three of these invariants vanish exactly, the fourth would also vanish. However, as Jarlskog mentioned, this assertion is not valid in the present case. In fact, this assertion is not valid whenever one generation decouples from the other three generations (zeros in the CKM matrix).

Using the small mass expansion is quite different from taking mass degeneracy limits mathematically. If one has mass degeneracy, extra freedom in the quark mixing matrix must be taken into account. In Ref. [7], the author also treated exact degeneracy differently to avoid possible singularity. Nevertheless, in our real world, we do not have any two quarks with the same mass, so it is reasonable to consider only the smallness of quarks but not degeneracy, though these two ways seem similar physically. It should be further noted that, if one applies $t - t'$ degeneracy, then $J'(2,3,4)$ and $J'(1,3,4)$ do not seem to vanish, which seems paradoxical. This can be traced, however, to Eq. (10), where the mass-squared difference appears in the denominator in defining the projection operators, and the second equality cannot apply in the $t - t'$ degeneracy limit. To address this issue, rather than flipping the definition of primed versus unprimed objects, we could inspect the behavior of $b - b'$ degeneracy limit instead of $t - t'$. One immediately sees that, if one maintains $m_{b'} - m_b > m_s > m_d$ while letting $m_s \to 0$, then taking $b' - b$ degeneracy limit, all four Jarlskog invariants would properly vanish, hence the previous paradox is an artefact of choosing to decompose down-type Jarlskog invariants. But we then see that "$t-t'$ degeneracy" (that mimic true $b-b'$ degeneracy) indeed cannot be applied. We therefore gain an insight that, while massless degeneracy of the first two generations can simultaneously be applied for up- and down-type quarks [because of vanishing mass protection in reaching second equality of Eq. (10)], this is not so for the degeneracy of the massive third and fourth generations.

In the previous section, we considered only the case with small $|V_{td}|$ and $|V_{t'd}|$. Now let us consider more scenarios when some of the elements in the CKM matrix are extremely small, leading to some vanishing triangle areas. First, let us consider the case where the fourth generation is totally decoupled from the first three generations. One then expects an effective three-generation theory, and the CPV effect should be the same as in SM3. Because of the decoupling, all rotation angles which link the fourth generation and lower generations are zero. That is,

$$V_{14} = V_{24} = V_{34} = V_{41} = V_{42} = V_{43} = 0,$$

and it follows that any $A^{u_1u_2}_{d_1d_2}$ that contains 4 in its label is zero. Λ_{234} is also zero because every term in Λ_{234} contains at least one zero factor. The four invariants then become

$$\begin{aligned} &J'(2,3,4) = J'(1,3,4) = J'(1,4,2) = 0, \\ &J'(1,2,3) = -v'(1,2,3)v(1,2,3)A^{23}_{23}, \end{aligned} \tag{24}$$

which is exactly what we have in SM3, and there is no CPV effect induced by the fourth generation, as expected.

But if the fourth generation exists, it is hard to conceive that it decouples from all other generations. Consider the case where the fourth generation decouples from the first two generations, that is,

$$V_{14} = V_{24} = V_{41} = V_{42} = 0,$$

then the only nonvanishing triangle areas in Eq. (12) are A^{23}_{23} and A^{34}_{34}. One can show from Eqs. (B1) and (B2) that one must have either $A^{23}_{23} = A^{34}_{34} = 0$ or $V_{34} = V_{43} = 0$. The first solution means there is no CPV at all, and the third generation also decouples from the first two generations, which contradicts experimental observation. The second solution, on the other hand, means that the fourth generation decouples also from the third generation, which is the previous scenario we have just discussed. This result shows that if the fourth generation does exist, it must either couple with at least two lower generations, or must fully decouple from all three lower generations.

However, even if the fourth generation is present and couples to all other generations, it is still possible that we have only one CPV phase. Consider, for instance, having b' decoupled from u and c, which gives $V_{14} = V_{24} = 0$. Then any triangle areas $A^{u_1u_2}_{d_1d_2}$ with a "4" in lower indices and an "1" or a "2" in upper indices will vanish. In addition, there are other triangle areas that would also vanish by using the unitarity condition,

$$A^{34}_{14,42,34} = -A^{14}_{14,42,34} - A^{24}_{14,42,34} = 0. \tag{25}$$

The only nonvanishing triangle areas used in Eq. (12) are A^{23}_{23}, A^{34}_{23}, and A^{42}_{23}. But Eq. (B1) with $(u_1, u_2, u_3, u_4) = (1, 2, 3, 4)$ and $(1, 3, 2, 4)$ gives

$$(|V_{34}|^2 - |V_{44}|^2)A^{34}_{23} = 0, \qquad |V_{34}|^2 A^{23}_{23} = -|V_{44}|^2 A^{42}_{23}. \tag{26}$$

Provided that $|V_{34}|^2 \neq 0$ and $|V_{34}|^2 \neq |V_{44}|^2$, there is only one degree of freedom in triangle areas, hence only one CPV phase. All the Jarlskog invariants are then proportional to this area, and the leading effect is

$$\begin{aligned} J'(2,3,4) \sim &-(m^2_{t'} - m^2_t)^2(m^2_c - m^2_u)(m^2_{b'} - m^2_s) \\ &\times (m^2_{b'} - m^2_b)(m^2_b - m^2_s)|V_{tb'}|^2 A^{ct}_{sb}, \end{aligned} \tag{27}$$

$$\begin{aligned} J'(1,3,4) \sim &(m^2_{t'} - m^2_t)^2(m^2_c - m^2_u)(m^2_{b'} - m^2_d)(m^2_{b'} - m^2_b) \\ &\times (m^2_b - m^2_d)|V_{tb'}|^2 A^{ct}_{sb}, \end{aligned} \tag{28}$$

which is smaller than Eq. (16) due to the factor m^2_c, but it is still enhanced by $\sim 10^{10}$ when compared with Eq. (1). One sees that if the fourth generation does not totally decouple from the other three, it will leave its fingerprint on some CPV process(es).

For other possible scenarios, one can follow the same recipe we used. First, the dependence of triangle areas are determined by the relations in Eqs. (B1) and (B2). Then, inserting these relations into Eqs. (12) and (13), one can identify the leading effect in the corresponding scenario. Finally, one should note that it is very unlikely to have any exact zero in the quark mixing matrix from theoretical perspective, and certainly not experimentally either. These cases allow us to see the asymptotic behavior of the leading effect.

V. CONCLUSION

The formula in Eq. (2), as if involving just 2-3-4 generations in a four-generation world, would be enhanced above the three-generation Jarlskog invariant J of Eq. (1) by an astounding 10^{15} or so. This is because the dependence on the small mass-squared differences between the two lightest generations get replaced by heavier masses at the weak scale. The purpose of our study is to check to what extent Eq. (2) is the leading CPV effect in the four-generation standard model.

We chose the more convenient starting point of Jarlskog's extension to four invariants in SM4. As we always maintain physical finite values for quark masses and CKM mixing elements, this is equivalent in SM4 to the more complicated Gronau, Kfir, and Loewy approach. Through algebraic manipulations, the more tedious of which are relegated to the Appendices, we arrive at the general results of Eq. (12), which depend on an algebraic function Λ_{234} defined in Eq. (13). With full generality, this does not offer too much insight. We then invoked the hierarchy of physical quark masses, i.e. the aforementioned smallness of the first two generations masses on the weak scale, to eliminate two Jarlskog invariants, $J'(1, 2, 3)$ and $J'(1, 4, 2)$, as subleading, as well as simplify Λ_{234}. Invoking the empirical condition of small rotations, that off-diagonal elements in V are not larger than $|V_{us}|$, does not simplify further the result of Eq. (14). One needs further knowledge of patterns of CKM elements (analogous to the mass hierarchy). Because of recent hints in $b \to s$ processes, this cannot yet be concluded. Instead, we found the relation of Eq. (19) between triangle areas would hold, given that $b \to d$ transitions seem to conform with the three-generation standard model. The resulting Eq. (21) largely confirms the suggestion of Eq. (2). In fact, one could have taken a much more efficient approach, for the purpose of identifying the leading effect, by making small mass expansion from the outset in Eq. (10), and arrive at Eq. (23). This retains all features of the proposed $J^{sb}_{2,3,4}$ in Eq. (2), keeping to $m_t^2/m_{t'}^2$ order, as well as order of CKM elements, which could be as large as 0.1. We therefore see that though $J^{sb}_{2,3,4}$ in Eq. (2) does seem to be the leading term in the presence of quark mass hierarchies and small rotation angles—which is our world—there should be a myriad of subleading terms that are perhaps only 10 times smaller.

In the course of our study, we also uncovered the apparent phenomenological condition for $J^{sb}_{2,3,4}$ in Eq. (2) to be the leading term. Current data suggest that CPV in $b \to s$ transitions, notably for mixing-dependent CPV in $B_s \to J/\psi\phi$, could be sizable, despite the B factory confirmation of consistency with a three-generation source for $b \to d$ transitions (notably for mixing-dependent CPV in $B_d \to J/\psi K_S$). Thus, V_{td} and $V_{t'd}$ seem subdued compared with V_{ts} and $V_{t's}$ in strength, respectively. In this case, we were able to derive Eq. (21) which is extremely close to Eq. (2), except for very minor differences. We thus conclude that, in general the claim of a large enhancement by fourth-generation masses is true, although there would be several terms comparable to $J^{sb}_{2,3,4}$ in Eq. (2). If, however, we do discover sizable CPV effect in $B_s \to J/\psi\phi$ that is much enhanced over SM3 expectations, then indeed $J^{sb}_{2,3,4}$ of Eq. (2), or $J'(2, 3, 4)$ of Eq. (21), is the single leading term. But there would still be subleading terms that could be just an order of magnitude less in strength, depending on the strength of associated CKM elements.

APPENDIX A: SOME ALGEBRA

We start from Eq. (10)

$$J'(a, b, c) = v'(a, b, c) \sum_{i,j,k=1}^{4} m_i^2 m_j^2 m_k^2 \times \mathrm{Im}(V_{ia}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{ka}^*). \quad (A1)$$

By replacing every term that contains m_1^2 in the above summation by the unitarity condition, the summation becomes

$$\sum_{i,j,k=2}^{4} (m_i^2 - m_1^2)(m_j^2 - m_1^2)(m_k^2 - m_1^2) \times \mathrm{Im}(V_{ia}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{ka}^*), \quad (A2)$$

where now the sum is overall possible i, j, k in the set $\{2, 3, 4\}$. Note that since a, b, c are all different, there exists no term like $\delta_{aa} = 1$ in Eq. (A2).

We can apply a similar trick to the down-type indices of V. Consider the case $a = 1$ and b, c are chosen differently from $\{2, 3, 4\}$

$$\begin{aligned}
\mathrm{Im}(V_{i1}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{k1}^*) &= -\mathrm{Im}(V_{i2}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{k2}^*) - \mathrm{Im}(V_{i3}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{k3}^*) - \mathrm{Im}(V_{i4}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{k4}^*) \\
&\quad + \mathrm{Im}(V_{ib}^* V_{jb}V_{jc}^* V_{kc})\delta_{ik} \\
&= -\mathrm{Im}(V_{ib}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{kb}^*) - \mathrm{Im}(V_{ic}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{kc}^*) - \mathrm{Im}(V_{id}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{kd}^*) \\
&\quad + \mathrm{Im}(V_{ib}^* V_{jb}V_{jc}^* V_{kc})\delta_{ik} \\
&= -\mathrm{Im}(V_{id}V_{ib}^* V_{jb}V_{jc}^* V_{kc}V_{kd}^*) + \mathrm{Im}(V_{ib}^* V_{jb}V_{jc}^* V_{kc})\delta_{ik} + |V_{ib}|^2 A_{bc}^{jk} - |V_{kc}|^2 A_{bc}^{ij}, \quad (A3)
\end{aligned}$$

where d is taken to be different from a, b, c, and the second equality follows from replacing 2, 3, 4 by b, c, d, by reordering the first three terms. Real factors are taken out in the third equality, and we also used

$$A_{bc}^{ij} = \mathrm{Im}[(V_{ib}V_{ic}^*)^* V_{jb}V_{jc}^*]. \quad (A4)$$

Substituting Eqs. (A2) and (A3) back into Eq. (A1), we have

$$J'(1,b,c) = v'(1,b,c)\Big\{\sum_{i,j=2}^{4}(m_i^2-m_1^2)^2(m_j^2-m_1^2)A_{bc}^{ij} - \sum_{i,j,k=2}^{4}(m_i^2-m_1^2)(m_j^2-m_1^2)(m_k^2-m_1^2)\mathrm{Im}(V_{id}V_{ib}^*V_{jb}V_{jc}^*V_{kc}V_{kd}^*) + \sum_{i,j,k=2}^{4}(m_i^2-m_1^2)(m_j^2-m_1^2)(m_k^2-m_1^2)(|V_{ib}|^2A_{bc}^{jk}-|V_{kc}|^2A_{bc}^{ij})\Big\}. \tag{A5}$$

The third summation would vanish, since the upper indices of A are antisymmetric; hence, each component will cancel one another.

Define now

$$\Lambda_{dbc} = -\sum_{i,j,k=2}^{4}(m_i^2-m_1^2)(m_j^2-m_1^2)(m_k^2-m_1^2) \times \mathrm{Im}(V_{id}V_{ib}^*V_{jb}V_{jc}^*V_{kc}V_{kd}^*). \tag{A6}$$

We note that the indices of A and Λ are antisymmetric, and hence we have

$$(m_i^2-m_1^2)^2(m_j^2-m_1^2)A_{bc}^{ij} + (m_j^2-m_1^2)^2(m_i^2-m_1^2)A_{bc}^{ji} = -v(1,i,j)A_{bc}^{ij}. \tag{A7}$$

Thus, the four Jarlskog invariants in SM4 can be written as

$$\begin{aligned} J'(2,3,4) &= -v'(2,3,4)\Lambda_{234},\\ J'(1,3,4) &= v'(1,3,4)[\Lambda_{234} - v(1,3,4)A_{34}^{34} - v(1,2,3)A_{34}^{23} - v(1,4,2)A_{34}^{42}],\\ J'(1,2,3) &= v'(1,2,3)[\Lambda_{234} - v(1,3,4)A_{23}^{34} - v(1,2,3)A_{23}^{23} - v(1,4,2)A_{23}^{42}],\\ J'(1,4,2) &= v'(1,4,2)[\Lambda_{234} - v(1,3,4)A_{42}^{34} - v(1,2,3)A_{42}^{23} - v(1,4,2)A_{42}^{42}]. \end{aligned} \tag{A8}$$

Λ_{234} can be expressed further in terms of triangle areas. From Eq. (A6), we consider the following quantity:

$$-(m_i^2-m_1^2)(m_j^2-m_1^2)(m_k^2-m_1^2)\mathrm{Im}(V_{ib}V_{ic}^*V_{jc}V_{jd}^*V_{kd}V_{kb}^*) = -\mathcal{M}_1^{ijk}\omega_{bcd}^{ijk}, \tag{A9}$$

where we have defined the mass prefactor $\mathcal{M}_1^{ijk} \equiv (m_i^2 - m_1^2)(m_j^2-m_1^2)(m_k^2-m_1^2)$ and the imaginary part of six CKM matrix elements $\omega_{bcd}^{ijk} \equiv \mathrm{Im}(V_{ib}V_{ic}^*V_{jc}V_{jd}^*V_{kd}V_{kb}^*)$. Note that ω_{bcd}^{ijk} has the following properties:

$$\sum_{i=2}^{4}\omega_{bcd}^{ijk} = -\omega_{bcd}^{1jk}, \qquad \omega_{bcd}^{ijk} = \omega_{cdb}^{jki} = \omega_{dbc}^{kij}, \qquad \omega_{bcd}^{iii} = 0, \qquad \omega_{bcd}^{iji} = |V_{ib}|^2A_{cd}^{ij}. \tag{A10}$$

Λ_{bcd} could be written as the following summation:

$$\Lambda_{bcd} = \sum_{i,j,k=2}^{4} -\mathcal{M}_1^{ijk}\omega_{bcd}^{ijk}. \tag{A11}$$

Note that from the second property of Eq. (A10), $\Lambda_{bcd} = \Lambda_{dbc} = \Lambda_{cdb}$, while the first property of Eq. (A10) implies,

$$\sum_{i,j,k=2}^{4}\omega_{bcd}^{ijk} = \sum_{j,k=2}^{4} -\omega_{bcd}^{1jk} = \sum_{k=2}^{4}\omega_{bcd}^{11k} = -\omega_{bcd}^{111} = 0. \tag{A12}$$

Combining Eqs. (A11) and (A12), we have

$$\Lambda_{bcd} = \sum_{i,j,k=2}^{4} -(\mathcal{M}_1^{ijk} - \mathcal{M}_1^{234})\omega_{bcd}^{ijk}. \tag{A13}$$

Since the upper indices of $\mathcal{M}$ are symmetric and $\omega_{bcd}^{iii} = 0$, $(\mathcal{M}_1^{ijk} - \mathcal{M}_1^{234})\omega_{bcd}^{ijk} = 0$ if (i, j, k) are taken to be all different or all the same from $\{2, 3, 4\}$. Thus, Eq. (A13) could be written as

$$\begin{aligned}\Lambda_{bcd} = {}& [(\mathcal{M}_1^{234} - \mathcal{M}_1^{232})(\omega_{bcd}^{232} + \omega_{cdb}^{232} + \omega_{dbc}^{232}) \\ &+ (\mathcal{M}_1^{234} - \mathcal{M}_1^{323})(\omega_{bcd}^{323} + \omega_{cdb}^{323} + \omega_{dbc}^{323})] \\ &+ \text{other two cyclic permutations of } (2,3,4),\end{aligned} \tag{A14}$$

where we have used the symmetric property of the upper indices of $\mathcal{M}$ and also the second property of Eq. (A10). Using now the fourth property of Eq. (A10) and express terms like ω_{bcd}^{232} in terms of triangle areas with real factors, Λ_{bcd} becomes

$$\begin{aligned}\Lambda_{bcd} = {}& [(m_2^2-m_1^2)(m_3^2-m_1^2)(m_4^2-m_2^2) \\ &\times (|V_{2b}|^2A_{cd}^{23} + |V_{2c}|^2A_{db}^{23} + |V_{2d}|^2A_{bc}^{23}) \\ &- (m_2^2-m_1^2)(m_3^2-m_1^2)(m_4^2-m_3^2) \\ &\times (|V_{3b}|^2A_{cd}^{23} + |V_{3c}|^2A_{db}^{23} + |V_{3d}|^2A_{bc}^{23})] \\ &+ \text{other two cyclic permutations of } (2,3,4).\end{aligned} \tag{A15}$$

Substituting $(2, 3, 4)$ for (b, c, d) one obtains Eq. (13).

APPENDIX B: RELATIONS BETWEEN TRIANGLE AREAS

Since there are only three independent phases in the CKM matrix in SM4, one expects there exists some relations among the nine triangle areas. For example, one can obtain $J'(1, 3, 4)$ directly from $J'(2, 3, 4)$ by exchanging the indices 1 and 2, rather than using the approach we presented. In this case, one would get two different expressions for $J'(1,3,4)$. By comparing the two, one would find nontrivial relations of some of the triangle areas.

There are in fact six relations, consisting of three up-type relations,

$$[(|V_{u_1 n}|^2 - |V_{u_2 n}|^2)A^{u_1 u_2}_{lm} + (|V_{u_3 n}|^2 - |V_{u_4 n}|^2)A^{u_3 u_4}_{lm}] + \text{cyclic permu. of } (l,m,n) = 0, \quad \text{(B1)}$$

and three down-type relations,

$$[(|V_{nd_1}|^2 - |V_{nd_2}|^2)A^{lm}_{d_1 d_2} + (|V_{nd_3}|^2 - |V_{nd_4}|^2)A^{lm}_{d_3 d_4}] + \text{cyclic permu. of } (l,m,n) = 0, \quad \text{(B2)}$$

where u_1 to u_4 (or d_1 to d_4 for down-type) are taken all differently from $\{1,2,3,4\}$, and l, m, n are also taken all differently from $\{1,2,3,4\}$ Different choices of (l,m,n) would in fact give equivalent relations, so throughout this Appendix we will regard (l,m,n) as given labels, say (2, 3, 4), without loss of generality.

Let us present a direct proof of these six relations. First, we consider the up-type relations. We define the left-hand side of Eq. (B1) to be

$$\begin{aligned}\mathcal{R}_{u_1u_2u_3u_4} = [&-\mathrm{Im}(V_{u_1 n}V^*_{u_1 n}V_{u_1 l}V^*_{u_1 m}V_{u_2 m}V^*_{u_2 l})\\ &-\mathrm{Im}(V_{u_2 n}V^*_{u_2 n}V_{u_2 l}V^*_{u_2 m}V_{u_1 m}V^*_{u_1 l})\\ &-\mathrm{Im}(V_{u_3 n}V^*_{u_3 n}V_{u_3 l}V^*_{u_3 m}V_{u_4 m}V^*_{u_4 l})\\ &-\mathrm{Im}(V_{u_4 n}V^*_{u_4 n}V_{u_4 l}V^*_{u_4 m}V_{u_3 m}V^*_{u_3 l})]\\ &+\text{other two cyclic permutations of } (l,m,n),\end{aligned} \quad \text{(B3)}$$

where we put back the real factors into $\mathrm{Im}(\cdots)$. With the unitarity condition, we can replace $V_{u_1 n}V^*_{u_1 l}$ by $-V_{u_2 n}V^*_{u_2 l} - V_{u_3 n}V^*_{u_3 l} - V_{u_4 n}V^*_{u_4 l}$, and substitute this into Eq. (B3). For example, the first term in Eq. (B3) would become

$$\begin{aligned}&-\mathrm{Im}(V_{u_1 n}V^*_{u_1 n}V_{u_1 l}V^*_{u_1 m}V_{u_2 m}V^*_{u_2 l})\\ &= |V_{u_2 l}|^2\mathrm{Im}(V_{u_1 n}V^*_{u_1 m}V_{u_2 m}V^*_{u_2 n})\\ &\quad + \mathrm{Im}(V_{u_3 l}V^*_{u_3 n}V_{u_1 n}V^*_{u_1 m}V_{u_2 m}V^*_{u_2 l})\\ &\quad + \mathrm{Im}(V_{u_4 l}V^*_{u_4 n}V_{u_1 n}V^*_{u_1 m}V_{u_2 m}V^*_{u_2 l})\\ &= |V_{u_2 l}|^2 A^{u_1 u_2}_{mn} + \omega^{u_3 u_1 u_2}_{lnm} + \omega^{u_4 u_1 u_2}_{lnm},\end{aligned} \quad \text{(B4)}$$

where $\omega^{u_3 u 1 u_2}_{lnm}$ is defined in Eq. (A9). Applying the unitarity condition to every term in Eq. (B3), we have

$$\begin{aligned}\mathcal{R}_{u_1u_2u_3u_4} = [&|V_{u_2 l}|^2 A^{u_1 u_2}_{mn} - |V_{u_1 l}|^2 A^{u_1 u_2}_{mn} + |V_{u_4 l}|^2 A^{u_3 u_4}_{mn}\\ &- |V_{u_3 l}|^2 A^{u_3 u_4}_{mn} + \omega^{u_3 u_1 u_2}_{lnm} + \omega^{u_4 u_1 u_2}_{lnm}\\ &+ \omega^{u_3 u_2 u_1}_{lnm} + \omega^{u_4 u_2 u_1}_{lnm} + \omega^{u_1 u_3 u_4}_{lnm} + \omega^{u_2 u_3 u_4}_{lnm}\\ &+ \omega^{u_1 u_4 u_3}_{lnm} + \omega^{u_2 u_4 u_3}_{lnm}]\\ &+ \text{other two cyclic permutations of } (l,m,n).\end{aligned} \quad \text{(B5)}$$

The first four terms in the brackets in the right-hand side together with the cyclic permutations amount to $-\mathcal{R}_{u_1u_2u_3u_4}$. And due to the second property of Eq. (A10), the cyclic permutations on ω's lower labels can be moved to its upper labels when all cyclic permutations are summed, so the left 24 ω's can be written as $\sum_{(i,j,k)'}\omega^{ijk}_{lnm}$ where $(i,j,k)'$ are taken all differently from $\{u_1, u_2, u_3, u_4\}$, which is equivalent to the set $\{1,2,3,4\}$. So now we have

$$\begin{aligned}2\mathcal{R}_{u_1u_2u_3u_4} &= \sum_{(i,j,k)'}\omega^{ijk}_{lnm}\\ &= \sum_{i,j,k=1}^{4}\omega^{ijk}_{lnm} - \sum_{i,j=1}^{4}(\omega^{iij}_{lnm} + \omega^{iji}_{lnm} + \omega^{jii}_{lnm}).\end{aligned} \quad \text{(B6)}$$

In the second equality, we allow all possible i, j, k in the summation and subtract back the extra terms. However, from the first property of Eq. (A10), $\sum_{a=1}^{4}\omega^{abc}_{lnm} = 0$, so all terms in the right-hand side of Eq. (B6) vanish when summing over j from 1 to 4. Finally, we have

$$\mathcal{R}_{u_1u_2u_3u_4} = 0, \quad \text{(B7)}$$

which proves Eq. (B1). The down-type relations Eq. (B2) can be derived similarly.

As mentioned in [18], in SM4 there are nine independent CKM triangle areas, but the relations shown in this appendix seem to reduce the number of independent CKM triangle areas to at most three. There is in fact no conflict. Provided the magnitude of each CKM mixing matrix element is known, the degree of freedom of the CKM triangles can be further reduced to at most three, which equals the number of physical phases, as shown in this appendix. This result satisfies our intuition because one needs rotation angles and phases to describe the mixing matrix. So if one does not know the magnitude of matrix elements, the degree of freedom of triangle areas would be larger than the number of physical phases, but when those rotation angles are known, one should be able to express some CKM triangle areas in terms of others like what we proposed in Eqs. (B1) and (B2).

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