Pseudoscalar mesons in the $SU(3)$ linear sigma model with Gaussian functional approximation

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We study the $SU(3)$ linear sigma model for the pseudoscalar mesons in the Gaussian functional approximation (GFA). We use the $SU(3)$ linear sigma model Lagrangian with nonet scalar and pseudoscalar mesons, including symmetry breaking terms. In the GFA, we take the Gaussian Ansatz for the ground state wave function and apply the variational method to minimize the ground state energy. We derive the gap equations for the dressed meson masses, which are actually just variational parameters in the GFA method. We use the Bethe-Salpeter equation for meson-meson scattering, which provides the masses of the physical nonet mesons. We construct the projection operators for the flavor $SU(3)$ in order to work out the scattering T matrix in an efficient way. In this paper, we discuss the properties of the Nambu-Goldstone bosons in various limits of the chiral $U_L(3) \times U_R(3)$ symmetry.

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I. INTRODUCTION

The masses and properties of the $SU(3)$ scalar mesons are long standing puzzles in hadron-nuclear physics related to the underlying chiral symmetry of QCD. It is also very interesting to describe the properties of these $SU(3)$ scalar and pseudoscalar mesons at finite temperature and density. To this end it is important to study the $U_L(3) \times U_R(3)$ symmetric linear sigma model in the nonperturbative Gaussian functional approximation (GFA) [1,2]. The linear sigma model is a strongly interacting renormalizable quantum field theory; due to the size of the self-interaction coupling constant(s) the perturbative approximations seem to be inapplicable. Therefore, a nonperturbative approximation, such as the Gaussian functional one, that is equivalent to the resummation of certain infinite classes of Feynman diagrams that are unitary and causal [1–3] is called for.

The chiral $U_L(3) \times U_R(3)$ symmetry in the $SU(3)$ linear sigma model [4] is both spontaneously and explicitly broken, which means that some pseudoscalar mesons are Nambu-Goldstone (NG) bosons, i.e. with vanishing masses in the chiral limit. Straightforward solutions to the gap equations in the Gaussian wave functional approximation yield nonzero meson masses even in the chiral limit [1,2]; however, proof of the NG theorem used to be an open problem for over 30 years [1,5]. The first solution to this problem in the $O(2)$ symmetric sigma model was based on the Bethe-Salpeter (BS) equation [6], but other proof soon followed [7]. The first proof was straightforwardly extended to $O(4) \simeq SU_L(2) \times SU_R(2)$ in Ref. [8] and was finally proven in the general $O(N)$ case in Ref. [5]. The $SU(3)$ linear sigma model has a chiral symmetry that corresponds to the $SU_L(3) \times SU_R(3)$ subgroup of the (explicitly broken) $O(18)$ symmetry, that depends on the specifics of the (symmetry breaking) parameters of the model, and thus readily fit into this framework, but the Nambu-Goldstone theorem has never been explicitly verified in the various limits of the chiral $U_L(3) \times U_R(3)$ Lagrangian.

There are also several influential studies of the thermal properties of various spinless mesons that are based on the Gaussian approximation, both in the two-flavor $SU(2)$ [9,10] and the three-flavor $SU(3)$ cases [11], but again without taking into account the Bethe-Salpeter equation. Therefore, these studies do not obey the NG theorem in the chiral limit and as such are ill-suited for the study of chiral symmetry restoration.

As for the $SU(3)$ case, there are many studies of the properties of $U(3)$ mesons in the mean field, or the Born approximation [12,13] and in the Gaussian approximation [11]. The extension of the chiral $SU(2)$ model to the chiral $SU(3)$ model is not trivial, because there are several different self-interaction terms (three rather than one in the simplest $SU(2)$ case, but one of them, the λ_2, is generally expected to be (much) smaller than the first one λ_1 [14]), so we have to develop the necessary mathematical tools to deal with the scattering Bethe-Salpeter equation (T matrix) for the $U(3)$ mesons [12,13,15].

It is important to explicitly work out the NG bosons for various cases of the chiral Lagrangian, so as to verify which pseudoscalar mesons are NG bosons, before applying this formalism to nonzero temperature and/or density. The question of $U_A(1)$ symmetry breaking also looms large over this endeavor, so we pay special attention to the flavor-singlet-octet mixing.

As this is a complicated method applied to a difficult problem, and many missteps have been made in the past,

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we take a step-by-step approach. We look first at the chiral limit: even here there are some nontrivial cases, such as when $\lambda_2 = 0$ and c=0, (many) new naively unexpected Nambu-Goldstone bosons appear beyond the "elementary fields" that already exist in the Lagrangian—they are "composite" (bound state) NG bosons that correspond to the broken $O(18)$ symmetry rather than the $U_L(3) \times U_R(3)$ one that has (at most) nine NG bosons. This formation of composite NG bosons demonstrates the nonperturbative nature and the respect of the underlying symmetries by the GFA method. We then turn on explicit chiral symmetry breaking term $h_0 \neq 0$, but with good $SU(3)$ symmetry. We show how this explicit symmetry breaking term influences the masses of the pseudoscalar mesons to lowest (linear) approximation.

In this paper, we present the necessary mathematical expressions necessary for the application of the Gaussian functional approximation, defined in Sec. III, to the $SU(3)$ linear sigma model introduced in Sec. II. In Sec. IV, we provide the expressions for the Bethe-Salpeter equations in various channels using the $SU(3)$ projection operators developed in Sec. V. In Sec. VI, we verify explicitly the NG theorem for various cases and identify which are the NG bosons. In Sec. VII, we briefly discuss the role of explicit symmetry breaking terms by taking the simplest case. Section VIII is devoted to a summary of this paper.

II. THE $SU(3)$ LINEAR SIGMA MODEL

To understand the masses of scalar and pseudoscalar mesons, we employ the $SU(3)$ linear sigma model [4,12,13,16] and use the GFA. In this section, we briefly review the $SU(3)$ linear sigma model and work out the mass gap equations in the mean field approximation.

The Lagrangian density of the $U_L(3) \times U_R(3)$ linear sigma model is given by

$$\begin{aligned}\mathcal{L}(\Phi) = {}& \mathrm{Tr}(\partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi) - \lambda_1 [\mathrm{Tr}(\Phi^\dagger \Phi)]^2 \\ & - \lambda_2 \mathrm{Tr}(\Phi^\dagger \Phi)^2 + c[\mathrm{Det}(\Phi) + \mathrm{Det}(\Phi^\dagger)] \\ & + \mathrm{Tr}[H(\Phi + \Phi^\dagger)]. \end{aligned} \tag{1}$$

The meson field matrix Φ is a complex 3×3 matrix of the scalar and pseudoscalar meson nonets,

$$\Phi = T_a \phi_a = T_a(\sigma_a + i\pi_a), \tag{2}$$

where σ_a are the scalar fields and π_a are the pseudoscalar fields. $T_a = \lambda_a/2$ are the generators of $U(3)$, where λ_a are the Gell-Mann matrices with $\lambda_0 = \sqrt{\frac{2}{3}}\mathbf{1}$. The 3×3 matrix H breaks the chiral symmetry explicitly and is chosen as

$$H = T_a h_a, \tag{3}$$

where h_a are nine (external) $SU(3)$ symmetry breaking parameters. Only three (diagonal) ones, $a = (0, 3, 8)$, are relevant and the two, $a = (0, 8)$, are the dominant ones. In this paper, we only study the case $h_0 \neq 0$, and so $SU(3)$ symmetry is conserved. We need to know at least the order of magnitude of the coupling constants. Here we may use the results of Ref. [11] as a (rough) guide to the expected values of the coupling constants: to first approximation we expect $\lambda_1 \simeq 50$, $\lambda_2 \simeq 1.5$, and if we define $c = \lambda_3 f_\pi$, we find $\lambda_3 \simeq 50$. Thus, we see that this is indeed a strongly coupled system and that we need a nonperturbative approximation.

The generators of $U(3)$ satisfy the (anti)commutation relations:

$$[\lambda_a, \lambda_b] = 2i f_{abc}\lambda_c, \quad \text{and} \quad \{\lambda_a, \lambda_b\} = 2d_{abc}\lambda_c, \tag{4}$$

where d_{abc} and f_{abc} "structure constants" are defined to contain the 0 index. The values of $SU(3)$ structure constants are provided in any good textbook and in review articles [17]. Those f structure constants with a zero among its a, b, c indices are zero, and those d structure constants containing 0 in its a, b, c indices are only nonzero for $d_{0ab} = \sqrt{2/3}\delta_{ab}$, with a $b = 1, \ldots, 8$.

By inserting the Φ field into the Lagrangian, the following Lagrangian is obtained:

$$\begin{aligned}\mathcal{L}(\sigma_a, \pi_a) = {}& \tfrac{1}{2}[\partial_\mu \sigma_a \partial^\mu \sigma_a + \partial_\mu \pi_a \partial^\mu \pi_a] \\ & - \tfrac{1}{2}m^2(\sigma_a\sigma_a + \pi_a\pi_a) \\ & + \mathcal{G}_{abc}(\sigma_a\sigma_b\sigma_c - 3\pi_a\pi_b\sigma_c) \\ & - 2\mathcal{H}_{abcd}\sigma_a\sigma_b\pi_c\pi_d - \tfrac{1}{3}\mathcal{F}_{abcd}(\sigma_a\sigma_b\sigma_c\sigma_d \\ & + \pi_a\pi_b\pi_c\pi_d) + h_a\sigma_a. \end{aligned} \tag{5}$$

The coefficients $\mathcal{G}_{abc}$, $\mathcal{F}_{abcd}$ and $\mathcal{H}_{abcd}$ are given by

$$\begin{aligned}\mathcal{G}_{abc} = \frac{c}{6}\bigg[& d_{abc} - \frac{3}{2}(\delta_{a0}d_{0bc} + \delta_{b0}d_{a0c} + \delta_{c0}d_{ab0}) \\ & + \frac{9}{2}d_{000}\delta_{a0}\delta_{b0}\delta_{c0}\bigg], \end{aligned} \tag{6}$$

$$\begin{aligned}\mathcal{F}_{abcd} = {}& \frac{\lambda_1}{4}(\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}) \\ & + \frac{\lambda_2}{8}(d_{abn}d_{ncd} + d_{adn}d_{nbc} + d_{acn}d_{nbd}), \end{aligned} \tag{7}$$

$$\begin{aligned}\mathcal{H}_{abcd} = {}& \frac{\lambda_1}{4}\delta_{ab}\delta_{cd} \\ & + \frac{\lambda_2}{8}(d_{abn}d_{ncd} + f_{acn}f_{nbd} + f_{bcn}f_{nad}). \end{aligned} \tag{8}$$

Considering the shift of the vacuum expectation values of $\sigma_a = \bar{\sigma}_a + \sigma'_a$, the Lagrangian can be written as

$$\mathcal{L}(\sigma_a, \pi_a) = \tfrac{1}{2}[\partial_\mu\sigma_a\partial^\mu\sigma_a + \partial_\mu\pi_a\partial^\mu\pi_a - \sigma_a(m_S^2)_{ab}\sigma_b - \pi_a(m_P^2)_{ab}\pi_b] + (\mathcal{G}_{abc} - \tfrac{4}{3}\mathcal{F}_{abcd}\bar{\sigma}_d)\sigma_a\sigma_b\sigma_c - 3(\mathcal{G}_{abc} + \tfrac{4}{3}\mathcal{H}_{abcd}\bar{\sigma}_d)\pi_a\pi_b\sigma_c - 2\mathcal{H}_{abcd}\sigma_a\sigma_b\pi_c\pi_d - \tfrac{1}{3}\mathcal{F}_{abcd}(\sigma_a\sigma_b\sigma_c\sigma_d + \pi_a\pi_b\pi_c\pi_d) - U(\bar{\sigma}), \quad (9)$$

where we have just written σ_a instead of σ'_a for simplicity of writing. The potential term $U(\bar{\sigma})$ is the tree-approximation potential and $\bar{\sigma}_a$ is determined at the tree level. The tree-level potential is

$$U(\bar{\sigma}_a) = \frac{m^2}{2}\bar{\sigma}_a^2 - \mathcal{G}_{abc}\bar{\sigma}_a\bar{\sigma}_b\bar{\sigma}_c + \frac{1}{3}\mathcal{F}_{abcd}\bar{\sigma}_a\bar{\sigma}_b\bar{\sigma}_d\bar{\sigma}_d - h_a\bar{\sigma}_a. \quad (10)$$

The mean field $\bar{\sigma}_a$ is obtained by the variation.

$$\frac{\partial U(\bar{\sigma}_a)}{\partial\bar{\sigma}_a} = m^2\bar{\sigma}_a - 3\mathcal{G}_{abc}\bar{\sigma}_b\bar{\sigma}_c + \frac{4}{3}\mathcal{F}_{abcd}\bar{\sigma}_b\bar{\sigma}_c\bar{\sigma}_d - h_a = 0, \quad (11)$$

and the "tree-level" masses of the scalar and pseudoscalar mesons are given by

$$(m_S^2)_{ab} = m^2\delta_{ab} - 6\mathcal{G}_{abc}\bar{\sigma}_c + 4\mathcal{F}_{abcd}\bar{\sigma}_c\bar{\sigma}_d, \qquad (m_P^2)_{ab} = m^2\delta_{ab} + 6\mathcal{G}_{abc}\bar{\sigma}_c + 4\mathcal{H}_{abcd}\bar{\sigma}_c\bar{\sigma}_d. \quad (12)$$

In the general $SU(3)$ symmetry breaking case, the 0–8 off-diagonal components of these matrices are not zero. The physical states must have a diagonal mass matrix, and we have to diagonalize the mass matrices.

III. GAUSSIAN FUNCTIONAL APPROXIMATION

The GFA [2,6,8] is the method based on assuming the ground state solution is a Gaussian functional around the mean field. We get the effective potential by acting the Hamiltonian on the ground state with scalar mesons having vacuum expectation values.

First, the Schrödinger equation in the functional formalism is given as

$$H|0\rangle = E|0\rangle, \quad (13)$$

where H is the total Hamiltonian and E is the corresponding energy for the wave function $|0\rangle$. The effective potential can be obtained by

$$E = \int d^3x\langle 0|\mathcal{H}|0\rangle, \quad (14)$$

where the Hamiltonian density is obtained through the Legendre transformation as

$$\mathcal{H}(\sigma_a, \pi_a) = -\frac{1}{2}\frac{\delta^2}{\delta\sigma_a^2} + \frac{1}{2}(\nabla\sigma_a)^2 - \frac{1}{2}\frac{\delta^2}{\delta\pi_a^2} + \frac{1}{2}(\nabla\pi_a)^2 + \frac{1}{2}m^2(\sigma_a^2 + \pi_a^2) - \mathcal{G}_{abc}(\sigma_a\sigma_b\sigma_c - 3\pi_a\pi_b\sigma_c) + 2\mathcal{H}_{abcd}\sigma_a\sigma_b\pi_c\pi_d + \frac{1}{3}\mathcal{F}_{abcd}(\sigma_a\sigma_b\sigma_c\sigma_d + \pi_a\pi_b\pi_c\pi_d) - h_a\sigma_a, \quad (15)$$

and the ground state wave functional as a Gaussian function is

$$|0\rangle = N\exp[-\tfrac{1}{4}(\sigma_a - \bar{\sigma}_a)G_{ab}^{-1}(m_\sigma)(\sigma_b - \bar{\sigma}_b) - \tfrac{1}{4}\pi_a G_{ab}^{-1}(m_\pi)\pi_b]. \quad (16)$$

Here, N is the normalization factor. The mass propagator is written as

$$G_{ab}(x, y) = \frac{1}{2}\delta_{ab}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2 + m_a^2}}e^{i\vec{k}(\vec{x}-\vec{y})}. \quad (17)$$

Finally the effective potential can be calculated as

$$\varepsilon = \langle 0|\mathcal{H}|0\rangle, \quad (18)$$

$$= \tfrac{1}{2}m^2\bar{\sigma}_a^2 + \tfrac{1}{4}\{G_{ab}^{-1}(m_\sigma) + G_{ab}^{-1}(m_\pi)\} + \tfrac{1}{2}(m^2 - m_{\sigma_a}^2)G_{ab}(m_\sigma) + \tfrac{1}{2}(m^2 - m_{\pi_a}^2)G_{ab}(m_\pi) - \mathcal{G}_{abc}\{\bar{\sigma}_a\bar{\sigma}_b\bar{\sigma}_c + 3\bar{\sigma}_a(G_{bc}(m_\sigma) - G_{bc}(m_\pi))\} + 2\mathcal{H}_{abcd}\{G_{ab}(m_\pi)\bar{\sigma}_c\bar{\sigma}_d + G_{ab}(m_\pi)G_{cd}(m_\sigma)\} + \tfrac{1}{3}\mathcal{F}_{abcd}\{\bar{\sigma}_a\bar{\sigma}_b\bar{\sigma}_c\bar{\sigma}_d + 6\bar{\sigma}_a\bar{\sigma}_b G_{cd}(m_\sigma) + 3G_{ab}(m_\sigma)G_{cd}(m_\sigma) + 3G_{ab}(m_\pi)G_{cd}(m_\pi)\} - h_a\bar{\sigma}_a. \quad (19)$$

The gap equations for scalar and pseudoscalar mesons are obtained by applying the variational principle with respect to meson masses, $\frac{\partial\varepsilon}{\partial m_a} = 0$ and then the masses are given as

$$(m_S^2)_{ab} = m^2\delta_{ab} - 6\mathcal{G}_{abc}\bar{\sigma}_c + 4\mathcal{F}_{abcd}\bar{\sigma}_c\bar{\sigma}_d + 4\mathcal{F}_{abcd}G_{cd}(m_\sigma) + 4\mathcal{H}_{abcd}G_{cd}(m_\pi), \qquad (m_P^2)_{ab} = m^2\delta_{ab} + 6\mathcal{G}_{abc}\bar{\sigma}_c + 4\mathcal{H}_{abcd}\bar{\sigma}_c\bar{\sigma}_d + 4\mathcal{H}_{abcd}G_{cd}(m_\sigma) + 4\mathcal{F}_{abcd}G_{cd}(m_\pi). \quad (20)$$

The equation following from the variation of the energy density with respect to the mean field value $\frac{\partial\varepsilon}{\partial\bar{\sigma}_a} = 0$ is given by

$$h_a = m^2\bar{\sigma}_a - 3\mathcal{G}_{abc}[\bar{\sigma}_b\bar{\sigma}_c + G_{bc}(m_\sigma) - G_{bc}(m_\pi)] + 4\mathcal{H}_{abcd}\bar{\sigma}_b G_{cd}(m_\pi) + \tfrac{1}{3}\mathcal{F}_{abcd}[4\bar{\sigma}_b\bar{\sigma}_c\bar{\sigma}_d + 12\bar{\sigma}_b G_{cd}(m_\sigma)]. \quad (21)$$

These equations provide the masses and the mean field values of the meson fields.

IV. T MATRIX FOR MESON-MESON SCATTERING

As explained in Refs. [5,6] one should work out the T matrix for the determination of the pseudoscalar mesons in order to fulfill the NG theorem. We work out the σ-π scattering for pseudoscalar mesons. The interaction kernel in the σ-π channel is written as

$$\begin{aligned} -iV_{acbd} &= -i2\mathcal{H}_{acbd}\cdot 2\cdot 2 - i3\Big(\mathcal{G}_{bea} + \frac{4}{3}\mathcal{H}_{beaf}\bar{\sigma}_f\Big)2 \\ &\quad\times \frac{i}{s-m_e^2}(-i)3\Big(\mathcal{G}_{edc} + \frac{4}{3}\mathcal{H}_{edcg}\bar{\sigma}_g\Big)2 \\ &= -i\Big[8\mathcal{H}_{acbd} + 36\Big(\mathcal{G}_{bea} + \frac{4}{3}\mathcal{H}_{beaf}\bar{\sigma}_f\Big) \\ &\quad\times \frac{1}{s-m_e^2}\Big(\mathcal{G}_{edc} + \frac{4}{3}\mathcal{H}_{edcg}\bar{\sigma}_g\Big)\Big]. \end{aligned} \quad (22)$$

With this interaction kernel we can get the T matrix as

$$\begin{aligned} -iT_{abcd} &= -iV_{abcd} - iV_{abef}i\Pi_{ef}(-iV_{efcd}) + \ldots \\ &= -i(V_{abcd} + V_{abef}\Pi_{ef}T_{efcd} + \ldots). \end{aligned} \quad (23)$$

Therefore, what we need to solve is the scattering matrix

$$T_{abcd} = V_{abcd} + V_{abef}\Pi_{ef}T_{efcd}. \quad (24)$$

The polarization term for meson masses m_a and m_b is

$$i\Pi_{ab}(p^2) = \int \frac{i}{(k-p)^2 - m_a^2 + i\varepsilon}\frac{i}{k^2 - m_b^2 + i\varepsilon}\frac{d^4k}{(2\pi)^4}. \quad (25)$$

To work out the polarization function Π_{ab}, first let us work out the $p^2 = 0$ case. In this case, we can write

$$\begin{aligned} \Pi_{ab}(0) &= i\int\Big(\frac{1}{k^2 - m_a^2 + i\varepsilon} - \frac{1}{k^2 - m_b^2 + i\varepsilon}\Big)\frac{1}{m_a^2 - m_b^2} \\ &\quad\times\frac{d^4k}{(2\pi)^4} = \frac{I_0(m_a^2) - I_0(m_b^2)}{m_a^2 - m_b^2}. \end{aligned} \quad (26)$$

Here, we can write the integral as

$$\begin{aligned} I_0(m^2) &= i\int\frac{1}{k^2 - m^2 + i\varepsilon}\frac{d^4k}{(2\pi)^4} = i\int\frac{1}{k_0^2 - \vec{k}^2 - m^2 + i\varepsilon}\frac{d^4k}{(2\pi)^4} \\ &= i\int\frac{1}{(k_0 - \sqrt{\vec{k}^2 + m^2} + i\varepsilon)(k_0 + \sqrt{\vec{k}^2 + m^2} + i\varepsilon)}\frac{d^4k}{(2\pi)^4} = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2 + m^2}} = \frac{1}{4\pi^2}\int_0^\Lambda\frac{k^2dk}{\sqrt{k^2 + m^2}} \\ &= \frac{1}{8\pi^2}m^2[x_3\sqrt{1 + x_3^2} - \log|x_3 + \sqrt{1 + x_3^2}|], \end{aligned} \quad (27)$$

where $x_3 = \Lambda/m$. We take $\Lambda \sim 1$ GeV to be fixed as a parameter of the model. We may take thefour-dimensional cutoff by transforming the above integral as $k_0 = ik_4$:

$$I_0(m^2) = i\int\frac{1}{k^2 - m^2 + i\varepsilon}\frac{d^4k}{(2\pi)^4} = \int\frac{1}{k^2 + m^2}\frac{d^4k}{(2\pi)^4} = \frac{1}{2\pi^2}\int\frac{k^3dk}{k^2 + m^2} = \frac{1}{4\pi^2}m^2[x_4^2 - \log|1 + x_4^2|], \quad (28)$$

where $x_4 = \Lambda/m$.

We write here the case for $m_a = m_b = m$, which is written in the paper of Nakamura *et al.* [8]. The general case has to be worked out in the same frame.

$$\Pi_{aa}(s) = \Pi_{aa}(0) + \frac{s}{(4\pi)^2}(-1 + J_{aa}(s)), \quad (29)$$

where

$$J_{aa}(s) = \sqrt{\frac{4m^2}{s} - 1}\arcsin\sqrt{\frac{s}{4m^2}}, \quad (30)$$

for $\frac{s}{4m^2} < 1$, and

$$J_{aa}(s) = \sqrt{1 - \frac{4m^2}{s}}\Big[\log\Big(\sqrt{\frac{s}{4m^2}} + \sqrt{\frac{s}{4m^2} - 1}\Big) - i\frac{\pi}{2}\Big], \quad (31)$$

for $1 < \frac{s}{4m^2} < \infty$. We should work out the general case $m_a \neq m_b$, in the "dispersive" form [see Eqs. (33)]

$$\begin{aligned} I_{M\mu}(s) &= i\int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-M^2+i\epsilon][(k-P)^2-\mu^2+i\epsilon]} \\ &= I_{M\mu}(0) - \frac{s}{(4\pi)^2}K_{M\mu}(s) \\ &= \frac{1}{2\lambda_0}\left(\frac{\mu^2}{\mu^2-M^2}\right) - \frac{s}{(4\pi)^2}K_{M\mu}(s) \\ &= \frac{1}{2\lambda_0}\left(\frac{\mu^2}{\mu^2-M^2}\right) - \frac{s}{16\pi^3}\int\frac{dt}{t-s-i\epsilon}\mathrm{Im}K_{M\mu}(t), \end{aligned} \tag{32}$$

where $s = P^2$ and the real and imaginary parts are

$$\begin{aligned} \mathrm{Im}K_{M\mu}(s) &= \frac{1}{s}\mathrm{Im}I_{M\mu}(s) \\ &= \frac{\pi}{s}\sqrt{\left(1-\frac{(M-\mu)^2}{s}\right)\left(1-\frac{(M+\mu)^2}{s}\right)} \\ &\quad\times \theta(s-(M+\mu)^2), \\ \mathrm{Re}K_{M\mu}(s) &= \frac{2}{s}\bigg[\left(\frac{M^2-\mu^2}{2s}\right)\log\frac{M}{\mu} \\ &\quad + \frac{1}{2}\left(1+\left(\frac{M^2+\mu^2}{M^2-\mu^2}\right)\log\frac{M}{\mu}\right) \\ &\quad - \sqrt{\left(1-\frac{(M-\mu)^2}{s}\right)\left(1-\frac{(M+\mu)^2}{s}\right)} \\ &\quad \times \tanh^{-1}\sqrt{\frac{s-(M+\mu)^2}{s-(M-\mu)^2}}\bigg]. \end{aligned} \tag{33}$$

V. $SU(3)$ PROJECTION OPERATORS AND MIXING OPERATORS

To solve the scattering Eq. (24), we use the method of projection operators. The $SU(3)$ group structure for the $SU(3)$ sigma model is

$$\begin{aligned} (\mathbf{1}_F \oplus \mathbf{8}_F)\otimes(\mathbf{1}_F \oplus \mathbf{8}_F) = {} & \mathbf{1}_{(1)} \oplus \mathbf{8}_{x(1)} \oplus \mathbf{8}_{y(1)} \oplus \mathbf{1}_{(8)} \\ & \oplus \mathbf{8}_{S(8)} \oplus \mathbf{8}_{A(8)} \oplus \mathbf{27}_{S(8)} \oplus \mathbf{10}_{A(8)} \\ & \oplus \overline{\mathbf{10}}_{A(8)}. \end{aligned} \tag{34}$$

We can write out the corresponding projection operators with some manipulations:

$$\begin{aligned} P^{1(1)}_{abcd} &= \delta_{ab}\delta_{cd}|_{a,b,c,d=0}, \\ P^{8x(1)}_{abcd} &= \frac{3}{2}\sum_{n=1}^{8}(d_{abn}d_{cdn})|_{a,c=0,b,d=1\ldots8}, \\ P^{8y(1)}_{abcd} &= \frac{3}{2}\sum_{n=1}^{8}(d_{abn}d_{cdn})|_{a,c=1\ldots8,b,d=0}, \\ P^{1(8)}_{abcd} &= \frac{1}{8}\delta_{ab}\delta_{cd}|_{a,b,c,d=1\ldots8}, \\ P^{8S(8)}_{abcd} &= \frac{3}{5}\sum_{n=1}^{8}(d_{abn}d_{cdn})|_{a,b,c,d=1\ldots8} \\ P^{8A(8)}_{abcd} &= \frac{1}{3}\sum_{n=1}^{8}(f_{abn}f_{cdn}), \\ P^{27S(8)}_{abcd} &= \frac{1}{2}(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})|_{a,b,c,d=1\cdots8} - P^{1(8)}_{abcd} \\ &\quad - P^{8S(8)}_{abcd}, \\ P^{(10+\overline{10})A(8)}_{abcd} &= \frac{1}{2}(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})|_{a,b,c,d=1\ldots8} - P^{8A(8)}_{abcd}, \end{aligned} \tag{35}$$

where the indices a, b, c, d can be $0\ldots8$. When they are 0, they are related to $\mathbf{1}_F$ of Eq. (34); while when they are $1\ldots8$, they are related to $\mathbf{8}_F$ of Eq. (34). These projection operators satisfy

$$P^x_{abef}P^y_{efcd} = \delta^{xy}P^x_{abcd}. \tag{36}$$

We have used various relations for the derivation of the projection operators:

$$\begin{gathered} f_{aij}f_{bij} = 3\delta_{ab}, \qquad d_{aij}d_{bij} = \tfrac{5}{3}\delta_{ab}, \qquad d_{aij}f_{bij} = 0, \qquad d_{aij}\delta_{ij} = 0, \qquad d_{aij}d_{bjk}d_{cki} = -\tfrac{1}{2}d_{abc}, \\ f_{aij}f_{bjk}d_{cki} = -\tfrac{3}{2}d_{abc}, \qquad f_{aij}f_{bjk}f_{cki} = \tfrac{2}{3}f_{abc}, \qquad f_{aij}d_{bjk}d_{cki} = 0, \\ d_{abn}d_{cdn} + d_{acn}d_{bdn} + d_{adn}d_{bcn} = \tfrac{1}{3}(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}), \\ f_{acn}f_{bdn} + f_{adn}f_{bcn} - 3d_{abn}d_{cdn} = -(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}) + \delta_{ab}\delta_{cd}, \qquad f_{acn}d_{bdn} - f_{adn}f_{bcn} - f_{abn}f_{cdn} = 0, \\ d_{acn}d_{bdn} - d_{adn}d_{bcn} - f_{abn}f_{cdn} = \tfrac{1}{12}(\delta_{ad}\delta_{bc}-\delta_{ac}\delta_{bd}). \end{gathered} \tag{37}$$

Besides these projection operators, we also have several mixing operators. They are used to express the mixing between the singlet and octet mesons. We note that they are not projection operators, and so we use O to denote them. In the singlet channel, there are two operators:

$$O^{1(M1)}_{abcd} = \delta_{ab}\delta_{cd}|_{a,b=1\ldots8,c,d=0},$$
$$O^{1(M2)}_{abcd} = \delta_{ab}\delta_{cd}|_{a,b=0,c,d=1\ldots8}. \tag{38}$$

They provide the mixing of the singlet and octet mesons in the resultant singlet channel. While in the octet channel, there are six operators

$$O^{8(M1)}_{abcd} = \sum_{n=1}^{8} d_{abn}d_{cdn}|_{a=0,b,c,d=1\cdots8},$$
$$O^{8(M2)}_{abcd} = \sum_{n=1}^{8} d_{abn}d_{cdn}|_{a,b,d=1\ldots8,c=0},$$
$$O^{8(M3)}_{abcd} = \sum_{n=1}^{8} d_{abn}d_{cdn}|_{b=0,a,c,d=1\cdots8},$$
$$O^{8(M4)}_{abcd} = \sum_{n=1}^{8} d_{abn}d_{cdn}|_{a,b,c=1\ldots8,d=0},$$
$$O^{8(M5)}_{abcd} = \sum_{n=1}^{8} d_{abn}d_{cdn}|_{a,d=0,b,c=1\cdots8},$$
$$O^{8(M6)}_{abcd} = \sum_{n=1}^{8} d_{abn}d_{cdn}|_{b,c=0,a,d=1\ldots8}. \tag{39}$$

They provide the mixing of the singlet-octet mesons and the octet mesons in the resultant octet channel.

We modify the scattering Eq. (24) to be

$$T_{abcd} = V_{abcd} + V_{abef}\Pi_{ef}T_{efcd} = V_{abcd} + V_{aba'b'}\Pi_{a'b'c'd'}T_{c'd'cd}, \tag{40}$$

where $\Pi_{abcd} = \delta_{ac}\delta_{bd}\Pi_{ab}$. Then by using the projection operators as well as the mixing operators, we find that there is a unique expansion for V_{abcd} of the scattering of $\sigma\pi \to \sigma\pi$:

$$V_{abcd} = V_{1(1)}P^{1(1)}_{abcd} + V_{8x(1)}P^{8x(1)}_{abcd} + V_{8y(1)}P^{8y(1)}_{abcd} + V_{1(8)}P^{1(8)}_{abcd} + V_{8S(8)}P^{8S(8)}_{abcd} + V_{27S(8)}P^{27S(8)}_{abcd} + V_{8A(8)}P^{8A(8)}_{abcd} + V_{10A(8)}P^{(10+\overline{10})A(8)}_{abcd} + V_{1(M1)}O^{1(M1)}_{abcd} + V_{1(M2)}O^{1(M2)}_{abcd} + V_{8(M1)}O^{8(M1)}_{abcd} + V_{8(M2)}O^{8(M2)}_{abcd} + V_{8(M3)}O^{8(M3)}_{abcd} + V_{8(M4)}O^{8(M4)}_{abcd} + V_{8(M5)}O^{8(M5)}_{abcd} + V_{8(M6)}O^{8(M6)}_{abcd}, \tag{41}$$

and so does Π_{abcd}. Moreover, we find that for Π_{abcd} only the projection operators are enough, which means

$$\Pi_{1(Mi)} = \Pi_{8(Mi)} = 0. \tag{42}$$

Since we find that both V_{abcd} and Π_{abcd} can be expanded by the projection operators (35) and the mixing operators (38) and (39), we assume that the scattering matrix T_{abcd} can also be expanded by these operators:

$$T_{abcd} = T_{1(1)}P^{1(1)}_{abcd} + T_{8x(1)}P^{8x(1)}_{abcd} + T_{8y(1)}P^{8y(1)}_{abcd} + T_{1(8)}P^{1(8)}_{abcd} + T_{8S(8)}P^{8S(8)}_{abcd} + T_{27S(8)}P^{27S(8)}_{abcd} + T_{8A(8)}P^{8A(8)}_{abcd} + T_{10A(8)}P^{(10+\overline{10})A(8)}_{abcd} + T_{1(M1)}O^{1(M1)}_{abcd} + T_{1(M2)}O^{1(M2)}_{abcd} + T_{8(M1)}O^{8(M1)}_{abcd} + T_{8(M2)}O^{8(M2)}_{abcd} + T_{8(M3)}O^{8(M3)}_{abcd} + T_{8(M4)}O^{8(M4)}_{abcd} + T_{8(M5)}O^{8(M5)}_{abcd} + T_{8(M6)}O^{8(M6)}_{abcd}, \tag{43}$$

and then we can separate the scattering Eq. (40) into several equations in different channels. We discuss them in the following subsections.

A. Singlet channel

We have the following relations for all the operators:

$$O^{1}_{abef}O^{x}_{efcd} = 0, \tag{44}$$

where O^1 denotes $P^{1(1)}$, $P^{1(8)}$, $O^{1(Mi)}$ and $O^{1(M2)}$, and O^x denotes operators of other flavors. Therefore, we can write out the Eq. (40) in the singlet channels:

$$\begin{aligned}&T_{1(1)}P^{1(1)}_{abcd}+T_{1(8)}P^{1(8)}_{abcd}+T_{1(M1)}O^{1(M1)}_{abcd}+T_{1(M2)}O^{1(M2)}_{abcd}\\&=(V_{1(1)}P^{1(1)}_{abcd}+V_{1(8)}P^{1(8)}_{abcd}+V_{1(M1)}O^{1(M1)}_{aba'b'}+V_{1(M2)}O^{1(M2)}_{aba'b'})+(V_{1(1)}P^{1(1)}_{aba'b'}+V_{1(8)}P^{1(8)}_{aba'b'}+V_{1(M1)}O^{1(M1)}_{aba'b'}\\&\quad+V_{1(M2)}O^{1(M2)}_{aba'b'})(\Pi_{1(1)}P^{1(1)}_{a'b'c'd'}+\Pi_{1(8)}P^{1(8)}_{a'b'c'd'})(T_{1(1)}P^{1(1)}_{c'd'cd}+T_{1(8)}P^{1(8)}_{c'd'cd}+T_{1(M1)}O^{1(M1)}_{c'd'cd}+T_{1(M2)}O^{1(M2)}_{c'd'cd}).\end{aligned}\tag{45}$$

It can be simplified to

$$\begin{pmatrix}T_{1(1)} & 2\sqrt{2}T_{1(M2)}\\ 2\sqrt{2}T_{1(M1)} & T_{1(8)}\end{pmatrix}=\begin{pmatrix}V_{1(1)} & 2\sqrt{2}V_{1(M2)}\\ 2\sqrt{2}V_{1(M1)} & V_{1(8)}\end{pmatrix}+\begin{pmatrix}V_{1(1)} & 2\sqrt{2}V_{1(M2)}\\ 2\sqrt{2}V_{1(M1)} & V_{1(8)}\end{pmatrix}\begin{pmatrix}\Pi_{1(1)} & 0\\ 0 & \Pi_{1(8)}\end{pmatrix}\times\begin{pmatrix}T_{1(1)} & 2\sqrt{2}T_{1(M2)}\\ 2\sqrt{2}T_{1(M1)} & T_{1(8)}\end{pmatrix},$$

and its solution is

$$\begin{pmatrix}T_{1(1)} & 2\sqrt{2}T_{1(M2)}\\ 2\sqrt{2}T_{1(M1)} & T_{1(8)}\end{pmatrix}=\left(1-\begin{pmatrix}V_{1(1)} & 2\sqrt{2}V_{1(M2)}\\ 2\sqrt{2}V_{1(M1)} & V_{1(8)}\end{pmatrix}\begin{pmatrix}\Pi_{1(1)} & 0\\ 0 & \Pi_{1(8)}\end{pmatrix}\right)^{-1}\begin{pmatrix}V_{1(1)} & 2\sqrt{2}V_{1(M2)}\\ 2\sqrt{2}V_{1(M1)} & V_{1(8)}\end{pmatrix}.$$

B. Octet channel ($\mathbf{8}_{x(1)}$, $\mathbf{8}_{y(1)}$ and $\mathbf{8}_{S(8)}$)

We have the following relations for all the operators:

$$O^8_{abef}O^x_{efcd}=0,\tag{46}$$

where O^8 denotes $P^{8x(1)}$, $P^{8y(1)}$, $P^{8S(8)}$ and $O^{8(Mi)}$, and O^x denotes other operators. Therefore, we can similarly write out the scattering Eq. (40) in the octet channels. After some simplifications, it turns to be

$$\begin{pmatrix}T_{8x(1)} & \frac{2}{3}T_{8(M5)} & \frac{\sqrt{10}}{3}T_{8(M1)}\\ \frac{2}{3}T_{8(M6)} & T_{8y(1)} & \frac{\sqrt{10}}{3}T_{8(M3)}\\ \frac{\sqrt{10}}{3}T_{8(M1)} & \frac{\sqrt{10}}{3}T_{8(M4)} & T_{8S(8)}\end{pmatrix}=\begin{pmatrix}V_{8x(1)} & \frac{2}{3}V_{8(M5)} & \frac{\sqrt{10}}{3}V_{8(M1)}\\ \frac{2}{3}V_{8(M6)} & V_{8y(1)} & \frac{\sqrt{10}}{3}V_{8(M3)}\\ \frac{\sqrt{10}}{3}V_{8(M1)} & \frac{\sqrt{10}}{3}V_{8(M4)} & V_{8S(8)}\end{pmatrix}+\begin{pmatrix}V_{8x(1)} & \frac{2}{3}V_{8(M5)} & \frac{\sqrt{10}}{3}V_{8(M1)}\\ \frac{2}{3}V_{8(M6)} & V_{8y(1)} & \frac{\sqrt{10}}{3}V_{8(M3)}\\ \frac{\sqrt{10}}{3}V_{8(M1)} & \frac{\sqrt{10}}{3}V_{8(M4)} & V_{8S(8)}\end{pmatrix}$$
$$\times\begin{pmatrix}\Pi_{8x(1)} & 0 & 0\\ 0 & \Pi_{8y(1)} & 0\\ 0 & 0 & \Pi_{8S(8)}\end{pmatrix}\begin{pmatrix}T_{8x(1)} & \frac{2}{3}T_{8(M5)} & \frac{\sqrt{10}}{3}T_{8(M1)}\\ \frac{2}{3}T_{8(M6)} & T_{8y(1)} & \frac{\sqrt{10}}{3}T_{8(M3)}\\ \frac{\sqrt{10}}{3}T_{8(M1)} & \frac{\sqrt{10}}{3}T_{8(M4)} & T_{8S(8)}\end{pmatrix},$$

and its solution is

$$\begin{pmatrix}T_{8x(1)} & \frac{2}{3}T_{8(M5)} & \frac{\sqrt{10}}{3}T_{8(M1)}\\ \frac{2}{3}T_{8(M6)} & T_{8y(1)} & \frac{\sqrt{10}}{3}T_{8(M3)}\\ \frac{\sqrt{10}}{3}T_{8(M1)} & \frac{\sqrt{10}}{3}T_{8(M4)} & T_{8S(8)}\end{pmatrix}=\left(1-\begin{pmatrix}V_{8x(1)} & \frac{2}{3}V_{8(M5)} & \frac{\sqrt{10}}{3}V_{8(M1)}\\ \frac{2}{3}V_{8(M6)} & V_{8y(1)} & \frac{\sqrt{10}}{3}V_{8(M3)}\\ \frac{\sqrt{10}}{3}V_{8(M1)} & \frac{\sqrt{10}}{3}V_{8(M4)} & V_{8S(8)}\end{pmatrix}\begin{pmatrix}\Pi_{8x(1)} & 0 & 0\\ 0 & \Pi_{8y(1)} & 0\\ 0 & 0 & \Pi_{8S(8)}\end{pmatrix}\right)^{-1}$$
$$\times\begin{pmatrix}V_{8x(1)} & \frac{2}{3}V_{8(M5)} & \frac{\sqrt{10}}{3}V_{8(M1)}\\ \frac{2}{3}V_{8(M6)} & V_{8y(1)} & \frac{\sqrt{10}}{3}V_{8(M3)}\\ \frac{\sqrt{10}}{3}V_{8(M1)} & \frac{\sqrt{10}}{3}V_{8(M4)} & V_{8S(8)}\end{pmatrix}.$$

C. $\mathbf{8}_{A(8)}$, $(\mathbf{10}\oplus\overline{\mathbf{10}})_{A(8)}$ and $\mathbf{27}_{S(8)}$ channels

We have the following relations for all the operators:

$$P^{8A(8)}_{abef}O^x_{efcd}=0,\tag{47}$$

where O^x denotes other operators. Therefore, we can write out the scattering Eq. (40) in the $\mathbf{8}_{A(8)}$ channel:

$$T_{8A(8)}=V_{8A(8)}+V_{8A(8)}\Pi_{8A(8)}T_{8A(8)}.\tag{48}$$

We have the following relations for all the operators:

$$P^{(10+\overline{10})A(8)}_{abef}O^x_{efcd}=0,\tag{49}$$

where O^x denotes operators of other flavors. Therefore, we can write out the scattering Eq. (40) in the decuplet channel:

$$T_{10A(8)} = V_{10A(8)} + V_{10A(8)}\Pi_{10A(8)}T_{10A(8)}. \quad (50)$$

We have the following relations for all the operators:

$$P^{27S(8)}_{abef} O^x_{efcd} = 0, \quad (51)$$

where O^x denotes operators of other flavors. Therefore, we can write out the scattering Eq. (40) in the $\mathbf{27}_{S(8)}$ channel:

$$T_{27S(8)} = V_{27S(8)} + V_{27S(8)}\Pi_{27S(8)}T_{27S(8)}. \quad (52)$$

VI. THE NAMBU-GOLDSTONE THEOREM

To check whether there are Nambu-Goldstone bosons, we need to check whether the scattering matrix $T(s)$ has a pole at $s = 0$. In this section, we study the Nambu-Goldstone theorem, and verify the Nambu-Goldstone bosons in the $SU(3)$ linear sigma model. We assume that there is a $SU(3)$ symmetry, which means that only $\bar{\sigma}_0$ is nonzero ($\bar{\sigma}_i = 0$, for $i = 1\ldots8$). To further simplify our calculation, we only study the pseudoscalar channels ($\sigma\pi \to \sigma\pi$), where the pseudoscalar mesons propagate in the interaction kernel.

Since the calculations in this system is still not so easy, our analysis will be done step by step. First, we assume $c = \lambda_2 = 0$, and $\lambda_1 \neq 0$. In this case, we find that all the pseudoscalar mesons are Nambu-Goldstone bosons. Then we assume $\lambda_2 \neq 0$, and find that only one singlet and one octet pseudoscalar mesons remain Nambu-Goldstone bosons. Finally, we assume $c \neq 0$, which is the most general case conserving $SU(3)$ symmetry. We find that only one octet pseudoscalar mesons remain to be Nambu-Goldstone bosons.

A. Case I: $c = \lambda_2 = 0$

This is the first step. When $c = \lambda_2 = 0$, we find that there is no mixing between two flavor-singlet mesons and among the four flavor-octet mesons in the pseudoscalar channel, and we can expand the potential matrix V (σ-π scattering) by only using the nonmixing projection operators (35):

$$\begin{aligned} V_{abcd} &= V_{1(1)}P^{1(1)}_{abcd} + V_{8x(1)}P^{8x(1)}_{abcd} + V_{8y(1)}P^{8y(1)}_{abcd} \\ &\quad + V_{1(8)}P^{1(8)}_{abcd} + V_{8S(8)}P^{8S(8)}_{abcd} + V_{27S(8)}P^{27S(8)}_{abcd} \\ &\quad + V_{8A(8)}P^{8A(8)}_{abcd} + V_{10A(8)}P^{(10+\overline{10})A(8)}_{abcd}, \end{aligned} \quad (53)$$

where the coefficients V_i are calculated to be

$$\begin{aligned} V_{1(1)} &= 2\lambda_1 + \frac{4\lambda_1^2\bar{\sigma}_0^2}{s-(m_P^2)_{00}}, \\ V_{8x(1)} &= 2\lambda_1 + \frac{4\lambda_1^2\bar{\sigma}_0^2}{s-(m_P^2)_{ii}}, \\ V_{8y(1)} &= V_{1(8)} = V_{8S(8)} = V_{8A(8)} = V_{27S(8)} = V_{10A(8)} = 2\lambda_1. \end{aligned} \quad (54)$$

To expand Π_{ab} at the point $s = 0$, first we write out its four-index form following Eq. (26):

$$\begin{aligned} \Pi_{abcd}(s=0) &= \delta_{ac}\delta_{bd}\Pi_{ab}(s=0) \\ &= \delta_{ac}\delta_{bd}\frac{I_0(m_S^{aa}) - I_0(m_P^{bb})}{(m_S^2)_{aa} - (m_P^2)_{bb}}, \end{aligned} \quad (55)$$

and Π_{abcd} can also be expanded by using the projection operators (35):

$$\begin{aligned} \Pi_{abcd} &= \Pi_{1(1)}P^{1(1)}_{abcd} + \Pi_{8x(1)}P^{8x(1)}_{abcd} + \Pi_{8y(1)}P^{8y(1)}_{abcd} \\ &\quad + \Pi_{1(8)}P^{1(8)}_{abcd} + \Pi_{8S(8)}P^{8S(8)}_{abcd} + \Pi_{27S(8)}P^{27S(8)}_{abcd} \\ &\quad + \Pi_{8A(8)}P^{8A(8)}_{abcd} + \Pi_{10A(8)}P^{(10+\overline{10})A(8)}_{abcd}, \end{aligned} \quad (56)$$

and its solution is

$$\begin{aligned} \Pi_{1(1)} &= \frac{I_0(m_S^{00}) - I_0(m_P^{00})}{(m_S^2)_{00} - (m_P^2)_{00}}, \\ \Pi_{8x(1)} &= \frac{I_0(m_S^{00}) - I_0(m_P^{ii})}{(m_S^2)_{00} - (m_P^2)_{ii}}, \\ \Pi_{8y(1)} &= \frac{I_0(m_S^{ii}) - I_0(m_P^{00})}{(m_S^2)_{ii} - (m_P^2)_{00}}, \\ \Pi_{1(8)} &= \Pi_{8S(8)} = \Pi_{8A(8)} = \Pi_{27S(8)} = \Pi_{10A(8)} \\ &= \frac{I_0(m_S^{ii}) - I_0(m_P^{ii})}{(m_S^2)_{ii} - (m_P^2)_{ii}}. \end{aligned} \quad (57)$$

In the case of $c = 0$ and $\lambda_2 = 0$, the mass Eqs. (20) are diagonal, so we have

$$\begin{aligned} (m_S^2)_{00} &= m^2 + 3\lambda_1\bar{\sigma}_0^2 + 3\lambda_1 I_0(m_S^{00}) + 8\lambda_1 I_0(m_S^{ii}) \\ &\quad + \lambda_1 I_0(m_P^{00}) + 8\lambda_1 I_0(m_P^{ii}), \\ (m_S^2)_{ii} &= m^2 + \lambda_1\bar{\sigma}_0^2 + \lambda_1 I_0(m_S^{00}) + 10\lambda_1 I_0(m_S^{ii}) \\ &\quad + \lambda_1 I_0(m_P^{00}) + 8\lambda_1 I_0(m_P^{ii}), \end{aligned} \quad (58)$$

and

$$(m_P^2)_{00} = m^2 + \lambda_1\bar{\sigma}_0^2 + \lambda_1 I_0(m_S^{00}) + 8\lambda_1 I_0(m_S^{ii}) + 3\lambda_1 I_0(m_P^{00}) + 8\lambda_1 I_0(m_P^{ii}),$$
$$(m_P^2)_{ii} = m^2 + \lambda_1\bar{\sigma}_0^2 + \lambda_1 I_0(m_S^{00}) + 8\lambda_1 I_0(m_S^{ii}) + \lambda_1 I_0(m_P^{00}) + 10\lambda_1 I_0(m_P^{ii}), \quad (59)$$

where $i = 1, \ldots, 8$, and $(m_S^2)_{ii}$ denote $(m_S^{11})^2$, $(m_S^{22})^2$, etc. Here we note that, as a matter of fact at the tree-approximation level, the octet scalar mesons, the singlet pseudoscalar scalar meson and the octet pseudoscalar mesons all have the same mass:

$$(m_S^2)_{ii} = (m_P^2)_{00} = (m_P^2)_{ii} = m^2 + \lambda_1\bar{\sigma}_0^2. \quad (60)$$

We note that one possible consistent solution to the gap Eqs. (21) is that they all (still) have the same mass:

$$(m_S^2)_{ii} = (m_P^2)_{00} = (m_P^2)_{ii} = m^2 + \lambda_1\bar{\sigma}_0^2 + \lambda_1 I_0(m_S^{00}) + 19\lambda_1 I_0(m_S^{ii}). \quad (61)$$

This result is very interesting: whereas we expected to find pseudoscalar Nambu-Goldstone bosons, which have zero masses, we found that even the pseudoscalar mesons here have nonzero masses $(m_P^2)_{00} = (m_P^2)_{ii}$. This is the usual "problem" of the NG theorem in the Gaussian approximation.

From Eq. (21), we have

$$m^2 = -\lambda_1\bar{\sigma}_0^2 - 3\lambda_1 I_0(m_S^{00}) - 8\lambda_1 I_0(m_S^{ii}) - \lambda_1 I_0(m_P^{00}) - 8\lambda_1 I_0(m_P^{ii}). \quad (62)$$

Then using this equation together with Eqs. (58) and (59), we obtain

$$\begin{aligned}(m_S^2)_{00} &= 2\lambda_1\bar{\sigma}_0^2,\\ (m_S^2)_{ii} &= -2\lambda_1 I_0(m_S^{00}) + 2\lambda_1 I_0(m_S^{ii}),\\ (m_P^2)_{00} &= -2\lambda_1 I_0(m_S^{00}) + 2\lambda_1 I_0(m_P^{00}),\\ (m_P^2)_{ii} &= -2\lambda_1 I_0(m_S^{00}) + 2\lambda_1 I_0(m_P^{ii}).\end{aligned} \quad (63)$$

In order to check whether there are Nambu-Goldstone bosons, we only need to check if the following equations hold

$$V_i(s=0)\Pi_i(s=0) = 1, \quad (64)$$

because in this case there is no flavor mixing of T-matrix elements/scattering operators. If Eq. (64) holds, then the T-matrix elements subject to the following Bethe-Salpeter equation:

$$T_i(s) = V_i(s) + V_i(s)\Pi_i(s)T_i(s) \quad (65)$$

have a pole at $s = 0$. This means there are massless mesons propagating, and thus the Nambu-Goldstone bosons turn up.

Equation (64) can be easily checked when $c = \lambda_2 = 0$. By using Eqs. (63) and (57), we have

$$\Pi_{1(1)} = \frac{(m_P^2)_{00}}{-4\lambda_1\bar{\sigma}_0^2 + 2\lambda_1(m_P^2)_{00}},\qquad \Pi_{8x(1)} = \frac{(m_P^2)_{ii}}{-4\lambda_1\bar{\sigma}_0^2 + 2\lambda_1(m_P^2)_{ii}}, \quad (66)$$

and

$$\Pi_{8y(1)} = \Pi_{1(8)} = \Pi_{8S(8)} = \Pi_{8A(8)} = \Pi_{27S(8)} = \Pi_{10A(8)} = \frac{1}{2\lambda_1}, \quad (67)$$

Together with Eqs. (54) we have

$$V_i(s=0)\Pi_i(s=0) = 1 \quad (68)$$

for all the allowed flavor representations $\mathbf{1}_{(1)}$, $\mathbf{8}_{x(1)}$, $\mathbf{8}_{y(1)}$, $\mathbf{1}_{(8)}$, $\mathbf{8}_{S(8)}$, $\mathbf{8}_{A(8)}$, $(\mathbf{10} \oplus \overline{\mathbf{10}})_{A(8)}$, $\mathbf{27}_{S(8)}$.

As there are several flavor singlets and octets, it is not clear just how many NG bosons in these channels are independent. Yet, it is clear that there are at least $1 + 8 + 10 + 27 = 46$ distinct NG bosons when $c = 0$ and $\lambda_2 = 0$. That is (much) more than nine NG bosons expected in the general $U_L(3) \times U_R(3)$ linear sigma model, and more than 17 NG bosons when the $O(18)$ symmetry is broken down to $O(17)$. The explanation for the fact that there are more than 17 NG bosons is that the $O(18)$ may be dynamically broken down to a symmetry that is lower than $O(17)$, e.g. the $O(16)$ or even $O(15)$. In this sense the GFA approximation is substantially different from the Born, or the one-loop approximations, which are not known to lead to ground state(s) with "exotically broken" symmetry.

Thus, we have proved that all the expected pseudoscalar mesons are Nambu-Goldstone bosons when $c = 0$ and $\lambda_2 = 0$, but also that there are many more. As there are no "elementary" meson fields in the flavor $(\mathbf{10} \oplus \overline{\mathbf{10}})_{A(8)}$ and $\mathbf{27}_{S(8)}$-plets in the $SU(3)$ linear sigma model, we must conclude that these NG bosons are (zero mass) bound states of (massive) elementary boson fields. This goes to show that the GFA method is well and truly nonperturbative and capable of dynamically producing bound states even in exotic flavor channels, such as the $(\mathbf{10} \oplus \overline{\mathbf{10}})_{A(8)}$ and $\mathbf{27}_{S(8)}$. Of course, this does not mean that in the ground state of QCD there are exotic NG bosons, because the $c = 0$ and $\lambda_2 = 0$ conditions do not correspond to reality. Therefore, we discuss the $c = 0$ and $\lambda_2 \neq 0$ case next.

B. Case II: $c = 0$ and $\lambda_2 \neq 0$

When $c = 0$ and $\lambda_2 \neq 0$, the mixing between two singlet pseudoscalar mesons and among three octet pseudoscalar mesons exist, and we need to use the mixing operators. The scattering matrix V can be expanded by using the nonmixing projection operators (35) as well as the mixing operators (38) and (39), and the solution is

$$\begin{aligned}
V_{1(1)} &= 2\lambda_1 + \frac{2}{3}\lambda_2 + \frac{4}{9}\frac{(3\lambda_1+\lambda_2)^2\bar{\sigma}_0^2}{s-(m_P^2)_{00}},\\
V_{8x(1)} &= 2\lambda_1 + \frac{2}{3}\lambda_2 + \frac{4}{9}\frac{(3\lambda_1+\lambda_2)^2\bar{\sigma}_0^2}{s-(m_P^2)_{ii}},\\
V_{8y(1)} &= 2\lambda_1 + \frac{2}{3}\lambda_2 + \frac{4}{9}\frac{\lambda_2^2\bar{\sigma}_0^2}{s-(m_P^2)_{ii}},\\
V_{1(8)} &= 2\lambda_1 - \frac{2}{3}\lambda_2 + \frac{32}{9}\frac{\lambda_2^2\bar{\sigma}_0^2}{s-(m_P^2)_{00}},\\
V_{8S(8)} &= 2\lambda_1 - \frac{4}{3}\lambda_2 + \frac{10}{9}\frac{\lambda_2^2\bar{\sigma}_0^2}{s-(m_P^2)_{ii}},\\
V_{8A(8)} &= 2\lambda_1 + 6\lambda_2, \qquad V_{27S(8)} = 2\lambda_1 + 2\lambda_2,\\
V_{10A(8)} &= 2\lambda_1,\\
V_{1(M1)} &= V_{1(M2)} = \frac{2}{3}\lambda_2 + \frac{4}{9}\frac{(3\lambda_1\lambda_2+\lambda_2^2)\bar{\sigma}_0^2}{s-(m_P^2)_{00}},\\
V_{8(M1)} &= V_{8(M2)} = \lambda_2 + \frac{2}{3}\frac{(3\lambda_1\lambda_2+\lambda_2^2)\bar{\sigma}_0^2}{s-(m_P^2)_{ii}},\\
V_{8(M3)} &= V_{8(M4)} = \lambda_2 + \frac{2}{3}\frac{\lambda_2^2\bar{\sigma}_0^2}{s-(m_P^2)_{ii}},\\
V_{8(M5)} &= V_{8(M6)} = \lambda_2 + \frac{2}{3}\frac{(3\lambda_1\lambda_2+\lambda_2^2)\bar{\sigma}_0^2}{s-(m_P^2)_{ii}}.
\end{aligned} \tag{69}$$

We do the same procedure for Π_{abcd}, and the results are (after choosing $s = 0$)

$$\begin{aligned}
\Pi_{1(1)} &= \frac{I_0(m_S^{00}) - I_0(m_P^{00})}{(m_S^2)_{00} - (m_P^2)_{00}},\\
\Pi_{8x(1)} &= \frac{I_0(m_S^{00}) - I_0(m_P^{ii})}{(m_S^2)_{00} - (m_P^2)_{ii}},\\
\Pi_{8y(1)} &= \frac{I_0(m_S^{ii}) - I_0(m_P^{00})}{(m_S^2)_{ii} - (m_P^2)_{00}},\\
\Pi_{1(8)} &= \Pi_{8S(8)} = \Pi_{8A(8)} = \Pi_{27S(8)}\\
&= \Pi_{10A(8)} = \frac{I_0(m_S^{ii}) - I_0(m_P^{ii})}{(m_S^2)_{ii} - (m_P^2)_{ii}},\\
\Pi_{1(Mi)} &= \Pi_{8(Mi)} = 0.
\end{aligned} \tag{70}$$

So Π_{abcd} is still "diagonal."

In the case of $c = 0$ and $\lambda_2 \neq 0$, the masses are diagonal, and from Eqs. (20) we have

$$\begin{aligned}
(m_S^2)_{00} &= m^2 + 3\lambda_1\bar{\sigma}_0^2 + \lambda_2\bar{\sigma}_0^2 + (3\lambda_1+\lambda_2)I_0(m_S^{00})\\
&\quad + (8\lambda_1 + 8\lambda_2)I_0(m_S^{ii}) + \left(\lambda_1 + \frac{1}{3}\lambda_2\right)I_0(m_P^{00})\\
&\quad + \left(8\lambda_1 + \frac{8}{3}\lambda_2\right)I_0(m_P^{ii}),\\
(m_S^2)_{ii} &= m^2 + \lambda_1\bar{\sigma}_0^2 + \lambda_2\bar{\sigma}_0^2 + (\lambda_1+\lambda_2)I_0(m_S^{00})\\
&\quad + (10\lambda_1 + 5\lambda_2)I_0(m_S^{ii}) + (\lambda_1 + \tfrac{1}{3}\lambda_2)I_0(m_P^{00})\\
&\quad + (8\lambda_1 + \tfrac{17}{3}\lambda_2)I_0(m_P^{ii}),
\end{aligned} \tag{71}$$

and

$$\begin{aligned}
(m_P^2)_{00} &= m^2 + \lambda_1\bar{\sigma}_0^2 + \tfrac{1}{3}\lambda_2\bar{\sigma}_0^2 + (\lambda_1 + \tfrac{1}{3}\lambda_2)I_0(m_S^{00})\\
&\quad + (8\lambda_1 + \tfrac{8}{3}\lambda_2)I_0(m_S^{ii}) + (3\lambda_1+\lambda_2)I_0(m_P^{00})\\
&\quad + (8\lambda_1 + 8\lambda_2)I_0(m_P^{ii}),\\
(m_P^2)_{ii} &= m^2 + \lambda_1\bar{\sigma}_0^2 + \tfrac{1}{3}\lambda_2\bar{\sigma}_0^2 + (\lambda_1 + \tfrac{1}{3}\lambda_2)I_0(m_S^{00})\\
&\quad + (8\lambda_1 + \tfrac{17}{3}\lambda_2)I_0(m_S^{ii}) + (\lambda_1+\lambda_2)I_0(m_P^{00})\\
&\quad + (10\lambda_1 + 5\lambda_2)I_0(m_P^{ii}).
\end{aligned} \tag{72}$$

From Eq. (21), we have

$$\begin{aligned}
m^2 &= -\lambda_1\bar{\sigma}_0^2 - \frac{1}{3}\lambda_2\bar{\sigma}_0^2 - (3\lambda_1+\lambda_2)I_0(m_S^{00})\\
&\quad - (8\lambda_1 + 8\lambda_2)I_0(m_S^{ii}) - \left(\lambda_1 + \frac{1}{3}\lambda_2\right)I_0(m_P^{00})\\
&\quad - \left(8\lambda_1 + \frac{8}{3}\lambda_2\right)I_0(m_P^{ii}).
\end{aligned} \tag{73}$$

Then using this equation together with Eqs. (71) and (72), we obtain

$$\begin{aligned}
(m_S^2)_{00} &= 2\lambda_1\bar{\sigma}_0^2 + \tfrac{2}{3}\lambda_2\bar{\sigma}_0^2,\\
(m_S^2)_{ii} &= 2\lambda_1(I_0(m_S^{ii}) - I_0(m_S^{00})) + \tfrac{2}{3}\lambda_2\bar{\sigma}_0^2\\
&\quad + 3\lambda_2(I_0(m_P^{ii}) - I_0(m_S^{ii})),\\
(m_P^2)_{00} &= 2\lambda_1(I_0(m_P^{00}) - I_0(m_S^{00}))\\
&\quad + \tfrac{2}{3}\lambda_2(I_0(m_P^{00}) - I_0(m_S^{00}))\\
&\quad + \tfrac{16}{3}\lambda_2(I_0(m_P^{ii}) - I_0(m_S^{ii})),\\
(m_P^2)_{ii} &= 2\lambda_1(I_0(m_P^{ii}) - I_0(m_S^{00})) + \tfrac{2}{3}\lambda_2(I_0(m_P^{00}) - I_0(m_S^{00}))\\
&\quad + \tfrac{7}{3}\lambda_2(I_0(m_P^{ii}) - I_0(m_S^{ii})).
\end{aligned} \tag{74}$$

1. Singlet channel

Since the mixing exists, we need to use Eq. (46) derived earlier. After inserting the expressions of the masses and the polarization energies, V_i and Π_i, which are listed in Eqs. (74), (69), and (70), we can verify that at the kinematical point $s = 0$, we have

$$\left| 1 - \begin{pmatrix} V_{1(1)} & 2\sqrt{2}V_{1(M2)} \\ 2\sqrt{2}V_{1(M1)} & V_{1(8)} \end{pmatrix} \begin{pmatrix} \Pi_{1(1)} & 0 \\ 0 & \Pi_{1(8)} \end{pmatrix} \right| = 0, \tag{75}$$

with the meaning that the determinant of the matrix within the vertical bars is zero. Therefore, there are Nambu-Goldstone bosons. Since there is a mixing between the two singlet pseudoscalar mesons, $T_{1(1)}$, $T_{1(8)}$, $T_{1(M1)}$ and $T_{1(M2)}$ all have a pole at $s = 0$. However, we can verify that only one of the two eigenvalues is 0. This means that only one of the singlet pseudoscalar meson is a Nambu-Goldstone boson.

2. Octet channel ($\mathbf{8}_{x(1)}$, $\mathbf{8}_{y(1)}$ and $\mathbf{8}_{S(8)}$)

We calculate the solution Eq. (47) when $c = 0$ and $\lambda_2 \neq 0$, and find that at the point $s = 0$, we have

$$\left| 1 - \begin{pmatrix} V_{8x(1)} & \frac{2}{3}V_{8(M5)} & \frac{\sqrt{10}}{3}V_{8(M1)} \\ \frac{2}{3}V_{8(M6)} & V_{8y(1)} & \frac{\sqrt{10}}{3}V_{8(M3)} \\ \frac{\sqrt{10}}{3}V_{8(M1)} & \frac{\sqrt{10}}{3}V_{8(M4)} & V_{8S(8)} \end{pmatrix} \times \begin{pmatrix} \Pi_{8x(1)} & 0 & 0 \\ 0 & \Pi_{8y(1)} & 0 \\ 0 & 0 & \Pi_{8S(8)} \end{pmatrix} \right| = 0, \tag{76}$$

where, again the vertical bars denote the determinant of the (3×3) matrix within. Therefore, these equations describe massless Nambu-Goldstone bosons. Moreover, we can verify that only one of the three eigenvalues is zero. This is difficult to prove analytically even with the aid of algebraic manipulation programs. Therefore, we randomly choose the values for the relevant parameters (coupling constants), and confirm this result. So we obtain the result that only one octet and one singlet of pseudoscalar mesons are Nambu-Goldstone bosons in this case. That agrees with the conventional result in the Born approximation, although the pseudoscalar spectral functions in the GFA [18] contain (much) more structure than a simple Dirac delta function, see, e.g. [3].

3. $\mathbf{8}_{A(8)}$, $(\mathbf{10} \oplus \overline{\mathbf{10}})_{A(8)}$ and $\mathbf{27}_{S(8)}$ channels

For the $\mathbf{8}_{A(8)}$ channel, we have

$$1 - V_{8A(8)}\Pi_{8A(8)} = \lambda_2 \frac{\bar{\sigma}_0^2 + (I_0(m_S^{00})) - I_0(m_P^{00})) - 10(I_0(m_S^{ii}) - I_0(m_P^{ii}))}{(3\lambda_1 - \lambda_2)(I_0(m_S^{ii}) - I_0(m_P^{ii})) + \lambda_2(\bar{\sigma}_0^2 + I_0(m_S^{00}) - I_0(m_P^{00}))}.$$

For the decuplet channel, we have

$$1 - V_{10A(8)}\Pi_{10A(8)} = \lambda_2 \frac{\bar{\sigma}_0^2 + (I_0(m_S^{00})) - I_0(m_P^{00})) - (I_0(m_S^{ii}) - I_0(m_P^{ii}))}{(3\lambda_1 - \lambda_2)(I_0(m_S^{ii}) - I_0(m_P^{ii})) + \lambda_2(\bar{\sigma}_0^2 + I_0(m_S^{00}) - I_0(m_P^{00}))}.$$

For the $\mathbf{27}_{S(8)}$ channel, we have

$$1 - V_{27S(8)}\Pi_{27S(8)} = \lambda_2 \frac{\bar{\sigma}_0^2 + (I_0(m_S^{00})) - I_0(m_P^{00})) - 4(I_0(m_S^{ii}) - I_0(m_P^{ii}))}{(3\lambda_1 - \lambda_2)(I_0(m_S^{ii}) - I_0(m_P^{ii})) + \lambda_2(\bar{\sigma}_0^2 + I_0(m_S^{00}) - I_0(m_P^{00}))}.$$

Therefore, none of them are Nambu-Goldstone bosons, as expected.

C. Case III: $c \neq 0 \neq \lambda_2$

In this case we assume that both c and λ_2 are nonzero. This is the most general case that conserves the $SU_L(3) \times SU(3)_R$ symmetry. The scattering matrix T can be expanded by using the nonmixing projection operators (35) as well as the mixing operators (38) and (39), and its solution is

$$
\begin{aligned}
V_{1(1)} &= 2\lambda_1 + \frac{2}{3}\lambda_2 + \frac{2}{9}\frac{2(3\lambda_1+\lambda_2)^2\bar{\sigma}_0^2 + 2\sqrt{6}(3\lambda_1+\lambda_2)c\bar{\sigma}_0 + 3c^2}{s-(m_P^2)_{00}},\\
V_{8x(1)} &= 2\lambda_1 + \frac{2}{3}\lambda_2 + \frac{1}{18}\frac{8(3\lambda_1+\lambda_2)^2\bar{\sigma}_0^2 - 4\sqrt{6}(3\lambda_1+\lambda_2)c\bar{\sigma}_0 + 3c^2}{s-(m_P^2)_{ii}},\\
V_{8y(1)} &= 2\lambda_1 + \frac{2}{3}\lambda_2 + \frac{1}{18}\frac{8\lambda_2^2\bar{\sigma}_0^2 - 4\sqrt{6}\lambda_2 c\bar{\sigma}_0 + 3c^2}{s-(m_P^2)_{ii}},\\
V_{1(8)} &= 2\lambda_1 - \frac{2}{3}\lambda_2 + \frac{4}{9}\frac{8\lambda_2^2\bar{\sigma}_0^2 - 4\sqrt{6}\lambda_2 c\bar{\sigma}_0 + 3c^2}{s-(m_P^2)_{00}},\\
V_{8S(8)} &= 2\lambda_1 - \frac{4}{3}\lambda_2 + \frac{5}{9}\frac{2\lambda_2^2\bar{\sigma}_0^2 + 2\sqrt{6}\lambda_2 c\bar{\sigma}_0 + 3c^2}{s-(m_P^2)_{ii}},\\
V_{8A(8)} &= 2\lambda_1 + 6\lambda_2, \qquad V_{27S(8)} = 2\lambda_1 + 2\lambda_2, \qquad V_{10A(8)} = 2\lambda_1,\\
V_{1(M1)} &= V_{1(M2)} = \frac{2}{3}\lambda_2 + \frac{1}{9}\frac{4(3\lambda_1\lambda_2+\lambda_2^2)\bar{\sigma}_0^2 - \sqrt{6}(3\lambda_1-\lambda_2)c\bar{\sigma}_0 - 3c^2}{s-(m_P^2)_{00}},\\
V_{8(M1)} &= V_{8(M2)} = \lambda_2 + \frac{1}{6}\frac{4(3\lambda_1\lambda_2+\lambda_2^2)\bar{\sigma}_0^2 + \sqrt{6}(6\lambda_1+\lambda_2)c\bar{\sigma}_0 - 3c^2}{s-(m_P^2)_{ii}},\\
V_{8(M3)} &= V_{8(M4)} = \lambda_2 + \frac{1}{6}\frac{4\lambda_2^2\bar{\sigma}_0^2 + \sqrt{6}\lambda_2 c\bar{\sigma}_0 - 3c^2}{s-(m_P^2)_{ii}},\\
V_{8(M5)} &= V_{8(M6)} = \lambda_2 + \frac{1}{12}\frac{8(3\lambda_1\lambda_2+\lambda_2^2)\bar{\sigma}_0^2 - 2\sqrt{6}(3\lambda_1+2\lambda_2)c\bar{\sigma}_0 + 3c^2}{s-(m_P^2)_{ii}}.
\end{aligned} \tag{77}
$$

We go through the same procedure for Π_{ab}, and the results are (after setting $s = 0$)

$$
\begin{aligned}
\Pi_{1(1)} &= \frac{I_0(m_S^{00}) - I_0(m_P^{00})}{(m_S^2)_{00} - (m_P^2)_{00}}, \qquad \Pi_{8x(1)} = \frac{I_0(m_S^{00}) - I_0(m_P^{ii})}{(m_S^2)_{00} - (m_P^2)_{ii}},\\
\Pi_{8y(1)} &= \frac{I_0(m_S^{ii}) - I_0(m_P^{00})}{(m_S^2)_{ii} - (m_P^2)_{00}},\\
\Pi_{1(8)} &= \Pi_{8S(8)} = \Pi_{8A(8)} = \Pi_{27S(8)} = \Pi_{10A(8)} = \frac{I_0(m_S^{ii}) - I_0(m_P^{ii})}{(m_S^2)_{ii} - (m_P^2)_{ii}},\\
\Pi_{1(\mathrm{mix})} &= \Pi_{8x(\mathrm{mix})} = \Pi_{8y(\mathrm{mix})} = \Pi_{8z(\mathrm{mix})} = 0,
\end{aligned} \tag{78}
$$

which is the same as the previous case.

In the case of $c \neq 0$ and $\lambda_2 \neq 0$, the masses are still diagonal, and from Eqs. (20) we have

$$
\begin{aligned}
(m_S^2)_{00} &= m^2 + 3\lambda_1\bar{\sigma}_0^2 + \lambda_2\bar{\sigma}_0^2 + (3\lambda_1+\lambda_2)I_0(m_S^{00}) + (8\lambda_1+8\lambda_2)I_0(m_S^{ii}) + \left(\lambda_1 + \frac{1}{3}\lambda_2\right)I_0(m_P^{00}) + \left(8\lambda_1 + \frac{8}{3}\lambda_2\right)I_0(m_P^{ii})\\
&\quad - \sqrt{\frac{2}{3}}c\bar{\sigma}_0,\\
(m_S^2)_{ii} &= m^2 + \lambda_1\bar{\sigma}_0^2 + \lambda_2\bar{\sigma}_0^2 + (\lambda_1+\lambda_2)I_0(m_S^{00}) + (10\lambda_1+5\lambda_2)I_0(m_S^{ii}) + \left(\lambda_1 + \frac{1}{3}\lambda_2\right)I_0(m_P^{00}) + \left(8\lambda_1 + \frac{17}{3}\lambda_2\right)I_0(m_P^{ii})\\
&\quad + \sqrt{\frac{1}{6}}c\bar{\sigma}_0,
\end{aligned} \tag{79}
$$

and

$$
\begin{aligned}
(m_P^2)_{00} &= m^2 + \lambda_1\bar{\sigma}_0^2 + \frac{1}{3}\lambda_2\bar{\sigma}_0^2 + \left(\lambda_1 + \frac{1}{3}\lambda_2\right)I_0(m_S^{00}) + \left(8\lambda_1 + \frac{8}{3}\lambda_2\right)I_0(m_S^{ii}) + (3\lambda_1 + \lambda_2)I_0(m_P^{00}) \\
&\quad + (8\lambda_1 + 8\lambda_2)I_0(m_P^{ii}) + \sqrt{\frac{2}{3}}c\bar{\sigma}_0, \\
(m_P^2)_{ii} &= m^2 + \lambda_1\bar{\sigma}_0^2 + \frac{1}{3}\lambda_2\bar{\sigma}_0^2 + \left(\lambda_1 + \frac{1}{3}\lambda_2\right)I_0(m_S^{00}) + \left(8\lambda_1 + \frac{17}{3}\lambda_2\right)I_0(m_S^{ii}) + (\lambda_1 + \lambda_2)I_0(m_P^{00}) \\
&\quad + (10\lambda_1 + 5\lambda_2)I_0(m_P^{ii}) - \sqrt{\frac{1}{6}}c\bar{\sigma}_0.
\end{aligned} \tag{80}
$$

From Eq. (21), we have

$$
\begin{aligned}
m^2 &= -\lambda_1\bar{\sigma}_0^2 - \frac{1}{3}\lambda_2\bar{\sigma}_0^2 + \sqrt{\frac{1}{6}}c\bar{\sigma}_0 - (3\lambda_1 + \lambda_2)I_0(m_S^{00}) - (8\lambda_1 + 8\lambda_2)I_0(m_S^{ii}) - \left(\lambda_1 + \frac{1}{3}\lambda_2\right)I_0(m_P^{00}) \\
&\quad - \left(8\lambda_1 + \frac{8}{3}\lambda_2\right)I_0(m_P^{ii}) + \sqrt{\frac{1}{6}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{00}) - I_0(m_P^{00})) - 2\sqrt{\frac{2}{3}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{ii}) - I_0(m_P^{ii})),
\end{aligned} \tag{81}
$$

Therefore, by using this equation together with Eqs. (79) and (80), we obtain

$$
\begin{aligned}
(m_S^2)_{00} &= 2\lambda_1\bar{\sigma}_0^2 + \frac{2}{3}\lambda_2\bar{\sigma}_0^2 - \sqrt{\frac{1}{6}}c\bar{\sigma}_0 + \sqrt{\frac{1}{6}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{00}) - I_0(m_P^{00})) - 2\sqrt{\frac{2}{3}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{ii}) - I_0(m_P^{ii})), \\
(m_S^2)_{ii} &= 2\lambda_1(I_0(m_S^{ii}) - I_0(m_S^{00})) + \sqrt{\frac{2}{3}}c\bar{\sigma}_0 + \frac{2}{3}\lambda_2\bar{\sigma}_0^2 + 3\lambda_2(I_0(m_P^{ii}) - I_0(m_S^{ii})) + \sqrt{\frac{1}{6}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{00}) - I_0(m_P^{00})) \\
&\quad - 2\sqrt{\frac{2}{3}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{ii}) - I_0(m_P^{ii})), \\
(m_P^2)_{00} &= 2\lambda_1(I_0(m_P^{00}) - I_0(m_S^{00})) + \sqrt{\frac{3}{2}}c\bar{\sigma}_0 + \frac{2}{3}\lambda_2(I_0(m_P^{00}) - I_0(m_S^{00})) + \frac{16}{3}\lambda_2(I_0(m_P^{ii}) - I_0(m_S^{ii})) \\
&\quad + \sqrt{\frac{1}{6}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{00}) - I_0(m_P^{00})) - 2\sqrt{\frac{2}{3}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{ii}) - I_0(m_P^{ii})), \\
(m_P^2)_{ii} &= 2\lambda_1(I_0(m_P^{ii}) - I_0(m_S^{00})) + \frac{2}{3}\lambda_2(I_0(m_P^{00}) - I_0(m_S^{00})) + \frac{7}{3}\lambda_2(I_0(m_P^{ii}) - I_0(m_S^{ii})) + \sqrt{\frac{1}{6}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{00}) - I_0(m_P^{00})) \\
&\quad - 2\sqrt{\frac{2}{3}}\frac{c}{\bar{\sigma}_0}(I_0(m_S^{ii}) - I_0(m_P^{ii})).
\end{aligned}
$$

1. Singlet channel

We calculate the solution Eq. (46) when $c \neq 0$ and $\lambda_2 \neq 0$, and find that at the point $s = 0$, we have

$$
\left| 1 - \begin{pmatrix} V_{1(1)} & 2\sqrt{2}V_{1(M2)} \\ 2\sqrt{2}V_{1(M1)} & V_{1(8)} \end{pmatrix} \begin{pmatrix} \Pi_{1(1)} & 0 \\ 0 & \Pi_{1(8)} \end{pmatrix} \right| \neq 0, \tag{82}
$$

where, again the vertical bars denote the determinant of the (2×2) matrix within. Therefore, we have verified that the singlet pseudoscalar meson is not a Nambu-Goldstone boson any more, due to the $U_A(1)$ symmetry breaking interaction constant $c \neq 0$.

2. Octet channel ($\mathbf{8}_{x(1)}$, $\mathbf{8}_{y(1)}$ and $\mathbf{8}_{S(8)}$)

We calculate the solution Eq. (47) when $c \neq 0$ and $\lambda_2 \neq 0$, and find that at the point $s = 0$, we have

$$
\left| 1 - \begin{pmatrix} V_{8x(1)} & \frac{2}{3}V_{8(M5)} & \frac{\sqrt{10}}{3}V_{8(M1)} \\ \frac{2}{3}V_{8(M6)} & V_{8y(1)} & \frac{\sqrt{10}}{3}V_{8(M3)} \\ \frac{\sqrt{10}}{3}V_{8(M1)} & \frac{\sqrt{10}}{3}V_{8(M4)} & V_{8S(8)} \end{pmatrix} \times \begin{pmatrix} \Pi_{8x(1)} & 0 & 0 \\ 0 & \Pi_{8y(1)} & 0 \\ 0 & 0 & \Pi_{8S(8)} \end{pmatrix} \right| = 0, \tag{83}
$$

where, again the vertical bars denote the determinant of the (3×3) matrix within. This is again difficult to prove analytically, so we again randomly choose the values for the relevant parameters, and obtain this result. Moreover, we can verify that only one of its three eigenvalues is zero. So we obtain the expected, yet nontrivial result that only one octet of pseudoscalar mesons are Nambu-Goldstone bosons.

VII. EXPLICITLY BROKEN CHIRAL SYMMETRY AND DASHEN'S FORMULA

When the chiral symmetry is explicitly broken, the NG theorem turns into a relation between the chiral symmetry breaking parameter and the NG boson mass, as first discussed by Dashen [19]. The NG theorem in the chiral limit has already been addressed in the Gaussian approximation and equivalent formalisms in Refs. [6,7]. Here, we turn to the nonchiral case.

As shown in Ref. [6] in the chiral limit the Nambu-Goldstone particle appears as a zero-mass pole in the T matrix in the pseudoscalar channel. Next we look at the zero CM energy $s = 0$ polarization function $V_\pi(0)\Pi_\pi(0)$ in the nonchiral case $h_0 = \varepsilon \neq 0$. For simplicity's sake we only study this in the $\lambda_2 = c = 0$ case (so as not to have to deal with complications associated with channel mixing(s) in the flavor-singlet and octet channels). Now the gap Eq. (21) becomes

$$m^2 = \frac{\epsilon}{\bar{\sigma}_0} - \lambda_1 \bar{\sigma}_0^2 - 3\lambda_1 I_0(m_S^{00}) - 8\lambda_1 I_0(m_S^{ii}) - \lambda_1 I_0(m_P^{00}) - 8\lambda_1 I_0(m_P^{ii}), \tag{84}$$

together with (58) and (59), we obtain

$$\begin{aligned} (m_S^2)_{00} &= \frac{\epsilon}{\bar{\sigma}_0} + 2\lambda_1 \bar{\sigma}_0^2, \\ (m_S^2)_{ii} &= \frac{\epsilon}{\bar{\sigma}_0} - 2\lambda_1 I_0(m_S^{00}) + 2\lambda_1 I_0(m_S^{ii}), \\ (m_P^2)_{00} &= \frac{\epsilon}{\bar{\sigma}_0} - 2\lambda_1 I_0(m_S^{00}) + 2\lambda_1 I_0(m_P^{00}), \\ (m_P^2)_{ii} &= \frac{\epsilon}{\bar{\sigma}_0} - 2\lambda_1 I_0(m_S^{00}) + 2\lambda_1 I_0(m_P^{ii}). \end{aligned} \tag{85}$$

We work out the BS equation for the flavor-singlet and octet channels. The polarization function is worked out in the flavor-singlet channel $\mathbf{1}_{(1)}$ as

$$V_{1(1)}(0)\Pi_{1(1)}(0) = 1 - \frac{\epsilon}{\bar{\sigma}_0} \frac{(m_S^2)_{00}}{(m_P^2)_{00}((m_S^2)_{00} - (m_P^2)_{00})} + \mathcal{O}(\epsilon^2), \tag{86}$$

and we obtain the similar result for the flavor-octet channel $\mathbf{8}_{x(1)}$:

$$V_{8x(1)}(0)\Pi_{8x(1)}(0) = 1 - \frac{\epsilon}{\bar{\sigma}_0} \frac{(m_S^2)_{00}}{(m_P^2)_{ii}((m_S^2)_{00} - (m_P^2)_{ii})} + \mathcal{O}(\epsilon^2), \tag{87}$$

as well as the positive-parity channel flavor-octet consisting of two scalar mesons. Since $V_\pi(0)\Pi_\pi(0) \simeq 1 - \mu^{-2}\frac{\varepsilon}{v}$, we see that the pole in the s-channel propagator has moved away from zero momentum. In order to find the mass, we must take into account the residue at the pole; thus we find $m_\pi^2 = \frac{\varepsilon}{v} + \mathcal{O}(\varepsilon^2)$, just as in the Born approximation. Here $h_0 = \varepsilon$ is the explicit symmetry breaking parameter and the "pion" symbol π denotes the complete set of 17 pseudo-NG bosons, whose mass should be small if this linearized approximation is to hold. This result is valid only for "small" values of the explicit chiral symmetry breaking parameters, such as that of the $SU(2)_L \times SU_R(2)$ symmetry breaking that is responsible for the pion's mass.

Of course, with $h_0 \neq 0$ it is possible to have an explicit breaking of the $O(18)$ symmetry down to the explicitly conserved, yet spontaneously broken $O(17)$ symmetry. In that case there will remain several massless NG bosons. For instance, in the other flavor-singlet channel $\mathbf{1}_{(8)}$ made up of flavor-octet mesons, we have

$$V_{1(8)}(0)\Pi_{1(8)}(0) = 1, \tag{88}$$

and the same result holds for the $\mathbf{8}_{y(1)}$, $\mathbf{8}_{S(8)}$, $\mathbf{8}_{A(8)}$, $\mathbf{27}_{S(8)}$ and $(\mathbf{10} \oplus \overline{\mathbf{10}})_{A(8)}$ channels. Of course, once one turns on $\lambda_2 \neq 0$ and/or $c \neq 0$ all of these NG bosons acquire masses, as derived in Sec. VI B. The NG mesons also acquire masses when one explicitly breaks the $O(17)$, $SU(3)$ or $SU(2)$ symmetries, e.g. by including $h_8 \neq 0$ and/or $h_3 \neq 0$. These masses can be evaluated by means of Dashen's formula so long as the explicit symmetry breaking is small, which is not the case for realistic values of $h_8 \neq 0$; $c \neq 0$. Therefore, this result is practically useful only for the (isotriplet) pion masses, but not for the kaons and the η and η' mesons. This is perhaps as far as one can go using only analytic methods.

The next step, to be taken in our next paper, will be to numerically solve the gap and Bethe-Salpeter equations with an explicitly broken $SU(3)$ symmetry, so as to reproduce the experimental pseudoscalar masses and their weak decay constants and thus to fix all of the free parameters in this model in the Gaussian approximation.

VIII. CONCLUSION

We have studied the NG theorem for the pseudoscalar mesons in the $U(3)_L \times U_R(3)$ linear sigma model. We have constructed the ground state wave function in the GFA. At this level, all the scalar mesons and the pseudoscalar mesons acquire finite masses by the minimal spontaneous symmetry breaking $\sigma_0 \neq 0$. Hence, the NG theorem is not satisfied at the GFA level.

Hence, we have developed a method to work out the Bethe-Salpeter equation for the scattering T matrix of mesons. To this end, it was important to work out the projection operators in order to separate various $SU(3)$ channels in the T matrix. We have then explicitly worked out the BS equations for the pseudoscalar mesons in the general case of the $U(3)_L \times U_R(3)$ linear sigma model. Since the verification of the NG theorem is quite complicated and tricky, we have decided to work out the NG theorem step by step.

First, we studied the $\lambda_1 \neq 0$ and $\lambda_2 = c = 0$ case. In this case, we verified that the NG bosons appear in the usual flavor-nonet channel, where the NG bosons are present at the mean field, or the Born approximation level. Additionally, we have found new composite NG bosons in certain other flavor channels that correspond to the breaking of the extended $O(18)$ symmetry down to a lower symmetry.

Then we studied the case with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, but $c = 0$. In this case, we found the usual flavor-nonet of NG bosons. We then studied the case when all the coupling constant in the Lagrangian are nonzero. In this case, we found only the flavor-octet pseudoscalar mesons as the NG bosons: the ninth pseudoscalar meson acquires a non-zero-mass and thus is not an NG boson any more. Of course, $c \neq 0$ corresponds to the explicit $U_A(1)$ symmetry breaking, that affects the η and η' mesons, and is comparable with, or perhaps even larger than the explicit breaking of the $SU(3)_L \times SU_R(3)$ symmetry.

We have discussed another simple case in order to examine how low-mass pseudo-NG bosons emerge due to the explicit chiral symmetry breaking: when the Lagrangian has just one small explicit chiral symmetry breaking parameter $h_0 = \varepsilon \neq 0$. There we confirmed that Dashen's result for pseudo-NG boson masses hold in the Gaussian approximation. This result is valid only for small values of the explicit chiral symmetry breaking parameters, such as that of the $SU(2)_L \times SU_R(2)$ symmetry breaking that is responsible for the pion's mass.

In this paper, we have analyzed the appearance of NG bosons for various cases in the chiral $U(3)_L \times U_R(3)$ linear sigma model Lagrangian. We have also studied the effect of the small explicit chiral symmetry breaking term to provide a small mass to the pseudoscalar bosons. This result is practically useful only for the (isotriplet of) pions, but not for the kaons and the η and η' mesons.

This is perhaps as far as one can possibly go using only analytic methods. The next step, to be taken in our next paper, will be to numerically solve the gap and Bethe-Salpeter equations with an explicitly broken $SU(3)$ symmetry, so as to reproduce the experimental pseudoscalar masses and their weak decay constants and thus to fix all of the free parameters in this model in the Gaussian approximation. Then it will be possible and (very) interesting to calculate the spectra of scalar bosons in the $SU(3)$ chiral sigma model.

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