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Restudy on the wave functions of $\Upsilon(nS)$ states in the light-front quark model and the radiative decays of $\Upsilon(nS) \to \eta_b + \gamma$

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The light-front quark model has been applied to calculate the transition matrix elements of heavy hadron decays. However, it is noted that using the traditional wave functions of the light-front quark model given in the literature, the theoretically determined decay constants of the $\Upsilon(nS)$ obviously contradict the data. This implies that the wave functions must be modified. Keeping the orthogonality among the nS states and fitting their decay constants, we obtain a series of the wave functions for $\Upsilon(nS)$. Based on these wave functions and by analogy with the hydrogen atom, we suggest a modified analytical form for the $\Upsilon(nS)$ wave functions. Using the modified wave functions, the obtained decay constants are close to the experimental data. Then we calculate the rates of radiative decays of $\Upsilon(nS) \to \eta_b + \gamma$. Our predictions are consistent with the experimental data on decays $\Upsilon(3S) \to \eta_b + \gamma$ within the theoretical and experimental errors.

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I. INTRODUCTION

Since the relativistic and higher-order α_s corrections are less important for bottomonia than for any other $q\bar{q}$ systems, studies on bottomonia may offer more direct information about the hadron configuration and application of the perturbative QCD. The key problem is how to deal with the hadronic transition matrix elements which are fully governed by the nonperturbative QCD effects. Many phenomenological models have been constructed and applied. Each of them has achieved relative successes, but since none of them are based on any well-established underlying theories, their model parameters must be obtained by fitting data. By doing so, some drawbacks of the model are exposed when dealing with different phenomenological processes. Thus one needs to continuously modify the model or refit its parameters, if not completely negate it. The light-front quark model (LFQM) is one such model. It has been applied to calculate the hadronic transitions and is generally considered successful. The model contains a Gaussian-type wave function whose parameters should be determined in a certain way.

The Gaussian-type wave function was recommended by the authors of Refs. [1,2], and most frequently, the wave function for the harmonic oscillator is adopted, which we refer to as the traditional LFQM wave function. As we employed the traditional LFQM wave functions to calculate the branching ratios of $\Upsilon(nS) \to \eta_b + \gamma$, some obvious contradictions between the theoretical predictions and experimental data emerged. Namely, the predicted $\mathcal{B}(\Upsilon(2S) \to \eta_b + \gamma)$ was one order larger than the experimental upper bound [3]. Moreover, as one carefully investigates the wave functions, one would face a serious problem. If the traditional wave functions were employed, the decay constants of $\Upsilon(nS)$ (f_V) would increase for higher n. It obviously contradicts the experimental data and the physics picture, which tells us that the decay constant of an nS state is proportional to its wave function at the origin, which manifests the probability that the two constituents spatially merge, so for excited states the probability should decrease. Thus the decay constants should be smaller when n is larger. The experimental data confirm this trend. But the theoretical calculations with the traditional wave functions result in an inverse order. To overcome these problems, one may adopt different model parameters (β) by fitting individual n decay constants as done in [3,4], but the orthogonality among the nS states is broken. In this work, we try to modify the harmonic oscillator functions and introduce an explicit n-dependent form for the wave functions. Keeping the orthogonality among the nS states $(n = 1, \ldots, 5)$, we modify the LFQM wave functions. By fitting the decay constants of $\Upsilon(nS)$, the concerned model parameters are fixed.

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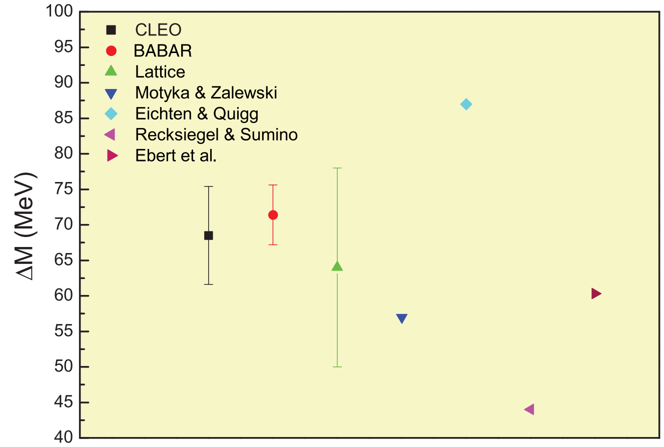

FIG. 1 (color online). ΔM coming from different experimental measurement and theoretical work.

Besides fitting the decay constants of the $\Upsilon(nS)$ family, one should test the applicability of the model in other processes. We choose the radiative decays of $\Upsilon(nS) \to \eta_b + \gamma$ as the probe. As a matter of fact, those radiative decays are of great significance for understanding the hadronic structure of the bottomonia family.

Indeed, the spin-triplet state of bottomonia $\Upsilon(nS)$ and the P states $\chi_b(nP)$ were discovered decades ago; however, the singlet state η_b evaded detection for a long time, even though much effort was made. Much phenomenological research on η_b has been done by some groups [5–12]. Different theoretical approaches result in different level splitting $\Delta M = \Upsilon(1S) - \eta_b(1S)$ (see Fig. 1). In [5] the authors used an improved perturbative QCD approach to get $\Delta M = 44$ MeV; using the potential model suggested in [13], Eichten and Quigg estimated $\Delta M = 87$ MeV [6]; in Ref. [7] the authors selected a nonrelativistic Hamiltonian with spin-dependent corrections to study the spectra of heavy quarkonia and got $\Delta M = 57$ MeV; the lattice prediction is $\Delta M = 51$ MeV [8], whereas the lattice result calculated in Ref. [9] was $\Delta M = 64 \pm 14$ MeV. Ebert *et al.* [10] directly studied spectra of heavy quarkonia in the relativistic quark model and gave $m_{\eta_b} = 9.400$ GeV. The dispersion of the values may imply that there exist some ambiguities in our understanding of the structures of the $b\bar{b}$ family.

The *BABAR* Collaboration [14] first measured $\mathcal{B}(\Upsilon(3S) \to \gamma\eta_b) = (4.8 \pm 0.5 \pm 0.6) \times 10^{-4}$, and determined $m_{\eta_b} = 9388.9^{+3.1}_{-2.3} \pm 2.7$ MeV, $\Delta M = 71.4^{+3.1}_{-2.3} \pm 2.7$ MeV in 2008. New data, $m_{\eta_b} = 9394.2^{+4.8}_{-4.9} \pm 2.0$ MeV and $\mathcal{B}(\Upsilon(2S) \to \gamma\eta_b) = (3.9 \pm 1.1^{+1.1}_{-0.9}) \times 10^{-4}$, were released in 2009 [15]. More recently, the CLEO Collaboration [16] confirmed the observation of η_b using a database of 6×10^6 $\Upsilon(3S)$ decays and assuming $\Gamma(\eta_b) \approx 10$ MeV; they obtained $\mathcal{B}(\Upsilon(3S) \to \gamma\eta_b) = (7.1 \pm 1.8 \pm 1.1) \times 10^{-4}$, $m_{\eta_b} = 9391.8 \pm 6.6 \pm 2.0$ MeV, and the hyperfine splitting $\Delta M = 68.5 \pm 6.6 \pm 2.0$ MeV, whereas, using a database with 9×10^6 $\Upsilon(2S)$ decays, they obtained $\mathcal{B}(\Upsilon(2S) \to \gamma\eta_b) < 8.4 \times 10^{-4}$ at the 90% confidential level. It is noted that the data of the two collaborations are in agreement on m_{η_b}, but the central values of $\mathcal{B}(\Upsilon(3S) \to \gamma\eta_b)$ are different. However, if the experimental errors are taken into account, the difference is within 1 standard deviation.

Some theoretical works [17–19] are devoted to accounting for the experimental results. In Ref. [10] the authors studied these radiative decays and estimated $\mathcal{B}(\Upsilon(3S) \to \eta_b + \gamma) = 4 \times 10^{-4}$, $\mathcal{B}(\Upsilon(2S) \to \eta_b + \gamma) = 1.5 \times 10^{-4}$, and $\mathcal{B}(\Upsilon(1S) \to \eta_b + \gamma) = 1.1 \times 10^{-4}$ with the mass $m_{\eta_b} = 9.400$ GeV. Their results for m_{η_b} and $\mathcal{B}(\Upsilon(3S) \to \eta_b + \gamma)$ are close to the data. The authors of Ref. [20] systematically investigated the magnetic dipole transition $V \to P\gamma$ in the LFQM [1,2,21,22]. In the QCD-motivated approach there are several free parameters, i.e., the quark mass and β in the wave function (the notation of β was given in the aforementioned literatures), which are fixed by the variational principle; then $\mathcal{B}(\Upsilon(1S) \to \eta_b + \gamma)$ was calculated, and the central value is $8.4(\text{or}\,7.7) \times 10^{-4}$.[1] It is also noted that the mass of $m_{\eta_b} = 9.657(\text{or}\,9.295)$ GeV presented in Ref. [20] deviates from the data listed before, so we are going to refix the parameter β in other ways; namely, we fix the parameter β by fitting the data.

Since experimentally m_{η_b} is determined by $\mathcal{B}(\Upsilon(nS) \to \eta_b + \gamma)$ and a study of the radiative decays can offer us much information about the characteristics of η_b, one should carefully investigate the transition within a relatively reliable theoretical framework. That is the aim of the present work; namely, we will evaluate the hadronic matrix element in terms of our modified LFQM.

This paper is organized as follows. In Sec. II we discuss how to modify the traditional wave functions in the LFQM. We present the formula to calculate the form factors for $V \to P\gamma$ in the LFQM and give numerical results in Sec. III. Section IV is devoted to our conclusion and a discussion.

II. THE MODIFIED WAVE FUNCTIONS FOR THE RADIALLY EXCITED STATES

When the LFQM is employed to calculate the decay constants and form factors, one needs the wave functions of the concerned hadrons. In most cases, the wave functions of the harmonic oscillator are adopted. In the works [1,2,20–23], only the wave function of the radial ground state is needed, but when in the processes under consideration radially excited states are involved, their wave functions should also be available. In [24,25], the traditional wave functions φ for the $1S$ and $2S$ states in configuration space from the harmonic oscillator are given as

[1]The different values correspond to the different potentials adopted in the calculations.

$$\varphi^{1S}(r) = \left(\frac{\beta^2}{\pi}\right)^{3/4} \exp\left(-\frac{1}{2}\beta^2\mathbf{r}^2\right),$$
$$\varphi^{2S}(r) = \left(\frac{\beta^2}{\pi}\right)^{3/4} \exp\left(-\frac{1}{2}\beta^2\mathbf{r}^2\right)\frac{1}{\sqrt{6}}(3 - 2\beta^2\mathbf{r}^2). \tag{1}$$

In order to maintain the orthogonality among nS states, the parameter β in the above two functions is the same. The wave functions for other nS states can be found in Appendix .

The decay constants of the nS states are directly proportional to the wave function at the origin,

$$f_V \propto \varphi(r = 0). \tag{2}$$

If we simply adopt the wave functions of the harmonic oscillator for all of them, as we do for the $1S$ state, then we find the wave functions at the origin; i.e. $\varphi(r = 0)$ (see the Appendix for details) rises with an increase of n (the principle quantum number), which means the decay constants would increase for larger n. For example, from Eq. (1) the ratio of wave functions of the $2S$ and $1S$ states at the origin is $3/\sqrt{6} > 1$.

The decay constants f_V of $\Upsilon(nS)$ are extracted from the processes $\Gamma(\Upsilon(nS) \to e^+e^-)$ with

$$\Gamma(V \to e^+e^-) = \frac{4\pi}{27}\frac{\alpha^2}{m_V} f_V^2, \tag{3}$$

where V represents $\Upsilon(nS)$ and m_V is its mass. Using the experimental data from PDG [26], we obtain the experimental values for f_V which are listed in Table I. Obviously, the decay constant becomes smaller as n becomes larger.

In the LFQM, the formula for calculating the vector meson decay constant is given by [1,2]

$$f_V = \frac{\sqrt{N_c}}{4\pi^3 M}\int dx \int d^2k_\perp \frac{\phi(nS)}{\sqrt{2x(1-x)}\tilde{M}_0} \times \left[xM_0^2 - m_1(m_1 - m_2) - k_\perp^2 + \frac{m_1 + m_2}{M_0 + m_1 + m_2}k_\perp^2\right], \tag{4}$$

TABLE I. The decay constants of $\Upsilon(nS)$ (in units of MeV). The column "f_Υ^T" represents the theoretical predictions with the traditional wave function in the LFQM. The column "f_Υ^M" represents the prediction with our modified wave function, and the values in parentheses are the corresponding values with $m_b = 4.8$ GeV as input. (The other values correspond to $m_b = 5.2$ GeV.)

nS	$f_\Upsilon^{\exp}$	f_Υ^T	f_Υ^M
$1S$	715 ± 5	715 ± 5	715 ± 5 (715 ± 5)
$2S$	497 ± 5	841 ± 7	497 ± 5 (498 ± 5)
$3S$	430 ± 4	925 ± 8	418 ± 5 (419 ± 4)
$4S$	340 ± 19	993 ± 8	378 ± 4 (397 ± 4)
$5S$	369 ± 42	1040 ± 9	349 ± 4 (351 ± 4)

where $m_1 = m_2 = m_b$, and other notations are collected in the Appendix. In the calculation we set $m_b = 5.2$ GeV following [20], and the decay constant of $\Upsilon(1S)$ is used to determine the parameter β_Υ as the input. We obtain $\beta_\Upsilon = 1.257 \pm 0.006$ GeV, corresponding to $f_{\Upsilon(1S)}^{\exp} = 715 \pm 5$ MeV. In order to illustrate the dependence of our results on m_b, we reset $m_b = 4.8$ GeV to repeat our calculation; then by fitting the same data, we fix $\beta_\Upsilon = 1.288 \pm 0.006$ GeV, and all the results are clearly shown in the following tables. The f_Υ^T's in Table I are the decay constants calculated in the traditional wave functions. These results expose an explicit contradictory trend. Thus, our calculation indicates that if the traditional wave functions are used, the obtained decay constants of $\Upsilon(nS)$ would sharply contradict the experimental data.

As aforementioned, the wave functions must be modified. Our strategy is to establish a new Gaussian-type wave function which is different from that of the harmonic oscillator. When modifying the wave functions, several principles must be respected:

(1) The wave function of $1S$ should not change because its application for dealing with various processes has been tested and the results indicate that it works well.
(2) The number of nodes of nS should not be changed.
(3) A factor may be added to the wave functions which should uniquely depend on n in analogy to the wave function of the hydrogen-like atoms, which is written as $R_n(r) = P_n^{\text{hydr}}(r)e^{-(Zr/na_0)}$, where $P_n(r)$ is a polynomial, Z is the atomic number, and a_0 is the Bohr radius.
(4) Using the new Gaussian-type wave function, the contradiction for the decay constants can be solved.

In the LFQM, we only need the wave functions in momentum space. Fourier transformation gives us the corresponding forms in momentum space; see the Appendix for details. The $1S$ wave function remains and is used to fix the model parameter. Now let us investigate the wave function of $2S$. According to the analogy of the hydrogen-like atom, we introduce a factor g_2 which represents the n dependence of the exponential in the wave function of $2S$; thus the wave function of $2S$ is changed to

$$\psi_M^{2S}(\mathbf{p}^2) = \left(\frac{\pi}{\beta^2}\right)^{3/4} \exp\left(-g_2\frac{\mathbf{p}^2}{2\beta}\right)\left(a + b\frac{\mathbf{p}^2}{\beta^2}\right), \tag{5}$$

where the subscript M denotes the modified function. Then by requiring it to be orthogonal to that of $1S$ and normalizing the wave function, we determine the parameters a and b in the modified wave function of $2S$. With this new wave function of $2S$, we demand the theoretical decay constant to be consistent with data, so g_2 should fall into a range determined by the experimental errors. Next, we obtain the modified wave function of $3S$, and those for $4S$ and $5S$ as well. In this case the modified wave functions of the nS states are more complicated than the traditional ones.

We have gained a series of numerical g_n's by the principles we discussed above. Next, we wish to guess an analytical factor g_n which is close to the numerical values of the series. We find that if $g_n = n^{\delta}(\delta = 1/1.82)$ is set, we almost recover the numerical series. Thus the wave function of the nS state in momentum space can be written as

$$\psi_M^{nS}(\mathbf{p}^2) = P_n(\mathbf{p}^2)\exp\left(-n^{\delta}\frac{\mathbf{p}^2}{2\beta^2}\right), \tag{6}$$

where $P_n(\mathbf{p}^2)$ is a polynomial in $\mathbf{p}^2$. The corresponding wave function of the nS state in configuration space can be written as

$$\psi_M^{nS}(r) = P'_n(\mathbf{r}^2)\exp\left(-\frac{\beta^2\mathbf{r}^2}{2n^{\delta}}\right). \tag{7}$$

Comparing with the case of the hydrogen-like atoms, the nS-wave functions are written as

$$R_{n0}(r) = P_n^{\mathrm{hydr}}(r)\exp\left(\frac{-Zr}{na_0}\right) \tag{8}$$

in the configuration space, where $P_n^{\mathrm{hydr}}(r)$ is a polynomial in r. The factor $1/n$ in the exponential power is obtained by solving the Schrödinger equation, where only the Coulomb potential exists. To modify the wave functions we get the factors numerically for all the nS states, then "guess" its analytical form. In the LFQM, the factor $1/n^{\delta}$ is introduced to fit the experimental data for nS decay constants. This analytical form is definitely not derived from an underlying theory, such as that for the hydrogen atom; thus the dependence on n is only an empirical expression. But we are sure that if the model is correct and our guess is reasonable, it should be obtained from QCD (maybe nonperturbative QCD). It is noted that the experimental errors are large, so that other forms for g_n might also be possible. The theoretical estimation of the decay constants of $\Upsilon(nS)$ (f_{Υ}^{M}) are also presented in Table I. The modified wave functions seem to work well, and they could be used for evaluating $\mathcal{B}(\Upsilon(nS)\to\eta_b+\gamma)$.

III. THE TRANSITION OF $\Upsilon(nS)\to\eta_b+\gamma$

In this section, we calculate the branching ratios of $\Upsilon(nS)\to\eta_b+\gamma$ in terms of the modified wave functions derived in the above section.

A. Formulation of $\Upsilon(nS)\to\eta_b+\gamma$ in the LFQM

The Feynman diagrams describing $\Upsilon(nS)\to\eta_b+\gamma$ are plotted in Fig. 2. The transition amplitude of $\Upsilon(nS)\to\eta_b+\gamma$ can be expressed in terms of the form factor $\mathcal{F}_{\Upsilon(nS)\to\eta_b}(q^2)$, which is defined as [20,21]

$$\begin{aligned}&\langle\eta_b(\mathcal{P}')|J_{\mathrm{em}}^{\mu}|\Upsilon(\mathcal{P},h)\rangle\\&\quad= ie\varepsilon^{\mu\nu\rho\sigma}\epsilon_{\nu}(\mathcal{P},h)q_{\rho}\mathcal{P}_{\sigma}\mathcal{F}_{\Upsilon(nS)\to\eta_b}(q^2),\end{aligned} \tag{9}$$

where $\mathcal{P}$ and $\mathcal{P}'$ are the four-momenta of $\Upsilon(nS)$ and η_b. $q=\mathcal{P}-\mathcal{P}'$ is the four-momentum of the emitted photon and $\epsilon_{\nu}(\mathcal{P},h)$ denotes the polarization vector of $\Upsilon(nS)$ with helicity h. For applying the LFQM, we first let the photon be virtual, i.e. leave its mass shell $q^2=0$ in the unphysical region of $q^2<0$. Then $\mathcal{F}_{\Upsilon(nS)\to\eta_b}(q^2)$ can be obtained in the $q^+=0$ frame with $q^2=q^+q^- - \mathbf{q}_{\perp}^2 = -\mathbf{q}_{\perp}^2<0$. Then we just analytically extrapolate $\mathcal{F}_{\Upsilon(nS)\to\eta_b}(\mathbf{q}_{\perp}^2)$ from the spacelike region to the timelike region ($q^2\geq 0$). By taking the limit $q^2\to 0$, one obtains $\mathcal{F}_{\Upsilon(nS)\to\eta_b}(q^2=0)$.

By means of the light-front quark model, one can obtain the expression of the form factor $\mathcal{F}_{\Upsilon(nS)\to\eta_b}(q^2)$ [20]:

$$\mathcal{F}_{\Upsilon(nS)\to\eta_b}(q^2) = e_b I(m_1,m_2,q^2) + e_b I(m_2,m_1,q^2), \tag{10}$$

where e_b is the electrical charge for the bottom quark, $m_1=m_2=m_b$, and

$$\begin{aligned}I(m_1,m_2,q^2) = \int_0^1\frac{dx}{8\pi^3}\int d^2\mathbf{k}_{\perp}\frac{\phi(x,\mathbf{k'}_{\perp})\phi(x,\mathbf{k}_{\perp})}{x_1\tilde{M}_0\tilde{M}'_0}\\ \times\left\{\mathcal{A}+\frac{2}{\mathcal{M}_0}\left[\mathbf{k}_{\perp}^2-\frac{(\mathbf{k}_{\perp}\cdot\mathbf{q}_{\perp})^2}{\mathbf{q}_{\perp}^2}\right]\right\},\end{aligned} \tag{11}$$

where $\mathcal{A}=x_2m_1+x_1m_2$, $x=x_1$, and the other variables in Eq. (11) are defined in the Appendix. In the covariant

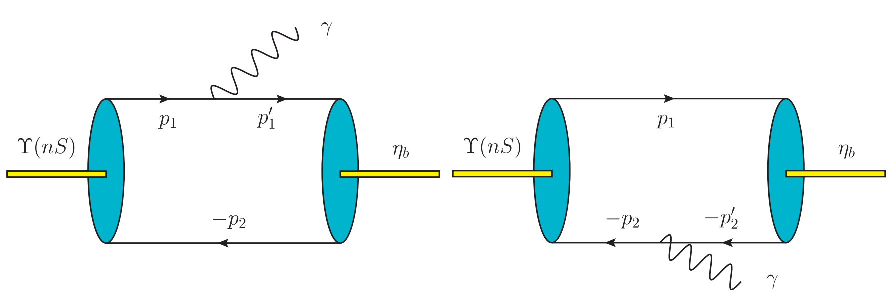

FIG. 2 (color online). Feynman diagrams depicting the radiative decay $\Upsilon(nS)\to\eta_b+\gamma$.

light-front quark model, the authors of [21] obtained the same form factor $\mathcal{F}_{\Upsilon(nS)\to\eta_b}(\mathbf{q}^2)$. The decay width for $\Upsilon(nS)\to\eta_b+\gamma$ is easily achieved,

$$\Gamma(\Upsilon(nS)\to\eta_b+\gamma)=\frac{\alpha}{3}\left[\frac{m^2_{\Upsilon(nS)}-m^2_{\eta_b}}{2m_{\Upsilon(nS)}}\right]^3\mathcal{F}^2_{\Upsilon(nS)\to\eta_b}(0), \tag{12}$$

where α is the fine-structure constant and $m_{\Upsilon(nS)}$ and m_{η_b} are the masses of $\Upsilon(nS)$ and η_b, respectively.

B. Numerical results

Now we begin to evaluate the transition rates of $\Upsilon(2S)\to\eta_c+\gamma$ with the modified wave functions. We still use the values of $m_b=5.2$ GeV and $\beta_\Upsilon=1.257\pm0.006$ GeV given in the last section. The parameter β_{η_b} is unknown; we determine it from the $\Upsilon(2S)\to\gamma\eta_b$ process. Comparing with the data $\mathcal{B}(\Upsilon(2S)\to\gamma\eta_b)=3.9\times10^{-4}$ [15], we obtain $\beta_{\eta_b}=1.246\pm0.005$ GeV, which is consistent with our expectation; namely, it is close to the value of $\beta_\Upsilon=1.257$ GeV. Under the heavy quark limit, they should be exactly equal, and the deviation must be of order $\mathcal{O}(1/m_b)$, which is small [27]. With these parameters, we can calculate the branching ratios $\mathcal{B}(\Upsilon(1S)\to\eta_b+\gamma)$, $\mathcal{B}(\Upsilon(3S)\to\eta_b+\gamma)$, $\mathcal{B}(\Upsilon(4S)\to\eta_b+\gamma)$, and $\mathcal{B}(\Upsilon(5S)\to\eta_b+\gamma)$. The numerical results are presented in the column "$\mathcal{B}_{\rm I}^M$" of Table II. Indeed, the b-quark mass is an uncertain parameter which cannot be directly measured, and in some literatures, different values for the b-quark mass have been adopted. To see how sensitive the result would be to the b-quark mass, we also present the numerical results with $m_b=4.8$ GeV, $\beta_\Upsilon=1.288\pm0.006$ GeV, and $\beta_{\eta_b}=1.287\pm0.005$ GeV in the column "$\mathcal{B}_{\rm II}^M$" of Table II. The results in the column "$\mathcal{B}^T$" of Table II are obtained with the traditional wave functions. Apparently, as the modified wave functions are employed, the theoretical predictions on the branching ratios of the radiative decays are much improved; namely, deviations from the data are diminished. About , Some comments are given about the numerical results, as follows:

(1) Comparing the results shown in column $\mathcal{B}_{\rm I}^M$ with those in column $\mathcal{B}_{\rm II}^M$, we find that they are not sensitive to m_b.

(2) For the decay $\Upsilon(1S)\to\eta_b+\gamma$, our prediction of the branching ratio is about 2.0×10^{-4}. This mode should be observed soon in the coming experiment. Our prediction is consistent with the results of Refs. [10,20]. The branching ratio is not sensitive to β_{η_b}, but it is sensitive to the mass splitting ΔM. This is easy to understand. The decay width is proportional to $(\Delta M)^3$; thus as ΔM is small, i.e., the masses of initial and daughter mesons are close to each other, any small change of m_{η_b} can lead to a remarkable difference in the theoretical prediction on the branching ratio. Thus the accurate measurement on $\mathcal{B}(\Upsilon(1S)\to\eta_b+\gamma)$ will be a great help to determine the mass of m_{η_b}.

(3) The process of $\Upsilon(2S)\to\eta_b+\gamma$ is used as an input to determine the parameter of η_b. The prediction of $\Upsilon(3S)\to\eta_b+\gamma$ is in accordance with the experimental data by an order of magnitude. After taking into account the experimental and theoretical errors, they can be consistent. This result could be of relatively large errors, because we only use four parameters $(m_b,\beta_\Upsilon,\beta_{\eta_b},\alpha)$ to determine five decay constants and three branching ratios for $\Upsilon(1S,2S,3S)\to\eta_b+\gamma$, and all of them possess certain errors.

(4) The branching ratios for the processes $\Upsilon(4S)\to\eta_b+\gamma$ and $\Upsilon(5S)\to\eta_b+\gamma$ are at the order of 10^{-8}; they will be nearly impossible to observe in the near future if there are not other mechanisms to enhance them.

(5) As an application, we predict the decay constant of η_b in terms of the model parameters we obtained above. We calculate the branching ratio of $\mathcal{B}(\Upsilon(2S)\to\gamma\eta_b)$ in the LFQM. By fitting data we fix the concerned model parameters for η_b, and then with them we predict the decay constant of η_b in the same framework of the LFQM [1,2]. In the calculations, the b-quark mass m_b and β_{η_b} are input parameters.

TABLE II. The branching ratios of $\Upsilon(nS)\to\gamma\eta_b$. In the column "$\mathcal{B}_{\rm I}^M$," $m_b=5.2$ GeV, $\beta_\Upsilon=1.257\pm0.006$ GeV, and $\beta_{\eta_b}=1.246\pm0.005$ GeV. In the column "$\mathcal{B}_{\rm II}^M$," $m_b=4.8$ GeV, $\beta_\Upsilon=1.288\pm0.006$ GeV, and $\beta_{\eta_b}=1.287\pm0.005$ GeV. In the column "$\mathcal{B}^T$," $m_b=5.2$ GeV, $\beta_\Upsilon=1.257\pm0.006$ GeV, and $\beta_{\eta_b}=1.249\pm0.005$ GeV.

Decay mode	$\mathcal{B}_{\rm I}^M$	$\mathcal{B}_{\rm II}^M$	$\mathcal{B}^T$	Experiment
$\Upsilon(1S)\to\eta_b+\gamma$	$(1.94\pm0.41)\times10^{-4}$	$(2.24\pm0.47)\times10^{-4}$	$(1.94\pm0.42)\times10^{-4}$	...
$\Upsilon(2S)\to\eta_b+\gamma$	$(3.90\pm1.49)\times10^{-4}$	$(3.90\pm1.49)\times10^{-4}$	$(3.90\pm1.49)\times10^{-4}$	$(3.9\pm1.1^{+1.1}_{-0.9})\times10^{-4}$ [15]
$\Upsilon(3S)\to\eta_b+\gamma$	$(1.87\pm0.71)\times10^{-4}$	$(1.68\pm0.72)\times10^{-4}$	$(1.05\pm0.40)\times10^{-5}$	$(4.8\pm0.5\pm0.6)\times10^{-4}$ [14]
				$(7.1\pm1.8\pm1.1)\times10^{-4}$ [16]
$\Upsilon(4S)\to\eta_b+\gamma$	$(8.81\pm3.32)\times10^{-8}$	$(7.82\pm3.35)\times10^{-8}$	$(2.25\pm0.88)\times10^{-10}$	...
$\Upsilon(5S)\to\eta_b+\gamma$	$(1.17\pm0.43)\times10^{-8}$	$(1.02\pm0.45)\times10^{-8}$	$(1.57\pm0.52)\times10^{-12}$	...

To show how sensitive the results are to the parameters, we use the two sets of input parameters given above, and the corresponding results are as follows: $f_{\eta_b} = 567$ MeV when $m_b = 5.2$ GeV and $\beta_{\eta_b} = 1.246$ GeV; $f_{\eta_b} = 604$ MeV when $m_b = 4.8$ GeV and $\beta_{\eta_b} = 1.287$ GeV. For a comparison, we deliberately change only m_b while keeping β_{η_b} unchanged to repeat the calculation, and obtain $f_{\eta_b} = 591$ MeV when $m_b = 5.2$ GeV and $\beta_{\eta_b} = 1.287$ GeV. It is noted that f_{η_b} is more sensitive to β_{η_b} than m_b.

IV. CONCLUSION

The LFQM has been successful in phenomenological applications. It is believed that it could be a reasonable model for dealing with the hadronic transitions where the nonperturbative QCD effects dominate. However, it seems that the wave function adopted in the previous literature has to be modified. As we study the decay constant of $\Upsilon(nS)$, we find that there exists a sharp contradiction between the theoretical prediction and data as long as the traditional harmonic oscillator wave functions are employed. Namely, the larger n is, the larger the predicted decay constant would be. It obviously contradicts the physics picture that for higher radially excited states, the wave function at the origin should be smaller than the lower ones. But the old wave functions would result in an inverse tendency. If enforcing all the decay constants of $\Upsilon(nS)$ to be fitted to the data in terms of the traditional wave functions, the orthogonality among all the nS states must be abandoned, but this is not acceptable according to the basic principle of quantum mechanics.

Thus we modify the wave functions of the radially excited states based on the common principles. Namely, we keep the orthogonality among the wave functions and their proper normalization. Moreover, we require the wave functions $\varphi_M(r)$ at the origin $r = 0$ to be consistent with the data; i.e. the decay constants for higher n must be smaller than those of the lower states. Concretely, we modify the exponential function in the wave functions by demanding that the power not be universal for all n's, but that it be dependent on n. Concretely we add a numerical factor g_n into $\exp(g_n \frac{-\mathbf{p}^2}{2\beta^2})$, and by fitting the data of the decay constants of $\Upsilon(nS)$, we obtain a series of numbers of g_n. Within a reasonable error range, we approximate g_n as $g(n) = n^\delta$ and calculate the value for δ. This is an alternative way, which is different from that adopted in Ref. [20], to fix the parameter.

With the modified wave functions of $\Upsilon(nS)$, we calculate the branching ratios of $\Upsilon(nS) \to \eta_b + \gamma$ in the LFQM. First, by fitting the well-measured central value of $\mathcal{B}(\Upsilon(2S) \to \eta_b + \gamma)$ [15], we obtain the parameter β_{η_b}. By the effective heavy quark theory, in the heavy quark limit the spin singlet and triplet of the $b\bar{b}$ system should degenerate; namely, the parameters of $\beta_{\Upsilon(1S)}$ and β_{η_b} should be very close. Our numerical result confirms this requirement.

Then we estimate the other $\Upsilon(nS) \to \eta_b + \gamma$. The order of magnitudes of our numerical results is consistent with data. Even though the predicted branching ratios still do not precisely coincide with the data, the result is much improved. The branching ratios of the processes $\Upsilon(4S) \to \eta_b + \gamma$ and $\Upsilon(5S) \to \eta_b + \gamma$ are predicted to be at the order of 10^{-8}. They will be difficult to measure in the future as long as there is no new physical mechanism to greatly enhance them.

By studying the radiative decay of $\Upsilon(nS) \to \eta_b + \gamma$, we can learn much about the hadronic structure of η_b. Even though much effort has been made to explore the spin singlet η_b, in Ref. [26], η_b was still omitted from the summary table. In fact, the determination of the mass of η_b is made via the radiative decays of $\Upsilon(nS) \to \eta_b + \gamma$ [14], and the recent data show $m_{\eta_b} = 9388.9^{+3.1}_{-2.3}(\text{stat}) \pm 2.7(\text{syst})$ MeV with the $\Upsilon(3S)$ data and $m_{\eta_b} = 9394.2^{+4.8}_{-4.9}(\text{stat}) \pm 2.0(\text{syst})$ MeV with the $\Upsilon(2S)$ data [15]. Penin [28] reviewed the progress for determining the mass of η_b and indicated that the accurate theoretical prediction of m_{η_b} would be a great challenge. Indeed, determining the wave function of η_b would be even more challenging. We carefully study the transition rates of the radiative decays, which would help one to extract information about m_{η_b}. The transition rate of $\Upsilon(1S) \to \eta_b + \gamma$ is very sensitive to the mass splitting $\Delta M = m_{\Upsilon(1S)} - m_{\eta_b}$ due to the phase space constraint; thus an accurate measurement of the radiative decay may be more useful to learn the spin dependence of the bottomonia.

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APPENDIX

1. The radial wave functions

The traditional wave functions ϕ in configuration space from the harmonic oscillator [24] are

$$\begin{aligned}
\varphi^{1S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\beta^2\mathbf{r}^2\right),\\
\varphi^{2S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\beta^2\mathbf{r}^2\right)\frac{1}{\sqrt{6}}(3-2\beta^2\mathbf{r}^2),\\
\varphi^{3S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\beta^2\mathbf{r}^2\right)\sqrt{\frac{2}{15}}\left(\frac{15}{4}-5\beta^2\mathbf{r}^2+\beta^4\mathbf{r}^4\right),\\
\varphi^{4S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\mathbf{r}^2\beta^2\right)\frac{1}{12\sqrt{35}}(-105+210\mathbf{r}^2\beta^2-84\mathbf{r}^4\beta^4+8\mathbf{r}^6\beta^6),\\
\varphi^{5S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\mathbf{r}^2\beta^2\right)\frac{1}{72\sqrt{70}}(945-2520\beta^2\mathbf{r}^2+1512\beta^4\mathbf{r}^4-288\beta^6\mathbf{r}^6+16\beta^8\mathbf{r}^8),
\end{aligned} \tag{A1}$$

and their Fourier transformations are

$$\begin{aligned}
\psi^{1S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\frac{\mathbf{p}^2}{\beta^2}\right),\\
\psi^{2S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\frac{\mathbf{p}^2}{\beta^2}\right)\frac{1}{\sqrt{6}}\left(3-2\frac{\mathbf{p}^2}{\beta^2}\right),\\
\psi^{3S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\frac{\mathbf{p}^2}{\beta^2}\right)\sqrt{\frac{2}{15}}\left(\frac{15}{4}-5\frac{\mathbf{p}^2}{\beta^2}+\frac{\mathbf{p}^4}{\beta^4}\right),\\
\psi^{4S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\frac{\mathbf{p}^2}{\beta^2}\right)\frac{1}{12\sqrt{35}}\left(-105+210\frac{\mathbf{p}^2}{\beta^2}-84\frac{\mathbf{p}^4}{\beta^4}+8\frac{\mathbf{p}^6}{\beta^6}\right),\\
\psi^{5S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\frac{\mathbf{p}^2}{\beta^2}\right)\frac{1}{72\sqrt{70}}\left(945-2520\frac{\mathbf{p}^2}{\beta^2}+1512\frac{\mathbf{p}^4}{\beta^4}-288\frac{\mathbf{p}^6}{\beta^6}+16\frac{\mathbf{p}^8}{\beta^8}\right).
\end{aligned} \tag{A2}$$

The modified wave functions φ_M in configuration space are defined as

$$\begin{aligned}
\varphi_M^{1S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\beta^2\mathbf{r}^2\right),\\
\varphi_M^{2S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2\times 2^\delta}\beta^2\mathbf{r}^2\right)(a_2-b_2\beta^2\mathbf{r}^2),\\
\varphi_M^{3S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2\times 3^\delta}\beta^2\mathbf{r}^2\right)(a_3-b_3\beta^2\mathbf{r}^2+c_3\beta^4\mathbf{r}^4),\\
\varphi_M^{4S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2\times 4^\delta}\mathbf{r}^2\beta^2\right)(-a_4+b_4\mathbf{r}^2\beta^2-c_4\mathbf{r}^4\beta^4+d_4\mathbf{r}^6\beta^6),\\
\varphi_M^{5S}(r) &= \left(\frac{\beta^2}{\pi}\right)^{3/4}\exp\left(-\frac{1}{2\times 5^\delta}\mathbf{r}^2\beta^2\right)(a_5-b_5\beta^2\mathbf{r}^2+c_5\beta^4\mathbf{r}^4-d_5\beta^6\mathbf{r}^6+e_5\beta^8\mathbf{r}^8)
\end{aligned} \tag{A3}$$

with coefficients, which are irrational numbers that are kept to five digits after the decimal point.

n	a_n	b_n	c_n	d_n	e_n
2	0.728 17	0.408 57	...	...	...
3	0.629 20	0.541 38	0.067 12	...	...
4	0.578 34	0.618 87	0.128 38	0.006 14	...
5	0.547 47	0.676 21	0.183 32	0.015 58	0.000 38

The corresponding modified wave functions in momentum space are

$$
\begin{aligned}
\psi_M^{1S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{1}{2}\frac{\mathbf{p}^2}{\beta^2}\right),\\
\psi_M^{2S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{2^\delta}{2}\frac{\mathbf{p}^2}{\beta^2}\right)\left(a_2' - b_2'\frac{\mathbf{p}^2}{\beta^2}\right),\\
\psi_M^{3S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{3^\delta}{2}\frac{\mathbf{p}^2}{\beta^2}\right)\left(a_3' - b_3'\frac{\mathbf{p}^2}{\beta^2} + c_3'\frac{\mathbf{p}^4}{\beta^4}\right),\\
\psi_M^{4S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{4^\delta\mathbf{p}^2}{2\beta^2}\right)\left(-a_4' + b_4'\frac{\mathbf{p}^2}{\beta^2} - c_4'\frac{\mathbf{p}^4}{\beta^4} + d_4'\frac{\mathbf{p}^6}{\beta^6}\right),\\
\psi_M^{5S}(\mathbf{p}^2) &= \left(\frac{1}{\beta^2\pi}\right)^{3/4}\exp\left(-\frac{5^\delta}{2}\frac{\mathbf{p}^2}{\beta^2}\right)\left(a_5' - b_5'\frac{\mathbf{p}^2}{\beta^2} + c_5'\frac{\mathbf{p}^4}{\beta^4} - d_5'\frac{\mathbf{p}^6}{\beta^6} + e_5'\frac{\mathbf{p}^8}{\beta^8}\right)
\end{aligned} \tag{A4}
$$

with coefficients.

n	a_n'	b_n'	c_n'	d_n'	e_n'
2	1.886 84	1.549 43	...	...	...
3	2.537 64	5.674 31	1.856 52	...	...
4	3.1439	12.589 84	10.051 13	1.889 15	...
5	3.674 93	22.582 05	31.066 66	13.517 92	1.704 76

2. Some notations in the LFQM

The incoming (outgoing) meson in Fig. 2 has the momentum $P^{(\prime)} = p_1{}^{(\prime)} + p_2$, where $p_1{}^{(\prime)}$ and p_2 are the momenta of the off-shell quark and antiquark, and

$$
\begin{aligned}
&p_1^+ = x_1 P^+, \qquad p_2^+ = x_2 P^+, \qquad p_{1\perp} = x_1 P_\perp + k_\perp, \qquad p_{2\perp} = x_2 P_\perp - k_\perp,\\
&p_1'^+ = x_1 P^+, \qquad p_2'^+ = x_2 P^+, \qquad p'_{1\perp} = x_1 P'_\perp + k'_\perp, \qquad p'_{2\perp} = x_2 P'_\perp - k'_\perp
\end{aligned}
$$

with $x_1 + x_2 = 1$, where x_i and $k_\perp(k'_\perp)$ are internal variables. M_0 and $\tilde{M}_0$ are defined as

$$
M_0^2 = \frac{k_\perp^2 + m_1^2}{x_1} + \frac{k_\perp^2 + m_2^2}{x_2}, \qquad \tilde{M}_0 = \sqrt{M_0^2 - (m_1 - m_2)^2}.
$$

The wave functions ϕ_M are transformed into

$$
\begin{aligned}
\phi_M(1S) &= 4\left(\frac{\pi}{\beta^2}\right)^{3/4}\sqrt{\frac{\partial k_z}{\partial x}}\exp\left(-\frac{k_z^2 + k_\perp^2}{2\beta^2}\right),\\
\phi_M(2S) &= 4\left(\frac{\pi}{\beta^2}\right)^{3/4}\sqrt{\frac{\partial k_z}{\partial x}}\exp\left(-\frac{2^\delta}{2}\frac{k_z^2 + k_\perp^2}{\beta^2}\right)\left(a_2' - b_2'\frac{k_z^2 + k_\perp^2}{\beta^2}\right),\\
\phi_M(3S) &= 4\left(\frac{\pi}{\beta^2}\right)^{3/4}\sqrt{\frac{\partial k_z}{\partial x}}\exp\left(-\frac{3^\delta}{2}\frac{k_z^2 + k_\perp^2}{\beta^2}\right)\left(a_3' - b_3'\frac{k_z^2 + k_\perp^2}{\beta^2} + c_3'\frac{(k_z^2 + k_\perp^2)^2}{\beta^4}\right),\\
\phi_M(4S) &= 4\left(\frac{\pi}{\beta^2}\right)^{3/4}\sqrt{\frac{\partial k_z}{\partial x}}\exp\left(-\frac{4^\delta}{2}\frac{k_z^2 + k_\perp^2}{\beta^2}\right)\left(-a_4' + b_4'\frac{k_z^2 + k_\perp^2}{\beta^2} - c_4'\frac{(k_z^2 + k_\perp^2)^2}{\beta^4} + d_4'\frac{(k_z^2 + k_\perp^2)^3}{\beta^6}\right),\\
\phi_M(5S) &= 4\left(\frac{\pi}{\beta^2}\right)^{3/4}\sqrt{\frac{\partial k_z}{\partial x}}\exp\left(-\frac{5^\delta}{2}\frac{k_z^2 + k_\perp^2}{\beta^2}\right)\left(a_5' - b_5'\frac{k_z^2 + k_\perp^2}{\beta^2} + c_5'\frac{(k_z^2 + k_\perp^2)^2}{\beta^4} - d_5'\frac{(k_z^2 + k_\perp^2)^3}{\beta^6} + e_5'\frac{(k_z^2 + k_\perp^2)^4}{\beta^8}\right).
\end{aligned} \tag{A5}
$$

More information can be found in Ref. [2].

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