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Fluctuations destroying long-range order in SU(2) Yang-Mills theory

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We study lattice SU(2) Yang-Mills theory with dimension $d \geq 4$. The model can be expressed as a $(d-1)$-dimensional O(4) nonlinear σ model in a d-dimensional heat bath. As is well known, the nonlinear σ model alone shows a phase transition. If the quark confinement is a consequence of the absence of a phase transition for the Yang-Mills theory, then the fluctuations of the heat bath must destroy the long-range order of the nonlinear σ model. In order to clarify whether this is true, we replace the fluctuations of the heat bath with Gaussian random variables, and obtain a Langevin equation which yields the effective action of the nonlinear σ model by analyzing the Fokker-Planck equation. It turns out that the fluctuations indeed destroy the long-range order of the nonlinear σ model within a mean-field approximation estimating a critical point, whereas for the corresponding U(1) gauge theory, the phase transition to the massless phase remains against the fluctuations.

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I. INTRODUCTION

We study Euclidean SU(2) Yang-Mills theory on the hypercubic lattice $\mathbb{Z}^d$ with dimension $d \geq 4$. It is widely believed that[1] the gauge theory shows a quark confinement phase with a mass gap for all the values of the coupling in dimensions $d = 4$. On the other hand, the corresponding U(1) gauge theory in dimensions $d = 4$ is proven to show the existence of a deconfining transition to a massless phase [2,3]. Thus it is expected that there exists a crucial difference between SU(2) and U(1) gauge theories.

In this paper, we explore the origin of this difference. For this purpose, we go back to the paper by Durhuus and Fröhlich [4]. They showed that the d-dimensional Yang-Mills system can be interpreted as many $(d-1)$-dimensional nonlinear σ models which are stacked up in the dth direction and coupled through $(d-1)$-dimensional external Yang-Mills fields.[2] When we view one of the $(d-1)$-dimensional nonlinear σ models, the system can be interpreted as a $(d-1)$-dimensional nonlinear σ model in a d-dimensional heat bath. When we turn off the interaction between the nonlinear σ model and the heat bath, the nonlinear σ model becomes the standard O(4) nonlinear σ model because the gauge group SU(2) is homeomorphic to the 3-sphere $\mathbb{S}^3$. As is well known, the O(4) nonlinear σ model is proven to show a phase transition [7] in dimensions greater than or equal to 3. This implies that, if the quark confinement is a consequence of the absence of a phase transition for the Yang-Mills theory, then the fluctuations of the external Yang-Mills fields must destroy the long-range order of the O(4) nonlinear σ model.

The effective action of the $(d-1)$-dimensional nonlinear σ model can be derived by integrating out the degrees of freedom of the heat bath. However, carrying out the integration is very difficult. Instead of doing so, we replace the fluctuations of the external Yang-Mills fields with Gaussian random variables. Within this approximation, the spins of the nonlinear σ model can be interpreted as "particles" which move on $\mathbb{S}^3$, acted on by the two-body force and the random forces. Namely, the dynamics of the particles obeys a Langevin equation [8]. As is well known, a Langevin dynamics yields a Fokker-Planck equation which describes the time evolution of the distribution of the particles. In the present system, the effective action of the nonlinear σ model can be derived from the steady state solution to the corresponding Fokker-Planck equation. In the effective action so obtained, the attractive potential between the two particles is modified by the fluctuations of the external Yang-Mills fields.

We show that the height and the width of the barrier of the attractive potential depend on the coupling constant of the Yang-Mills theory. Roughly speaking, the critical value of the coupling constant for the phase transition to a massless phase can be estimated by the height and the width of the barrier of the attractive potential. Therefore the critical value becomes a function of the coupling constant. In consequence, we obtain that within a certain mean-field approximation, the critical value is always strictly less than the value of the coupling constant itself for weak couplings. This implies that the critical value must be equal to zero; i.e., there is no phase transition to a massless phase for nonzero coupling constants.

On the other hand, the corresponding U(1) gauge theory shows that the attractive potential does not depend on the coupling constant for weak coupling constants within the same approximation. Namely, the fluctuations of the external Yang-Mills fields do not affect the critical behavior of the O(2) nonlinear σ model.

This paper is organized as follows. In the next section, we express SU(2) Yang-Mills theory in the form of the

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[1]See, for example, Ref. [1].
[2]See also related articles [5,6].

O(4) nonlinear σ model with a large heat bath, following Durhuus and Fröhlich [4]. In Sec. III, we obtain the Langevin equation for the particles moving on $\mathbb{S}^3$, by replacing the fluctuations of the heat bath with Gaussian random variables. In the standard procedure, the Langevin equation yields the Fokker-Planck equation for the distribution of the particles. In Sec. IV, a steady state solution to the Fokker-Planck equation is obtained. The result immediately yields the effective action of the nonlinear σ model. Further, we show that the phase transition of the O(4) nonlinear σ model disappears, owing to the fluctuations, within a mean-field approximation for the effective action so obtained. In Sec. V, we apply the same method to the corresponding U(1) gauge theory, and show that the phase transition to the massless phase remains against the fluctuations.

II. YANG-MILLS THEORY AS A σ MODEL IN A HEAT BATH

Let Λ be a sublattice of $\mathbb{Z}^d$. The SU(2) gauge field on Λ is a map from the oriented links or nearest neighbor pairs $\langle \mathbf{q}, \mathbf{q}' \rangle$ of sites $\mathbf{q}, \mathbf{q}'$ of the lattice Λ to the Lie group $G =$ SU(2),

$$\langle \mathbf{q}, \mathbf{q}' \rangle \mapsto U_{\mathbf{q}\mathbf{q}'} \in G, \tag{2.1}$$

obeying

$$U_{\mathbf{q}'\mathbf{q}} = (U_{\mathbf{q}\mathbf{q}'})^{-1}. \tag{2.2}$$

Let γ be an oriented path which is written $\gamma = \langle \mathbf{q}_1, \mathbf{q}_2 \rangle \times \langle \mathbf{q}_2, \mathbf{q}_3 \rangle \cdots \langle \mathbf{q}_{n-1}, \mathbf{q}_n \rangle$ with the oriented links $\langle \mathbf{q}_i, \mathbf{q}_{i+1} \rangle$ of the neighboring sites $\mathbf{q}_i$, $\mathbf{q}_{i+1}$, for $i = 1, 2, \ldots, n-1$. When $\mathbf{q}_1 = \mathbf{q}_n$, the path γ is a loop. For an oriented path γ, we write

$$U_\gamma = U_{\mathbf{q}_1\mathbf{q}_2} U_{\mathbf{q}_2\mathbf{q}_3} \cdots U_{\mathbf{q}_{n-1}\mathbf{q}_n}. \tag{2.3}$$

The Euclidean action of pure Yang-Mills theory on the lattice $\Lambda \subset \mathbb{Z}^d$ is given by

$$\mathcal{A}_d^{\rm YM}(\Lambda) := -\frac{1}{2} \sum_{p \subset \Lambda} \mathrm{Re\,Tr} U_{\partial p}, \tag{2.4}$$

where p denotes an oriented plaquette (unit square) of Λ, and ∂p is the oriented loop formed by the four sides of p. The orientation of the loop ∂p obeys the orientation of the plaquette p. The expectation value is given by

$$\langle \cdots \rangle_\Lambda := Z_\Lambda^{-1} \int \prod_{b \subset \Lambda} dU_b (\cdots) \exp[-\beta \mathcal{A}_d^{\rm YM}(\Lambda)] \tag{2.5}$$

with the inverse temperature β and the normalization Z_Λ, where b is a link in Λ and dU_b is the Haar measure of the gauge group $G =$ SU(2).

Following Durhuus and Fröhlich [4], we use the relation between the d-dimensional Yang-Mills action and a $(d-1)$-dimensional nonlinear σ model. The coordinates of a lattice site $\mathbf{q}$ are denoted $(x^{(1)}, x^{(2)}, \ldots, x^{(d-1)}, x^{(d)}) = (\mathbf{i}, x^{(d)})$ with $\mathbf{i} = (x^{(1)}, \ldots, x^{(d-1)}) \in \mathbb{Z}^{d-1}$. Write $\Lambda_\tau = \Lambda \cap \{\mathbf{q}: x^{(d)} = \tau\}$ for the $(d-1)$-dimensional hyperplane, and $\Lambda^0 = \Lambda \cap \mathbb{Z}^{d-1} \times \{0\}$ for the projection onto the $\mathbb{Z}^{d-1}$ lattice. Let $U_{\mathbf{ij}}^h(\tau)$ denote the gauge field $U_{\mathbf{q}\mathbf{q}'}$ assigned to the link $\langle \mathbf{q}, \mathbf{q}' \rangle$ in Λ_τ with $\mathbf{q} = (\mathbf{i}, \tau)$ and $\mathbf{q}' = (\mathbf{j}, \tau)$, and $U_{\mathbf{i}}^v(\tau)$ the gauge field $U_{\mathbf{q}\mathbf{q}'}$ with $\mathbf{q} = (\mathbf{i}, \tau)$ and $\mathbf{q}' = (\mathbf{i}, \tau+1)$. The former are called horizontal gauge fields localized at $x^{(d)} = \tau$, and the latter are called vertical gauge fields localized in the slice $[\tau, \tau+1]$. Now the Yang-Mills action can be rewritten as

$$\begin{aligned}\mathcal{A}_d^{\rm YM}(\Lambda) = &-\frac{1}{2}\sum_\tau \sum_{p \subset \Lambda_\tau} \mathrm{Re\,Tr} U_{\partial p}^h \\ &-\frac{1}{2}\sum_\tau \sum_{\langle \mathbf{i}, \mathbf{j} \rangle \subset \Lambda^0} \mathrm{Re\,Tr} U_{\mathbf{i}}^v(\tau)^{-1} U_{\mathbf{ij}}^h(\tau) \\ &\times U_{\mathbf{j}}^v(\tau) U_{\mathbf{ji}}^h(\tau+1).\end{aligned} \tag{2.6}$$

The first term in the right-hand side is a sum of Yang-Mills actions which depend on the horizontal gauge fields in a $(d-1)$-dimensional hyperplane at $x^{(d)} = \tau$. As to the second term, the vertical gauge fields in different slices are not coupled to each other. Therefore the second double sum of (2.6) is written as a sum of an action of a $(d-1)$-dimensional nonlinear σ model for the vertical gauge fields. The explicit form of the action in the slice $[\tau, \tau+1]$ is given by

$$\begin{aligned}&\mathcal{A}_{d-1}^{\sigma}(\Lambda^0; U^h(\tau), U^h(\tau+1)) \\ &= -\frac{1}{2} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle \subset \Lambda^0} \mathrm{Re\,Tr} U_{\mathbf{i}}^v(\tau)^{-1} U_{\mathbf{ij}}^h(\tau) U_{\mathbf{j}}^v(\tau) U_{\mathbf{ji}}^h(\tau+1)\end{aligned} \tag{2.7}$$

in the external gauge fields, $U^h(\tau) = \{U_{\mathbf{ij}}^h(\tau)\}$ and $U^h(\tau+1) = \{U_{\mathbf{ij}}^h(\tau+1)\}$.

Let $\mathbb{S}^3$ denote the 3-sphere. In order to express the gauge fields in terms of spins $\mathbf{S} \in \mathbb{S}^3$, we use the homeomorphism φ: $\mathbb{S}^3 \to$ SU(2) which is defined by [4]

$$\varphi(\mathbf{S}) = \varphi(S^{(0)}, S^{(1)}, S^{(2)}, S^{(3)}) = \begin{pmatrix} S^{(0)} + iS^{(3)} & -S^{(1)} + iS^{(2)} \\ S^{(1)} + iS^{(2)} & S^{(0)} - iS^{(3)} \end{pmatrix} \tag{2.8}$$

with the radius $(S^{(0)})^2 + (S^{(1)})^2 + (S^{(2)})^2 + (S^{(3)})^2 = 1$. Then the interaction potential V_{12} between two spins $\mathbf{S}_1$ and $\mathbf{S}_2$ in the nonlinear σ model (2.7) can be written

$$V_{12} = -\tfrac{1}{2} \mathrm{Re\,Tr} \varphi(\mathbf{S}_1)^{-1} \varphi(\boldsymbol{\sigma}_1) \varphi(\mathbf{S}_2) \varphi(\boldsymbol{\sigma}_2)^{-1}, \tag{2.9}$$

where we have written $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ for the external horizontal gauge fields. When the external gauge fields $\boldsymbol{\sigma}_\ell$ take the vacuum configurations, $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2 = (1, 0, 0, 0)$, the interaction becomes that of the O(4) nonlinear σ model in $(d-1)$ dimensions as

$$V_{12} = -\frac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}\varphi(\mathbf{S}_1)^{-1}\varphi(\mathbf{S}_2) = -\mathbf{S}_1\cdot\mathbf{S}_2 = -\sum_{k=0}^{3} S_1^{(k)}S_2^{(k)}. \quad (2.10)$$

As is well known, the O(4) nonlinear σ model shows a long-range order of spins at low temperatures in three or higher dimensions [7]. The long-range order leads to the perimeter law of the decay of the Wilson loop [4]. The perimeter law implies the deconfinement of quarks. If the confinement of quarks indeed occurs in the SU(2) gauge theory, the fluctuations of the external gauge fields around the vacuum must destroy the long-range order of the O(4) nonlinear σ model.

In order to take account of the fluctuations around the vacuum configuration of the external gauge fields, we approximate $\boldsymbol{\sigma}_\ell$ as

$$\boldsymbol{\sigma}_\ell = (\sqrt{1-|\hat{\boldsymbol{\sigma}}_\ell|^2}, \hat{\boldsymbol{\sigma}}_\ell) \approx (1, \hat{\boldsymbol{\sigma}}_\ell) \quad (2.11)$$

with small fluctuations,

$$\hat{\boldsymbol{\sigma}}_\ell = (\sigma_\ell^{(1)}, \sigma_\ell^{(2)}, \sigma_\ell^{(3)}), \quad \text{for } \ell = 1, 2. \quad (2.12)$$

We write $\delta\boldsymbol{\sigma}_\ell = (0, \hat{\boldsymbol{\sigma}}_\ell)$. Then the two-body potential is written

$$V_{12} \approx -\mathbf{S}_1\cdot\mathbf{S}_2 - \tfrac{1}{2}\mathrm{Re}\,\mathrm{Tr}\varphi(\mathbf{S}_1)^{-1}\varphi'(\delta\boldsymbol{\sigma}_1)\varphi(\mathbf{S}_2) - \tfrac{1}{2}\mathrm{Re}\,\mathrm{Tr}\varphi(\mathbf{S}_1)^{-1}\varphi(\mathbf{S}_2)\varphi'(-\delta\boldsymbol{\sigma}_2), \quad (2.13)$$

dropping the second order[3] in the fluctuations $\delta\boldsymbol{\sigma}_\ell$. Here we have written

$$\varphi'(\delta\boldsymbol{\sigma}) = \begin{pmatrix} i\sigma^{(3)} & -\sigma^{(1)} + i\sigma^{(2)} \\ \sigma^{(1)} + i\sigma^{(2)} & -i\sigma^{(3)} \end{pmatrix}. \quad (2.14)$$

The right-hand side of (2.13) can be written

$$V_{12} \approx V_0 + V_{\mathrm{R}} \quad (2.15)$$

with

$$V_0 = -\mathbf{S}_1\cdot\mathbf{S}_2 \quad (2.16)$$

and

$$V_{\mathrm{R}} = -\sqrt{2}\hat{\boldsymbol{\sigma}}_+\cdot(\hat{\mathbf{S}}_1\times\hat{\mathbf{S}}_2) - \sqrt{2}\hat{\boldsymbol{\sigma}}_-\cdot(S_1^{(0)}\hat{\mathbf{S}}_2 - S_2^{(0)}\hat{\mathbf{S}}_1), \quad (2.17)$$

where

$$\hat{\boldsymbol{\sigma}}_\pm = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\sigma}}_2 \pm \hat{\boldsymbol{\sigma}}_1), \quad (2.18)$$

and

$$\hat{\mathbf{S}}_\ell = (S_\ell^{(1)}, S_\ell^{(2)}, S_\ell^{(3)}), \qquad \ell = 1, 2. \quad (2.19)$$

Thus the present system can be expressed as the O(4) nonlinear σ model in the heat bath. The interaction between the nonlinear σ model and the heat bath is given by V_{R}.

III. LANGEVIN DYNAMICS FOR TWO PARTICLES ON $\mathbb{S}^3$

If we can integrate out the degrees of freedom of the heat bath, then we can obtain the effective action of the nonlinear σ model. However, it is a very difficult problem. Instead, we replace the fluctuations of the external gauge fields with Gaussian random variables. Then, the spins of the σ model can be interpreted as the particles which move on $\mathbb{S}^3$, acted on by the two-body force and the random forces.

In order to derive the effective two-body interaction between two spins of the σ model within this approximation, we first introduce the Langevin equation for the two particles. We write $\hat{x}_\ell = (x_\ell^{(1)}, x_\ell^{(2)}, x_\ell^{(3)})$, $\ell = 1, 2$, for the local coordinates of the two 3-spheres $\mathbb{S}^3$. Then the Langevin equation [8] is given by

$$\frac{d}{dt}x_\ell^{(i)} = F_{0,\ell}^{(i)} + F_{\mathrm{R},\ell}^{(i)}, \qquad \ell = 1, 2; \qquad i = 1, 2, 3, \quad (3.1)$$

with the forces $F_{0,\ell}^{(i)}$, $F_{\mathrm{R},\ell}^{(i)}$, which are given by the gradient[4] of the potentials as

$$F_{0,\ell}^{(i)} = -g_\ell^{ij}\partial_{j,\ell}V_0 \quad (3.2)$$

and

$$F_{\mathrm{R},\ell}^{(i)} = -g_\ell^{ij}\partial_{j,\ell}V_{\mathrm{R}}, \quad (3.3)$$

where ${g^{ij}}_\ell$ is the matrix inverse of the metric tensor $g_{ij,\ell}$ for the particle ℓ, and we have used the Einstein summation convention and written

$$\partial_{i,\ell} = \frac{\partial}{\partial x_\ell^{(i)}}. \quad (3.4)$$

Let $\rho_t(\hat{x}_1, \hat{x}_2)$ be the distribution of the two particles on $\mathbb{S}^3\times\mathbb{S}^3$. The expectation value of the function $f(\hat{x}_1, \hat{x}_2)$ on $\mathbb{S}^3\times\mathbb{S}^3$ at time t is given by

[3]The contributions of the second order of the fluctuations $\delta\boldsymbol{\sigma}_\ell$ give order of temperature $T = \beta^{-1}$ in the potential V_{12}. Therefore one can expect that the contributions of the second order slightly modify the coupling constants of the interaction potentials at low temperatures.

[4]See, for example, Ref. [9].

$$\langle f\rangle_t := \int_{\mathbb{S}^3\times\mathbb{S}^3} f(\hat{x}_1,\hat{x}_2)\rho_t(\hat{x}_1,\hat{x}_2)d\mu_1 d\mu_2, \tag{3.5}$$

where we have written

$$d\mu_\ell = \sqrt{\det g_\ell}\,dx_\ell^{(1)}dx_\ell^{(2)}dx_\ell^{(3)} \quad \text{for } \ell = 1, 2. \tag{3.6}$$

For a small $\Delta t > 0$, the following relation must hold:

$$\langle f\rangle_{t+\Delta t} = \mathbb{E}\int_{\mathbb{S}^3\times\mathbb{S}^3} f(\hat{x}_1(t+\Delta t),\hat{x}_2(t+\Delta t)) \times \rho_t(\hat{x}_1,\hat{x}_2)d\mu_1 d\mu_2 + \mathcal{O}((\Delta t)^2), \tag{3.7}$$

where $\mathbb{E}$ stands for the average over the fluctuations $\hat{\boldsymbol{\sigma}}_\ell$, $\ell = 1, 2$, and $\hat{x}_\ell(t+\Delta t)$ is the solution of the Langevin equation (3.1) with the initial conditions $\hat{x}_\ell(t) = \hat{x}_\ell$ at time t. As usual, we assume that, for the short interval $[t, t+\Delta t]$, the fluctuations $\hat{\sigma}_\ell^{(i)}$ are constant, and they satisfy

$$\mathbb{E}[\sigma_\ell^{(i)}] = 0, \qquad \mathbb{E}[\sigma_\ell^{(i)}\sigma_\ell^{(j)}] = \frac{\alpha}{\Delta t}\delta^{ij}, \qquad \text{and} \quad \mathbb{E}[\sigma_1^{(i)}\sigma_2^{(j)}] = \frac{\alpha'}{\Delta t}\delta^{ij}, \tag{3.8}$$

where α and α' are nonnegative constants, and δ^{ij} is the Kronecker delta. Physically, a natural assumption is that α and α' satisfy the condition $\alpha > \alpha' > 0$. From the relation between the fluctuations and the temperature of the heat bath, both α and α' are proportional to the temperature β^{-1} of the heat bath.

From the Langevin equation (3.1), we have

$$x_\ell^{(i)}(s) - x_\ell^{(i)}(t) = \int_t^s dt' \frac{dx_\ell^{(i)}(t')}{dt} = \int_t^s dt' F_\ell^{(i)}(\tilde{x}(t')), \tag{3.9}$$

where we have written $F_\ell^{(i)} = F_{0,\ell}^{(i)} + F_{R,\ell}^{(i)}$ and $\tilde{x}(t) = (\hat{x}_1(t), \hat{x}_2(t))$. Using this relation, we obtain

$$F_\ell^{(i)}(\tilde{x}(t')) = F_\ell^{(i)}(\tilde{x}(t)) + \sum_{m,k}\frac{\partial F_\ell^{(i)}(\tilde{x}(t))}{\partial x_m^{(k)}}\int_t^{t'} dt'' F_m^{(k)}(\tilde{x}(t'')) + \cdots. \tag{3.10}$$

Combining these, the expansion with respect to Δt is derived as

$$x_\ell^{(i)}(t+\Delta t) = x_\ell^{(i)}(t) + F_\ell^{(i)}(\tilde{x}(t))\Delta t + \frac{1}{2}\sum_{m,k}\frac{\partial F_\ell^{(i)}(\tilde{x}(t))}{\partial x_m^{(k)}} F_m^{(k)}(\tilde{x}(t))(\Delta t)^2 + \cdots. \tag{3.11}$$

Substituting this into (3.7) and using (3.8), the order of Δt yields

$$\int_M d\mu f(\tilde{x})\frac{\partial\rho_t(\tilde{x})}{\partial t} = \int_M d\mu \sum_{\ell,i}\frac{\partial f(\tilde{x})}{\partial x_\ell^{(i)}}F_{0,\ell}^{(i)}(\tilde{x})\rho_t(\tilde{x}) + \frac{\Delta t}{2}\int_M d\mu \sum_{\ell,i;m,j}\frac{\partial^2 f(\tilde{x})}{\partial x_\ell^{(i)}\partial x_m^{(j)}}\mathbb{E}[F_{\mathrm{R},\ell}^{(i)}(\tilde{x})F_{\mathrm{R},m}^{(j)}(\tilde{x})]\rho_t(\tilde{x}) + \frac{\Delta t}{2}\int_M d\mu \sum_{\ell,i;n,k}\frac{\partial f(\tilde{x})}{\partial x_\ell^{(i)}}\mathbb{E}\left[\frac{\partial F_{\mathrm{R},\ell}^{(i)}(\tilde{x})}{\partial x_n^{(k)}}F_{\mathrm{R},n}^{(k)}(\tilde{x})\right]\rho_t(\tilde{x}), \tag{3.12}$$

where we have written $M = \mathbb{S}^3\times\mathbb{S}^3$ and $d\mu = d\mu_1 d\mu_2$. Since this equation holds for any function f, we can derive the equation of the time evolution for the distribution ρ_t, i.e., the Fokker-Planck equation.

To this end, consider first the first term in the right-hand side of (3.12). Note that

$$\sum_i \frac{\partial f(\tilde{x})}{\partial x_\ell^{(i)}}F_{0,\ell}^{(i)}(\tilde{x})\rho_t(\tilde{x}) = \sum_i \frac{1}{\sqrt{\det g_\ell}}\frac{\partial}{\partial x_\ell^{(i)}}\sqrt{\det g_\ell}F_{0,\ell}^{(i)}(\tilde{x})f(\tilde{x})\rho_t(\tilde{x}) - \sum_i f(\tilde{x})\frac{1}{\sqrt{\det g_\ell}}\frac{\partial}{\partial x_\ell^{(i)}}\sqrt{\det g_\ell}F_{0,\ell}^{(i)}(\tilde{x})\rho_t(\tilde{x}) = \mathrm{div}_\ell[F_{0,\ell}(\tilde{x})f(\tilde{x})\rho_t(\tilde{x})] - f(\tilde{x})\,\mathrm{div}_\ell[F_{0,\ell}(\tilde{x})\rho_t(\tilde{x})], \tag{3.13}$$

where div_ℓ stands for the divergence for the particle ℓ. Combining this with the divergence theorem,[5]

$$\int_{\mathbb{S}^3} d\mu_\ell\,\mathrm{div}_\ell v_\ell = 0, \tag{3.14}$$

for a vector field v_ℓ on $\mathbb{S}^3$, the first term in the right-hand side of (3.12) is written as

$$\sum_{\ell,i}\int_M d\mu(\partial_{i,\ell}f)F_{0,\ell}^{(i)}\rho_t = -\sum_\ell\int_M d\mu f\,\mathrm{div}_\ell(F_{0,\ell}\rho_t). \tag{3.15}$$

As to the second and third terms in the right-hand side of (3.12), we must compute the second moments of the random forces. But one can treat these terms in the same way as in the above. The details are given in Appendix A. As a result, the Fokker-Planck equation is given by

[5] See, for example, Theorem 5.11 in Chap. II of Ref. [9].

$$\frac{\partial \rho_t}{\partial t} = -\sum_\ell \mathrm{div}_\ell (F_{0,\ell}\rho_t) + (\alpha + \alpha') \times \sum_\ell \{\Delta_\ell \rho_t - \mathrm{div}_\ell[\xi_\ell \,\mathrm{div}_\ell(\xi_\ell \rho_t)]\} - (\alpha + \alpha')\{\mathrm{div}_1[\boldsymbol{\eta}_1 W \cdot \mathrm{div}_2(\boldsymbol{\eta}_2\rho_t)] + \mathrm{div}_2[\boldsymbol{\eta}_2 W \cdot \mathrm{div}_1(\boldsymbol{\eta}_1 \rho_t)]\} - 2\alpha' \sum_{m,n} \mathrm{div}_m[\hat{\boldsymbol{\zeta}}_m \cdot \mathrm{div}_n(\hat{\boldsymbol{\zeta}}_n \rho_t)], \tag{3.16}$$

where Δ_ℓ is the Laplacian for the particle ℓ, and we have written $W = \mathbf{S}_1 \cdot \mathbf{S}_2$; the vector fields ξ_ℓ, $\boldsymbol{\eta}_\ell$, and $\hat{\boldsymbol{\zeta}}_\ell$ are given by

$$\xi^i_\ell := g^{ij}_\ell \partial_{j,\ell} W, \tag{3.17}$$

$$\boldsymbol{\eta}^i_\ell := g^{ij}{}_\ell \partial_{j,\ell} \mathbf{S}_\ell, \tag{3.18}$$

and

$$\hat{\boldsymbol{\zeta}}^i_\ell := g^{ij}{}_\ell \partial_{j,\ell}(S_1^{(0)}\hat{\mathbf{S}}_2 - S_2^{(0)}\hat{\mathbf{S}}_1) \tag{3.19}$$

for $i = 1, 2, 3$ and $\ell = 1, 2$. Here the vectors $\boldsymbol{\eta}^i_\ell$ have four components like $\mathbf{S}_\ell$, and $\hat{\boldsymbol{\zeta}}^i_\ell$ have three components like $\hat{\mathbf{S}}_\ell$. This Fokker-Planck equation can be written

$$\frac{\partial \rho_t}{\partial t} = -\mathrm{div} J \quad \text{with} \quad \mathrm{div} J = \mathrm{div}_1 J_1 + \mathrm{div}_2 J_2 \tag{3.20}$$

in terms of the current $J = (J_1, J_2)$, which is given by

$$J^i{}_\ell = g^{ij}{}_\ell J_{j,\ell} \tag{3.21}$$

with

$$J_{j,1} = -(\partial_{j,1}V_0)\rho_t - (\alpha + \alpha')\{\partial_{j,1}\rho_t - [(\partial_{j,1}W)\mathrm{div}_1(\xi_1\rho_t) + W(\partial_{j,1}\mathbf{S}_1)\cdot \mathrm{div}_2(\boldsymbol{\eta}_2\rho_t)]\} + 2\alpha' \hat{\boldsymbol{\zeta}}_{j,1} \cdot [\mathrm{div}_1(\hat{\boldsymbol{\zeta}}_1\rho_t) + \mathrm{div}_2(\hat{\boldsymbol{\zeta}}_2\rho_t)] \tag{3.22}$$

and with $J_{j,2}$ given by exchanging the subscripts 1 and 2 in $J_{j,1}$. Here we have written

$$\hat{\boldsymbol{\zeta}}_{i,\ell} := \partial_{i,\ell}(S_1^{(0)}\hat{\mathbf{S}}_2 - S_2^{(0)}\hat{\mathbf{S}}_1). \tag{3.23}$$

IV. A STEADY STATE FOR THE FOKKER-PLANCK DYNAMICS

The effective potential V_{eff} between the two particles is derived from a steady distribution $\rho_t = \rho$ for the Fokker-Planck equation (3.20), as in (4.7) below. For a steady distribution $\rho_t = \rho$, the Fokker-Planck equation (3.20) becomes $\mathrm{div} J = 0$. In order to obtain the solution near the north pole, $\mathbf{S}_\ell = (1, 0, 0, 0)$, for $\ell = 1, 2$, we introduce the local coordinates (x_ℓ, y_ℓ, z_ℓ) for $\ell = 1, 2$, as

$$\mathbf{S}_\ell = \left(\sqrt{1 - x_\ell^2 - y_\ell^2 - z_\ell^2}, x_\ell, y_\ell, z_\ell\right). \tag{4.1}$$

We write

$$\mathbf{r} = (x, y, z) = (x_1 - x_2, y_1 - y_2, z_1 - z_2) \tag{4.2}$$

and

$$\mathbf{R} = (X, Y, Z) = (x_1 + x_2, y_1 + y_2, z_1 + z_2). \tag{4.3}$$

We also write $r = |\mathbf{r}|$ and $R = |\mathbf{R}|$. In order to solve the partial differential equation $\mathrm{div} J = 0$, we employ the Cauchy-Kowalevski–type expansion[6] with respect to small x_ℓ, y_ℓ, z_ℓ.

Let us compute the x component $J_{x,1}$ of the current J_1 for particle 1. Note that

$$V_0 = -\mathbf{S}_1 \cdot \mathbf{S}_2 = -1 + \tfrac{1}{2} r^2 + \tfrac{1}{8}(\mathbf{r}\cdot\mathbf{R})^2 + \cdots. \tag{4.4}$$

Immediately,

$$\frac{\partial V_0}{\partial x_1} = x + \frac{1}{4}(\mathbf{r}\cdot\mathbf{R})x + \frac{1}{4}(\mathbf{r}\cdot\mathbf{R})X + \cdots. \tag{4.5}$$

Therefore, the first term of $J_{x,1}$ of (3.22) becomes

$$-(\partial_{x,1}V_0)\rho = [-x - \tfrac{1}{4}(\mathbf{r}\cdot\mathbf{R})x - \tfrac{1}{4}(\mathbf{r}\cdot\mathbf{R})X + \cdots]\rho. \tag{4.6}$$

In order to treat the rest of the terms of $J_{x,1}$, we assume that the steady state solution $\rho_t = \rho$ of $\mathrm{div} J = 0$ has the form

$$\rho = \exp[-\beta V_{\mathrm{eff}}], \tag{4.7}$$

where V_{eff} is the effective potential to be determined, and β is the inverse temperature of the heat bath. Both α and α' are proportional to the temperature β^{-1}, as mentioned in the preceding section. The effective potential V_{eff} must be vanishing for $\mathbf{r} = 0$ because the two-body potential (2.13) becomes constant irrespective of the external fluctuations for $\mathbf{S}_1 = \mathbf{S}_2$. From this and taking account of the spherical and exchange symmetries, we assume that the effective potential V_{eff} can be expended as

$$V_{\mathrm{eff}} = C_{20}r^2 + C_{40}r^4 + C_{22}r^2R^2 + C'_{22}(\mathbf{r}\cdot\mathbf{R})^2 + \cdots, \tag{4.8}$$

where C_{20}, C_{40}, C_{22}, and C'_{22} are the coefficients to be determined. In the following, we take α and α' to be small, and ignore the order of α and α'.

For small x_ℓ, y_ℓ, z_ℓ, the current $J_{x,1}$ is written

[6]See, for example, Sec. D of Chap. 1 in Ref. [10].

$$J_{x,1} = \left[-x - \frac{1}{4}(\mathbf{r}\cdot\mathbf{R})x - \frac{1}{4}(\mathbf{r}\cdot\mathbf{R})X\right]\rho - (\alpha - \alpha')\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\rho$$
$$+ (\alpha + \alpha')\left[x\left(x\frac{\partial\rho}{\partial x_1} + y\frac{\partial\rho}{\partial y_1} + z\frac{\partial\rho}{\partial z_1}\right) + x\left(x_2\frac{\partial\rho}{\partial x_2} + y_2\frac{\partial\rho}{\partial y_2} + z_2\frac{\partial\rho}{\partial z_2}\right) - \frac{r^2}{2}\frac{\partial\rho}{\partial x_2}\right]$$
$$+ 2\alpha'\left[-x_1\left(x\frac{\partial\rho}{\partial x_1} + y\frac{\partial\rho}{\partial y_1} + z\frac{\partial\rho}{\partial z_1}\right) + x_2\left(x_1\frac{\partial\rho}{\partial x_1} + y_1\frac{\partial\rho}{\partial y_1} + z_1\frac{\partial\rho}{\partial z_1}\right) - \left(\frac{3}{2}x + \frac{1}{2}X\right)\left(x_2\frac{\partial\rho}{\partial x_2} + y_2\frac{\partial\rho}{\partial y_2} + z_2\frac{\partial\rho}{\partial z_2}\right)\right.$$
$$\left. - r_2^2\frac{\partial\rho}{\partial x_1} + \frac{1}{2}(r_1^2 + r_2^2)\frac{\partial\rho}{\partial x_2}\right] + \cdots. \tag{4.9}$$

The derivation is given in Appendix B. Let us substitute ρ of (4.7) into this right-hand side, assuming the expansion (4.8) for the effective potential. First of all, since the leading order which is proportional to $x\exp[-\beta V_{\rm eff}]$ must be vanishing, we have

$$4\beta(\alpha - \alpha')C_{20} = 1. \tag{4.10}$$

Since we can choose

$$\beta = \frac{1}{\alpha - \alpha'} \tag{4.11}$$

without loss of generality, we have

$$C_{20} = \tfrac{1}{4}. \tag{4.12}$$

Using these, we get

$$-(\alpha - \alpha')\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\exp[-\beta V_{\rm eff}] = \left(\frac{\partial V_{\rm eff}}{\partial x_1} - \frac{\partial V_{\rm eff}}{\partial x_2}\right)\exp[-\beta V_{\rm eff}] \tag{4.13}$$

with

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)V_{\rm eff} = x + 8C_{40}r^2x + 4C_{22}R^2x + 4C'_{22}(\mathbf{r}\cdot\mathbf{R})X + \cdots. \tag{4.14}$$

Moreover, we have

$$\left(x\frac{\partial}{\partial x_1} + y\frac{\partial}{\partial y_1} + z\frac{\partial}{\partial z_1}\right)\rho = \left(-\frac{1}{2}\beta r^2 + \cdots\right)\exp[-\beta V_{\rm eff}], \tag{4.15}$$

$$\left(x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + z_1\frac{\partial}{\partial z_1}\right)\rho = \left[-\frac{1}{4}\beta r^2 - \frac{1}{4}\beta(\mathbf{r}\cdot\mathbf{R}) + \cdots\right]\exp[-\beta V_{\rm eff}], \tag{4.16}$$

$$\left(x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2} + z_2\frac{\partial}{\partial z_2}\right)\rho = \left[-\frac{1}{4}\beta r^2 + \frac{1}{4}\beta(\mathbf{r}\cdot\mathbf{R}) + \cdots\right]\exp[-\beta V_{\rm eff}], \tag{4.17}$$

$$-\frac{r^2}{2}\frac{\partial\rho}{\partial x_2} = \left[-\frac{\beta}{4}xr^2 + \cdots\right]\exp[-\beta V_{\rm eff}], \tag{4.18}$$

and

$$-r_2^2\frac{\partial\rho}{\partial x_1} + \frac{1}{2}(r_1^2 + r_2^2)\frac{\partial\rho}{\partial x_2} = \frac{\beta}{4}x[r^2 + R^2 - (\mathbf{r}\cdot\mathbf{R})]\exp[-\beta V_{\rm eff}] + \cdots. \tag{4.19}$$

Substituting these into (4.9), we obtain

$$J_{x,1}\exp[\beta V_{\rm eff}] = [8C_{40} - 1]r^2x + \frac{\alpha'\beta}{2}[r^2X - (\mathbf{r}\cdot\mathbf{R})x] + \left[4C_{22} + \frac{\alpha'\beta}{2}\right]R^2x + \left[4C'_{22} - \frac{(\alpha + \alpha')\beta}{4}\right](\mathbf{r}\cdot\mathbf{R})X + \cdots. \tag{4.20}$$

From $\mathrm{div}J = 0$, the coefficients must satisfy the relations

$$5(8C_{40} - 1) + \alpha'\beta = 0 \tag{4.21}$$

and

$$3\left[4C_{22} + \frac{\alpha'\beta}{2}\right] + \left[4C'_{22} - \frac{(\alpha + \alpha')\beta}{4}\right] = 0. \tag{4.22}$$

Using these relations, the current $J_{x,1}$ can be written

$$J_{x,1} = \left\{-\frac{\alpha'\beta}{5}r^2x + \frac{\alpha'\beta}{2}[r^2X - (\mathbf{r}\cdot\mathbf{R})x] + A[R^2x - 3(\mathbf{r}\cdot\mathbf{R})X]\right\}\exp[-\beta V_{\rm eff}] + \cdots \tag{4.23}$$

with the constant

$$A = 4C_{22} + \frac{\alpha'\beta}{2}, \tag{4.24}$$

which we cannot determine in the present method. Clearly one notices that in $\mathrm{div}J$, the other terms appear,

$$\tfrac{1}{5}\alpha'\beta^2r^4 \quad \text{and} \quad -A\beta[r^2R^2 - 3(\mathbf{r}\cdot\mathbf{R})^2]. \tag{4.25}$$

These are higher order contributions in powers of the local coordinates but of order β. Since the equation $\mathrm{div}J = 0$ must hold, this implies that there must exist some terms of order of β in the effective potential $V_{\rm eff}$ so as to cancel the above terms of (4.25).

When both of the coefficients C_{22} and C'_{22} depend on β, the corresponding terms may appear in the expansion. In this case, from (4.22), we have

$$C_{22} \sim C\beta \quad \text{and} \quad C'_{22} \sim -3C\beta \tag{4.26}$$

with some constant C for a large β. Substituting these into $V_{\rm eff}$, we have

$$V_{\rm eff} \sim \tfrac{1}{4}r^2 + C_{40}r^4 + C\beta[r^2R^2 - 3(\mathbf{r}\cdot\mathbf{R})^2]. \tag{4.27}$$

This leads to instability of binding of the two particles because the value of R^2 is expected to become larger than order β^{-1} in the thermal equilibrium. Thus we require that both C_{22} and C'_{22} are of order 1.

In consequence, we need the following terms in the effective potential $V_{\rm eff}$:

$$C_{60}r^6, \qquad C_{42}r^4R^2, \qquad C'_{42}r^2(\mathbf{r}\cdot\mathbf{R})^2. \tag{4.28}$$

Here all the coefficients, C_{60}, C_{42}, C'_{42}, are proportional to β for a large β. In the same way as in the above, we can determine these coefficients as

$$\begin{aligned} C_{60} &= -\frac{3!}{7!}\alpha'\beta^2, \\ C_{42} &= \frac{1}{56}A\beta, \quad \text{and} \quad C'_{42} = -\frac{3}{56}A\beta \end{aligned} \tag{4.29}$$

so as to cancel the above terms (4.25) which appear in div$J$. As a result, the dominant contributions in the effective potential $V_{\rm eff}$ are given by

$$V_{\rm eff} \sim \frac{1}{4}r^2 - \frac{3!}{7!}\alpha'\beta^2r^6 + \frac{1}{56}A\beta r^2[r^2R^2 - 3(\mathbf{r}\cdot\mathbf{R})^2] \tag{4.30}$$

for a large β because the second, third, and fourth terms in the right-hand side of (4.8) do not affect the critical behavior.

Now we discuss the critical behavior of the $(d-1)$-dimensional σ model with the above two-body interaction $V_{\rm eff}$. Consider first the case of $A = 0$. Namely, the effective potential is given by

$$V_{\rm eff} \sim \frac{1}{4}r^2 - \frac{3!}{7!}\alpha'\beta^2r^6 \tag{4.31}$$

for small r and large β. The second term lowers the potential barrier. Within a mean-field approximation [11], the critical temperature $T_{\rm C}$ can be estimated by the volume and the height of the potential well. More precisely, $T_{\rm C} \sim$ (volume) $\times$ (height). In the present case, the width w and the height h of the effective potential $V_{\rm eff}$ are estimated as

$$w \sim (\lambda\beta)^{-1/4}, \qquad h \sim (\lambda\beta)^{-1/2}, \tag{4.32}$$

where we have written

$$\lambda = 12 \times \frac{3!}{7!}\alpha'\beta. \tag{4.33}$$

Therefore the critical temperature $T_{\rm C}$ is estimated as

$$T_{\rm C} \sim w^3 \times h \sim (\lambda\beta)^{-5/4}. \tag{4.34}$$

This is lower than β^{-1} for small temperature $T = \beta^{-1}$. This implies that the true critical temperature must be equal to zero.

In the case of $A \neq 0$, the third term in the right-hand side of (4.30) may heighten the potential barrier if R^2 does not take a small value. But it is impossible that the term heightens the potential barrier in all the directions of $\mathbf{r}$. Thus we reach the same conclusion, $T_{\rm C} = 0$.

Let us make the following two remarks:

(1) Our argument can be applied to the systems in arbitrary dimensions. Therefore a reader might think that our method suggests no phase transition for non-Abelian lattice gauge theory also in five or higher dimensions. On this point, we should remark the following: We used the two-body approximation, considering only a single plaquette. When dealing with two plaquettes within our method, three- and four-body interactions would appear in the effective potential for the nonlinear σ model. The resulting interactions may change the conclusion of this section. Namely, a high-dimensional system may exhibit a phase transition. Actually, in five or higher dimensions, the effect of the three- or four-body interactions may not be ignored because the number of the neighboring plaquettes for a fixed plaquette becomes large, compared to low-dimensional systems. However, taking account of such interactions is not so easy.

(2) Consider the O(4) nonlinear σ model on the three-dimensional lattice with the effective two-body interaction which we obtained. Then the correlation length of the model leads to an estimate of the string tension [4,5]. Does the scaling limit so obtained give the standard continuum? This problem must be very important. But it is very difficult to compute the low temperature asymptotics of the correlation length for such a weakly attractive potential.

V. DIFFERENCE BETWEEN U(1) AND SU(2) GAUGE THEORIES

Let us look at the difference between U(1) and SU(2) gauge theories.

For this purpose, we apply the present method to the Abelian case $G = \mathrm{U}(1)$. In the case, the gauge field U_b on a link b is written

$$U_b = \exp[i\theta_b] \tag{5.1}$$

in terms of the angle variable $\theta_b \in [0, 2\pi)$. Therefore the two-body interaction V_{12} between θ_1 and θ_2 is written

$$V_{12} = -\cos(\theta_1 - \theta_2 + \sigma_1 - \sigma_2), \tag{5.2}$$

where σ_1 and σ_2 are the angle variables of the external fields. We write $\theta = \theta_1 - \theta_2$ and $\delta\sigma = \sigma_1 - \sigma_2$, and assume that $\delta\sigma$ is a small fluctuation. Under this assumption, the potential can be approximated as

$$V_{12} \approx -\cos\theta + \delta\sigma \sin\theta. \tag{5.3}$$

Then the Langevin equation is given by

$$\frac{d\theta}{dt} = -\sin\theta - \delta\sigma\cos\theta. \tag{5.4}$$

As usual, we assume

$$\mathbb{E}[(\delta\sigma)^2] = \frac{\alpha}{\Delta t} \tag{5.5}$$

for a small Δt. In the same way as in the SU(2) case, we obtain the Fokker-Planck equation,

$$\frac{\partial \rho_t}{\partial t} = \left[\frac{\partial}{\partial\theta}\sin\theta + \frac{\alpha}{2}\frac{\partial}{\partial\theta}\sin\theta\cos\theta + \frac{\alpha}{2}\frac{\partial^2}{\partial\theta^2}\cos^2\theta\right]\rho_t. \tag{5.6}$$

For a steady state $\rho_t = \rho$, we have

$$\left[\sin\theta + \frac{\alpha}{2}\sin\theta\cos\theta + \frac{\alpha}{2}\frac{\partial}{\partial\theta}\cos^2\theta\right]\rho = 0. \tag{5.7}$$

One can easily find the solution

$$\rho = \begin{cases} (\cos\theta)^{-1}\exp[-2\alpha^{-1}/\cos\theta] & \text{for } -\pi/2 < \theta < \pi/2 \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Since the diffusion disappears at $\theta = \pm\pi/2$ in the right-hand side of (5.4), the particle cannot move beyond the points. Clearly, we have

$$\rho \sim \text{const}\exp[-\alpha^{-1}\theta^2] \tag{5.9}$$

for a small θ. Thus there is no term which is proportional to α^{-1} or higher powers of α^{-1} in the effective potential, and the critical behavior can be expected to be the same as the standard O(2) nonlinear-σ model. This is consistent with the rigorous result of [2,3].

APPENDIX A: DERIVATION OF THE FOKKER-PLANCK EQUATION

Consider first the case with $\alpha' = 0$ in (3.8). We introduce σ^{ij}, satisfying $\sigma^{ji} = -\sigma^{ij}$, with

$$(\sigma^{01}, \sigma^{02}, \sigma^{03}) = (\sigma_+^{(1)}, \sigma_+^{(2)}, \sigma_+^{(3)}) \quad \text{and} \\ (\sigma^{23}, \sigma^{31}, \sigma^{12}) = (\sigma_-^{(1)}, \sigma_-^{(2)}, \sigma_-^{(3)}). \tag{A1}$$

Then the random potential V_R of (2.17) can be written

$$V_\mathrm{R} = -\frac{1}{\sqrt{2}}\varepsilon_{ijk\ell}\sigma^{ij}S_1^{(k)}S_2^{(\ell)}, \tag{A2}$$

where $\varepsilon_{ijk\ell}$ is completely antisymmetric and satisfies $\varepsilon_{0123} = +1$, and we have used the Einstein summation convention. From $\alpha' = 0$, we have

$$\mathbb{E}[\sigma^{\alpha\beta}\sigma^{mn}] = \frac{\alpha}{\Delta t}(\delta^{\alpha m}\delta^{\beta n} - \delta^{\alpha n}\delta^{\beta m}). \tag{A3}$$

Using (A2) and (A3), we obtain

$$\begin{aligned} \mathbb{E}[(\partial_{\ell,1}V_\mathrm{R})(\partial_{k,1}V_\mathrm{R})] &= \frac{1}{2}\mathbb{E}[\varepsilon_{\alpha\beta\gamma\delta}\sigma^{\alpha\beta}(\partial_{\ell,1}S_1^{(\gamma)})S_2^{(\delta)}\varepsilon_{mnst}\sigma^{mn}(\partial_{k,1}S_1^{(s)})S_2^{(t)}] \\ &= \frac{\alpha}{2\Delta t}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{mnst}(\delta^{\alpha m}\delta^{\beta n} - \delta^{\alpha n}\delta^{\beta m})(\partial_{\ell,1}S_1^{(\gamma)})S_2^{(\delta)}(\partial_{k,1}S_1^{(s)})S_2^{(t)} \\ &= \frac{2\alpha}{\Delta t}\sum_{\gamma,\delta}[(\partial_{\ell,1}S_1^{(\gamma)})(\partial_{k,1}S_1^{(\gamma)})S_2^{(\delta)}S_2^{(\delta)} - (\partial_{\ell,1}S_1^{(\gamma)})S_2^{(\gamma)}(\partial_{k,1}S_1^{(\delta)})S_2^{(\delta)}]. \end{aligned} \tag{A4}$$

Using the metric

$$g_{ij,\ell} = \frac{\partial \mathbf{S}_\ell}{\partial x_\ell^{(i)}}\cdot\frac{\partial \mathbf{S}_\ell}{\partial x_\ell^{(j)}} \tag{A5}$$

of $\mathbb{S}^3$ for the particle ℓ, the above result is written

$$\mathbb{E}[(\partial_{\ell,1}V_\mathrm{R})(\partial_{k,1}V_\mathrm{R})] = \frac{2\alpha}{\Delta t}[g_{\ell k,1} - (\partial_{\ell,1}W)(\partial_{k,1}W)] \tag{A6}$$

and

$$\mathbb{E}[(\partial_{\ell,2}V_\mathrm{R})(\partial_{k,2}V_\mathrm{R})] = \frac{2\alpha}{\Delta t}[g_{\ell k,2} - (\partial_{\ell,2}W)(\partial_{k,2}W)], \tag{A7}$$

where we have written $W = \mathbf{S}_1\cdot\mathbf{S}_2$. Similarly, we have

$$\mathbb{E}[(\partial_{k,1}\partial_{j,1}V_R)(\partial_{\ell,1}V_R)] = \frac{2\alpha}{\Delta t}\sum_{\gamma,\delta}\left[\frac{\partial^2 S_1^{(\gamma)}}{\partial x_1^{(k)}\partial x_1^{(j)}}\frac{\partial S_1^{(\gamma)}}{\partial x_1^{(\ell)}}S_2^{(\delta)}S_2^{(\delta)} - \frac{\partial^2 S_1^{(\gamma)}}{\partial x_1^{(k)}\partial x_1^{(j)}}S_2^{(\gamma)}\frac{\partial S_1^{(\delta)}}{\partial x_1^{(\ell)}}S_2^{(\delta)}\right]. \quad \text{(A8)}$$

Combining this with

$$\sum_\gamma \frac{\partial^2 S_1^{(\gamma)}}{\partial x_1^{(k)}\partial x_1^{(j)}}\frac{\partial S_1^{(\gamma)}}{\partial x_1^{(\ell)}} = \Gamma^m_{kj,1}g_{m\ell,1}, \quad \text{(A9)}$$

we obtain

$$\mathbb{E}[(\partial_{k,1}\partial_{j,1}V_R)(\partial_{\ell,1}V_R)] = \frac{2\alpha}{\Delta t}[\Gamma^m_{kj,1}g_{m\ell,1} - (\partial_{k,1}\partial_{j,1}W)(\partial_{\ell,1}W)], \quad \text{(A10)}$$

where $\Gamma^m_{k\ell,1}$ are the Christoffel symbols [9]. In the same way, we get

$$\mathbb{E}[(\partial_{\ell,1}V_R)(\partial_{k,2}V_R)] = -\frac{2\alpha}{\Delta t}W(\partial_{\ell,1}\partial_{k,2}W) \quad \text{(A11)}$$

and

$$\mathbb{E}[(\partial_{k,2}\partial_{j,1}V_R)(\partial_{\ell,2}V_R)] = -\frac{2\alpha}{\Delta t}(\partial_{k,2}W)(\partial_{j,1}\partial_{\ell,2}W). \quad \text{(A12)}$$

Using (A6), we have

$$\mathbb{E}[F^{(i)}_{R,1}F^{(j)}_{R,1}] = \mathbb{E}[g^{i\ell}{}_1(\partial_{\ell,1}V_R)g^{jk}{}_1(\partial_{k,1}V_R)] = \frac{2\alpha}{\Delta t}g^{i\ell}{}_1 g^{jk}{}_1[g_{\ell k,1} - (\partial_{\ell,1}W)(\partial_{k,1}W)] = \frac{2\alpha}{\Delta t}(g^{ij}{}_1 - \xi^i_1\xi^j_1), \quad \text{(A13)}$$

where ξ^i_ℓ is the vector field which is given by (3.17). From (A6) and (A10), we obtain

$$\begin{aligned}\sum_k \mathbb{E}\left[\frac{\partial F^{(i)}_{R,1}}{\partial x^k_1}F^{(k)}_{R,1}\right] &= \mathbb{E}[(\partial_{k,1}g^{ij}{}_1\partial_{j,1}V_R)(g^{k\ell}{}_1\partial_{\ell,1}V_R)]\\ &= (\partial_{k,1}g^{ij}{}_1)g^{k\ell}{}_1\mathbb{E}[(\partial_{j,1}V_R)(\partial_{\ell,1}V_R)] + g^{ij}{}_1 g^{k\ell}{}_1\mathbb{E}[(\partial_{k,1}\partial_{j,1}V_R)(\partial_{\ell,1}V_R)]\\ &= \frac{2\alpha}{\Delta t}(\partial_{k,1}g^{ij}{}_1)g^{k\ell}{}_1[g_{j\ell,1} - (\partial_{j,1}W)(\partial_{\ell,1}W)] + \frac{2\alpha}{\Delta t}g^{ij}{}_1 g^{k\ell}{}_1[\Gamma^m_{kj,1}g_{m\ell,1} - (\partial_{k,1}\partial_{j,1}W)(\partial_{\ell,1}W)]\\ &= \frac{2\alpha}{\Delta t}[\partial_{j,1}g^{ij}{}_1 + g^{ij}{}_1\Gamma^k_{kj,1} - (\partial_{k,1}\xi^i_1)\xi^k_1] = \frac{2\alpha}{\Delta t}\left[\frac{1}{\sqrt{\det g_1}}\partial_{j,1}g^{ij}{}_1\sqrt{\det g_1} - (\partial_{k,1}\xi^i_1)\xi^k_1\right],\end{aligned} \quad \text{(A14)}$$

where we have used[7]

$$\Gamma^k_{kj,1} = \frac{1}{\sqrt{\det g_1}}\partial_{j,1}\sqrt{\det g_1}. \quad \text{(A15)}$$

In the same way, the relations (A11) and (A12) yield

$$\mathbb{E}[F^{(i)}_{R,1}F^{(j)}_{R,2}] = g^{i\ell}_1 g^{jk}_2\mathbb{E}[(\partial_{\ell,1}V_R)(\partial_{k,2}V_R)] = -\frac{2\alpha}{\Delta t}g^{i\ell}{}_1 g^{jk}{}_2 W(\partial_{\ell,1}\partial_{k,2}W) \quad \text{(A16)}$$

and

$$\sum_k \mathbb{E}\left[\frac{\partial F^{(i)}_{R,1}}{\partial x^k_2}F^{(k)}_{R,2}\right] = \mathbb{E}[(\partial_{k,2}g^{ij}{}_1\partial_{j,1}V_R)(g^{k\ell}{}_2\partial_{\ell,2}V_R)] = g^{ij}{}_1 g^{k\ell}{}_2\mathbb{E}[(\partial_{k,2}\partial_{j,1}V_R)(\partial_{\ell,2}V_R)] = -\frac{2\alpha}{\Delta t}g^{ij}{}_1 g^{k\ell}{}_2(\partial_{k,2}W)(\partial_{j,1}\partial_{\ell,2}W), \quad \text{(A17)}$$

respectively. The contribution from the two random forces $F_{R,\ell}$ with the same indexes $\ell = 1$ in the right-hand side of (3.12) becomes

$$\begin{aligned}I_{11} :&= \frac{\Delta t}{2}\left\{\sum_{i,j}\int_M d\mu\frac{\partial^2 f}{\partial x_1^{(i)}\partial x_1^{(j)}}\mathbb{E}[F^{(i)}_{R,1}F^{(j)}_{R,1}] + \sum_{i,k}\int_M d\mu\frac{\partial f}{\partial x_1^{(i)}}\mathbb{E}\left[\frac{\partial F^{(i)}_{R,1}}{\partial x_1^{(k)}}F^{(k)}_{R,1}\right]\right\}\rho_t\\ &= \alpha\sum_{i,j}\int_M d\mu\frac{\partial^2 f}{\partial x_1^{(i)}\partial x_1^{(j)}}(g^{ij}{}_1 - \xi^i_1\xi^j_1)\rho_t + \alpha\sum_i\int_M d\mu\frac{\partial f}{\partial x_1^{(i)}}\left[\frac{1}{\sqrt{\det g_1}}\partial_{j,1}g^{ij}{}_1\sqrt{\det g_1} - (\partial_{k,1}\xi^i_1)\xi^k_1\right]\rho_t,\end{aligned} \quad \text{(A18)}$$

where we have used (A13) and (A14). Note that

[7] See, for example, Sec. 7 of Chap. I of Ref. [12].

$$\int_M d\mu\Big[\sum_{i,j} g^{ij}{}_1 \frac{\partial^2 f}{\partial x_1^{(i)}\partial x_1^{(j)}} + \sum_i\Big(\frac{1}{\sqrt{\det g_1}}\partial_{j,1} g^{ij}{}_1\sqrt{\det g_1}\Big)\frac{\partial f}{\partial x_1^{(i)}}\Big]\rho_t$$
$$= \int_M d\mu\Big(\frac{1}{\sqrt{\det g_1}}\partial_{j,1} g^{ij}{}_1\sqrt{\det g_1}\partial_{i,1} f\Big)\rho_t = \int_M d\mu(\Delta_1 f)\rho_t = \int_M d\mu f(\Delta_1\rho_t), \tag{A19}$$

where the second equality follows from the property[8] of the Laplacian Δ_ℓ. The rest of the contributions in the right-hand side of (A18) are computed as

$$\int_M d\mu\Big[\sum_{i,j}\frac{\partial^2 f}{\partial x_1^{(i)}\partial x_1^{(j)}}\xi_1^i\xi_1^j + \sum_i\frac{\partial f}{\partial x_1^{(i)}}(\partial_{k,1}\xi_1^i)\xi_1^k\Big]\rho_t$$
$$= \int_M d\mu\Big[\frac{1}{\sqrt{\det g_1}}\partial_{i,1}\sqrt{\det g_1}(\partial_{j,1} f)\xi_1^i\xi_1^j\rho_t - (\partial_{j,1} f)\frac{1}{\sqrt{\det g_1}}\partial_{i,1}\sqrt{\det g_1}\xi_1^i\xi_1^j\rho_t\Big] + \int_M d\mu(\partial_{j,1} f)(\partial_{i,1}\xi_1^j)\xi_1^i\rho_t$$
$$= -\int_M d\mu(\partial_{j,1} f)\xi_1^j\frac{1}{\sqrt{\det g_1}}\partial_{i,1}\sqrt{\det g_1}\xi_1^i\rho_t = -\int_M d\mu(\partial_{j,1} f)\xi_1^j\,\mathrm{div}_1[\xi_1\rho_t]$$
$$= -\int_M d\mu\frac{1}{\sqrt{\det g_1}}\partial_{j,1}\sqrt{\det g_1}\xi_1^j f\,\mathrm{div}_1(\xi_1\rho_t) + \int_M d\mu f\frac{1}{\sqrt{\det g_1}}\partial_{j,1}\sqrt{\det g_1}\xi_1^j\,\mathrm{div}_1(\xi_1\rho_t)$$
$$= \int_M d\mu f\,\mathrm{div}_1[\xi_1\,\mathrm{div}_1(\xi_1\rho_t)], \tag{A20}$$

where we have used the divergence theorem (3.14). Substituting this and (A19) into (A18), we obtain

$$I_{11} = \alpha\int_M d\mu f\{\Delta_1\rho_t - \mathrm{div}_1[\xi_1\,\mathrm{div}_1(\xi_1\rho_t)]\}. \tag{A21}$$

Next consider the contribution from the two random forces $F_{\mathrm{R},\ell}$ with different indices, $\ell = 1$ and $\ell = 2$, in the right-hand side of (3.12). Using (A16) and (A17), we obtain

$$I_{12} := \frac{\Delta t}{2}\Big\{\sum_{i,j}\int_M d\mu\frac{\partial^2 f}{\partial x_1^{(i)}\partial x_2^{(j)}}\mathbb{E}[F_{\mathrm{R},1}^{(i)}F_{\mathrm{R},2}^{(j)}] + \sum_{i,k}\int_M d\mu\frac{\partial f}{\partial x_1^{(i)}}\mathbb{E}\Big[\frac{\partial F_{\mathrm{R},1}^{(i)}}{\partial x_2^{(k)}}F_{\mathrm{R},2}^{(k)}\Big]\Big\}\rho_t$$
$$= -\alpha\int_M d\mu\sum_{i,j}\frac{\partial^2 f}{\partial x_1^{(i)}\partial x_2^{(j)}}g^{i\ell}{}_1 g^{jk}{}_2 W(\partial_{\ell,1}\partial_{k,2}W)\rho_t - \alpha\int_M d\mu\sum_i\frac{\partial f}{\partial x_1^{(i)}}g^{ij}{}_1 g^{k\ell}{}_2(\partial_{k,2}W)(\partial_{j,1}\partial_{\ell,2}W)\rho_t$$
$$= -\alpha\int_M d\mu\frac{1}{\sqrt{\det g_2}}\partial_{j,2}\sqrt{\det g_2}g^{jk}{}_2(\partial_{i,1} f)g^{i\ell}{}_1(\partial_{\ell,1}\partial_{k,2}W)W\rho_t + \alpha\int_M d\mu(\partial_{i,1} f)$$
$$\times\frac{1}{\sqrt{\det g_2}}\partial_{j,2}\sqrt{\det g_2}g^{jk}{}_2 g^{i\ell}{}_1(\partial_{\ell,1}\partial_{k,2}W)W\rho_t - \alpha\int_M d\mu(\partial_{i,1} f)g^{ij}{}_1 g^{k\ell}{}_2(\partial_{k,2}W)(\partial_{j,1}\partial_{\ell,2}W)\rho_t$$
$$= \alpha\int_M d\mu(\partial_{i,1} f)g^{i\ell}{}_1 W\frac{1}{\sqrt{\det g_2}}\partial_{j,2}\sqrt{\det g_2}g^{jk}{}_2(\partial_{\ell,1}\partial_{k,2}W)\rho_t, \tag{A22}$$

where we have used the divergence theorem (3.14). Recalling $W = \mathbf{S}_1\cdot\mathbf{S}_2$, we have

$$\partial_{\ell,1}\partial_{k,2}W = (\partial_{\ell,1}\mathbf{S}_1)\cdot(\partial_{k,2}\mathbf{S}_2). \tag{A23}$$

Substituting this into the above result, we get

[8] See, for example, Corollary 5.13 in Chap. II of Ref. [9].

$$I_{12} = \alpha \int_M d\mu (\partial_{i,1} f) g^{i\ell}{}_1 (\partial_{\ell,1} \mathbf{S}_1) W \cdot \frac{1}{\sqrt{\det g_2}} \partial_{j,2} \sqrt{\det g_2} g^{jk}{}_2 (\partial_{k,2} \mathbf{S}_2) \rho_t = \alpha \int_M d\mu (\partial_{i,1} f) \boldsymbol{\eta}_1^i W \cdot \mathrm{div}_2(\boldsymbol{\eta}_2 \rho_t)$$
$$= \alpha \int_M d\mu \frac{1}{\sqrt{\det g_1}} \partial_{i,1} \sqrt{\det g_1} \boldsymbol{\eta}_1^i f W \cdot \mathrm{div}_2(\boldsymbol{\eta}_2 \rho_t) - \alpha \int_M d\mu f \frac{1}{\sqrt{\det g_1}} \partial_{i,1} \sqrt{\det g_1} \boldsymbol{\eta}_1^i W \cdot \mathrm{div}_2(\boldsymbol{\eta}_2 \rho_t)$$
$$= -\alpha \int_M d\mu f\, \mathrm{div}_1[\boldsymbol{\eta}_1 W \cdot \mathrm{div}_2(\boldsymbol{\eta}_2 \rho_t)], \tag{A24}$$

where $\boldsymbol{\eta}_\ell^i$ is given by (3.18). From (3.12), (3.15), (A18), (A21), (A22), and (A24), we obtain the Fokker-Planck equation,

$$\frac{\partial \rho_t}{\partial t} = -\sum_\ell \mathrm{div}_\ell (F_{0,\ell} \rho_t) + \alpha \sum_\ell \{\Delta_\ell \rho_t - \mathrm{div}_\ell [\xi_\ell\, \mathrm{div}_\ell (\xi_\ell \rho_t)]\} - \alpha \{\mathrm{div}_1[\boldsymbol{\eta}_1 W \cdot \mathrm{div}_2(\boldsymbol{\eta}_2 \rho_t)] + \mathrm{div}_2[\boldsymbol{\eta}_2 W \cdot \mathrm{div}_1(\boldsymbol{\eta}_1 \rho_t)]\}, \tag{A25}$$

for $\alpha' = 0$.

Next consider the case with $\alpha' \neq 0$. To begin with, we note that

$$\mathbb{E}[\sigma_+^{(i)} \sigma_+^{(j)}] = \frac{1}{2} \mathbb{E}[(\sigma_2^{(i)} + \sigma_1^{(i)})(\sigma_2^{(j)} + \sigma_1^{(j)})] = \frac{1}{2} \{\mathbb{E}[\sigma_2^{(i)} \sigma_2^{(j)}] + \mathbb{E}[\sigma_1^{(i)} \sigma_1^{(j)}] + \mathbb{E}[\sigma_2^{(i)} \sigma_1^{(j)}] + \mathbb{E}[\sigma_1^{(i)} \sigma_2^{(j)}]\} = \frac{\alpha + \alpha'}{\Delta t} \delta^{ij}. \tag{A26}$$

Similarly,

$$\mathbb{E}[\sigma_-^{(i)} \sigma_-^{(j)}] = \frac{\alpha - \alpha'}{\Delta t} \delta^{ij}. \tag{A27}$$

Further, we have

$$\mathbb{E}[\sigma_+^{(i)} \sigma_-^{(j)}] = \tfrac{1}{2} \mathbb{E}[(\sigma_2^{(i)} + \sigma_1^{(i)})(\sigma_2^{(j)} - \sigma_1^{(j)})] = \tfrac{1}{2} \{\mathbb{E}[\sigma_2^{(i)} \sigma_2^{(j)}] - \mathbb{E}[\sigma_1^{(i)} \sigma_1^{(j)}] - \mathbb{E}[\sigma_2^{(i)} \sigma_1^{(j)}] + \mathbb{E}[\sigma_1^{(i)} \sigma_2^{(j)}]\} = 0. \tag{A28}$$

Since we can write

$$\mathbb{E}[\sigma_-^{(i)} \sigma_-^{(j)}] = \frac{\alpha + \alpha'}{\Delta t} \delta^{ij} - \frac{2\alpha'}{\Delta t} \delta^{ij}, \tag{A29}$$

it is sufficient to calculate the corrections from the second term in this right-hand side, by replacing α with $\alpha + \alpha'$ in the above result (A25).

In (A13), the correction to $\mathbb{E}[g^{i\ell}{}_1 (\partial_{\ell,1} V_\mathrm{R}) g^{jk}{}_1 (\partial_{k,1} V_\mathrm{R})]$ is given by

$$-\frac{4\alpha'}{\Delta t} \hat{\boldsymbol{\zeta}}_1^i \cdot \hat{\boldsymbol{\zeta}}_1^j, \tag{A30}$$

where $\hat{\boldsymbol{\zeta}}_\ell^i$ is given by (3.19). Similarly, the correction to $\mathbb{E}[(\partial_{k,1} g^{ij}{}_1 \partial_{j,1} V_\mathrm{R})(g^{k\ell}{}_1 \partial_{\ell,1} V_\mathrm{R})]$ in (A14) is given by

$$-\frac{4\alpha'}{\Delta t} (\partial_{k,1} \hat{\boldsymbol{\zeta}}_1^i) \cdot \hat{\boldsymbol{\zeta}}_1^k. \tag{A31}$$

Therefore the same calculations as those from (A18)–(A21) yield the correction

$$-2\alpha'\, \mathrm{div}_1[\hat{\boldsymbol{\zeta}}_1 \cdot \mathrm{div}_1(\hat{\boldsymbol{\zeta}}_1 \rho_t)] \tag{A32}$$

in the right-hand side of the Fokker-Planck equation (A25).

In (A16), the correction to $\mathbb{E}[g^{i\ell}{}_1 (\partial_{\ell,1} V_\mathrm{R}) g_2^{jk} (\partial_{k,2} V_\mathrm{R})]$ is given by

$$-\frac{4\alpha'}{\Delta t} \hat{\boldsymbol{\zeta}}_1^i \cdot \hat{\boldsymbol{\zeta}}_2^j. \tag{A33}$$

Further, the correction to $\mathbb{E}[(\partial_{k,2} g^{ij}{}_1 \partial_{j,1} V_\mathrm{R})(g^{k\ell}{}_2 \partial_{\ell,2} V_\mathrm{R})]$ in (A17) is given by

$$-\frac{4\alpha'}{\Delta t} (\partial_{k,2} \hat{\boldsymbol{\zeta}}_1^i) \cdot \hat{\boldsymbol{\zeta}}_2^k. \tag{A34}$$

Therefore similar calculations to those from (A22)–(A24) yield the correction

$$-2\alpha'\, \mathrm{div}_1[\hat{\boldsymbol{\zeta}}_1 \cdot \mathrm{div}_2(\hat{\boldsymbol{\zeta}}_2 \rho_t)], \tag{A35}$$

in the right-hand side of the Fokker-Planck equation (A25). In consequence, the Fokker-Planck equation is given by

$$\frac{\partial \rho_t}{\partial t} = -\sum_\ell \mathrm{div}_\ell (F_{0,\ell} \rho_t) + (\alpha + \alpha') \sum_\ell \{\Delta_\ell \rho_t - \mathrm{div}_\ell [\xi_\ell\, \mathrm{div}_\ell (\xi_\ell \rho_t)]\} - (\alpha + \alpha') \{\mathrm{div}_1[\boldsymbol{\eta}_1 W \cdot \mathrm{div}_2(\boldsymbol{\eta}_2 \rho_t)] + \mathrm{div}_2[\boldsymbol{\eta}_2 W \cdot \mathrm{div}_1(\boldsymbol{\eta}_1 \rho_t)]\} - 2\alpha' \sum_{m,n} \mathrm{div}_m [\hat{\boldsymbol{\zeta}}_m \cdot \mathrm{div}_n (\hat{\boldsymbol{\zeta}}_n \rho_t)]. \tag{A36}$$

APPENDIX B: DERIVATION OF THE EXPANSION (4.9)

The metric $g_{ij,\ell}$ of $\mathbb{S}^3$ is computed as

$$g_{ij,\ell} = \begin{pmatrix} 1+\gamma_\ell x_\ell^2 & \gamma_\ell x_\ell y_\ell & \gamma_\ell x_\ell z_\ell \\ \gamma_\ell y_\ell x_\ell & 1+\gamma_\ell y_\ell^2 & \gamma_\ell y_\ell z_\ell \\ \gamma_\ell z_\ell x_\ell & \gamma_\ell z_\ell y_\ell & 1+\gamma_\ell z_\ell^2 \end{pmatrix} = \begin{pmatrix} 1+x_\ell^2 & x_\ell y_\ell & x_\ell z_\ell \\ y_\ell x_\ell & 1+y_\ell^2 & y_\ell z_\ell \\ z_\ell x_\ell & z_\ell y_\ell & 1+z_\ell^2 \end{pmatrix} + \cdots, \quad \text{(B1)}$$

where we have written

$$\gamma_\ell = \frac{1}{\sqrt{1-r_\ell^2}} \quad \text{with} \quad r_\ell = \sqrt{x_\ell^2+y_\ell^2+z_\ell^2}. \quad \text{(B2)}$$

Therefore, the inverse $g^{ij}{}_\ell$ is given by

$$g_\ell^{ij} = \begin{pmatrix} 1-x_\ell^2 & -x_\ell y_\ell & -x_\ell z_\ell \\ -y_\ell x_\ell & 1-y_\ell^2 & -y_\ell z_\ell \\ -z_\ell x_\ell & -z_\ell y_\ell & 1-z_\ell^2 \end{pmatrix} + \cdots. \quad \text{(B3)}$$

Using this, we have

$$(\partial_{x,1}W)\mathrm{div}_1(\xi_1\rho) = \frac{\partial \mathbf{S}_1\cdot\mathbf{S}_2}{\partial x_1}\frac{1}{\sqrt{\det g_1}}\partial_{i,1}\sqrt{\det g_1}\,g^{ij}{}_1(\partial_{j,1}\mathbf{S}_1\cdot\mathbf{S}_2)\rho = -xg^{ij}{}_1(\partial_{j,1}\mathbf{S}_1\cdot\mathbf{S}_2)\partial_{i,1}\rho + \cdots$$
$$= x\left(x\frac{\partial\rho}{\partial x_1} + y\frac{\partial\rho}{\partial y_1} + z\frac{\partial\rho}{\partial z_1}\right) + \cdots. \quad \text{(B4)}$$

Similarly,

$$W(\partial_{x,1}\mathbf{S}_1)\cdot\mathrm{div}_2(\boldsymbol{\eta}_2\rho) = W(\partial_{x,1}\mathbf{S}_1)\cdot g^{ij}{}_2(\partial_{j,2}\mathbf{S}_2)\partial_{i,2}\rho + \cdots = W(\partial_{x,1}\partial_{j,2}\mathbf{S}_1\cdot\mathbf{S}_2)g^{ij}{}_2\partial_{i,2}\rho + \cdots$$
$$= W\left\{\partial_{j,2}\left[-x - \frac{1}{2}(\mathbf{r}\cdot\mathbf{R})x_1 + \cdots\right]\right\}g^{ij}{}_2\partial_{i,2}\rho + \cdots$$
$$= Wg^{i1}{}_2\partial_{i,2}\rho + W\left[x_1x_2\frac{\partial\rho}{\partial x_2} + x_1y_2\frac{\partial\rho}{\partial y_2} + x_1z_2\frac{\partial\rho}{\partial z_2}\right] + \cdots$$
$$= \left(1-\frac{1}{2}r^2\right)\left[g^{11}{}_2\frac{\partial\rho}{\partial x_2} + g^{21}{}_2\frac{\partial\rho}{\partial y_2} + g^{31}{}_2\frac{\partial\rho}{\partial z_2}\right] + \left[x_1x_2\frac{\partial\rho}{\partial x_2} + x_1y_2\frac{\partial\rho}{\partial y_2} + x_1z_2\frac{\partial\rho}{\partial z_2}\right] + \cdots$$
$$= \frac{\partial\rho}{\partial x_2} - \frac{1}{2}r^2\frac{\partial\rho}{\partial x_2} - \left[x_2^2\frac{\partial\rho}{\partial x_2} + x_2y_2\frac{\partial\rho}{\partial y_2} + x_2z_2\frac{\partial\rho}{\partial z_2}\right] + \left[x_1x_2\frac{\partial\rho}{\partial x_2} + x_1y_2\frac{\partial\rho}{\partial y_2} + x_1z_2\frac{\partial\rho}{\partial z_2}\right] + \cdots$$
$$= \frac{\partial\rho}{\partial x_2} - \frac{1}{2}r^2\frac{\partial\rho}{\partial x_2} + x\left[x_2\frac{\partial\rho}{\partial x_2} + y_2\frac{\partial\rho}{\partial y_2} + z_2\frac{\partial\rho}{\partial z_2}\right] + \cdots. \quad \text{(B5)}$$

We write

$$\hat{\boldsymbol{\zeta}}_{i,\ell} = (\zeta_{i,\ell}^{(1)}, \zeta_{i,\ell}^{(2)}, \zeta_{i,\ell}^{(3)}). \quad \text{(B6)}$$

Note that

$$\zeta_{x,1}^{(a)} = \frac{\partial}{\partial x_1}(S_1^{(0)}S_2^{(a)} - S_2^{(0)}S_1^{(a)}) = \frac{-x_1}{\sqrt{1-r_1^2}}S_2^{(a)} - \sqrt{1-r_2^2}\frac{\partial S_1^{(a)}}{\partial x_1}. \quad \text{(B7)}$$

Therefore, we have

$$\hat{\boldsymbol{\zeta}}_{x,1} = \left(\frac{-x_1x_2}{\sqrt{1-r_1^2}} - \sqrt{1-r_2^2}, \frac{-x_1y_2}{\sqrt{1-r_1^2}}, \frac{-x_1z_2}{\sqrt{1-r_1^2}}\right) = \left(-x_1x_2 - \sqrt{1-r_2^2}, -x_1y_2, -x_1z_2\right) + \cdots. \quad \text{(B8)}$$

In the same way,

$$\hat{\boldsymbol{\zeta}}_{y,1} = \left(-y_1x_2, -y_1y_2 - \sqrt{1-r_2^2}, -y_1z_2\right) + \cdots \quad \text{(B9)}$$

and

$$\hat{\boldsymbol{\zeta}}_{z,1} = \left(-z_1x_2, -z_1y_2, -z_1z_2 - \sqrt{1-r_2^2}\right) + \cdots. \quad \text{(B10)}$$

From these results, we obtain

$$\hat{\boldsymbol{\zeta}}_{x,1}\cdot\hat{\boldsymbol{\zeta}}_{x,1} = 1 - r_2^2 + 2x_1x_2 + \cdots, \quad \text{(B11)}$$

$$\hat{\boldsymbol{\zeta}}_{x,1}\cdot\hat{\boldsymbol{\zeta}}_{y,1} = y_1x_2 + x_1y_2 + \cdots, \quad \text{(B12)}$$

and

$$\hat{\boldsymbol{\zeta}}_{x,1}\cdot\hat{\boldsymbol{\zeta}}_{z,1} = z_1x_2 + x_1z_2 + \cdots. \quad \text{(B13)}$$

Using these, we have

$$\begin{aligned}\hat{\zeta}_{x,1}\cdot \mathrm{div}_1(\hat{\zeta}_1\rho) &= \hat{\zeta}_{x,1}\cdot g^{ij}{}_1\hat{\zeta}_{j,1}\partial_{i,1}\rho+\cdots\\ &=(1-r_2^2+2x_1x_2)g^{i1}{}_1\partial_{i,1}\rho+(y_1x_2+x_1y_2)g^{i2}{}_1\partial_{i,1}\rho+(z_1x_2+x_1z_2)g^{i3}{}_1\partial_{i,1}\rho+\cdots\\ &=(1-r_2^2+2x_1x_2)\left[(1-x_1^2)\frac{\partial\rho}{\partial x_1}-x_1y_1\frac{\partial\rho}{\partial y_1}-x_1z_1\frac{\partial\rho}{\partial z_1}\right]+(y_1x_2+x_1y_2)\frac{\partial\rho}{\partial y_1}+(z_1x_2+x_1z_2)\frac{\partial\rho}{\partial z_1}+\cdots\\ &=\frac{\partial\rho}{\partial x_1}-r_2^2\frac{\partial\rho}{\partial x_1}-x_1\left(x\frac{\partial\rho}{\partial x_1}+y\frac{\partial\rho}{\partial y_1}+z\frac{\partial\rho}{\partial z_1}\right)+x_2\left(x_1\frac{\partial\rho}{\partial x_1}+y_1\frac{\partial\rho}{\partial y_1}+z_1\frac{\partial\rho}{\partial z_1}\right)+\cdots.\end{aligned}\tag{B14}$$

In the same way,

$$\hat{\zeta}_{x,2}=(x_1x_2+\sqrt{1-r_1^2},x_2y_1,x_2z_1)+\cdots,\tag{B15}$$

$$\hat{\zeta}_{y,2}=(y_2x_1,y_1y_2+\sqrt{1-r_1^2},y_2z_1)+\cdots,\tag{B16}$$

and

$$\hat{\zeta}_{z,2}=(z_2x_1,z_2y_1,z_1z_2+\sqrt{1-r_1^2})+\cdots.\tag{B17}$$

Combining these with (B8)–(B10), we obtain

$$\hat{\zeta}_{x,1}\cdot\hat{\zeta}_{x,2}=-(1-\tfrac{1}{2}r_1^2-\tfrac{1}{2}r_2^2+2x_1x_2)+\cdots,\tag{B18}$$

$$\hat{\zeta}_{x,1}\cdot\hat{\zeta}_{y,2}=-2x_1y_2+\cdots,\tag{B19}$$

and

$$\hat{\zeta}_{x,1}\cdot\hat{\zeta}_{z,2}=-2x_1z_2+\cdots.\tag{B20}$$

Using these, we have

$$\begin{aligned}\hat{\zeta}_{x,1}\cdot \mathrm{div}_2(\hat{\zeta}_2\rho) &= \hat{\zeta}_{x,1}\cdot g^{ij}{}_2\hat{\zeta}_{j,2}\partial_{i,2}\rho+\cdots\\ &=-\left(1-\frac{1}{2}r_1^2-\frac{1}{2}r_2^2+2x_1x_2\right)g^{i1}{}_2\partial_{i,2}\rho-2x_1y_2g^{i2}{}_2\partial_{i,2}\rho-2x_1z_2g^{i3}{}_2\partial_{i,2}\rho+\cdots\\ &=-g^{i1}{}_2\partial_{i,2}\rho+\frac{1}{2}(r_1^2+r_2^2)\frac{\partial\rho}{\partial x_2}-2x_1\left(x_2\frac{\partial\rho}{\partial x_2}+y_2\frac{\partial\rho}{\partial y_2}+z_2\frac{\partial\rho}{\partial z_2}\right)+\cdots\\ &=-\frac{\partial\rho}{\partial x_2}+\frac{1}{2}(r_1^2+r_2^2)\frac{\partial\rho}{\partial x_2}-\left(\frac{3}{2}x+\frac{1}{2}X\right)\left(x_2\frac{\partial\rho}{\partial x_2}+y_2\frac{\partial\rho}{\partial y_2}+z_2\frac{\partial\rho}{\partial z_2}\right)+\cdots.\end{aligned}\tag{B21}$$

Substituting (4.6), (B4), (B5), (B14), and (B21) into (3.22), we obtain the expansion (4.9).

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