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Quasidegenerate neutrinos in $SO(10)$

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We propose a specific ansatz for the structure of Yukawa matrices in $SO(10)$ models that lead to quasidegenerate neutrinos through the type-I seesaw mechanism. Consistency of this ansatz is demonstrated through detailed fits to fermion masses and mixing angles, all of which can be explained with reasonable accuracy in a model that uses the Higgs fields transforming as 10, 120, and $\overline{126}$ representations of $SO(10)$. The proposed ansatz is shown to follow from an extended model based on the three generations of the vectorlike fermions and an $O(3)$ flavor symmetry. Successful numerical fits are also discussed in earlier proposed models, which used a combination of the type-I and type-II seesaw mechanisms for obtaining quasidegenerate neutrinos. Large neutrino mixing angles emerge as a consequence of neutrino mass degeneracy in both these cases.

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I. INTRODUCTION

Experiments over the years have revealed the following:

(1) Two of the neutrino mixing angles are large as opposed to the small quark mixing angles.

(2) Neutrino mass hierarchy is milder compared to quarks, and the extreme case of all neutrinos being quasidegenerate is still an allowed possibility.

Several independent reasons have been advanced [1–3] to understand feature (1) of the fermion spectrum but it may be that its answer lies in (2). Large mixing angles become quite natural if neutrinos are almost degenerate. They remain undefined in the exact degenerate limit. A small perturbation that leads to differences in neutrino masses can also stabilize all or some of the mixing angles to large values. Such theory, which predicts quasidegeneracy, has a built-in mechanism to explain large mixing angles. We present an $SO(10)$-based unified description of fermion masses and mixing leading to hierarchical charged fermions and quasidegenerate neutrino masses.

$SO(10)$ models provide a natural framework for understanding neutrino masses because of the seesaw mechanisms [1] inherent in them. Neutrino masses arise in these models from two separate sources either from the vacuum expectation value of the left-handed triplet (type-II) or from the right-handed triplet (type-I) Higgs. It was pointed out [4,5] long ago that the combination of these two sources provides an interesting framework for understanding quasidegeneracy of neutrinos. In this approach, some flavor symmetry leads to a degenerate type-II contribution, and its breaking in the Dirac neutrino masses then leads to departure from degeneracy through the type-I contribution. This is realizable if the type-II contribution dominates over the type-I, which is not always the case [6,7]. An alternative possibility is that both degeneracy and its breaking arise from a single source, namely, the type-I seesaw mechanism. This, however, requires a peculiar structure for the right-handed (RH) neutrino mass matrix M_R. It has been pointed out that the required structure can arise from the "Dirac screening" [8] or more generally from the application of the minimal flavor violation [9] hypothesis to the leptonic sector [10].

While these possibilities are known, there does not exist a detailed study of all fermion masses and mixing in the context of realistic $SO(10)$ models with quasidegenerate neutrinos, and we address this question using (A) a type-I mechanism alone and (B) a combination of type-I and type-II mechanisms.

We use supersymmetric $SO(10)$ as our basic framework. Fermion masses arise in renormalizable $SO(10)$ models through their couplings to Higgs fields transforming as 10, $\overline{126}$, and 120 representations. One needs at least two of these fields to get fermion mixing, and the minimal model with 10 and $\overline{126}$ has attracted a lot of attention [2,6,7,11]. There have been studies of models with an additional 120 also [12–14]. In our context, we find that all three Higgs representations are needed to obtain satisfactory fits to fermion masses and mixing. Starting with a supersymmetric $SO(10)$, an effective minimal supersymmetric standard model (MSSM) is obtained by assuming fine-tuning, which keeps only two Higgs doublets lights. The final fermion mass matrices obtained after $SO(10)$ and $SU(2)_L \times U(1)$ breaking can be parametrized as [13,14]

$$M_d = H + F + G, \qquad M_u = r(H + sF + t_u G),$$
$$M_l = H - 3F + t_l G, \qquad M_D = r(H - 3sF + t_D G),$$
$$M_L = r_L F, \qquad M_R = r_R^{-1} F, \tag{1}$$

where the matrices H and F are complex symmetric and G

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is an antisymmetric matrix in generation space. We follow the same conventions as used in [14]. H, F, and G arise from the fermionic Yukawa couplings to the 10, $\overline{126}$, and 120 Higgs fields, respectively. r, s, t_u, t_l, t_D, r_L, and r_R are complex parameters. The light neutrino mass matrix is given by

$$\mathcal{M}_\nu = r_L F - r_R M_D F^{-1} M_D^T \equiv \mathcal{M}_\nu^{II} + \mathcal{M}_\nu^{I}. \quad (2)$$

It is known that the above fermion mass structure allows different mixing patterns for quarks and neutrinos if a type-II seesaw mechanism dominates [2,11]. Consider the limit in which the contribution of the 10-plet H dominates. In this limit, all the charged fermions are diagonalized by the same matrix and the Cabibbo-Kobayashi-Maskawa (CKM) matrix becomes proportional to identity. In the same limit, neutrino mixing with the type-II dominance is governed by F in Eq. (2) leading to nontrivial leptonic mixing. In fact, if only H dominates the charged fermion masses, then one can obtain b-τ unification, which in turn drives the large atmospheric mixing [2]. The existing fits [7,14] to fermion masses and mixing with type-II dominance are for the hierarchical neutrino masses. A degenerate neutrino spectrum can be obtained in this approach with an additional assumption:

$$F = c_0 I, \quad (3)$$

with I denoting a 3×3 identity matrix. The subdominant type-I contribution can then lead to the quark mixing and neutrino mass differences.

The realization of the attractive type-II dominated scenario was found difficult in the context of the minimal model [6,7]. It was found that parameter space favored by the overall fit to fermion masses suppresses the type-II contribution compared to the type-I. This motivates us to study degenerate neutrinos in the context of a purely type-I seesaw mechanism. A general framework to obtain quasi-degenerate neutrinos in a type-I seesaw was recently discussed [10] and following it we impose

$$F = aH^2. \quad (4)$$

Since H is a symmetric matrix it can be diagonalized by a unitary matrix: $U^T H U = D_H$, where D_H is a diagonal matrix with real elements. Without loss of generality, we can express the mass matrices in (1) in an $SO(10)$ basis with a diagonal H. This basis is obtained from Eq. (1) by the replacement $H \to D_H$ and

$$H^2 \to D_H V^* D_H, \quad (5)$$

where D_H is a diagonal matrix with real elements. G retains its antisymmetric form, and we use the same notation for it and for various mass matrices in the new basis. $V = U^T U$ in Eq. (5) is a symmetric unitary matrix that can be parametrized [15] in terms of two angles and three phases.

Before we present the detailed fits, let us look at the implications of the ansatz Eq. (4) qualitatively.

(i) Correct b-τ unification and second generation masses are obtained if a dominant contribution to the charged fermion masses comes from H with a subdominant contribution from F and G. Retaining only the H contribution, the ansatz, Eq. (4) implies that

$$\mathcal{M}_\nu^I = -r_R M_D F^{-1} M_D^T \approx -\frac{r^2 r_R}{a} V + \cdots, \quad (6)$$

where the $\cdots$ terms arise from the $\overline{126}$ and 120 contributions to the Dirac mass matrix M_D. The CKM matrix is unity in this limit while the neutrino mixing is determined from V. The diagonalization of V leads [15] to $\theta_{23} = \phi$, $\theta_{12} = \frac{\theta}{2}$, and $\theta_{13} = 0$ where the angles θ_{ij} are angles defined in the standard parametrization of the leptonic mixing matrix and ϕ, θ enter into the definition of V [15]. Thus ansatz in Eq. (4) can lead to a correct description of the quark and leptonic mixing angles to zeroth order without requiring the type-II dominance as is commonly done.

(ii) If H in the original basis was real, then V entering Eq. (5) would be unity. In this case, all the fermion mixing vanish in the absence of the 120 contribution. Thus complex couplings and CP violation prove to be important in understanding large neutrino mixing within this approach.

The contributions from $\overline{126}$ and 120-plets induce nonzero quark mixing angles and perturb Eq. (6):

$$\mathcal{M}_\nu^I(M_X) = -\frac{r_R r^2}{a}(V - 6saD_H + t_D(GD_H^{-1}V - VD_H^{-1}G) + \mathcal{O}(s^2, t_D^2)). \quad (7)$$

$\mathcal{M}_\nu^I(M_X)$ corresponds to an effective dimension five operator induced after integration of the right-handed neutrino fields. Assuming that the heavy mass scale is close to the grand unified theory (GUT) scale and neglecting the effect of the Dirac neutrino couplings in the renormalization group (RG) evolution, the radiatively corrected low scale neutrino mass matrix is given by [1]

$$\mathcal{M}_{\nu f}(M_Z) = I_\tau \mathcal{M}_{\nu f}(M_X) I_\tau^\dagger, \quad (8)$$

where $I_\tau \approx \mathrm{Diag}(1, 1, 1 + \epsilon_\tau)$, $\epsilon_\tau \approx -\frac{1}{\cos^2\beta}\frac{m_\tau^2}{16\pi^2 v^2}\ln\frac{M_X}{M_Z}$, and $\mathcal{M}_{\nu f}$ denotes the neutrino mass matrix in the flavor basis.

II. NUMERICAL FITS: TYPE-I SEESAW

We now discuss detailed fits to fermion masses and mixing based on the ansatz (4) and the fermion mass matrices, Eq. (1). The latter are defined at the GUT scale M_X. We use as our input the quark and lepton masses obtained at M_X in the MSSM for $\tan\beta = 10$, $M_{\rm SUSY} =$ 1 TeV, and $M_{\rm GUT} = 2 \times 10^{16}$ GeV. The input values in the quark sector are given in [14,16]. We include the RG evolution in neutrino mass matrix as follows. Using the charged lepton mass matrix at M_X, we numerically determine $\mathcal{M}_{\nu f}(M_X)$. The low scale neutrino mass matrix in Eq. (8) is then numerically determined and used to obtain the observable neutrino masses and mixing angles. For neutrino masses and lepton mixings, we use the updated low energy values given in [17].

We do the χ^2 fitting to check the viability of the model as previously done in [7,13,14]. In this case we have a total of 25 real parameters (3 in D_H, 5 in V, 6 in G, real r, complex s, a, t_u, t_l, and t_D), which are fitted over 16 observables (9 charged fermion masses, 4 CKM parameters, 2 leptonic mixing angles, and $\Delta m^2_{\rm sol}/\Delta m^2_{\rm atm}$). Lepton mixings and $\Delta m^2_{\rm sol}/\Delta m^2_{\rm atm}$ are independent of the overall neutrino mass ($m_0 = |\frac{r_R r^2}{a}|$) appearing in Eq. (7). m_0 sets the overall neutrino mass scale and can be determined from the fit using the observed value of $\Delta m^2_{\rm atm}$. Our definition of χ^2 allows only the solution with $\Delta m^2_{\rm sol}\cos 2\theta > 0$ as required by experiments. We also set $r = \frac{m_t}{m_b}$ and minimize χ^2 with respect to the remaining 24 parameters. The results of the minimization are displayed as solutions (1) and (2) in Table I. We obtained the best fit value of $\chi^2 = 2.038$ corresponding to the solution (1) for which all the observables are fitted within $\lesssim 0.9\sigma$. Solution (2) is also acceptable, which fits all observables within $\lesssim 0.7\sigma$ with the exception of the down quark mass m_d. We also include in Table I the values of the Majorana phases obtained at the minimum.

θ_{13} has not been included in our definition of χ^2 and its initial value was zero. This becomes nonzero but remains small in both the solutions displayed. However, almost the entire allowed range in θ_{13} is compatible with reasonable fits to other fermion masses as shown by both the solutions. All the solutions displayed in Table I predict large CP violating leptonic phase.

The values of m_0 determined using the observed value of $\Delta m^2_{\rm atm}$ are seen from Table I to be $\gg \Delta m^2_{\rm atm}$ showing the consistency of our ansatz. This arises as a result of Eq. (7) and the smallness of s, t_D. The m_0 in turn determine the heaviest RH neutrino mass scale [see Eq. (1) and ansatz (4)],

$$M_3 \approx r_R^{-1}|a|m_b^2 \approx \frac{r^2}{m_0} m_b^2 \approx 1.3 \times 10^{13}\ \text{GeV},$$

in case of solution (1). Here we used, $m_0 = \frac{r_R r^2}{|a|}$. Thus the RH neutrino mass falls below the GUT scale for this particular solution.

Let us now illustrate how the ansatz (4) can be obtained in a model from a flavor symmetry. A simple flavor symmetry to be used is $O(3)$ under which three generations of the 16-plet ψ transform as triplets. The $O(3)$ breaking is introduced through a complex flavon field η transforming as spin 2. We need to introduce three generations of vectorlike multiplets $\Psi_V + \Psi_{\bar V}$ transforming as $(16, 3) + (\overline{16}, 3)$ under $SO(10) \times O(3)$ and a $U(1)_X$ symmetry in order to realize Eq. (4). The X charges of $(\psi, \Psi_V, \Psi_{\bar V}, \eta, \phi_{10}, \phi_{\overline{126}})$ are chosen, respectively, as $(x, y, -y, 1/2(y-x), -(x+y), -2y)$ with $x \neq y$. The general superpotential invariant under $SO(10) \times O(3) \times U(1)_X$ can be written as

$$W = M\Psi_{\bar V}\Psi_V + \beta\Psi_V\Psi_V\phi_{\overline{126}} + \gamma\Psi_V\psi\phi_{10} + \frac{\delta}{M_P}\Psi_{\bar V}\eta^2\psi + \frac{\delta'}{M_P}{\rm Tr}\eta^2\Psi_{\bar V}\psi + \cdots. \quad (9)$$

The $O(3)$ and $U(1)_X$ breaking originates in the above superpotential only from the Planck scale effects through the vacuum expectation value (vev) of the flavon field η. The last two terms are the only terms that determine both the 10 and $\overline{126}$ Yukawa couplings once the heavy vectorlike fields are integrated out. The dotted terms correspond to terms suppressed by M_P^2. Here, the mass M of the vectorlike pair and the scale of the vev of η lie above the GUT scale. The effective theory after integration of the vectorlike field is represented by

$$W_{\rm eff} \approx \beta\psi\xi^2\psi\phi_{\overline{126}} + \gamma\psi\xi\psi\phi_{10}, \quad (10)$$

where

$$\xi_{ab} \equiv \frac{\delta}{MM_P}\left(\eta^2_{ab} + \frac{\delta'}{\delta}Tr\eta^2\delta_{ab}\right)$$

and $a, b = 1, 2, 3$ refer to the $O(3)$ index. This effective superpotential is also $SO(10) \times O(3) \times U(1)_X$ invariant. The Yukawa coupling H is proportional to the $\langle\xi\rangle$ and is a general complex symmetric matrix. The F is related to the square of H and satisfies the ansatz in Eq. (4). The coupling to the 120 field can be generated by introducing a flavon field χ with the $U(1)_X$ charge $-2x$ and transforming as a triplet of $O(3)$. This leads to the Yukawa coupling matrix G through the coupling

$$\psi\frac{\chi}{M_P}\psi\phi_{120}.$$

A detailed model along this line will require study of the details of the vacuum structure of the potential involving η, χ, and possibly additional fields for generating the right structure of the Yukawa couplings H, G.

III. NUMERICAL FITS: TYPE-II SEESAW

We now turn to the numerical discussion of the ansatz (3) in which the contribution of $\overline{126}$ to fermion masses is assumed to be $O(3)$ invariant. The $O(3)$ breaking arises from the H and G contributions, which lead to departure from degeneracy through the type-I seesaw. We shall not specify how this breaking occurs [18]. Such an ansatz for the type-II contribution was considered [5] in the specific context of $SO(10)$. Detailed fits to fermion masses with

TABLE I. Best fit solutions for fermion masses and mixing obtained assuming the type-I seesaw dominance [solutions (1) and (2)] and type-II seesaw dominance [solution (3)]. Various observables and their pulls obtained at the minimum are shown (see text for details). Notations and conventions used here are the same as in [14]. The boldfaced quantities are predictions of the respective solutions.

No.	Observables	Sol. 1 Fitted value	Sol. 1 Pull	Sol. 2 Fitted value	Sol. 2 Pull	Sol. 3 Fitted value	Sol. 3 Pull
1	m_d [MeV]	0.653 677	−0.917 861	0.207 819	**−2.005 32**	0.868 041	−0.395 023
2	m_s [MeV]	17.5885	−0.386 821	21.6923	0.402 361	12.2829	**−1.407 14**
3	m_b [GeV]	1.111 31	0.418 721	1.058 32	−0.046 348	1.256 34	**1.691 41**
4	m_u [MeV]	0.462 718	0.084 789 6	0.450 825	0.005 499 32	0.450 489	0.003 261 1
5	m_c [GeV]	0.210 603	0.013 684 9	0.211 727	0.069 565 4	0.210 393	0.003 245 03
6	m_t [GeV]	63.6891	−0.832 404	67.6155	−0.658 038	102.325	0.883 371
7	m_e [MeV]	0.358 503	0.009 696 91	0.358 506	0.020 678 2	0.358 502	0.005 031 07
8	m_μ [MeV]	75.6719	0.007 345 14	75.6711	−0.008 306 4	75.6709	−0.011 180 9
9	m_τ [GeV]	1.292 19	−0.008 144 29	1.292 23	0.021 840 4	1.292 17	−0.024 457 6
10	$\frac{\Delta m^2_{\rm sol}}{\Delta m^2_{\rm atm}}$	0.030 351 4	0.050 109	0.030 323 7	0.037 787 7	0.030 253 8	0.006 594 21
11	m_0 [eV]	**0.31**	⋯	**0.17**	⋯	**0.36**	⋯
12	$\sin\theta^q_{12}$	0.224 205	−0.059 210 2	0.224 306	0.003 594 73	0.224 154	−0.091 312 5
13	$\sin\theta^q_{23}$	0.035 130 8	0.023 704	0.035 042 6	−0.044 117 3	0.035 143 6	0.033 571
14	$\sin\theta^q_{13}$	0.003 193 36	−0.013 286 7	0.003 158 71	−0.082 589 7	0.003 261 99	0.123 983
15	$\sin^2\theta^l_{12}$	0.319 801	−0.061 907 9	0.321 124	0.018 777 4	0.321 168	0.021 467 3
16	$\sin^2\theta^l_{23}$	0.481 942	0.313 909	0.436 492	−0.178 126	0.439 779	−0.142 55
17	$\sin^2\theta^l_{13}$	**0.019 526 6**	⋯	**0.002 881 76**	⋯	**0.035 683 6**	⋯
18	$\delta_{\rm CKM}$ [°]	67.7227	0.247 333	56.4935	−0.134 071	49.7146	−0.429 864
19	$\delta_{\rm PMNS}$ [°]	**53.98**	⋯	**−66.99**	⋯	**−25.33**	⋯
20	α_1 [°]	**146.55**	⋯	**−59.31**	⋯	**137.71**	⋯
21	α_2 [°]	**−89.88**	⋯	**162.41**	⋯	**−33.44**	⋯
	χ^2		2.038		4.684		6.0

recent data are, however, not presented in these works. We assume H, G to have the most general form and choose to work in a basis with a diagonal H. In this basis, Eq. (3) gets changed to

$$F = c_0 V, \tag{11}$$

where V is a unitary symmetric matrix. In this basis, the charged fermion mass matrices can be obtained from Eq. (1) by replacing H with diagonal D_H, and F with $c_0 V$. The neutrino mass matrix, Eq. (2) can be written in the same basis as

$$M_\nu = m_0(V - \epsilon M_D V^* M_D^T). \tag{12}$$

The parameter ϵ controls the contribution from a type-I seesaw, which induces splittings in neutrino masses.

We use these equations to fit all the fermion masses and mixing using the previous procedure. Results corresponding to the minimal case are displayed as solution (3) in Table I. The best fit solution we obtained here corresponds to $\chi^2 = 6.0$, which is acceptable for 16 data points from a statistical point of view and all the observables except m_b and m_s are fitted with less than 1σ accuracy. The obtained fit in the type-II case is, however, not as good as in the case of a pure type-I seesaw combined with the ansatz (4). As before, the m_0 sets the overall neutrino mass scale, which is determined to be $\sim$0.36 eV, using the atmospheric scale and fits shown in Table I. Numerical fits also lead to $\epsilon \approx 2 \times 10^{-6}$ GeV^{-2}. Since the scale of M_D is set by the top mass, the type-I contribution relative to the type-II is given by $\epsilon m_t^2 \sim 10^{-2}$, and the type-II contribution dominates as assumed. Now the overall scale of the RH neutrino mass is given by [see Eq. (1) and ansatz (3)]

$$M_3 \approx \frac{1}{m_0 \epsilon} \approx 1.1 \times 10^{15} \text{ GeV},$$

which is close to the GUT scale, unlike the minimal models with type-II dominance but hierarchical neutrinos [6,7]. The increase in M_3 here is linked to the degeneracy of neutrinos. The atmospheric neutrino mass scale in models with type-II seesaw and hierarchical neutrinos is typically given by

$$\Delta m^2_{\rm atm} \sim \frac{v^4}{M_3^2},$$

while in the present case it arises from the combination of type-I and type-II contributions and is scaled by

$$\Delta m^2_{\rm atm} \sim m_0 \frac{v^2}{M_3},$$

leading to a higher M_3 compared to a purely type-II dominated scenario.

IV. SUMMARY

Obtaining a unified description of vastly different patterns of quark and lepton spectra is a challenging task. This becomes more so if neutrinos are quasidegenerate. We have shown here that it is indeed possible to obtain such a description starting from the fermionic mass structure, Eq. (1) that can arise in a general $SO(10)$ model. We considered two distinct possibilities based on purely type-I and the other based on the mixture of type-I and type-II seesaw mechanisms. Both these possibilities can lead to quasidegenerate spectra if they are supplemented, respectively, with ansatz (4) and (3). We have shown through the detailed numerical analysis that these ansatz are capable of explaining the entire fermionic spectrum and not just the quasidegenerate neutrinos. Moreover, the origin of large leptonic mixing here is linked to the quasidegenerate structure determined by the matrix V, providing yet another reason why quark and leptonic mixing angles are so different in spite of underlying unified mass structure.

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