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Magnetic expansion of Nekrasov theory: The SU(2) pure gauge theory

Wei He* and Yan-Gang Miao†

Department of Physics, Nankai University, Tianjin 300071, China

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It is recently claimed by Nekrasov and Shatashvili that the $\mathcal{N}=2$ gauge theories in the Ω background with $\epsilon_1=\hbar$, $\epsilon_2=0$ are related to the quantization of certain algebraic integrable systems. We study the special case of SU(2) pure gauge theory; the corresponding integrable model is the A_1 Toda model, which reduces to the sine-Gordon quantum mechanics problem. The quantum effects can be expressed as the WKB series written analytically in terms of hypergeometric functions. We obtain the magnetic and dyonic expansions of the Nekrasov theory by studying the property of hypergeometric functions in the magnetic and dyonic regions on the moduli space. We also discuss the relation between the electric-magnetic duality of gauge theory and the action-action duality of the integrable system.

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I. INTRODUCTION

The nonperturbative properties of quantum field theories have been one of the most active research subjects during the past few decades; we have known a lot of information through various analytical or numerical methods. The four-dimensional Yang-Mills theory stands as one of the few most attractive field models. One of the milestones of studying the supersymmetric gauge theories is the Seiberg-Witten solution of the four-dimensional $\mathcal{N}=2$ gauge theory [1], which results in a fully analytic understanding of a large class of supersymmetric gauge theories. Their solution is based on typical features of supersymmetric gauge theories, i.e. the holomorphic structure of prepotential and the electric-magnetic duality of gauge theory. By analyzing the vacuum structure of the moduli space and the related monodromy problem, Seiberg and Witten discovered that the low energy physics of the gauge theory is encoded in a geometric object, an elliptic curve, the prepotential can be obtained through the periods of a holomorphic differential one form along the two conjugate homology cycles. The periods can be written as hypergeometric functions on the moduli space; they manifest the electric-magnetic duality in a very explicit way: the electric-magnetic duality group of the gauge theory is the same as the discontinuous reparametrization group of the elliptic curve. The solution is valid on the whole moduli space. In some regions the theory is a weakly coupled electric theory, in some regions the electric theory is strongly coupled, but it can be reformulated as a weakly coupled magnetic theory. By choosing suitable quantities as the fundamental degrees of freedom, we can either expand the effective action in terms of the electric fields or in terms of the magnetic (or dyonic) fields. Subsequent works have extended the solution to the $\mathcal{N}=2$ theory with more general gauge groups and with matters; it is also found that these solution can be interpreted in the context of string theory, see review [2,3].

The original work of Seiberg and Witten is reinterpreted in [4] from a different viewpoint. The hard part of solving the $\mathcal{N}=2$ gauge theory is the sum of the instanton contributions, but the multi-instanton measure on moduli space grows very complicated as the number of instantons increases, only the first few multi-instanton contributions have been calculated directly. The problem is solved through the localization technique; this can be achieved only after embedding the $\mathcal{N}=2$ gauge theory into the so-called Ω background [4–9]. The Ω background is a twist of the $\mathbb{R}^4$ bundle characterized by two complex parameters ϵ_1, ϵ_2. The partition function of this theory can be expressed in a compact form as a contour integral and can be analytically performed to arbitrary order in the instanton expansion. The instanton part of the Seiberg-Witten theory $\mathcal{F}^{\rm inst}(a, m, q)$ can be obtained through the Nekrasov partition function by the limit $\epsilon_1=-\epsilon_2=\hbar\rightarrow 0$.

The Nekrasov theory not only gives Seiberg-Witten theory a more mathematically solid explanation, and the theory is also important by itself, its rich structure is still largely unknown. One of the still mysterious aspects of Nekrasov theory is its modular property, i.e. the electric-magnetic duality property. In the original Seiberg-Witten formulation of the solution, the electric-magnetic duality is manifestly realized; the hypergeometric function is well defined on the whole moduli space coordinated by u, and we can get asymptotic expansion near $u=\infty$ and $\pm\Lambda^2$ which correspond to electric region and magnetic (dyonic) region, respectively. But the Nekrasov theory is defined in the electric region. Its partition function does not directly depend on the moduli space coordinate u, and it is not clear how the electric-magnetic duality works. It is interesting to find a way to study the magnetic (or dyonic) expansion of Nekrasov theory. Some earlier works concern this problem appear in [10]; the authors study the Nekrasov theory with

*weihe@nankai.edu.cn
†miaoyg@nankai.edu.cn

$\epsilon_1 = -\epsilon_2 = \lambda$ which is related to topological string theory and matrix model [11]. The general case of ϵ_1, ϵ_2 remains unknown.

A rather interesting discovery is recently presented in [12,13], where the relation between the Nekrasov theory and the quantization of algebraic integrable system is established. The correspondence between $\mathcal{N} = 2$ gauge theories and the classical integrable systems has been extensively studied soon after Seiberg-Witten theory. It was noted in [14,15] that the Seiberg-Witten solution of gauge theory is related to the classical integrable system; more precisely, the Seiberg-Witten curve of the gauge theory is identical to the spectral curve of the classical integrable system if we suitably identify physical quantities on the two sides, see review [16,17]. In [12], the authors develop the correspondence to the quantum level, they claim that the Ω-background twisted gauge theory with $\epsilon_1 = \epsilon$, $\epsilon_2 = 0$ is related to the quantization of the corresponding classical integrable system, and the ϵ-parameter of the gauge theory is identified with the Plank constant $\hbar$. Here the "quantization" refers to the integrable system side; on the gauge theory side it corresponds to higher order ϵ twist expansion. Some evidence supporting the correspondence is presented in [18,19]; they consider the special case of SU(2) pure Yang-Mills theory which is related to the A_1 Toda integrable system; the A_1 Toda system reduces to the sine-Gordon quantum mechanics problem on the complex plane. It is shown that the energy spectrum and wave function of the quantized mechanical model give the results consistent with the requirement of Nekrasov theory. Discussion on more general cases is presented in [20].

In [18], the authors found that the higher order $\hbar$ corrections can be obtained via acting on certain higher order differential operators on the leading order result, i.e. the Seiberg-Witten solution. This fact indicates that the $\hbar$ corrections can be also expressed compactly in hypergeometric functions which are valid on the whole moduli space. We use this observation to study the magnetic expansion of Nekrasov theory with $\epsilon_1 = \hbar$, $\epsilon_2 = 0$, by expanding higher order contour integrals in the magnetic region on the moduli space. Dyonic expansion is also obtained; it manifests a similar pattern with the magnetic case. In fact, the magnetic and dyonic expansions are related by a $\mathbb{Z}_2$ symmetry, therefore we mainly discuss the electric-magnetic duality of the system.

II. HIGHER ORDER CONTOUR INTEGRALS

In the Seiberg-Witten solution of $\mathcal{N} = 2$ SU(2) pure gauge theory the quantum moduli space is a complex quantity u of mass dimension two on which there are three singularities at $u = \infty$, $\pm\Lambda^2$ which correspond to the electric region and magnetic (dyonic) region, respectively. Λ denotes the dynamical generated scale of the gauge theory. For simplicity we set the scale $\Lambda = 1$ and it can be restored by dimensional analysis at last. The moduli space is the quotient of the upper half plane H by $\Gamma(2)$, where $\Gamma(2)$ is subgroup of $SL(2, \mathbb{Z})$ congruent to 1 modulo 2. The low energy effective action is described by the Seiberg-Witten curve whose moduli space is exactly $H/\Gamma(2)$:

$$y^2 = (x^2 - 1)(x - u) \tag{1}$$

and the corresponding Seiberg-Witten differential

$$d\lambda(u, x) = \frac{\sqrt{2}}{2\pi}\frac{\sqrt{x-u}}{\sqrt{x^2-1}}dx. \tag{2}$$

Then $a^{(0)}$ and $a_D^{(0)}$ are integrals of $d\lambda$ along the conjugate circles α and β, respectively,

$$a^{(0)} = \oint_\alpha d\lambda, \qquad a_D^{(0)} = \oint_\beta d\lambda, \tag{3}$$

where $a^{(0)}$ is the vacuum expectation value of the scalar field, and $a_D^{(0)}$ is the dual quantity. On the moduli space, the two cycles α and β correspond to the integral contours encircling branch point pairs $(-1, +1)$ and $(+1, u)$, respectively. The result can be written in terms of hypergeometric function,

$$\begin{aligned} a^{(0)}(u) &= \sqrt{2(u+1)}F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{u+1}\right) \\ a_D^{(0)}(u) &= \frac{i}{2}(u-1)F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u}{2}\right). \end{aligned} \tag{4}$$

They are well defined on the whole moduli space.

At each singularity electric (or magnetic/dyonic) particles become massless. If we treat the corresponding massless particles as fundamental degrees of freedoms, the theory is weakly coupled in the region near the singularity. Near $u = \infty$, the theory is weakly coupled electric theory, and the low energy effective prepotential $\mathcal{F}^{(0)}$ is obtained from

$$a_D^{(0)} = \frac{\partial}{\partial a^{(0)}}\mathcal{F}^{(0)}. \tag{5}$$

Near $u = 1$, the electric theory is strongly coupled. The electric-magnetic duality works in the sense that, if the theory is reformulated in terms of the dual magnetic fields, it is weakly coupled. The dual prepotential $\mathcal{F}_D^{(0)}$ can be obtained from

$$a^{(0)} = \frac{\partial}{\partial a_D^{(0)}}\mathcal{F}_D^{(0)}. \tag{6}$$

A similar mechanism works for the dyonic region near $u = -1$.

The Nekrasov theory can be viewed as the quantized version of the Seiberg-Witten theory. The partition function $Z(a, m, \epsilon_1, \epsilon_2, \Lambda)$ can be written in terms of the prepotential $\mathcal{F}(a, m, \epsilon_1, \epsilon_2, \Lambda)$:

$$Z(a, m, \epsilon_1, \epsilon_2, \Lambda) = \exp\frac{\mathcal{F}(a, m, \epsilon_1, \epsilon_2, \Lambda)}{\epsilon_1\epsilon_2}, \qquad (7)$$

where a is related to the vacuum expectation value of scalar fields, m denotes the masses of matter fields, and $\mathcal{F}(a, m, \epsilon_1, \epsilon_2, \Lambda)$ is a regular function in the limit $\epsilon_1 \to 0$, $\epsilon_2 \to 0$. We are interested in the case of $\epsilon_1 = \hbar$, $\epsilon_2 = 0$ and with no matter. It is shown in [18,19] that the modul a and the prepotential can be expanded as

$$\begin{aligned} a(a^{(0)}) &= a^{(0)} + \hbar^2 a^{(1)}(a^{(0)}) + \hbar^4 a^{(2)}(a^{(0)}) + \cdots \\ \mathcal{F}(a, \hbar) &= \mathcal{F}^{(0)}(a) + \hbar^2 \mathcal{F}^{(1)}(a) + \hbar^4 \mathcal{F}^{(2)}(a) + \cdots, \end{aligned} \qquad (8)$$

where the superscript (0) indicates the "classical" quantities and the superscript (i), $i \geq 1$ indicates the "quantum" corrected ones. Note that the function variable of $\mathcal{F}$ is a rather than $a^{(0)}$. A dual variable of a can be defined by

$$a_D = \frac{\partial}{\partial a}\mathcal{F} \qquad (9)$$

and expanded as

$$a_D = a_D^{(0)}(a) + \hbar^2 a_D^{(1)}(a) + \hbar^4 a_D^{(2)}(a) + \cdots. \qquad (10)$$

In the limit $\hbar \to 0$, only the leading order remains and it is just the Seiberg-Witten theory. The higher order $\hbar$ corrections are explained as effects of Ω twist in the gauge theory side, and as quantization on the dual integrable system side. In [18] the authors find $a^{(i)}(a^{(0)})$ and $a_D^{(i)}(a)$, $i \geq 1$ can be obtained from $a^{(0)}(u)$ and $a_D^{(0)}(u)$ by acting on certain higher order differential operators on them. In the following, we will explain it and write the results in hypergeometric function.

The integrals of (3) can be written in another form. If we change the variable as $x = \cos\phi$, then the integrals become $-(2\pi)^{-1}\int\sqrt{2(u - \cos\phi)}d\phi$, which is the classical action integral $\int p(\phi)d\phi$ of the sine-Gordon action $\mathcal{L} = \dot{\phi}^2 - \cos\phi$ for a particle at the given "energy" u. In order to quantize the system, we are led to the Schrödinger equation

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{d\phi^2} + \cos\phi\right)\Psi(\phi) = u\Psi(\phi). \qquad (11)$$

When the system is quantized, the contour integrals (3) are lifted to the monodromies of the phase of the wave function along the circles α and β. Equation (11) is the Mathieu equation, and some properties of the corresponding gauge theory have been obtained by analyzing its periodic solution [19]. We apply WKB method and write the wave function as

$$\begin{aligned} \Psi(\phi) &= \exp\frac{i}{\hbar}\int^{\phi} P(\phi')d\phi' \\ &= \exp\frac{i}{\hbar}\int^{\phi}(P_0 + \hbar P_1 + \hbar^2 P_2 + \cdots)d\phi', \end{aligned} \qquad (12)$$

then we have

$$\begin{aligned} &P_0 = \sqrt{2(u - \cos\phi)}, \qquad P_1 = \frac{i}{2}(\ln P_0)', \\ &P_2 = -\frac{1}{8P_0}[2(\ln P_0)'' - ((\ln P_0)')^2], \qquad P_3 = \frac{i}{2}\left(\frac{P_2}{P_0}\right)', \\ &\ldots \end{aligned} \qquad (13)$$

where the prime denotes $\frac{\partial}{\partial\phi}$.

As P_1 and P_3 are total derivatives, their contour integrals are zero. Only the contour integrals of $P_0, P_2, P_4, \ldots$ give nonzero results; they are related to $a^{(0)}, a^{(1)}, a^{(2)}, \ldots$, respectively. The nonzero part of the P_2 contour integral gives

$$\begin{aligned} \oint_{\alpha,\beta} P_2 d\phi &= \frac{1}{32\sqrt{2}}\oint_{\alpha,\beta}\frac{\sin^2\phi - 4u\cos\phi + 4}{(u - \cos\phi)^{5/2}}d\phi \\ &= -\frac{1}{48\sqrt{2}}\oint_{\alpha,\beta}\frac{\cos\phi}{(u - \cos\phi)^{3/2}}d\phi \\ &= \frac{1}{48}(2ud_u^2 + d_u)\oint_{\alpha,\beta}\sqrt{2(u - \cos\phi)}d\phi, \end{aligned} \qquad (14)$$

where d_u denotes $\frac{d}{du}$. Using the formula

$$\frac{d}{dz}F(\alpha, \beta, \gamma; z) = \frac{\alpha\beta}{\gamma}F(\alpha + 1, \beta + 1, \gamma + 1; z), \qquad (15)$$

we get

$$\begin{aligned} a^{(1)} &= \frac{1}{48}(2ud_u^2 + d_u)a^{(0)}(u) \\ &= \frac{1}{24}\Bigg[(2(u + 1))^{-3/2}F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{u + 1}\right) \\ &\quad - 2(u - 1)(2(u + 1))^{-5/2}F\left(\frac{1}{2}, \frac{3}{2}, 2; \frac{2}{u + 1}\right) \\ &\quad - 6u(2(u + 1))^{-7/2}F\left(\frac{3}{2}, \frac{5}{2}, 3; \frac{2}{u + 1}\right)\Bigg] \end{aligned} \qquad (16)$$

$$\begin{aligned} a_D^{(1)} &= \frac{1}{48}(2ud_u^2 + d_u)a_D^{(0)}(u) \\ &= \frac{i}{96}\Bigg[F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1 - u}{2}\right) \\ &\quad - \frac{1}{16}(5u - 1)F\left(\frac{3}{2}, \frac{3}{2}, 3; \frac{1 - u}{2}\right) \\ &\quad + \frac{3}{16}u(u - 1)F\left(\frac{5}{2}, \frac{5}{2}, 4; \frac{1 - u}{2}\right)\Bigg]. \end{aligned} \qquad (17)$$

In a similar way, the third order contour integral is

$$\oint P_4 d\phi = \frac{1}{2^9 \times 45}(28u^2 d_u^4 + 120ud_u^3 + 75d_u^2)\oint P_0 d\phi. \qquad (18)$$

We will not give the full detail here because it is little long. Both $a^{(2)}$ and $a_D^{(2)}$ can be written in a similar form as that of (16) and (17) in terms of hypergeometric functions. We can either expand them near the point $u=\infty$ and $u=1$, or directly act on the forth order differential operator on the series expansion of $a^{(0)}(u)$ and $a_D^{(0)}(u)$; they will give the same result. This strategy can be applied to higher order contour integrals, which contains more complicated differential operators.

To obtain results (14) and (18), we have followed the trick of [18], although our differential operators are slightly different from theirs because we have set $\Lambda = 1$.

III. ELECTRIC EXPANSION

In the following three sections we will derive the prepotential in the electric region and the magnetic (dyonic) region. We will not list the full details of the procedure, we only explicitly give some series expansions which are interesting for our project. Some of them have been known before, but appear in different literatures, or derived through other methods. We use the newly discovered relation between the gauge theory and the integrable model to give a complete and consistent derivation. Some of our results, especially the magnetic (dyonic) expansion of the prepotential for the case $\epsilon_1=\hbar$, $\epsilon_2=0$, are new.

On the moduli space, $u=\infty$ corresponds to the electric region; near this point the massless excitations are weakly coupled Abelian U(1) electric fields. Expanding $a(u)$ and $a_D(u)$ near ∞, up to the $\hbar^4$ order we have

$$\begin{aligned} a(u) &= a^{(0)}(u) + \hbar^2 a^{(1)}(u) + \hbar^4 a^{(2)}(u) + \cdots \\ &= \sqrt{2u}\left[1 - \frac{1}{4}\left(\frac{1}{2u}\right)^2 - \frac{15}{64}\left(\frac{1}{2u}\right)^4 - \frac{105}{256}\left(\frac{1}{2u}\right)^6 - \frac{15\,015}{16\,384}\left(\frac{1}{2u}\right)^8 + \cdots\right] \\ &\quad - \frac{\hbar^2}{\sqrt{2u}}\left[\frac{1}{2^4}\left(\frac{1}{2u}\right)^2 + \frac{35}{2^7}\left(\frac{1}{2u}\right)^4 + \frac{1155}{2^{10}}\left(\frac{1}{2u}\right)^6 + \frac{75\,075}{2^{14}}\left(\frac{1}{2u}\right)^8 + \cdots\right] \\ &\quad - \frac{\hbar^4}{(2u)^{3/2}}\left[\frac{1}{2^6}\left(\frac{1}{2u}\right)^2 + \frac{273}{2^{10}}\left(\frac{1}{2u}\right)^4 + \frac{5005}{2^{11}}\left(\frac{1}{2u}\right)^6 + \frac{2\,297\,295}{2^8}\left(\frac{1}{2u}\right)^8 + \cdots\right]. \end{aligned} \tag{19}$$

This result has been obtained in [18,19], through different methods. Our method here follows, and simplifies, the one in [18]. From $a(u)$, the inverse series gives

$$\begin{aligned} 2u &= a^2 + \frac{1}{2}a^{-2} + \frac{5}{32}a^{-6} + \frac{9}{64}a^{-10} + \frac{1469}{8192}a^{-14} + \cdots + \hbar^2\left(\frac{1}{8}a^{-4} + \frac{21}{64}a^{-8} + \frac{55}{64}a^{-12} + \frac{18\,445}{8192}a^{-16} + \cdots\right) \\ &\quad + \hbar^4\left(\frac{1}{32}a^{-6} + \frac{219}{512}a^{-10} + \frac{1495}{512}a^{-14} + \frac{985\,949}{65\,536}a^{-18} + \cdots\right). \end{aligned} \tag{20}$$

The series expansion of $a_D(u) = a_D^{(0)} + \hbar^2 a_D^{(1)} + \hbar^4 a_D^{(2)}$ is lengthy, it contains many terms of the form $(c_1 + c_2 \ln 2 + c_3 \ln u)u^{k+(1/2)}$, $k=0,1,2,\ldots$. Substituting $u=u(a)$ into $a_D(u)$, we get the series expansion of $a_D(a)$ with very simple structure. The prepotential is obtained from $a_D = \frac{\partial}{\partial a}\mathcal{F}$:

$$\begin{aligned} \mathcal{F}(a,\hbar) &= \frac{i}{4\pi}\left[4a^2\left(\ln 2a - \frac{3}{2}\right) - \frac{1}{2}a^{-2} - \frac{5}{64}a^{-6} - \frac{3}{64}a^{-10} + \cdots\right] \\ &\quad + \hbar^2\frac{i}{4\pi}\left(\frac{1}{6}\ln a - \frac{1}{8}a^{-4} - \frac{21}{128}a^{-8} - \frac{55}{192}a^{-12} + \cdots\right) \\ &\quad + \hbar^4\frac{i}{4\pi}\left(\frac{1}{1440}a^{-2} - \frac{1}{32}a^{-6} - \frac{219}{1024}a^{-10} - \frac{1495}{1536}a^{-14} + \cdots\right). \end{aligned} \tag{21}$$

The results are consistent with other works. For example, in [18], the $\hbar^0$ and $\hbar^2$ order results of $\mathcal{F}(a,\hbar)$ have been obtained through the same method as here; in [19], the form of power series of $\mathcal{F}(a,\hbar)$ has been obtained through analyzing the Mathieu function. Here we derive the coefficients; in [21], direct gauge theory calculation gives the instanton part of the prepotential up to four instantons contribution; it is easy to check that our result is coincident with theirs if we set $\epsilon_1=\hbar$, $\epsilon_2=0$. It is also worth mentioning that, in [21], other choices such as $\epsilon_1=-\epsilon_2=\hbar$ or $\epsilon_1=\epsilon_2=\hbar$ will give different results. Not only the rational coefficients are different, the powers of a are also different. For example, for the case $\epsilon_1=-\epsilon_2$, the prepotential will be the one given in [10,11] which is different from (21). This fact explicitly indicates that the quantization of the integrable model we discuss here indeed corresponds to a special corner of the Nekrasov theory with $\epsilon_1=\hbar$, $\epsilon_2=0$.

IV. MAGNETIC EXPANSION

Now we have confidence that the WKB contour integrals of the integrable model indeed give the Nekrasov theory in the electric region. What we will do next is just to expand the WKB contour integrals in the magnetic region, i.e. near $u = 1$ on the moduli space. In the magnetic region, magnetic monopoles couple to the dual U(1) Abelian gauge fields as massive matter hypermultiplets. The effective action is obtained by integrating out all massive fields; their effects are encoded in the subleading terms of (28).

The motivation of studying magnetic expansion of Nekrasov function comes from two sides.

First, the Nekrasov theory is formulated in the electric region, where it can be explicitly expanded in terms of the electric quantity a; however, its magnetic expansion is much less known, although various dual quantities can be formally defined. A special corner of the parameter space $\epsilon_1 = -\epsilon_2$ has been investigated in [10], using results of the holomorphic anomaly equation of topological string theory. However, the general case is unknown. In this paper, through the relation with the integrable system, we can explore another corner with $\epsilon_1 = \hbar$, $\epsilon_2 = 0$.

Second, the electric-magnetic duality of the gauge theory has some interplay with duality of the integrable model. According to the discussion in [22], on the symplectic manifold $\mathcal{M}$ related to the integrable system, the Liouville's theorem states that the symplectic form ω has a normal form locally written in terms of coordinates (I, φ):

$$\omega = dI \wedge d\varphi, \tag{22}$$

where φ is the coordinate variable, and I is the action variable in the sense that the Hamiltonian is a function of only I: $H = h(I)$. For the classical integrable model which corresponds to the $\mathcal{N} = 2$ gauge theory, the symplectic manifold is the tangent space of the gauge theory moduli space, $\mathcal{M} = \mathcal{C} \times T$, where $\mathcal{C}$ is the complex plane related to the vacuum expectation value of the adjoint complex scalar field, and T is the torus related to the complex gauge coupling τ. The symplectic form is [15]

$$\omega = da^{(0)}(u) \wedge \frac{dx}{y(u,x)}, \tag{23}$$

where $y = y(u, x)$ is the Seiberg-Witten curve. The Hamiltonian of the integrable system is identified with the beta function of the prepotential of the gauge theory, and $a^{(0)}$ is the action variable.

The gauge theory has electric-magnetic duality which maps $\tau \to -\frac{1}{\tau}$ and $a^{(0)} \to a_D^{(0)}$, and we can formulate the theory as either electric theory or magnetic theory. Therefore, in the magnetic formulation, the symplectic structure discussed above is reformulated in the dual quantities. Near $u = \infty$, the gauge theory is a weakly coupled electric theory; the appropriate action variable is $a^{(0)}(u)$. While near $u = 1$ the gauge theory is a weakly coupled magnetic theory, the appropriate action variable is $a_D^{(0)}(u)$. On the u plane, we have $da^{(0)} \wedge da_D^{(0)} = 0$; therefore there exists a *potential* that maps $a^{(0)}$ and $a_D^{(0)}$ to each other:

$$a_D^{(0)} = \frac{\partial}{\partial a^{(0)}} \mathcal{F}^{(0)} \tag{24}$$

or

$$a^{(0)} = \frac{\partial}{\partial a_D^{(0)}} \mathcal{F}_D^{(0)}. \tag{25}$$

Depending on the electromagnetic frame we work in, we choose one of $\mathcal{F}^{(0)}$ and $\mathcal{F}_D^{(0)}$ as the potential. We say this integrable system manifests the *action-action duality.* $\mathcal{F}^{(0)}$ and $\mathcal{F}_D^{(0)}$ are dual to each other; they are the Seiberg-Witten *prepotential* of the gauge theory in the electric and magnetic region, respectively. The magnetic expansion of the Seiberg-Witten theory has been known [3], and investigating the quantum version is a natural next step.

If the classical electric-magnetic duality has a well-defined quantum correspondence, then there should exist a dual pair $\mathcal{F}(a, \hbar)$ and $\mathcal{F}_D(a_D, \hbar)$. For the special case of SU(2) pure Yang-Mills theory with $\epsilon_1 = \hbar$, $\epsilon_2 = 0$, the quantum correction can be expressed in terms of hypergeometric function through WKB series and can be analytically studied in the magnetic region. This provides a glimpse to the dual phase of the integrable system.

Expanding (4) near the magnetic point $u = 1$ (set $\sigma = u - 1$, therefore $d_u = d_\sigma$), and using the differential operators of (14) and (18), we get the asymptotic form of $\hat{a}_D = i a_D$ and a up to the order of $\hbar^4$:

$$\begin{aligned}\hat{a}_D(\sigma) &= \hat{a}_D^{(0)}(\sigma) + \hbar^2 \hat{a}_D^{(1)}(\sigma) + \hbar^4 \hat{a}_D^{(2)}(\sigma) + \cdots \\ &= -\frac{1}{2}\sigma + \frac{1}{32}\sigma^2 - \frac{3}{512}\sigma^3 + \frac{25}{16\,384}\sigma^4 + \cdots - \frac{\hbar^2}{2^7}\left(1 - \frac{5}{16}\sigma + \frac{35}{256}\sigma^2 - \frac{525}{8192}\sigma^3 + \cdots\right) \\ &\quad - \frac{\hbar^4}{2^{18}}\left(-17 + \frac{721}{32}\sigma - \frac{10\,941}{512}\sigma^2 + \frac{141\,757}{8192}\sigma^3 + \cdots\right).\end{aligned} \tag{26}$$

The inverse series gives

$$\sigma = -2\hat{a}_D + \frac{1}{4}\hat{a}_D^2 + \frac{1}{32}\hat{a}_D^3 + \frac{5}{512}\hat{a}_D^4 + \cdots + \frac{\hbar^2}{2^6}\left(-1 - \frac{3}{8}\hat{a}_D - \frac{17}{64}\hat{a}_D^2 - \frac{205}{1024}\hat{a}_D^3 + \cdots\right) + \frac{\hbar^4}{2^{17}}\left(9 + \frac{405}{16}\hat{a}_D + \frac{2943}{64}\hat{a}_D^2 + \frac{69\,001}{1024}\hat{a}_D^3 + \cdots\right). \quad (27)$$

In the magnetic region, the series expansion of $a(\sigma)$ contains many terms of the form $(c_1 + c_2 \ln 2 + c_3 \ln\sigma)\sigma^k$, $k = 0, 1, 2, \ldots$. Similar to the case of electric expansion, after substituting $\sigma = \sigma(\hat{a}_D)$ into $a(\sigma)$, we get the series expansion of $a(\hat{a}_D)$ with very simple structure. The dual prepotential $\mathcal{F}_D$ can be obtained from $a = \frac{\partial}{\partial a_D}\mathcal{F}_D$:

$$\mathcal{F}_D(a_D, \hbar) = \frac{1}{i\pi}\left[\frac{\hat{a}_D^2}{2}\ln\left(-\frac{\hat{a}_D}{2}\right) + 4\hat{a}_D - \frac{3}{4}\hat{a}_D^2 + \frac{1}{16}\hat{a}_D^3 + \frac{5}{512}\hat{a}_D^4 + \frac{11}{4096}\hat{a}_D^5 + \cdots\right] + \frac{\hbar^2}{i\pi 2^5}\left(\frac{1}{3}\ln\hat{a}_D - \frac{3}{2^3}\hat{a}_D - \frac{17}{2^7}\hat{a}_D^2 - \frac{205}{2^{10}\times 3}\hat{a}_D^3 + \cdots\right) + \frac{\hbar^4}{i\pi 2^{11}}\left(-\frac{7}{45}\hat{a}_D^{-2} + \frac{135}{2^9}\hat{a}_D + \frac{2943}{2^{13}}\hat{a}_D^2 + \cdots\right). \quad (28)$$

Some interesting features of the dual prepotential (28) can be compared to that appearing in [10], although they discuss a different corner of the Nekrasov theory with $\epsilon_1 = -\epsilon_2$ (see the Conclusion). Except the only two terms containing $\ln\hat{a}_D$, the quantum corrections are powers of $\hat{a}_D$, their coefficients are all rational numbers, the same as that in [10]. This fact serves a nontrivial examination of the result itself. If there were mistakes in the coefficients of expansions $\hat{a}_D(\sigma)$, $\sigma(\hat{a}_D)$ and $a(\sigma)$, then (28) would contain terms like $c_1 + c_2 \ln 2 + c_3 \ln\hat{a}_D$ in any other terms. In the $\hbar^4$ order correction of $\mathcal{F}_D$, the first two terms of order $\mathcal{O}(\hat{a}_D^{-1})$ and $\mathcal{O}(1)$ are absent. The same pattern happens in formula (2.33) of [10]. Actually, in their case the "gap" phenomenon happens for all higher genus corrections; we believe that in our case the gap phenomenon also persists to higher order $\hbar$ corrections.

Although no direct gauge theory calculation in the dual magnetic fields is available, however, from the experience of electric expansion, we have an explanation for the different terms of $\mathcal{F}_D$. Terms of order $\hat{a}_D^2\ln\hat{a}_D$ and $\hbar^2\ln\hat{a}_D, \hbar^4\hat{a}_D^{-2}, \ldots$ in the prepotential correspond to the classical and one loop contributions. Other terms correspond to integrating out multiparticle massive hypermultiplets, which are monopole particles in the original electric theory.

In [12,13], the prepotential of the gauge theory is identified with the Yang-Yang function [23] of the quantum Toda integrable model. The problem has two kinds of formulations, called type A and type B spectral problems. The type B problem is solved by the periodic Mathieu function with quantization condition [12]

$$\frac{1}{\hbar}\frac{\partial}{\partial a_D}\mathcal{F}_D = \frac{a}{\hbar} = n, \qquad n \in \mathbb{Z}. \quad (29)$$

This is studied in [19]. The type A problem is solved by the dual equation

$$\frac{i}{\hbar}\frac{\partial}{\partial a}\mathcal{F} = \frac{\hat{a}_D}{\hbar} = m, \qquad m \in \mathbb{Z}. \quad (30)$$

The two quantization conditions serve as the Bethe equations of the corresponding spectrum problems. However, the appearance of potential in (29) and (30) only serves as a conceptual definition, in practice, only $a(u) = \hbar n$ and $\hat{a}_D(\sigma) = \hbar m$ are used to compute the energy spectrum u. The type B's eigenvalue as a function of the quantum number n is given in (20) as series expansion; it can be expressed in a more compact form as the periodic solution of the Mathieu equation [19]. The type A's eigenvalue as a function of the quantum number m is given in (27). In the next section we will further explain its relation to the Mathieu equation. The two type problems are connected by the S duality $\tau \to -\frac{1}{\tau}$.

Therefore we have a clear picture about the role of the electric-magnetic duality on the side of the integrable model: it is the action-action duality [22] of the quantum integrable model that maps type A and type B spectrums to each other [12].

V. DYONIC EXPANSION

The untwisted $\mathcal{N} = 2$ SU(2) gauge theory has a global $\mathbb{Z}_2$ symmetry acting on the u plane by $u \to -u$. Under the $\mathbb{Z}_2$ symmetry the magnetic region at $u = \Lambda^2$ is mapped to the dyonic region at $u = -\Lambda^2$. At the dyonic point, the massless soliton particles are either charge $(n_e, n_m) = (1, -1)$ dyons or charge $(n_e, n_m) = (1, 1)$ dyons, depending on the direction from which we cross the wall of marginal stability and approach the dyonic point [24]. We choose the convention $(n_e, n_m) = (1, -1)$; therefore, the Seiberg-Witten solution behaves as $a - a_D \sim u + 1$ near $u = -1$. The electric-magnetic duality together with the $\mathbb{Z}_2$ symmetry generate the electric-magnetic-dyonic triality. In the following, we will give the dyonic expansion of the Nekrasov theory; it is related to the magnetic expansion by a $\mathbb{Z}_2$ symmetry of the Nekrasov theory.

The dyonic expansion is very similar to that of the magnetic expansion, therefore we only briefly report the main results. Setting $\varpi = u + 1$ we get

$$\begin{aligned} a_T &= a - a_D = a_T^{(0)} + \hbar^2 a_T^{(1)} + \hbar^4 a_T^{(2)} + \cdots \\ &= \frac{1}{2}\varpi + \frac{1}{32}\varpi^2 + \frac{3}{512}\varpi^3 + \frac{25}{16\,384}\varpi^4 + \cdots + \frac{\hbar^2}{2^7}\left(1 + \frac{5}{16}\varpi + \frac{35}{256}\varpi^2 + \frac{525}{8192}\varpi^3 + \cdots\right) \\ &\quad + \frac{\hbar^4}{2^{18}}\left(17 + \frac{721}{32}\varpi + \frac{10\,941}{512}\varpi^2 + \frac{141\,757}{8192}\varpi^3 + \cdots\right). \end{aligned} \tag{31}$$

The inverse series gives the eigenvalue of the corresponding quantum mechanics problem:

$$\begin{aligned} \varpi &= 2a_T - \frac{1}{4}a_T^2 - \frac{1}{32}a_T^3 - \frac{5}{512}a_T^4 + \cdots + \frac{\hbar^2}{2^6}\left(-1 - \frac{3}{2^3}a_T - \frac{17}{2^6}a_T^2 - \frac{205}{2^{10}}a_T^3 + \cdots\right) \\ &\quad + \frac{\hbar^4}{2^{17}}\left(-9 - \frac{405}{2^4}a_T - \frac{2943}{2^6}a_T^2 - \frac{69\,001}{2^{10}}a_T^3 + \cdots\right). \end{aligned} \tag{32}$$

We can substitute $\varpi = \varpi(a_T)$ into either $a(\varpi)$ or $a_D(\varpi)$, and the dyonic prepotential can be obtained from either $a(a_T) = \frac{\partial}{\partial a_T}\mathcal{F}_T$ or $a_D(a_T) = \frac{\partial}{\partial a_T}\mathcal{F}_T$. The two kinds of choice correspond to doing electric-dyonic duality and magnetic-dyonic duality, respectively. We choose electric-dyonic duality and get

$$\begin{aligned} \mathcal{F}_T(a_T,\hbar) &= \frac{1}{i\pi}\left[\frac{a_T^2}{2}\ln\left(-\frac{a_T}{16}\right) + 4a_T - \frac{3}{4}a_T^2 + \frac{1}{16}a_T^3 + \frac{5}{512}a_T^4 + \frac{11}{4096}a_T^5 + \cdots\right] \\ &\quad + \frac{\hbar^2}{i\pi 2^5}\left(-\frac{1}{3}\ln a_T + \frac{3}{2^3}a_T + \frac{17}{2^7}a_T^2 + \frac{205}{2^{10}\times 3}a_T^3 + \cdots\right) + \frac{\hbar^4}{i\pi 2^{11}}\left(-\frac{7}{45a_T^3} + \frac{135}{2^9}a_T + \frac{2943}{2^{13}}a_T^2 + \cdots\right). \end{aligned} \tag{33}$$

The dyonic prepotential is very close to that of magnetic, only differing by a $-\frac{3}{2}a_T^2\ln 2$ term and a minus sign in the $\hbar^2$ correction.

Now, we will discuss a relation between the magnetic (dyonic) expansion and the periodic solution of the Mathieu equation. Equation (11) can be rewritten as

$$\Psi''(z) + (A - 2B\cos 2z)\Psi(z) = 0 \tag{34}$$

with $A = \frac{8u}{\hbar^2}$, $B = \frac{4\Lambda^2}{\hbar^2}$, $2z = \phi$, and Λ restored. The periodic solution is marked by a quantum number ν; ν is an even integer for the solution with period π, an odd integer for the solution with period 2π. In [19] we have studied the eigenvalue problem of the periodic solution in the case of small $\frac{\sqrt{B}}{\nu}$ expansion, which corresponds to the electric expansion of gauge theory. Here, we will encounter the small $\frac{\nu}{\sqrt{B}}$ expansion.

The eigenvalue formulas (27) and (32) have very similar structure, restore Λ and set $\hat{a}_D = \hbar\frac{\nu}{2}$ in (27), and set $a_T = \hbar\frac{\nu}{2}$ in (32), with ν an even integer. Then (27) and (32) can be rewritten in terms of A, B, ν, and we find that they are asymptotic expansions of the following two more compact expressions:

$$\begin{aligned} A_\nu &= 2B - 4\nu\sqrt{B} + \frac{4\nu^2 - 1}{2^3} + \frac{4\nu^3 - 3\nu}{2^6\sqrt{B}} + \frac{80\nu^4 - 136\nu^2 + 9}{2^{12}B} + \frac{528\nu^5 - 1640\nu^3 + 405\nu}{2^{16}B^{3/2}} \\ &\quad + \frac{2016\nu^6 - 10\,080\nu^4 + 5886\nu^2 - 243}{2^{19}B^2} + \cdots \end{aligned} \tag{35}$$

for (27), and

$$\begin{aligned} A_\nu &= -2B + 4\nu\sqrt{B} - \frac{4\nu^2 + 1}{2^3} - \frac{4\nu^3 + 3\nu}{2^6\sqrt{B}} - \frac{80\nu^4 + 136\nu^2 + 9}{2^{12}B} - \frac{528\nu^5 + 1640\nu^3 + 405\nu}{2^{16}B^{\frac{3}{2}}} \\ &\quad - \frac{2016\nu^6 + 10\,080\nu^4 + 5886\nu^2 + 243}{2^{19}B^2} + \cdots \end{aligned} \tag{36}$$

for (32). Formulas (35) and (36) can be found in the formula (20.2.30) in [25]. Their notation w is related to ours by $w = 2\nu$. The terms of order $\frac{1}{B^2}$ come from the $\hbar^6$ correction; we have checked all other higher order terms which we do not explicitly list here for the sake of avoiding unnecessary lengthy. The two cases are related to each other by the $\mathbb{Z}_2$ symmetry $\nu \to i\nu$, $B \to -B$. To ensure the expansions make sense, we need the ratio of the adjacent terms to be small $\frac{\nu}{\sqrt{B}} \sim \frac{a_{D/T}}{\Lambda} \ll 1$, which is the physical requirement that the expansions are performed in the weakly coupled region of magnetic/dyonic theory.

The sine-Gordon model (11) is the quantum mechanics problem on the moduli space of the Seiberg-Witten theory. Its excitation spectrum can be classified into three species: the excitations in the bottom of the potential are "dyonic excitations"; the excitations near the mouth of the potential are "magnetic excitations"; the excitations far above the top of the potential are "electric excitations."

VI. CONCLUSION

The recently discovered relation between the Nekrasov gauge theory, i.e. $\mathcal{N} = 2$ gauge theory in the Ω background, and quantization of the algebraic integrable system is an exciting field that needs deeper understanding. In this paper we study the expansion of the SU(2) Nekrasov theory with $\epsilon_1 = \hbar$, $\epsilon_2 = 0$ on the whole moduli space, through its relation to the sine-Gordon quantum mechanics model. We focus on this relatively simple model because in this case many quantities can be explicitly calculated, however, it presents the basic features of the novel correspondence. The consistence of theses results on both sides, i.e. on the gauge theory side and the integrable model side, gives nontrivial support to the correspondence. Using the observation of [18], higher order quantum effects can be obtained by acting on certain higher order differential operators on the classical results, and can be compactly expressed in terms of the hypergeometric function. The hypergeometric function is well defined on the whole moduli space, therefore studying the property in the magnetic (dyonic) region is straightforward, by expanding the results near the magnetic (dyonic) points $u = \pm 1$. It is remarkable that the coefficients of the subleading terms of the prepotentials $\mathcal{F}$, $\mathcal{F}_D$, and $\mathcal{F}_T$ are all rational numbers. We stress here that the three prepotentials $\mathcal{F}$, $\mathcal{F}_D$, and $\mathcal{F}_T$ are the local asymptotic expansions of the same object that is globally well defined. It seems that some symmetries of the Seiberg-Witten theory, such as the $\Gamma(2)$ modular symmetry and the $\mathbb{Z}_2$ global symmetry, survive under the Ω twist with $\epsilon_1 = \hbar$, $\epsilon_2 = 0$.

The electric-magnetic duality of the gauge theory corresponds to the action-action duality on the integrable system side. The action-action duality exchanges the role of a and a_D (or a_T) and maps two kinds of spectrum problem to each other. The prepotential $\mathcal{F}$ of the gauge theory serves as the Yang-Yang function of the type A spectrum problem, and the dual prepotential $\mathcal{F}_D$ (and $\mathcal{F}_T$) serves as the Yang-Yang function of the type B spectrum problem. For the case of pure SU(2) Yang-Mills theory, the eigenvalue of the two types of problems are well-known results of the Mathieu equation.

In the four-dimensional Ω-deformed theory, the two parameters ϵ_1, ϵ_2 can take arbitrary complex value, the full structure of the theory is very rich. Some special corners of the parameter space have been studied in detail; they are often related to some other field theory models. We briefly list several cases that have appeared in the literature:

(i) The case of $\epsilon_1 = -\epsilon_2 = \lambda$ has been studied in its related contexts of topological string and matrix models, see for example [11]. In that case, the partition function takes the form

$$\mathcal{F} = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}^{(g)}(a). \tag{37}$$

The higher order correction terms $\mathcal{F}^{(g)}(a)$ correspond to the gravitational couplings to the Seiberg-Witten gauge theory. $\mathcal{F}^{(g)}(a)$ is the holomorphic limit of the quantity $F^{(g)}(\tau, \bar{\tau})$ of the type B topological sigma model which corresponds to the higher genus gravity correction. In [10], the authors study the SU(2) theory, using the $\Gamma(2) \subset SL(2, \mathbb{Z})$ (quasi)modular property of $F^{(g)}(\tau, \bar{\tau})$, the magnetic expansion of $\mathcal{F}_D^{(g)}(a_D)$ was obtained by the limit $\bar{\tau}_D = -\frac{1}{\bar{\tau}} \to \infty$.

(ii) The case of $\epsilon_1 = \hbar$, $\epsilon_2 = 0$ is related to the quantization of integrable systems, initiated in [12]. The present work investigates the particular SU(2) pure gauge theory using its connection with the integrable system. Comparing our results with that in [10], we know that the Nekrasov theory with $\epsilon_1 = -\epsilon_2 \neq 0$ and $\epsilon_1 \neq 0$, $\epsilon_2 = 0$ will give the same $\mathcal{F}^{(0)}(a)$ and $\mathcal{F}_D^{(0)}(a_D)$ which is the Seiberg-Witten theory, but for higher order corrections, $\mathcal{F}^{(g)}(a)$ and $\mathcal{F}_D^{(g)}(a_D)$ for $g \geq 1$, they two cases give different results.

(iii) The case $\epsilon_1 = \epsilon_2$ is explored in [26]. The corresponding gauge theory in the Ω background is identical to the physical theory defined on the Euclidean S^4, with $\epsilon_1 = \epsilon_2$ equal to the inverse of the radius of the four sphere.

(iv) Recently, a relation between the four-dimensional Ω-deformed theory and the two-dimensional Liouville conformal field theory (CFT) is discovered by Alday, Gaiotto, and Tachikawa [27]. It states that the Nekrasov partition function is identical to the correlation function of Liouville CFT on certain Riemann surfaces with punctures. In their case, the corresponding Nekrasov theory sits in the corner of $\epsilon_1 \cdot \epsilon_2 = 1$; the ϵ_1, ϵ_2 parameters are related to the central charge of the Liouville CFT.

(v) More recently, in [28], the authors try to embed the deformed gauge theory with general ϵ_1, ϵ_2 into topological string theory. Its partition function with generic ϵ_1, ϵ_2 satisfies an extended version of the holomorphic anomaly equation. Especially, they find that theory at $\epsilon_1 = -2\epsilon_2$ can be identified with an orientifold of the theory at $\epsilon_1 = -\epsilon_2$.

It seems that the Nekrasov theory is a very powerful structure that unifies several fields in an unexpected way. Exploring the general case of ϵ_1, ϵ_2, especially its modular property, largely remains untouched.

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