PHYSICAL REVIEW D **82,** 025006 (2010)

One-loop effective potential in $\mathcal{M}_4 \times T^2$ with and without 't Hooft flux

A. F. Faedo,[1] D. Hernández,[2] S. Rigolin,[3] and M. Salvatori[4]

[1]*INFN Sezione di Padova, Via Marzolo 8, I-35131 Padova, Italy*

[2]*Departamento de Física Teórica and Instituto de Física Teórica, Universidad Autónoma de Madrid, Cantoblanco, E-28049 Madrid, Spain*

[3]*Dipartimento di Fisica, Università di Padova and INFN Padova, Via Marzolo 8, I-35131 Padova, Italy*

[4]*Altacontrol SW, Madrid, Spain*

(Received 19 November 2009; published 13 July 2010)

We review the basic notions of compactification in the presence of a background flux. In extra-dimensional models with more than five dimensions, Scherk and Schwarz boundary conditions have to satisfy 't Hooft consistency conditions. Different vacuum configurations can be obtained, depending whether trivial or nontrivial 't Hooft flux is considered. The presence of the magnetic background flux provides, in addition, a mechanism for producing four-dimensional chiral fermions. Particularizing to the six-dimensional case, we calculate the one-loop effective potential for a $U(N)$ gauge theory on $\mathcal{M}_4 \times \mathcal{T}^2$. We first review the well-known results of the trivial 't Hooft flux case, where one-loop contributions produce the usual Hosotani dynamical symmetry breaking. Finally we applied our result for describing, for the first time, the one-loop contributions in the nontrivial 't Hooft flux case.

DOI: 10.1103/PhysRevD.82.025006 PACS numbers: 11.10.Kk, 11.15.Ex, 11.25.Mj

I. INTRODUCTION

There is still one sector completely unknown in the standard model (SM) of electroweak interactions: the Higgs sector. The Higgs boson must exist, either as an elementary particle or as a composite resonance.

In the SM, the Higgs boson is a scalar particle with the appropriate bilinear and quadrilinear self-interactions to drive the $SU(2)_{EW} \times U(1)_Y$ spontaneous symmetry breaking. All experimental available data agree in indicating that the mass of such a state should be of the order of the electroweak scale $v \sim \mathcal{O}(200)$ GeV. However, in the SM, the Higgs mass parameter is not protected by any symmetry and thus can, in principle, get corrections which are quadratically dependent on possible higher scales to which the Higgs boson is sensitive. Ultimately, the Higgs mass should be sensitive to the scale at which quantum gravity effects appear: the Planck scale, $M_{\rm Pl}$. Therefore, from the SM point of view, a Higgs mass at the electroweak scale appears "unnatural," This represents the essence of the SM hierarchy problem.

Various mechanisms have been devised in order to eliminate the quadratic sensitivity of the Higgs mass to the cutoff scale, like for example *supersymmetry*, *technicolor* [1] and *little Higgs* [2]. In this paper, conversely, we examine another particularly interesting mechanism, known as *gauge-Higgs unification* [3] where the lightness of the Higgs field is guaranteed by the gauge symmetry itself.

The main idea of gauge-Higgs unification is that a single higher-dimensional gauge field gives rise to all the four-dimensional (4D) bosonic degrees of freedom: the gauge bosons, from the ordinary four space-time components and the scalar bosons (and the Higgs fields among them) from the extra-dimensional ones. The essential point concerning the hierarchy problem solution is that, although the higher-dimensional gauge symmetry is globally broken by the compactification procedure, it always remains locally unbroken. Any local (sensitive to the UV physics) mass term for the scalars is then forbidden by the gauge symmetry and the Higgs mass has only a nonlocal and UV-finite origin.

After the first seminal paper [3] where a compactification on $M_4 \times S^2$ was studied, the gauge-Higgs unification idea has been mainly applied to the framework of gauge theories in non-simply connected space-time. When the space is non-simply connected, zero field strength configurations do not necessarily imply flat connection configurations. In these scenarios, in fact, nonintegrable (gauge-invariant) phases, associated to nontrivial Wilson loops, appear. These phases can be interpreted, from the 4D point of view, as vacuum expectation values (vevs) of the extra-dimensional gauge field (i.e. scalar) components. The minimum of the tree-level scalar potential does not depend on these vevs and, consequently, these phases are just free parameters that describe equivalent (classical) vacuum configurations of the theory. This degeneracy is lifted at the quantum level [4,5]. The quantum stable vacuum of the theory is obtained minimizing the one-loop effective potential. Depending on the matter content included in the specific model, the minimum of the scalar potential preserves, or not, the original symmetry group. If the minimum corresponds to vanishing phases (vevs) then the original symmetry is preserved. Conversely, if at the minimum some of the phases (vevs) are nontrivial then the gauge symmetry group is dynamically broken [6–8]. This mechanism, conventionally known as the *Hosotani mechanism*, can be used to reproduce the spontaneous electroweak symmetry breaking in the context of gauge-Higgs unification. Moreover, as the Wilson loop is a gauge-invariant nonlocal operator (with any power of the scalar components of the gauge fields) through this mechanism one obtains an operator for the Higgs mass that is automatically free from any UV-divergence [9,10].

This idea has been widely investigated in the context of $5D$ compactifications on $M_4 \times S^1(S^1/Z_2)$, with either flat [11] or warped extra dimension [12]. Some work has been done also in the context of $6D$ compactifications (with or without orbifolds) [13,14]. In all these models, the need of having compactification in presence of singularities [15] is mainly motivated by the necessity of obtaining $4D$ chiral fermions, starting from higher-dimensional theories [16].

Beside orbifold compactification, it is well known that $4D$ chiral theories can be obtained by compactifying in the presence of a background field, either a scalar field (*domain wall* scenarios) [17], either gauge—and eventually gravity—backgrounds with nontrivial field strength (*flux compactification*) [18].

The idea of obtaining chiral fermions in the presence of Abelian gauge and gravitational backgrounds was first proposed by Randjbar-Daemi, Salam, and Strathdee [18], on a $6D$ space-time with the two extra dimensions compactified on a sphere. This seminal idea was right away adapted to heterotic string constructions [19] and it is still nowadays deeply used in the framework of intersecting branes scenarios [20].

From the field theory point of view $6D$ compactification on $M_4 \times \mathcal{T}^2$ in the presence of a background flux, living in the extra-dimensions, has been studied in [21,22]. The typical framework one can consider is that of an $U(N)$ gauge theory in six dimensions, with a nonvanishing $U(N)$ background field strength living in the extra dimensions. As it is well known [18], the presence of an extra-dimensional stable magnetic flux, associated to the Abelian subgroup $U(1) \in U(N)$, induces chirality in four dimensions. However, there is no stable background flux associated to the non-Abelian field strength, since the $SU(N)$ gauge field is a flat connection on $\mathcal{T}^2$. Consequently any nonvanishing non-Abelian background field strength, introduced *ab initio*, can be gauged away [23]. The numerical proof of this statement is, however, technically quite difficult, requiring us to solve explicitly the Olesen-Nielsen instability [24] on the torus. This was done, for the first time, in [21] where the complete $4D$ tree-level scalar potential was numerically minimized including simultaneously Kaluza-Klein and Landau heavy modes.

Besides producing $4D$ chirality, the presence of a nonvanishing $U(1)$ flux also affects the non-Abelian part of the group, $SU(N) \in U(N)$, being connected to a topological quantity, conventionally known as the *non-Abelian 't Hooft flux* [25], and producing interesting $SU(N)$ symmetry breaking patterns. While the $SU(N)$ trivial 't Hooft flux case has been deeply analyzed in the literature, the field theory analysis and the phenomenological applications of the nontrivial (non-Abelian) 't Hooft flux has been explored only recently. In [21] an effective field theory approach was used to explicitly show the $SU(2)$ classical symmetry breaking pattern and the resulting gauge-scalar spectrum, for both the trivial and nontrivial 't Hooft non-Abelian flux. In [22] such an analysis was extended and generalized to the $SU(N)$ case. Recently, then, the symmetry breaking pattern of models with the simultaneous presence of orbifold and non-Abelian 't Hooft flux has been analyzed by [26]. Models with $N = 1$ supersymmetry have been also considered in [27].

The main motivation of this paper is to study, at one-loop level, the symmetry breaking patterns analyzed at tree-level in [21,22]. To do this we calculate the one-loop effective scalar potential in the presence of 't Hooft flux. In the case of trivial 't Hoof flux, one reduces to the well-known results already present in the literature (see for example [9] for a $6D$ example). There was, however, no calculation available up to now of how the Hosotani mechanism does work in the presence of nontrivial 't Hooft flux. This generalization is provided here.

The paper is organized as follows. In Sec. II we summarize the main aspects of a $6D$ theory in the presence of an $U(N)$ background living in the extra-dimensions. The symmetry breaking patterns obtained in the case of trivial and nontrivial t' Hooft flux are analyzed and the tree-level gauge and scalar spectrum are derived. In Sec. III we recall the main notions about chiral fermions in the presence of a background (magnetic) flux. We discuss the relation between 't Hooft flux and magnetic flux and we explicitly write the spectrum for fermions in the fundamental and adjoint representation. In Sec. IV we calculate the one-loop effective potential contribution of gauge, scalar, and fermionic sectors, for both trivial and nontrivial 't Hooft flux and we discuss some phenomenological consequences. Finally in sec. V we state our conclusions. In Appendix A we explicitly calculate the $U(N)$ wave functions in the fundamental representation, extending the usual derivation to the symmetry breaking mechanism presented in this paper. In Appendix A we briefly remind the general formalism for calculating the one-loop effective scalar potential using the heat function method.

II. $U(N)$ GAUGE THEORY ON $\mathcal{M}_4 \times \mathcal{T}^2$

Consider a $U(N)$ gauge theory on a $6D$ space-time[1] where the two extra dimensions are compactified on an orthogonal torus $\mathcal{T}^2$. To completely define a field theory on a torus one has to specify the periodicity conditions: that is, to describe how the fields transform under the shifts $y \longrightarrow y + \ell_a$, with ℓ_a being the vectors identifying the lattice

[1]Throughout the paper, with x and y we denote the ordinary and extra coordinates, respectively. Latin upper case indices M, N run over all the $6D$ space, whereas Greek and Latin lower case indices μ, ν, and a, b run over the four ordinary and the two extra-dimensions, respectively.

shifts along the a circle of length l_a. Let us denote with T_a the embeddings of these shifts in the fundamental representation of $U(N)$. The general periodicity conditions[2] for the gauge field A_M, that preserve $4D$ Poincaré invariance, read

$$\mathbf{A}_{\mathbf{M}}(x, y + \ell_a) = T_a(y)\mathbf{A}_{\mathbf{M}}(x, y)T_a^\dagger(y) + \frac{i}{g}T_a(y)\partial_M T_a^\dagger(y). \quad (1)$$

This equation is derived from the fact that while individual gauge fields may not be single-valued on the torus, any physical scalar quantity, like the Lagrangian, must be. The periodicity conditions in Eq. (1) are usually referred as Scherk-Schwarz (SS) boundary conditions [28].

The transition functions $T_a(y)$ (hereafter simply *twists*) in order to preserve the $4D$ Poincaré invariance, can only depend on the extra-dimensional coordinates y. Consistency with the geometry imposes the following $U(N)$ condition on the twists [25,29]:

$$T_1(y + \ell_2)T_2(y) = e^{i\theta}T_2(y + \ell_1)T_1(y). \quad (2)$$

This condition is obtained imposing that the value of the gauge field $A_M(y_1 + l_1, y_2 + l_2)$ has to be independent on the path which has been followed to reach the final point $(y_1 + l_1, y_2 + l_2)$ from the starting point (y_1, y_2), up to a constant element of the center of the group, which, for $U(N)$, is a phase.

One can easily verify that the inclusion of fields that transform in a representation sensitive to the center of the group, like, for example, the fundamental representation, imposes, in Eq. (2), the additional constraint $\theta = 0$. As we are interested in models with the simultaneous presence of fields in the adjoint and in the fundamental representation, throughout the paper we will set $\theta = 0$ in the consistency condition of Eq. (2).

The $U(N)$ twist matrices can be, locally, decomposed as the product of an element $e^{iv_a(y)} \in U(1)$ and an element $\mathcal{V}_a(y) \in SU(N)$ as follows:

$$T_a(y) = e^{iv_a(y)}\mathcal{V}_a(y). \quad (3)$$

Using this parametrization, the $U(N)$ consistency condition can be split in a $SU(N)$ and $U(1)$ part, respectively,

$$e^{2\pi i(m/N)}\mathcal{V}_1(y + \ell_2)\mathcal{V}_2(y) = \mathcal{V}_2(y + \ell_1)\mathcal{V}_1(y), \quad (4)$$

$$\Delta_2 v_1(y) - \Delta_1 v_2(y) = 2\pi\frac{m}{N}, \quad (5)$$

with $\Delta_a v_b(y) = v_b(y + \ell_a) - v_b(y)$. The exponential factor in Eq. (4) is simply an element of the center of $SU(N)$. The integer $m = 0, 1, \ldots, N - 1$ (modulo N) is a gauge-invariant quantity called the non-Abelian 't Hooft flux [25]. Furthermore, it coincides with the value of a quantized Abelian magnetic flux living on the torus, Eq. (5), or, in other words, with the first Chern class of $U(N)$ on $\mathcal{T}^2$.

[2]We consider here exclusively the case of internal automorphisms.

A. Boundary conditions vs background flux

Up to here we have discussed the general properties of a $6D$ $U(N)$ gauge theory with SS boundary conditions. We are interested now to particularize the discussion considering the specific set of $U(N)$ gauge field configurations characterized by a constant (background) field strength living in the extra dimensions and pointing in an arbitrary direction of the gauge space. The physical relevance of these configurations will be immediately clear in the following subsections.

Let us expand the $U(N)$ gauge field, $\mathbf{A}_{\mathbf{M}}$, in terms of the stationary background, B_M, and the fluctuation field, A_M, around it as

$$\mathbf{A}_{\mathbf{M}}(x, y) = B_M(x, y) + A_M(x, y) = B_a(y)\delta_{aM} + A_M(x, y). \quad (6)$$

The specific form of the background field in the previous equation is chosen to guarantee $4D$ Poincaré invariance. In the presence of such a background, the general SS periodicity conditions for the fluctuation and background fields read

$$A_M(x, y + \ell_a) = T_a(y)A_M(x, y)T_a^\dagger(y), \quad (7)$$

$$B_b(y + \ell_a) = T_a(y)\left[B_b(y) + \frac{i}{g}\partial_b\right]T_a^\dagger(y). \quad (8)$$

Following the definition of Eq. (3), we can write the periodicity conditions for the $U(1)$ and $SU(N)$ part of the fluctuation and background fields,[3] respectively, as

$$A_M^{(0)}(x, y + \ell_a) = A_M^{(0)}(x, y), \quad B_b^{(0)}(y + \ell_a) = B_b^{(0)}(y) + \frac{\sqrt{2N}}{g}\partial_b v_a(y), \quad (9)$$

$$A_M^{(k)}(x, y + \ell_a)\lambda_k = \mathcal{V}_a(y)A_M^{(k)}(x, y)\lambda_k\mathcal{V}_a^\dagger(y), \quad B_b^{(k)}(y + \ell_a)\lambda_k = \mathcal{V}_a(y)\left[B_b^{(k)}(y)\lambda_k + \frac{i}{g}\partial_b\right]\mathcal{V}_a^\dagger(y). \quad (10)$$

Notice however that neither the twists or the background flux are gauge invariant quantities and so the split between Eqs. (7) and (8) is purely conventional.

Not all the possible choices of background fields and boundary conditions are compatible. To illustrate this, let

[3]We use the following conventions for the $U(1)$ and $SU(N)$ generators, λ_0 and λ_k: $\lambda_0 = 1_N/\sqrt{2N}$ and $\mathrm{Tr}[\lambda_k\lambda_{k'}] = \frac{1}{2}\delta_{kk'}$, with $k, k' = 1, 2, \ldots, (N^2 - 1)$.

us discuss the simplest case of an $U(1)$ gauge theory [or the $U(1)$ sector of the $U(N)$ theory] and consider a constant background field strength:

$$B^{(0)}_{ab}(y) = \partial_a B^{(0)}_b - \partial_b B^{(0)}_a = \frac{\mathcal{F}}{g\mathcal{A}}, \qquad B^{(0)}_a(y) = -\frac{\mathcal{F}}{2g\mathcal{A}}\epsilon_{ab}y_b, \tag{11}$$

with $\mathcal{F}$ a dimensionless constant (flux) and $\mathcal{A}$ the area of the torus. Compatibility between Eq. (11) and the boundary conditions of Eq. (9) force $v_a(y)$ to be of the form:

$$v_a(y) = \frac{\mathcal{F}}{2\mathcal{A}}\epsilon_{ab}\ell_a y_b, \tag{12}$$

with $\mathcal{F} = 2\pi m$ from Eq. (5). It was shown by [5], that in the case of a $SU(N)$ gauge theory on $\mathcal{M}_4 \times \mathcal{S}^1$, starting from a compatible choice of background field and boundary conditions on the circle, it is always possible to go to a gauge in which either the twist is trivial or the background field is vanishing, the latter defined as the *symmetric gauge*. Moreover it was shown that in this gauge the $5D$ twist coincides with the Wilson loop and can be parametrized in terms of a nonintegrable, gauge-invariant phase: the SS phase [28]. This quantity, in the gauge in which the twist is trivial, appears instead as a background field component and can be interpreted, from the $4D$ point of view, as nonvanishing vev for the $4D$ scalar (gauge) field.

Similarly to what happens in the $5D$ case, it was shown in [22] that also for a $SU(N)$ gauge theory on $\mathcal{M}_4 \times \mathcal{T}^2$ it is always possible to choose a gauge, namely, the *symmetric gauge* in which the $SU(N)$ background field strength on the torus vanishes and the $SU(N)$ twist matrices are constant. Let us define the $U(N)$ Wilson line and Wilson loop around the a circle, respectively, as

$$W_a(y_f, y_i) = \mathcal{P}\exp\left\{ig\int_{y_i}^{y_f} dz^b B_b(z)\right\}T_a(y), \tag{13}$$

$$W_a(y, y+\ell_a) = \mathcal{P}\exp\left\{ig\int_{y}^{y+\ell_a} dz^b B_b(z)\right\}T_a(y) \equiv W_a, \tag{14}$$

where $\mathcal{P}$ stands for the path-ordered product. It is immediate to see that the $U(1)$ part of the twist automatically cancels in Eq. (14) with the exponential part of the Abelian background field, due to the condition of Eq. (9). Consequently in the symmetric gauge the following relations hold:

$$(\mathcal{V}_a(y))_{\rm sym} \equiv V_a = W_a(y, y+\ell_a), \qquad (B^{(k)}_{ab}(y))_{\rm sym} = 0. \tag{15}$$

Being the trace of Eq. (14) a gauge-invariant and y-independent quantity, one consequently ends up, in the $6D$ case, with two independent nonintegrable SS phases.

However, contrary to the lower dimensional case, in the $6D$ case, the symmetry of the classical vacua depends on an additional gauge-invariant quantity, the 't Hooft non-Abelian flux. The relation of the non-Abelian 't Hooft flux and the existence of a background (Abelian) magnetic flux can be immediately understood calculating the trace of the $U(1)$ part of the Abelian background field strength and using the Abelian periodicity condition of Eq. (9):

$$\begin{aligned}\frac{g}{N}\int_{\mathcal{T}^2} d^2y\,{\rm Tr}[B_{12}(y)] &= g\int_{\mathcal{T}^2} d^2y\frac{(\partial_1 B^{(0)}_2(y) - \partial_2 B^{(0)}_1(y))}{\sqrt{2N}}\\ &= \int dy_2\partial_2 v_1(y) - \int dy_1 \partial_1 v_2(y)\\ &= [\Delta_2 v_1(y) - \Delta_1 v_2(y)] = \frac{2\pi m}{N}.\end{aligned} \tag{16}$$

That is, the 't Hooft consistency condition of Eq. (5) implies the quantization of the Abelian magnetic flux in terms of the non-Abelian 't Hooft flux m.

B. Trivial 't Hooft flux: $m = 0$

The spectrum can be easily discussed in the *symmetric gauge*. For the $m = 0$ case, Eq. (4) tell us that the two V_a matrices commute and consequently can be parametrized as

$$V_a = e^{2\pi i(\alpha_a\cdot H)}, \qquad \alpha_a\cdot H \equiv \sum_{\rho=1}^{N-1}\alpha^\rho_a H_\rho, \tag{17}$$

with H_ρ the $(N-1)$ generators of the Cartan subalgebra of $SU(N)$. The periodicity condition, and consequently the classical vacua, are characterized by $2(N-1)$ real continuous parameters, $0 \le \alpha^\rho_a < 1$. These parameters are nonintegrable phases, which arise only in a topologically nontrivial space and cannot be gauged away. When all the α^ρ_a are vanishing the initial symmetry is unbroken. At the classical level α^ρ_a are undetermined. Their values are dynamically determined at the quantum level [4,5] where a rank-preserving symmetry breaking can occur. This dynamical and spontaneous symmetry breaking mechanism is conventionally known as the *Hosotani mechanism*.

In order to write down the explicit expression for the (tree-level) mass spectrum of the $4D$ gauge and scalar components of the $6D$ gauge field one can introduce the Cartan-Weyl basis for the $SU(N)$ generators. In addition to the $(N-1)$ generators of the Cartan subalgebra, H_ρ, one defines $N(N-1)$ nondiagonal generators, E_r, such that the following commutation relations are satisfied:

$$[H_\rho, H_\sigma] = 0, \qquad [H_\rho, E_r] = q^\rho_r E_r. \tag{18}$$

In this basis, the V_a act in a diagonal way, that is

$$V_a H_\rho V_a^\dagger = H_\rho, \qquad V_a E_r V_a^\dagger = e^{2\pi i(\alpha_a\cdot q_r)}E_r, \tag{19}$$

and the four-dimensional mass spectrum reads simply:

$$m^2_{(k)} = 4\pi^2 \sum_{a=1}^{2} (n_a + \alpha_a \cdot q_k)^2 \frac{1}{l_a^2}, \qquad n_a \in \mathbb{Z}, \tag{20}$$

with k here labeling the $(N^2 - 1)$ $SU(N)$ gauge (scalar) components. For a gauge (scalar) field component A^ρ_M, associated to a generator belonging to the Cartan subalgebra, H_ρ, one has $q_\rho = (0, \ldots, 0)$ and the spectrum reduces to the ordinary Kaluza-Klein (KK) one. For a gauge (scalar) field component A^r_M associated to the nondiagonal generators, E_r, one has, instead, $q_r \neq (0, \ldots, 0)$ and the mass spectrum is consequently shifted by a factor proportional to the nonintegrable phases α^ρ_a. When all the $\alpha^\rho_a \neq 0$, then only the gauge field components associated to the generators of the Cartan subalgebra are massless. Therefore, the symmetry breaking induced by the commuting twists, V_a, does not lower the rank of $SU(N)$. This result is the one generally reported by the literature.

One can easily generalize these results to the $U(N)$ case adding an extra diagonal generator, $H_0 = 1_N/\sqrt{2N}$. Obviously H_0 commute with all the twists V_a and consequently A^0_M always remains unbroken. The maximal symmetry breaking pattern that can be achieved in the $m = 0$ case, for an $U(N)$ gauge theory is given by

$$U(N) \sim U(1) \times SU(N) \to U(1) \times U(1)^{N-1} = U(1)^N. \tag{21}$$

This symmetry breaking mechanism is exactly the same Hosotani mechanism one is used to working with in a $5D$ framework.

C. Nontrivial 't Hooft flux: $m \neq 0$

In the $m \neq 0$ case, the twists V_a do not commute between themselves and so necessarily they induce a rank-reducing symmetry breaking. The most general solution of the consistency relation Eq. (4) can be parametrized as [22,23,30]

$$V_1 = \omega_1 P^{s_1} Q^{t_1}, \qquad V_2 = \omega_2 P^{s_2} Q^{t_2}. \tag{22}$$

Here s_a, t_a are integer parameters taking values between $0, \ldots, (N-1)$ (modulo N) and satisfying the constraint:

$$s_1 t_2 - s_2 t_1 = \tilde{m}. \tag{23}$$

P and Q are $SU(N)$ constant matrices given by

$$P \equiv P_{\tilde{N}} \otimes 1_{\mathcal{K}}, \qquad Q \equiv Q_{\tilde{N}} \otimes 1_{\mathcal{K}}. \tag{24}$$

In the previous equations we defined $\mathcal{K} \equiv \text{g.c.d.}(m, N)$, $\tilde{m} \equiv m/\mathcal{K}$, and $\tilde{N} \equiv N/\mathcal{K}$. The matrices $P_{\tilde{N}}$ and $Q_{\tilde{N}}$ are the following $\tilde{N} \times \tilde{N}$ matrices:

$$\begin{cases} (P_{\tilde{N}})_{jk} = e^{i\pi(\tilde{N}-1/\tilde{N})} \delta_{j,k-1} \\ (Q_{\tilde{N}})_{jk} = e^{-2\pi i(k-1/\tilde{N})} e^{i\pi(\tilde{N}-1)/\tilde{N}} \delta_{jk} \end{cases} \tag{25}$$
$$j, k = 1, 2, \ldots, \tilde{N},$$

satisfying the consistency conditions

$$\begin{aligned} P_{\tilde{N}} Q_{\tilde{N}} &= e^{-2\pi i(1/\tilde{N})} Q_{\tilde{N}} P_{\tilde{N}}, \\ (P_{\tilde{N}})^{\tilde{N}} &= (Q_{\tilde{N}})^{\tilde{N}} = e^{\pi i(\tilde{N}-1)}. \end{aligned} \tag{26}$$

When $\mathcal{K} = 1$, then $\tilde{N} = N$ and P and Q reduce to the usual elementary twist matrices defined by 't Hooft [25]. The matrices ω_a are constant elements of $SU(\mathcal{K}) \subset SU(N)$. They commute between themselves and with P and Q. Therefore ω_a can be parametrized in terms of generators H_j belonging to the Cartan subalgebra of $SU(\mathcal{K})$:

$$\omega_a = e^{2\pi i(\alpha_a \cdot H)}, \qquad \alpha_a \cdot H \equiv \sum_{\rho=1}^{\mathcal{K}-1} \alpha^\rho_a H_\rho. \tag{27}$$

Here α^ρ_a are $2(\mathcal{K}-1)$ real continuous parameters, $0 \leq \alpha^\rho_a < 1$. As in the $m = 0$ case, they are nonintegrable phases and their values must be dynamically determined at the quantum level producing a dynamical and spontaneous symmetry breaking.

Following [22] one can introduce a particular basis for the $SU(N)$ generators[4] that we are going to denote as $\tau_{(\rho,\sigma)}(\Delta, k_\Delta)$. To determine the $m \neq 0$ spectrum the action of the twists V_a on this basis is needed:

$$V_a \tau_{(\rho,\sigma)}(\Delta, k_\Delta) V_a^\dagger = e^{2\pi i/\tilde{N}(s_a\Delta + t_a k_\Delta) + 2\pi i(\alpha_a \cdot q_{(\rho,\sigma)})} \times \tau_{(\rho,\sigma)}(\Delta, k_\Delta). \tag{28}$$

In this basis, the $SU(\mathcal{K})$ generators that commute with P and Q are simply given by $\tau_{(\rho,\sigma)}(0, 0)$. In particular, the generators belonging to the Cartan subalgebra of $SU(\mathcal{K})$ are given by $H^\rho = \tau_{(\rho,\rho)}(0, 0)$. The following commutation relations are satisfied:

$$\begin{aligned} &[\tau_{(\rho,\rho)}(0, 0), \tau_{(\sigma,\sigma)}(0, 0)] = 0, \\ &[\tau_{(\tau,\tau)}(0, 0), \tau_{(\rho,\sigma)}(\Delta, k_\Delta)] = q^{(\rho,\sigma)}_\tau \tau_{(\rho,\sigma)}(\Delta, k_\Delta). \end{aligned}$$

From Eq. (28) one can easily obtain the $m \neq 0$ $4D$ mass spectrum:

$$m^2_{(\rho,\sigma)}(\Delta, k_\Delta) = 4\pi^2 \sum_{a=1}^{2} \left(n_a + \frac{1}{\tilde{N}}(s_a\Delta + t_a k_\Delta) + \alpha_a \cdot q_{(\rho,\sigma)} \right)^2 \frac{1}{l_a^2}, \qquad n_a \in \mathbb{Z} \tag{29}$$

Therefore, beside the usual KK mass term, there are other two additional contributions. The first one, quantized in terms of $1/\tilde{N}$, is a consequence of the nontrivial commutation rule of Eq. (26) between P and Q that induces the $SU(N) \to SU(\mathcal{K})$ symmetry breaking. Since s_a, t_a cannot be simultaneously zero, the spectrum described by Eq. (29)

[4] Here Δ and k_Δ are integers assuming values between $0, \ldots, (\tilde{N}-1)$, while ρ, σ take values between $1, \ldots, \mathcal{K}$, excluding the case $(\Delta = k_\Delta = 0, \rho = \sigma)$ in which ρ takes values between $1, \ldots, (\mathcal{K}-1)$. Readers interested in the details of such a basis should look to [22] for a detailed discussion.

always exhibits some (tree-level) degree of symmetry breaking. Given a set of s_a, t_a and for all the $\alpha_a^\rho = 0$ (that is $\omega_a = 1$), only the gauge bosons components associated to $\tau_{(\rho,\sigma)}(0,0)$, the generators of $SU(\mathcal{K})$, admit zero modes. This is an explicit symmetry breaking. The second contribution to the gauge mass is associated to the ω_a degrees of freedom and it depends on the continuous parameters α_a^ρ. For $\mathcal{K} > 1$ and all the nonintegrable phases $\alpha_a^\rho \neq 0$, the only massless modes correspond to the gauge bosons associated to the Cartan subalgebra of $SU(\mathcal{K})$. The symmetry breaking pattern induced by the ω_a produces a Hosotani symmetry breaking that does not lower the rank of $SU(\mathcal{K})$.

The maximal symmetry breaking pattern that can be achieved for an $U(N)$ gauge theory with matter fields in the fundamental is, in the $m \neq 0$ case, given by

$$U(N) \sim U(1) \times SU(N) \to U(1) \times U(1)^{\mathcal{K}-1} = U(1)^{\mathcal{K}}. \tag{30}$$

When $\mathcal{K} = 1$ the $SU(N)$ subgroup is completely broken and the only unbroken symmetry is the $U(1) \in U(N)$. This symmetry breaking pattern has no analogous in 5D frameworks and it is peculiar of higher-dimensional models where fluxes can be defined.

Two final comments on the spectrum properties are in order. First of all, it could appear from Eq. (29) that gauge boson (or scalar) masses depend on the specific choice of the two integer parameters s_a, t_a. However, one can explicitly prove that for a given $\tilde{m}$, any possible choice of s_a, t_a, satisfying the constraint of Eq. (23) gives the same boson (scalar) masses. This property will hold at the one-loop level too. Second, notice that in both the cases of trivial and nontrivial 't Hooft flux, the classical effective 4D spectrum depends on the gauge indices but not on the Lorentz ones. This implies that at the classical level the 4D scalar fields A_a are expected to be degenerate with the 4D gauge fields A_μ with the same gauge quantum numbers. We will see in Sec. IV that this degeneracy can be removed at the quantum level.

III. FERMIONS IN THE FUNDAMENTAL AND ADJOINT OF $U(N)$

We consider now the fermionic sector, reviewing how to introduce fermions and define 4D chirality in the presence of a $U(N)$ background flux.

Fermions transforming in the fundamental or in the adjoint representation of $U(N)$ obey the following periodicity conditions:

$$\Psi(x, y + \ell_a) = T_a(y)\Psi(x, y),$$
$$\Psi(x, y + \ell_a) = T_a(y)\Psi(x, y)T_a^\dagger(y),$$

where $T_a(y)$ must be, for gauge invariance, the same twists defined in Eq. (1). For definiteness we make use of the following representation of the Clifford algebra:

$$\Gamma^\mu = \gamma^\mu \otimes 1_2, \qquad \Gamma^5 = \gamma^5 \otimes i\sigma_1, \qquad \Gamma^6 = \gamma^5 \otimes i\sigma_2. \tag{31}$$

In six dimensions, chirality can be defined by means of the matrix

$$\Gamma_7 = \prod_M \Gamma^M = \gamma^5 \otimes \sigma_3, \qquad \mathcal{P}_{L,R} = \left(\frac{1 \mp \Gamma_7}{2}\right). \tag{32}$$

A 6D chiral fermion, in terms of 4D left and right Weyl spinors $\psi_{L,R}$, can be written as

$$\Psi_L = \mathcal{P}_L \Psi_{6D} = \begin{pmatrix} \psi_L \\ \chi_R \end{pmatrix}, \qquad \Psi_R = \mathcal{P}_R \Psi_{6D} = \begin{pmatrix} \psi_R \\ \chi_L \end{pmatrix}. \tag{33}$$

The Lagrangian for a 6D massless left fermion, in the fundamental and in the adjoint of $U(N)$ can be written, respectively, as

$$\mathcal{L}_f = i\bar{\Psi}_L \Gamma^M D_M \Psi_L, \qquad D_M = \partial_M - ig\delta_{M,a} B_a(y), \tag{34}$$

$$\begin{aligned} \mathcal{L}_f &= i\bar{\Psi}_L \Gamma^M \mathcal{D}_M \Psi_L, \\ \mathcal{D}_M &= \partial_M - ig\delta_{M,a}[B_a(y), \cdot]. \end{aligned} \tag{35}$$

D_M ($\mathcal{D}_M$) is the 6D covariant derivative in the fundamental (adjoint) representation, with respect to the $U(N)$ background. From Eq. (34) one obtains the following Klein-Gordon equations for the zero mode of the 4D Dirac spinors in the fundamental:

$$(\partial^2 - D_z D_{\bar{z}} + [D_z, D_{\bar{z}}])\chi_R = 0, \tag{36}$$

$$(\partial^2 - D_z D_{\bar{z}})\psi_L = 0, \tag{37}$$

with $D_z = (D_5 - iD_6)$, $D_{\bar{z}} = (D_5 + iD_6)$, and the commutator being

$$[D_z, D_{\bar{z}}] = 2gB_{56} = 2gB_{56}^{(0)}\lambda_0 + 2g\sum_{k=1}^{N^2-1} B_{56}^{(k)}\lambda_k. \tag{38}$$

The extra-dimensional derivative terms in Eqs. (36) and (37) can be interpreted as mass terms in four dimensions. Moreover, the presence of a nonvanishing commutator introduces a mass splitting between the 4D fermions of opposite chirality that, thus, cannot have, simultaneously, a massless 0-mode state [18].

Equivalent equations in the adjoint representation can be obtained by simply replacing D_M with $\mathcal{D}_M$, the mass splitting between fermions of different chirality now being

$$[\mathcal{D}_z, \mathcal{D}_{\bar{z}}] = 2g[B_{56}, \cdot] = 2g\sum_{k=1}^{N^2-1} B_{56}^{(k)}[\lambda_k, \cdot]. \tag{39}$$

Therefore, as expected, the mass splitting for fermions in the adjoint is sensitive only to the non-Abelian part of the flux.

As argued in the previous section, all stable $SU(N)$ background configurations are trivial while the Abelian part is quantized and proportional to the 't Hooft flux m. The mass splitting for fermions in the fundamental and in the adjoint representation of $U(N)$ is consequently given by

$$[D_z, D_{\bar{z}}] = \frac{4\pi}{\mathcal{A}} \frac{m}{N}, \qquad [\mathcal{D}_z, \mathcal{D}_{\bar{z}}] = 0. \tag{40}$$

The previous equations reflect the well-known result that, for a nonvanishing 't Hooft flux, only fermions in the fundamental can be chiral while theories with only fermions in the adjoint of $U(N)$ must necessarily be vectorlike.

In short, in our context, the presence of chirality is directly related to the presence of the 't Hooft flux through the Abelian magnetic flux.

A. Fermions in the presence of trivial 't Hooft flux: $m = 0$

For $m = 0$ both the $SU(N)$ and $U(1)$ part of the twists, defined in Eqs. (4) and (5), separately commute. This means that it is possible to find a gauge, i.e. the *symmetric* gauge, where $\mathcal{V}_a^{\rm sym} = V_a$ is a constant matrix and $v_a^{\rm sym} = 0$. In this gauge, obviously, both the $SU(N)$ and the $U(1)$ background field strength vanish and no background magnetic flux is present. If the $SU(N)$ twist is not trivial, i.e. $V_a \neq 1_N$, then some of the original symmetry group is broken, as seen in the previous section, and the corresponding fermionic zero modes are lifted. However the $4D$ theory is not chiral.

In fact, let us start for definiteness with a $6D$ chiral fermion, Ψ_L in the fundamental of $SU(N)$. Identifying ψ_L and χ_R as the two chiral components of a $4D$ Dirac KK state, one obtains the following masses for the k-th component of the fundamental multiplet:

$$m^2_{n(k)} = m_5^2 + m_6^2 = 4\pi^2 \sum_{a=1}^{2} \frac{1}{l_a^2}(n_a + \alpha_a \cdot q_k)^2, \qquad n_a \in \mathbb{Z}, \tag{41}$$

with $H^j \Psi_{(k)} = q_k^j \Psi_{(k)}$. In the case of vanishing 't Hooft flux, there is no difference in the mass spectrum between fermions belonging to the fundamental or the adjoint of $U(N)$, other than the difference in the charges q_k.

B. Fermions in the presence of nontrivial 't Hooft flux: $m \neq 0$

Setting a nontrivial $SU(N)$ 't Hooft flux, along with a nontrivial $U(1)$ background, provides the conditions to have a chiral theory. Let us consider fermions in the fundamental representation of $U(N)$. $4D$ masses are given by the eigenvalues of the extra-dimensional operators, with eigenfunctions consistent with the imposed periodicity conditions:

$$(-D_z D_{\bar{z}} + [D_z, D_{\bar{z}}])\chi_R^p = m^2_{p(R)} \chi_R^p, \qquad \chi_R^p(y + \ell_a) = T_a(y)\chi_R^p(y), \tag{42}$$

$$(-D_z D_{\bar{z}})\psi_L^p = m^2_{p(L)} \psi_L^p, \qquad \psi_L^p(y + \ell_a) = T_a(y)\psi_L^p(y). \tag{43}$$

One should notice that while the operators act diagonally in the N-dimensional gauge space, the $N \times N$ matrices appearing in the boundary conditions are not diagonal and consequently they mix different components within the multiplet. With the following definition of creation and annihilation operators [31]:

$$a^\dagger = -\sqrt{\frac{N\mathcal{A}}{4\pi m}} D_z, \qquad a = \sqrt{\frac{N\mathcal{A}}{4\pi m}} D_{\bar{z}}, \tag{44}$$

it is immediate to show that the energy eigenstates are equally spaced, differing only in the presence of the zero-mode for the case of the left-handed field. Diagonalizing the $4D$ Lagrangian the following mass spectrum is obtained:

$$m^2_{p(R)} = \frac{4\pi m}{\mathcal{A}N}(p + 1), \qquad m^2_{p(L)} = \frac{4\pi m}{\mathcal{A}N} p \quad \text{with} \quad p \in \mathbb{N}, \tag{45}$$

that is, there is no massless eigenstate for the right-handed fermion. Notice the important fact that the SS phases are completely absorbed and do not show up in the spectrum. As a consequence, fermions in the fundamental representation will not help in solving the vacuum degeneracy, not even at the one-loop level.

One apparent oddity is that now there seems to be N solutions to the equations, one for each direction of the $SU(N)$ fundamental. However, we know that the remaining symmetry after the breaking is, at most, $SU(\mathcal{K})$. For the case $\alpha_a = 0$ it is not obvious how those N fermions arrange themselves in $SU(\mathcal{K})$ representations. Ultimately it is proven solving directly the equations, that only $\mathcal{K}$ independent degrees of freedom remain from the original N. These indeed organize in the fundamental of $SU(\mathcal{K})$. The full solution can be found in Appendix A.

Finally, we can address the possibility of having adjoint fermions. Clearly, these fermions are as "blind" to the 't Hooft flux as the gauge fields. For them, the matrices V_i, now written in the adjoint representation of $SU(N)$ commute. They will be generated by some element of the Cartan subalgebra $\alpha_a \cdot H$ of $SU(N)/Z_N$ which will also give rise to a SS-like mass term for a KK tower

$$m^2_{\text{adj}(\rho,\sigma)} = 4\pi^2 \sum_{a=1}^{2} \left(n_a + \frac{1}{\tilde{N}}(s_a \Delta + t_a k_\Delta) + \alpha_a \cdot q_{(\rho,\sigma)}\right)^2 \times \frac{1}{l_a^2}. \tag{46}$$

The symmetry group that remains is rank $(\mathcal{K}-1)$ and depends on the values of α_a. Notice that the fermions will arrange themselves in representations of the resulting group. In particular, if we start from fermions in the adjoint of $SU(N)$ and $\alpha^a = 0$, we end up with $\tilde{N}^2$ adjoint representations and $(\tilde{N}^2 - 1)$ trivial representations of $SU(\mathcal{K})$ in the compactified theory. One can explicitly verify that for a given $\tilde{m}$, any possible choice of $\{s_a, t_a\}$, satisfying the constraint of Eq. (23) gives the same fermion masses.

IV. ONE-LOOP EFFECTIVE POTENTIAL ON $\mathcal{M}_4 \times \mathcal{T}^2$

The favored approach for the calculation of the one-loop effective potential in the extra-dimensional framework [5,11] has been the direct computation through the formula

$$V_{\rm eff} = \frac{i}{2} \,{\rm Tr}\,{\rm lnDet}(D_M D^M). \tag{47}$$

The effective potential is obtained as a sum over all the eigenvalues of the quadratic $(4+d)$-dimensional operator, $D_M D^M$. Usually this entails an integral over continuous four-dimensional eigenvalues as well as a discrete sum over extra-dimensional ones.

In this work, instead, we will compute the one-loop effective action for a $U(N)$ gauge theory on $\mathcal{M}_4 \times \mathcal{T}^2$ using the heat kernel technique.[5] A brief introduction containing the main formulas is given in the appendices. The generality of this method permits computing directly in the complete higher-dimensional manifold rather than performing the dimensional reduction and summing over the resulting $4D$ degrees of freedom as is usual. In some circumstances, in particular, when discussing the ultraviolet properties of the theory, this is crucial [32] and to some extent has motivated our choice.

Since the heat kernel computation takes place explicitly in coordinate space, it results in a very useful instrument to distinguish contributions from local (ultraviolet sensitive) and nonlocal (ultraviolet insensitive) diagrams. The local contributions do not depend on the periodicity conditions and are invariant under all the original symmetries. Thus, they do not contribute to determine the symmetry breaking order parameters. Only nonlocal contributions will be relevant for symmetry breaking, which is then protected from ultraviolet divergences.

In any case we have found that, at least in the case of vanishing 't Hooft flux, the nonlocal pieces of the effective potential, computed in the complete manifold and in the reduced theory, do coincide. We find no reason to expect a change in this picture when adding nontrivial 't Hooft flux.

The details of the whole procedure are given in the appendices. For the main purposes of the following sections, it is enough to quote here the final result. After regularization, one obtains the following contributions to the one-loop effective action:

(i) Gauge bosons and ghosts:

$$\Gamma^{g+gh}_{(1)} = -4\frac{V^{4+2}}{\pi^3} \sum_{w_1,w_2\neq 0} \frac{{\rm Tr}(W_1^{w_1} W_2^{w_2})}{[(l_1 w_1)^2 + (l_2 w_2)^2]^3}. \tag{48}$$

The overall factor 4 is due to the fact that for a flat manifold and gauge background with zero field strength, the only effect of the ghosts is to reduce to four the possible polarizations of a $6D$ gauge boson.[6]

(ii) Matter fields in the representation $\mathcal{R}$ of $U(N)$

$$\Gamma^{f,s}_{(1)} = -\eta_{f,s}\frac{V^{4+2}}{\pi^3} \sum_{w_1,w_2\neq 0} \frac{{\rm Tr}_{\mathcal{R}}(W_1^{w_1} W_2^{w_2})}{[(l_1 w_1)^2 + (l_2 w_2)^2]^3}, \tag{49}$$

where $\eta_f = -4$ and $\eta_s = 2$ for Weyl fermions and complex scalars, respectively.

Here, V^{4+2} is the $6D$ volume, Tr denotes the trace over the chosen $U(N)$ representation and $W_a \equiv W_a(y, y)$ is the Wilson loop. We also find that fields in representations sensitive to the 't Hooft flux, as, for example, the fundamental one, do not help in removing the degeneracy among the infinity of $U(N)$ vacua. This can be clearly seen for fermions in the fundamental representation if one observes that the spectrum in Eq. (45) does not contain any dependence on the SS phases. This in turn implies that the one-loop effective action is independent of such parameters and therefore the contribution is only a (divergent) constant, that is, vacuum energy.

Summarizing, only representations for which the commutator of covariant derivatives is zero help in removing the degeneracy among the infinity of $U(N)$ vacua. While in the case of trivial 't Hooft flux, $m = 0$, all representations fall in this category, for nontrivial 't Hooft flux, $m \neq 0$, only representations insensitive to the center of the $U(N)$ gauge group influence the determination of the true vacuum.

A. The $m \neq 0$ case in detail

We concentrate now on the one-loop effective potential for the case of nontrivial 't Hooft flux, $m \neq 0$. The main purpose here is to use the general formulas previously derived and point out similarities and differences with respect to the case, commonly treated in the literature, of trivial 't Hooft flux $m = 0$. In order to simplify the discussion, the background *symmetric* gauge is used. In such a gauge, indeed, the vacuum gauge configurations are trivial

[5] For an approach similar to the one used in this paper see, for instance, [6].

[6] The general quadratic fluctuation operators for gauge bosons and ghosts are $g_{\mu\nu}D^2 + R_{\mu\nu} - 2igF_{\mu\nu}$ and D^2, respectively.

and the $SU(N)$ part of the twists are constant matrices coinciding with the Wilson loops, see Eq. (15).

It is possible to show that the discrete part of the Wilson loops only affects the overall scale of the one-loop effective action but not its shape. Consider, for example, the contribution due to gauge and ghost fluctuating fields. In this case, the trace appearing in Eq. (48) can be reduced to

$$\mathrm{Tr}[V_1^{w_1}V_2^{w_2}] = \sum_{\rho,\sigma}\omega_1^{w_1}\omega_2^{w_2} \cdot \sum_{k_\Delta,\Delta} e^{2\pi i/\tilde{N}[(s_1w_1+s_2w_2)\Delta+(t_1w_1+t_2w_2)k_\Delta]}. \tag{50}$$

Furthermore one can easily prove that

$$\sum_{k_\Delta,\Delta} e^{2\pi i/\tilde{N}[(s_1w_1+s_2w_2)\Delta+(t_1w_1+t_2w_2)k_\Delta]} = \begin{cases}\tilde{N}^2 & \text{if } w_1=\tilde{N}n_1, w_2=\tilde{N}n_2 \\ 0 & \text{otherwise}\end{cases}.$$

Therefore the effective potential contribution for gauge and ghosts is simply

$$\Gamma_{(1)}^{g+gh} = -4\tilde{N}^2\frac{V^{4+2}}{\pi^3}\sum_{n_1,n_2\neq 0}\frac{\mathrm{Tr}[\omega_1^{\tilde{N}n_1}\omega_2^{\tilde{N}n_2}]}{[(\tilde{N}l_1n_1)^2+(\tilde{N}l_2n_2)^2]^3}. \tag{51}$$

From the previous result one can notice that the effective potential depends only on the continuous parameters contained in the twists and on $\tilde{m}$, but it does not depend on the specific choice made for the discrete parameters s_a, t_a compatible with the constraint in Eq. (23). Consequently, the resulting one-loop gauge mass spectrum will depend only on the value of the SS phases and on $\tilde{m}$. Also at one-loop level, two different sets of s_a, t_a (for a fixed $\tilde{m}$) represent only different parametrizations of the same vacuum. Concerning gauge and ghost contributions, Eq. (51) shows clearly also that, apart from an overall scale, a $U(N)$ theory with nontrivial 't Hooft flux m coincides with the case of a $U(\mathcal{K})\subset U(N)$ theory on a torus with lengths given by $L_a=\tilde{N}l_a$ and with commuting periodicity conditions given by $\omega_a^{\tilde{N}}$. This is the expected symmetry according to the previous tree-level analysis.

In order to make more explicit the previous statements, we discuss now the particular example of a symmetry breaking pattern $SU(N)\to SU(\mathcal{K})$ with $\mathcal{K}=2$.

Adopting the standard notation used in the literature [13], where only the $m=0$ case was treated, one can rewrite the one-loop effective potential contribution to gauge and ghost Γ^{g+gh} of Eq. (51) as

$$\Gamma_{(1)}^{g+gh} = -8\tilde{N}^2\frac{V^{4+2}}{\pi^3}\times\Bigg\{2\sum_{n_1,n_2=1}\frac{\cos(2\pi\tilde{N}n_1\alpha_1)\cos(2\pi\tilde{N}n_2\alpha_2)}{[(\tilde{N}l_1n_1)^2]+(\tilde{N}l_2n_2)^2]^3} + \sum_{n_1=1}\frac{\cos(2\pi\tilde{N}n_1\alpha_1)}{(\tilde{N}l_1n_1)^6}+\sum_{n_2=1}\frac{\cos(2\pi\tilde{N}n_2\alpha_2)}{(\tilde{N}l_2n_2)^6}\Bigg\}, \tag{52}$$

with the weights for the adjoint of $SU(2)$ equal to ± 1. As one expects, for an $SU(2)$ gauge group, the effective potential depends on the two SS (continuous) parameters: $\alpha 1$, $\alpha 2$. The only remnant of the original group and of the symmetry breaking driven by the nontrivial 't Hooft flux m is the presence of the coefficient $\tilde{N}=N/\mathcal{K}$. This term modifies the periodicity of the effective action and consequently it may change the location of the stable one-loop minima of the effective potential.

The effective potential for matter fields in any given $SU(2)$ representation can be obtained in a similar manner. Each positive weight q of the representation contributes to the effective potential with a term

$$\begin{aligned}\Gamma_{(1)}^{f,s} &= -2\eta_{f,s}\tilde{N}^2\frac{V^{4+2}}{\pi^3}\times\Bigg\{2\sum_{n_1,n_2=1}\frac{\cos(2q\pi\tilde{N}n_1\alpha_1)\cos(2q\pi\tilde{N}n_2\alpha_2)}{[(\tilde{N}l_1n_1)^2]+(\tilde{N}l_2n_2)^2]^3} + \sum_{n_1=1}\frac{\cos(2q\pi\tilde{N}n_1\alpha_1)}{(\tilde{N}l_1n_1)^6}+\sum_{n_2=1}\frac{\cos(2q\pi\tilde{N}n_2\alpha_2)}{(\tilde{N}l_2n_2)^6}\Bigg\}, \\ &\equiv \eta_{f,s}\Gamma_{(1)}^q,\end{aligned} \tag{53}$$

and the total contribution is found as a sum over q. The effective potential is periodic in α_i with period $1/\tilde{N}$ or $2/\tilde{N}$, depending on whether q is integer or half-integer. Notice that the gauge/ghosts contribution, Eq. (52), could have been obtained from Eq. (53) setting $q=1$ (adjoint representation), with an additional factor of 2 with respect to the complex scalar field contribution accounting for the extra degrees of freedom [see Eqs. (48) and (49)].

Since the potential is periodic we can easily find its extrema. In particular the point ($\alpha_1=0$, $\alpha_2=0$) is stable. In Fig. 1 we show, for exemplification, the effective potential for a theory with an original $SU(4)$ gauge symmetry broken down explicitly to $SU(2)$ by a $m=2$ 't Hooft flux and including, as matter fields, one $6D$ Weyl fermion in the **5** representation of $SU(2)$ and one $6D$ complex scalar in the adjoint representation of $SU(2)$. In plotting the result, for definiteness, we assumed $l_1=l_2=l$ and set the volume factor $\pi^3l^6=1$. The effective potential depicted in Fig. 1 has a minimum for $\alpha_1=\alpha_2=0.1184$. For this value of the SS parameters α_i the $SU(2)$ symmetry is dynamically broken to $U(1)$ by the usual (rank-preserving) Hosotani mechanism.

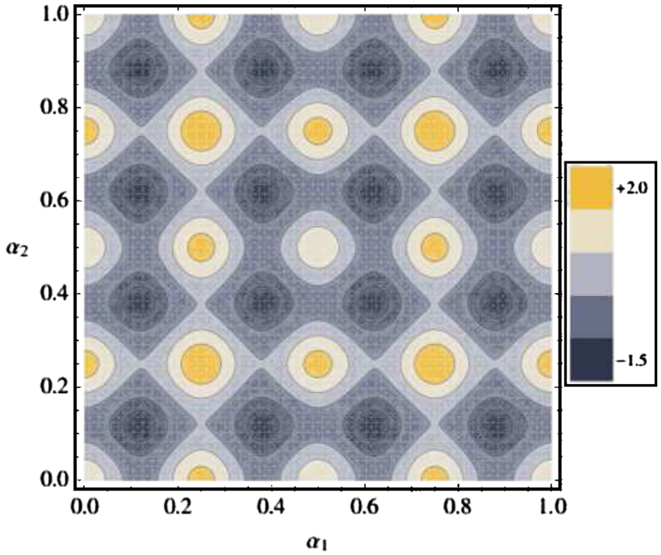

FIG. 1 (color online). Effective potential for the toy model discussed, as function of the phases α_1, α_2. Lighter (darker) regions indicate maximum (minimum) of the effective potential.

This constitutes an example of the double symmetry breaking mechanism outlined in Sec. II. The first symmetry breaking is explicit and can be thought as the mechanism breaking the grand unified theory symmetry group to the SM gauge group, while the second dynamical (spontaneous) symmetry breaking could be seen as the EW symmetry breaking of the SM. An intrinsic problem of this mechanism is due to the fact that the two scales at which these breakings occur are connected to the same geometry factor $M \approx 1/l$. Nevertheless the spontaneous symmetry breaking depends explicitly on the weight of the fields in the $SU(2)$ representation, while the 't Hooft breaking does not. So higher weights provide smaller values for the phases α_i and consequently a smaller value for the EW symmetry breaking scale.

B. Reducible adjoint representations

We want to exemplify the results previously obtained focusing on those representations that are not sensitive to the center of the group, considering, in particular, tensor products of adjoint representations. Let $i_R, j_R, \ldots$ be indices running from 1 to the dimension R of such representation. With $|i_R\rangle$ we represent the states that diagonalize the action of the boundary conditions V_a. In the case of the adjoint representation, the indices and the parameters appearing in Eq. (28), ρ, σ, Δ, k_Δ, are now functions of i_R and the equation has consequently to be rewritten as

$$V_a|i_{\rm adj}\rangle = \exp\left\{\frac{2\pi i}{\tilde{N}}(s_a\Delta^{(i)} + t_a k_\Delta^{(i)}) + 2\pi i(\alpha_a \cdot q_{(\rho_i,\sigma_i)})\right\} \times |i_{\rm adj}\rangle. \quad (54)$$

The important fact is that the action of the V_a over the product of any number of adjoint representations is already diagonal because of this choice of basis. If we take, for definiteness, the case of the product of two adjoint representations it turns out that, although clearly we can not say which combination of $|i_{\rm adj}\rangle|j_{\rm adj}\rangle$ belongs to this or that irreducible representation, we can nevertheless say how they transform under the simple diagonal action of the V_a, namely,

$$\begin{aligned} V_a(|i_{\rm adj}\rangle|j_{\rm adj}\rangle) &= V_a|i_{\rm adj}\rangle \times V|j_{\rm adj}\rangle \\ &= \exp\left\{\frac{2\pi i}{\tilde{N}}(s_a(\Delta^{(i)} + \Delta^{(j)}) + t_a(k_\Delta^{(i)} + k_\Delta^{(j)})) \right. \\ &\quad \left. + 2\pi i(\alpha_a \cdot (q_{(\rho_i,\sigma_i)} + q_{(\rho_j,\sigma_j)}))\right\}|i_{\rm adj}\rangle|j_{\rm adj}\rangle. \end{aligned} \quad (55)$$

In other words, we can obtain the spectrum without the need of identifying each field with its irreducible representation. The spectrum for the matter fields belonging to the product of two adjoint representations reads

$$\begin{aligned} m_{i,j}^2 = 4\pi^2 \sum_{a=1}^{2}\bigg(& n_a + \frac{1}{\tilde{N}}(s_a(\Delta^{(i)} + \Delta^{(j)}) + t_a(k_\Delta^{(i)} + k_\Delta^{(j)})) \\ & + \alpha_a \cdot (q_{(\rho_i,\sigma_i)} + q_{(\rho_j,\sigma_j)})\bigg)^2. \end{aligned} \quad (56)$$

After the symmetry breaking, the residual symmetry of the theory is $SU(\mathcal{K})$. The masses coming from each $SU(\mathcal{K})$ representation can be clearly identified by their weights, formed by adding the weights of the adjoint $q_{(\rho_i,\sigma_i)} + q_{(\rho_j,\sigma_j)} = q_{ij}$.

As the effective potential is a function of the mass eigenvalues alone, Eq. (56) tells us that in order to find the effective potential for a product of adjoints it is sufficient to substitute in the contribution of a single field the sum of all the adjoint representations in the product. This can be implemented rigorously in our formalism by allowing the Green function to wear two pairs of gauge indices, one pair for each adjoint representation.

The main point here is that, although we have in fact deduced the contribution to the effective potential of the reducible representation formed by the product of two adjoints, one can identify each term with one of the irreducible components through their weights. Therefore we argue that the contribution to the effective potential of any irreducible representation that can be obtained as a component of some product of adjoints is completely determined by the representation weights and given by the formula Eq. (51) where the ω's carry the weight information.

C. Scalar fields mass splitting

An interesting aspect of the one-loop analysis is related to the radiative contribution to the masses of the 4D scalars that arise from the extra components of the gauge fields. It is well known [21] that gauge and scalar masses obtained through a non singular toroidal compactification are degenerate. In fact, regardless of the Lorentz indices, the square masses are given by the eigenvalues of the operator

$$m^2 \equiv -\mathcal{D}^2 = -(\mathcal{D}_1^2 + \mathcal{D}_2^2), \tag{57}$$

where $\mathcal{D}_a$ are the covariant derivatives with respect to a fixed stable background compatible with the periodicity conditions. As seen before, the covariant derivatives for an adjoint representation always satisfy $[\mathcal{D}_1, \mathcal{D}_2] = 0$. The fact that the operator in Eq. (57) does not depend on the $4D$ Lorentz indices, implies that in the $4D$ effective theory, one should always find at least a scalar degenerate with any gauge field. The discussion of the scalar masses, however, is a delicate issue and it needs some additional comments.

In case of an unbroken gauge symmetry the extra-dimensional scalar fields A_a can be expanded in terms of usual Kaluza-Klein modes $A_a^{(\vec{n},k)}(x)$. Integrating over the torus surface, one can build the following combinations of the $4D$ scalar degrees of freedom:

$$A^{(\vec{n},k)}(x) = \frac{1}{\sqrt{m^2_{(\vec{n},k)}}}(m_{(n_1,k)}A_2^{(\vec{n},k)}(x) - m_{(n_2,k)}A_1^{(\vec{n},k)}(x)), \tag{58}$$

$$a^{(\vec{n},k)}(x) = \frac{1}{\sqrt{m^2_{(\vec{n},k)}}}(m_{(n_1,k)}A_1^{(\vec{n},k)}(x) + m_{(n_2,k)}A_2^{(\vec{n},k)}), \tag{59}$$

with k the index of the adjoint representation and $m_{(n_a,k)} = 2\pi n_a/l_a$ the usual KK mass term. It can be easily shown that while the field $A^{(\vec{n},k)}(x)$ takes a mass given by

$$m^2_{(\vec{n},k)}A^{(-\vec{n},k)}(x)A^{(\vec{n},k)}(x) = (m^2_{(n_1,k)} + m^2_{(n_2,k)})A^{(-\vec{n},k)}(x)A^{(\vec{n},k)}(x), \tag{60}$$

the orthogonal combination $a^{(\vec{n},k)}(x)$ remains massless.

The $4D$ scalars $a^{(\vec{n},k)}(x)$ are coupled to the $4D$ gauge fields by a derivative coupling. Having the quantum numbers of the current associated to the broken gauge symmetry the scalars $a^{(\vec{n},k)}$ can be seen as the pseudo-Goldstone bosons associated to the compactification symmetry breaking (from $6D$ to $4D$). The fields $a^{(\vec{n},k)}$ with $n \neq 0$ are absorbed by the corresponding KK gauge bosons that acquire a KK mass term leaving unchanged the counting of total degrees of freedom.

In case of nontrivial boundary conditions the previous formula can be straightforwardly modified and the corresponding mass terms, $m_{(n_a,k)}$, read from Eq. (20) or Eq. (29) depending on the value of the 't Hooft flux m. Notice that now the index k in Eqs. (60) and (59) runs over the indices of the Cartan-Weyl basis of Eq. (18) for the $m = 0$ case, while for the $m \neq 0$ case, k represents the set of indices $(\Delta, k_\Delta, \rho, \sigma)$ characterizing the basis in Eq. (28). For any broken symmetry there is a physical scalar field with a mass $m^2_{(\vec{n},k)} = m^2_{(n_1,k)} + m^2_{(n_2,k)} \neq 0$, degenerate with the associated gauge boson plus a massless pseudo-Goldstone boson. Instead, for gauge and scalar fields associated to generators of conserved symmetry, $m^2_{(0,k)} = 0$, and consequently there are two massless (and physical) scalars, $A^{(0,k)}(x)$ and $a^{(0,k)}(x)$ degenerate with the associated gauge field. In the $m = 0$ case, these zero modes arise from the scalars associated to the generators of the $SU(N)$ Cartan subalgebra while in the $m \neq 0$ case they are associated to the generators of the Cartan subalgebra of $SU(\mathcal{K}) \in SU(N)$.

However, the presence of such massless scalar degrees of freedom is, in general, an unwanted feature for obvious phenomenological reasons. Luckily, these scalars associated to the conserved symmetries receive a mass term from loop contributions. One can directly check this fact by taking the second derivative of the effective potential with respect to the continuous SS parameters α_i and evaluating it at the minimum. The reason why these masses are not forbidden by gauge invariance can be seen by writing all the gauge-invariant effective operators that can appear at one-loop level. Let us work for definiteness in the *symmetric gauge*. Then the fields $A^{(k)}(x, y)$, with k belonging to the Cartan subalgebra of $SU(N)$ [or $SU(\mathcal{K})$ if $m \neq 0$] are periodic in the extra dimensions. Gauge transformations $U = e^{i\beta\cdot H}$ with $\beta_k(x, y)$ a periodic function in the y coordinates preserve the residual gauge invariance

$$\begin{aligned} A_a^{(k)}(x, y) &\rightarrow (UA_a(x, y)U^\dagger)_k + \frac{i}{g}(U\partial_a U^\dagger)_k \\ &= A_a^{(k)}(x, y) - \frac{1}{g}\partial_a\beta_k(x, y). \end{aligned} \tag{61}$$

Now, the following class of operators:

$$O_n = c_n \mathrm{Tr}\bigg(\int dy_1 dy_2 A_a^{(k)}(x, y)\bigg)^n, \qquad \forall n \in \mathbb{N}, \tag{62}$$

are gauge invariant for any transformation of the form in Eq. (61) with periodic $\beta_k(x, y)$. In particular, the operator with $n = 2$, represents a gauge-invariant mass term for the scalar fields.

So, while in the tree-level Lagrangian, locality and gauge invariance forbid any mass terms for the $6D$ gauge bosons at one-loop order, instead, new nonlocal and gauge-invariant operators appear in the effective action, some of them playing the role of $4D$ scalar mass terms. For this to happen it is fundamental to work with non-simply connected manifolds. In the case of a space-time of the type $\mathcal{M}_4 \times \mathcal{T}^2$, the nonlocal operators are associated to the noncontractible cycles of $\mathcal{T}^2$ and they can only contain the extra-components of a $6D$ gauge boson, A_a. Therefore, only these can take a mass whereas the ordinary components A_μ do not.

V. CONCLUSIONS

The Hosotani mechanism is a very interesting symmetry breaking mechanism that arises in models defined in non-simply connected space-times, in which one has to specify the periodicity conditions of fields around the noncontractible cycles. It has been frequently applied in extra-dimensional model building to surrogate the SM electroweak symmetry breaking. While in $5D$ models, $M_4 \times S^1$, the Hosotani mechanism completely describes the symmetry breaking pattern, in higher-dimensional compactifications an additional ingredient has to be taken into account: the 't Hooft (non-Abelian) flux. This flux appears as a consistency condition once we impose that the value of the gauge field has to be independent of the path which has been followed to reach the starting point after wrapping the noncontractible loops, up to a constant element of the center of the group. For this to be nontrivial one clearly needs at least two non-simply connected extra dimensions and thus we have focused in the case of a two-torus, that is $M_4 \times \mathcal{T}^2$.

On the other hand, we have selected $U(N)$ as the gauge group for two phenomenological reasons. First, even when the 't Hooft flux is nonvanishing the theory admits the presence of fields in the fundamental representation. Second, since the 't Hooft flux is intimately related to the existence of a constant background magnetic flux for the $U(1) \subset U(N)$, it induces $4D$ chirality for fundamental fermions through the usual mechanism [18]. This is important because in $\mathcal{T}^2$ all stable $SU(N)$ background configurations are trivial [22] and therefore the non-Abelian piece of the group could not do the job.

In this scenario, the symmetry breaking pattern for a $U(N)$ gauge theory strongly depends on an integer parameter $m = 0, \ldots, N-1$. For trivial values of the 't Hooft flux, $m = 0$, one recovers the "usual" Hosotani mechanism with two different nonintegrable phases. This breaking is rank preserving because the Cartan subalgebra always remains unbroken. In the case of nonvanishing 't Hooft flux, $m \neq 0$, two different processes occur simultaneously: an explicit symmetry breaking associated to the nonvanishing flux and a spontaneous and dynamical one, associated to the Hosotani mechanism. The explicit breaking due to the flux can reduce the rank of the group and thus has a different phenomenology than the previous one.

In this paper we have, for the fist time, completely described the Hosotani mechanism in the presence of a nontrivial 't Hooft flux. In particular, we have calculated the mass spectrum both for the gauge fields and associated scalars and for fermions in different representations. Because of its sensitivity to the center of $U(N)$, the nature of the fermionic spectrum for the fundamental representation is peculiar. We have mentioned the possibility of obtaining chiral four-dimensional matter. The discussion of how fermions get masses and mix is, however, beyond the scope of this paper.

A well-known fact of the Hosotani mechanism is the degeneracy of the vacuum at tree level, and this is inherited in our model. A study of radiative corrections is therefore customary for obtaining both the true vacuum with the surviving symmetry and the values of the masses. With this aim, we have computed the one-loop effective potential for the general case of nonvanishing 't Hooft flux. We have found a very compact form in terms of the corresponding Wilson loops that can be particularized to the desired representation. Notice that for $m \neq 0$, matter in a representation sensitive to the center of the group does not help in removing the degeneracy since its contribution to the effective potential is a constant independent of the parameters that characterize the pattern of symmetry breaking.

It seems to us that the connection between the 't Hooft and the Hosotani mechanisms offers new and very interesting possibilities for model builders. In fact, in this framework, a double symmetry breaking can occur, without having the need to introduce any additional structure. The first symmetry breaking is explicit and can be thought, for example, as the mechanism breaking the grand unified theory symmetry to the SM gauge group, while the second dynamical (spontaneous) symmetry breaking can be seen as the EW symmetry breaking of the SM. Of course some extra work is needed in order to obtain a phenomenologically viable model.

APPENDIX A: WAVE FUNCTIONS IN FUNDAMENTAL REPRESENTATION

In this Appendix we explicitly compute the wave function of a field, belonging to the $U(N)$ fundamental representation and living on a $2D$ torus with specific $U(N)$ periodicity conditions represented by the twists $T_a(y)$. In particular, we expand previous results to include the case of nontrivial 't Hooft flux.

Let $\Psi^{(p)}(y)$ be the solution of the harmonic oscillator eigenvalue problem:

$$a^\dagger a \Psi^{(p)}(y) = p\Psi^{(p)}(y), \qquad p \in \mathbb{N}, \tag{A1}$$

with the creation and annihilation operators as defined in Eq. (44). The wave function $\Psi^{(p)}(y)$ satisfies the following periodicity conditions:

$$\Psi^{(p)}(y + \ell_a) = e^{\epsilon_{ab} i\pi(m/N)(y_b/l_b)} \omega_a P^{s_a} Q^{t_a} \Psi^{(p)}(y), \tag{A2}$$

where we have expressed the general $U(N)$ twists in the symmetric gauge in terms of the 't Hooft matrices P and Q, using the definitions in Eq. (22). As in the standard harmonic oscillator case, it is possible to compute first the zero mode, satisfying $a\Psi^{(0)}(y) = 0$ and, subsequently, obtain all the higher modes by recursively applying the creation operator, $a^\dagger$. In the rest of the Appendix we will uniquely concentrate in deriving the zero mode and consequently, from now on, we will drop the index 0. The wave function

$\Psi(y)$ can be decomposed in $\tilde{N}$ components:

$$\Psi(y) \equiv (\psi_1(y), \ldots, \psi_j(y), \ldots, \psi_{\tilde{N}}(y))^T,$$

where $\psi_j(y)$ are $\mathcal{K}$-dimensional vectors of components $\psi_j(y) \equiv (\psi_{j,1}(y), \ldots, \psi_{j,\mathcal{K}}(y))^T$. Equation (A2) written in components of the $\tilde{N}$ representation reads[7]:

$$\psi_j(y + \ell_1) = e^{i\pi(\tilde{m}/\tilde{N})(y_2/l_2)}\omega_1 e^{i\pi(1-\tilde{N}/\tilde{N})}e^{2\pi i1/\tilde{N}(j-1)}\psi_j(y), \tag{A3}$$

$$\psi_j(y + \ell_2) = e^{-i\pi(\tilde{m}/\tilde{N})(y_1/l_1)}\omega_2 e^{i\pi\tilde{m}(\tilde{N}-1/\tilde{N})}\psi_{j+\tilde{m}}(y). \tag{A4}$$

The standard trick to diagonalize such periodicity conditions consists in repeating $\tilde{N}$ times the fundamental shift of length l_a. Introducing the following (diagonal) $\mathcal{K} \times \mathcal{K}$ phase matrices:

$$e^{2\pi i\hat{\gamma}_1} = e^{i\pi(1-\tilde{N})}\omega_1^{\tilde{N}}, \qquad e^{2\pi i\hat{\gamma}_2} = e^{i\pi\tilde{m}(\tilde{N}-1)}\omega_2^{\tilde{N}}, \tag{A5}$$

and defining $L_a = \tilde{N}l_a$ and $d = \tilde{m}\,\tilde{N}$ we have the new periodicity conditions

$$\psi_j(y + \tilde{N}\ell_a) = e^{i\pi d\epsilon_{ab}(y_b/L_b)}e^{2\pi i\hat{\gamma}_a}\psi_j(y). \tag{A6}$$

The next step is finding the harmonic oscillator zero mode. A possible ansatz for the wave function $\psi_j(y)$, compatible with the periodicity condition along the direction y_1 is

$$\psi_j(y) = \sum_{n=-\infty}^{\infty} e^{i\pi d(y_1y_2/L_1L_2)}e^{2\pi i(y_1/L_1)(n+\hat{\gamma}_1)}C_{j,n}(y_2) \quad \text{for } j = 1, \ldots, \tilde{N}. \tag{A7}$$

Here $C_{j,n}(y_2)$ are $\mathcal{K}$ dimensional functions of the y_2 coordinate. Furthermore, the condition along the direction y_2, Eq. (A6) imposes that the coefficients $C_{j,n}(y_2)$ must satisfy

$$C_{j,n}(y_2 + L_2) = e^{2\pi i\hat{\gamma}_2}C_{j,n+d}(y_2). \tag{A8}$$

The explicit expression for the coefficients $C_{j,n}(y_2)$ is obtained substituting Eq. (A7) in $a\Psi = 0$ and solving the differential equation. We obtain

$$C_{j,n}(y_2) = e^{-(\pi d/L_1L_2)y_2^2}e^{-2\pi(n+\hat{\gamma}_1)y_2/L_1}A_{j,n}. \tag{A9}$$

The coefficients $A_{j,n}$ are then determined using Eq. (A8), implying

$$A_{j,n+d} = e^{-2\pi(L_2/L_1)(n+\hat{\gamma}_1+d/2)}e^{-2\pi i\hat{\gamma}_2}A_{j,n}, \tag{A10}$$

whose solution is

$$A_{j,n} = e^{-(\pi/d)(L_2/L_1)n^2}e^{-2\pi i(\hat{\gamma}_2-i(L_2/L_1)\hat{\gamma}_1)n/d}B_{j,n}, \quad \text{with} \quad B_{j,n+d} = B_{j,n}. \tag{A11}$$

[7]For definiteness, we will consider here the case $s_1 = t_2 = 0$, $t_1 = -1$, and $s_2 = \tilde{m}$. Any other choice of the coefficients s_a, t_a satisfying the constraint of Eq. (23) is of course equivalent.

Therefore, there exist only d independent solutions for the zero mode of each component ψ_j. We will characterize them by the integer number $q = 0, \ldots, d-1$. All in all, the lightest wave function j-th component can be written as

$$\psi_j(y) = \sum_{q=0}^{d-1} f_q(y)B_{j,q}, \qquad \sum_{q=0}^{d-1}|B_{j,q}|^2 = 1, \tag{A12}$$

where $B_{j,q}$ are, for each j, d arbitrary ($\mathcal{K}$ dimensional vector) coefficients, properly normalized, and $f_q(y)$ are the d independent ($\mathcal{K} \times \mathcal{K}$ matrix) eigenfunctions given by

$$f_q(y) = \left(\frac{2d}{L_1^3L_2}\right)^{1/4} e^{(\pi id/L_1L_2)y_2(y_1+iy_2)}e^{2\pi i\hat{\gamma}_1/L_1(y_1+iy_2)} \times \sum_{n=-\infty}^{\infty} e^{-\pi dL_2/L_1(n+q/d)^2} \times e^{-2\pi i(\hat{\gamma}_2-i(L_2/L_1)\hat{\gamma}_1-((y_1+iy_2)d/L_2))(n+q/d)}. \tag{A13}$$

Notice that at this state, the solutions $f_q(y)$ do not depend explicitly on the index $j = 1, \ldots, \tilde{N}$. They do depend, implicitly, on the index $k = 1, \ldots, \mathcal{K}$ through the phase matrices $\hat{\gamma}_a$ that are diagonal $\mathcal{K} \times \mathcal{K}$ matrices (in general not proportional to the identity).

We must now work backwards to recover the wave function on the original $l_1 \times l_2$ torus. Substituting the solution Eqs. (A12) and (A13) in Eqs. (A3) and (A4) we find that, in order to be compatible with the reduced torus, the following two conditions must be satisfied:

$$\sum_{q=0}^{d-1} e^{2\pi i(q/\tilde{N})}f_qB_{j,q} = e^{2\pi i(j-1/\tilde{N})}\sum_{q=0}^{d-1} f_qB_{j,q}, \tag{A14}$$

$$\sum_{q=0}^{d-1} f_qB_{j+\tilde{m},q} = \sum_{q=0}^{d-1} f_{q+\tilde{m}}B_{j,q}. \tag{A15}$$

The condition, Eq. (A14), is satisfied only if $q = q'\tilde{N} + j - 1$, with $q' = 0, 1, \ldots, \tilde{m} - 1$. So, as expected, there are only $\tilde{m}$ independent ($\mathcal{K}$ dimensional) solutions in the original torus instead of the $d = \tilde{m}\,\tilde{N}$ allowed in the extended one. Using Eq. (A14) and the fact that $q = q + d$ and $\tilde{N}/\tilde{m}$ cannot be integers, one obtains that Eq. (A15) is satisfied only if $B_{j,q} = B_q$, i.e. the B_q are j-independent constant ($\mathcal{K}$ dimensional) coefficients.

Finally, the zero-mode solution of the eigenvalue problem in Eq. (A1) with the periodicity conditions in Eq. (A3) and (A4) is given by

$$\Psi^{(0)}(y) = (\psi_1^{(0)}(y), \ldots, \psi_j^{(0)}(y), \ldots, \psi_{\tilde{N}}^{(0)}(y))^T,$$

$$\psi_j^{(0)}(y) = \sum_{q=0}^{\tilde{m}-1} f_{q\tilde{N}+j-1}(y)B_q.$$

There are in total m degrees of freedom. Notice that the explicit symmetry breaking $SU(N) \to SU(\mathcal{K})$ due to the 't Hooft flux is made explicit through the j-index dependence of the wave functions $f_{q\tilde{N}+j-1}(y)$, that localize the solutions at different points of the torus. In the case in which all $SU(\mathcal{K})$ continuous phases α_a are zero, these degrees of freedom form $\tilde{m}$ independent fundamental representations of $U(\mathcal{K})$: in this case indeed $(f_q)_{11} = (f_q)_{22} = \ldots = (f_q)_{\mathcal{K}\mathcal{K}}$. On the contrary, for nontrivial phases α_a, different entries of the fundamental $U(\tilde{N})$ representation may have a different wave function. Notice that the $U(\mathcal{K})$ breaking manifests itself only in the form of a wave function: the eigenvalues of the number operator $a^\dagger a$ (and consequently the effective $4D$ masses) are completely determined by the commutation rules in Eq. (40) and they do not depend on the $SU(\mathcal{K})$ continuous phases.

ACKNOWLEDGMENTS

We are indebted for very useful discussions to E. Alvarez, M. B. Gavela, and M. García-Pérez. The work of A. F. Faedo, D. Hernandez, and S. Rigolin has been partially supported by CICYT through the project FPA2006-05423 and by CAM through the project HEPHACOS, P-ESP-00346. D. Hernández acknowledges financial support from the MEC through FPU Grant No. AP-2005-3603. S. Rigolin acknowledges also the partial support of an Excellence Grant of Fondazione Cariparo and of the European Programme "Unification in the LHC era" under Contract No. PITN-GA-2009-237920 (UNILHC).

APPENDIX B: THE HEAT KERNEL AND THE EFFECTIVE ACTION: THE COMPUTATION

The heat kernel is a very efficient way of calculating quantum effects in field theories defined on general manifolds[8] and the reason lies in its intimate connection with the one-loop effective action. The latter is, in general, a divergent quantity that requires regularization and a very elegant way of doing so is using ζ-function techniques [34]. In that formalism, the basic equation for computing the one-loop ($\overline{\text{MS}}$) renormalized effective potential is

$$\Gamma^{\text{ren}}_{(1)}(\mu) = -\tfrac{1}{2}\zeta'_\Delta(0) - \tfrac{1}{2}\log\mu^2\zeta_\Delta(0), \qquad \text{(B1)}$$

where μ is an appropriate regularization scale. The ζ function is related to the heat kernel $G(t)$ by a Mellin transformation

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}G(t) = \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\int d^{4+d}x G(x,x;t). \qquad \text{(B2)}$$

It follows that the so-called heat function $G(x, x_0; t)$ is the only ingredient we need to compute the effective potential. This function is a solution to the heat equation $\Delta_x G(x, x_0; t) = -\frac{\partial}{\partial t}G(x, x_0; t)$ where Δ is the operator in the quadratic part of the action, usually resulting from the expansion around an arbitrary background field. In addition we have to impose suitable initial conditions $G(x, x_0; t = 0) = \delta^{4+d}(x - x_0)$. Calculating the heat function instead of the heat kernel will be necessary to capture the nonlocal nature of the contributions we are looking for.

The previous reasoning applies independently of the manifold considered. Now, suppose that the $\{y_a, a = 1, \ldots, d\}$ coordinates describe an extra-dimensional compact manifold. Then, at the level of the action, it is possible to expand the fields in harmonics of this manifold to get a four-dimensional theory with an infinite number of modes. Each of these KK modes has its own quadratic operator, for example, in our case

$$\Delta_n = -\partial_\mu\partial^\mu + M_n^2, \qquad \text{(B3)}$$

where M_n^2 are the eigenvalues of the operator acting on the extra-dimensional coordinates. This term is perceived in four dimensions as a mass, different for each mode. Hence, it is intuitive to compute the contribution $\Gamma^n_{(1)}$, associated to Eq. (B3), of a single mode to the effective potential and then add up the infinite tower, hoping that

$$\Gamma_{(1)} = \sum_n \Gamma^n_{(1)}. \qquad \text{(B4)}$$

For a finite number of fields, this relation is safe. Unfortunately, the case of an infinite number of modes is much more delicate. For instance, it has been observed several times [32] that, in general, the UV divergences and counterterms computed in the complete manifold $\delta\Gamma_{(1)}$ do not coincide with the ones obtained after summing the counterterms due to each particular mode, i.e.,

$$\delta\Gamma_{(1)} \neq \sum_n \delta\Gamma^n_{(1)}. \qquad \text{(B5)}$$

In this respect, we are not aware of precise statements about finite or nonlocal contributions to the effective action. Having this in mind, we will perform the computation according to the two prescriptions implicit in Eq. (B4).

Let us start with the right-hand side, that is, solving the heat equation for an operator of the form Eq. (B3) with the usual four-dimensional $\delta^4(x - x_0)$ as the initial condition. The form of the heat function for a flat Laplacian with a mass term is well known [33]. Using Eqs. (B1) and (B2), it leads to the effective action

$$\Gamma^n_{(1)\text{ren}} = -\frac{V^4}{(4\pi)^2}(M_n^2)^2\left(\frac{3}{4} - \frac{1}{2}\log\frac{M_n^2}{\mu^2}\right). \qquad \text{(B6)}$$

Up to this point, we have not particularized the form of the spectrum M_n^2, but we must in order to evaluate the infinite sum. However, it is easy to check that the nonlocal (and finite) contribution to the one-loop effective action

[8] We will consider here only the flat manifold case, but all the formalism can be easily extended to curved ones. See for example [33] for an extensive review on the subject.

comes only from 6D fields which have a vanishing covariant derivatives commutator and therefore are insensitive to the 't Hooft flux. On the contrary, fields in representations with a nonvanishing commutator give only a divergent constant, independent of the symmetry breaking parameters and irrelevant for determining the true vacuum. This should be clear from the absence of SS phases in the spectrum of fermions in the fundamental representation, Eq. (45).

Consequently, in the following we will concentrate only on the first type of 4D degrees of freedom. For such 4D fields, the tree-level squared masses read

$$M^2_{n(k)} = 4\pi^2 \sum_{a=1}^{2} (n_a + w_a^{(k)})^2 \frac{1}{l_a^2}, \tag{B7}$$

where (k) is a representation index and $w_a^{(k)}$ contains all continuous parameters characterizing the $U(N)$ vacua and appearing in the periodicity conditions and/or in the background (if we are not in the "symmetric gauge"). They are related to Wilson loops winding once the two noncontractible cycle of the torus as follows:

$$[W_a(y,y)]_{ik} = (\mathcal{P} e^{ig\int_y^{y+l_a} B_b dy^b} T_a)_{ik} \equiv e^{2\pi i w_a^{(k)}} \delta_{ik}. \tag{B8}$$

Summing the effective potential, Eq. (B6), for each 4D mode of the form in Eq. (B7) we are led to the evaluation of two series. For the sake of simplicity in the previous equations and in the following lines we drop the index (k) from the formulas. The first series gives

$$\sum_{n_1,n_2} \left(\sum_{a=1}^{2} (n_a + w_a)^2 \frac{4\pi^2}{l_a^2} \right)^2 = \frac{V^2}{2\pi} \frac{1}{\xi^{4+d/2}} \bigg|_{\xi=0}. \tag{B9}$$

We see that the first contribution to the 4D effective potential is independent of the continuous parameters appearing in the background and in the periodicity conditions. It yields a divergence proportional to the volume. The calculation of the second series proceeds in a similar way:

$$\sum_{n_1,n_2} \left(\sum_{a=1}^{2} (n_a + w_a)^2 \frac{4\pi^2}{l_a^2} \right)^2 \log \sum_{a=1}^{2} \frac{4\pi^2 (n_a + w_a)^2}{l_a^2 \mu^2} = -\frac{V^2}{2\pi} \int_0^\infty \frac{dt}{t^3} - \frac{64 V^2}{\pi} \sum_{m_1,m_2 \neq 0} \frac{W_1^{m_1} W_2^{m_2}}{[(l_1 m_1)^2 + (l_2 m_2)^2]^3}. \tag{B10}$$

The first term in the last line is the divergent contribution from the zero mode and, consequently, it is proportional to the volume but independent of the continuous parameters characterizing the $U(N)$ vacua. The second term is the finite contribution we are interested in.

Obviating the parameter-independent terms, the contribution to the one-loop effective action of a 6D degree of freedom with the 4D spectrum $M^2_{n(k)}$ of the form Eq. (B7) is

$$(\Gamma^{\text{ren}}_{(1)})_k = -\frac{V^{4+2}}{\pi^3} \sum_{m_1,m_2 \neq 0} \frac{\text{Tr}(W_1^{m_1} W_2^{m_2})}{[(l_1 m_1)^2 + (l_2 m_2)^2]^3}. \tag{B11}$$

Particularizing the trace to the desired representation of both Lorentz and gauge group indices one gets the effective potential used in the main body of the paper.

As a final check, we will repeat the computation but without any reference to the spectrum of the reduced theory, that is, solving directly the heat equation in 6D. As we have mentioned, trapping nonlocal physics with the heat kernel is not an easy task. For this reason, we will consider only the more tractable case of vanishing 't Hooft flux, where a "symmetric" gauge is fully accessible. In this particular gauge, the content of the theory is completely displaced to the nontrivial constant periodicity conditions while the background field can be switched off.

Our path to obtain the relevant contributions will be to reflect the desired periodicity of the torus in the initial conditions. For another attempt along similar lines see [26]. Consider the following ansatz for the extra-dimensional delta:

$$\delta^{\mathcal{T}_2}(y - y_0) \equiv \sum_{m_a=-\infty}^{\infty} \delta^2(y - y_0 + m \cdot \ell) T_1^{m_1} T_2^{m_2}, \tag{B12}$$

where we use $m \cdot \ell$ as the short-hand notation for the coordinate shift $m_1\ell_1 + m_2\ell_2$. The extra-dimensional coordinates, y_a, are defined in the fundamental domain of the torus, $y \in [0, l_a)$. The δ^2 appearing on the right-hand side is the usual Dirac delta defined in the covering space $\mathbb{R}^2$. The integers m_a are the winding numbers that account for how many times one has to wind around the cycle a in order to connect the coordinates y and $y + w \cdot \ell$ in the covering space. One gets a factor of the twist for each of these windings. Their presence in the initial condition ensures the desired periodicity of the heat function and therefore of the effective potential, as well as their gauge invariance. This expression makes sense since the twists are point independent and commute in the absence of flux.[9]

Now we can to solve the heat equation with the desired initial conditions. Let us consider the contribution to the one-loop effective potential due to a field in a generic representation of $U(N)$. In the symmetric gauge the operator is again a flat Laplacian and the heat function is guessed from the usual one incorporating the needed periodicity

[9]This ansatz is inspired in studies of the heat kernel in finite temperature field theories, in which Euclidean time is compactified into a circle. The heat function can be expressed as an infinite sum of zero temperature (that is, uncompactified) heat kernels as shown in [35]. Our initial condition is a generalization to nontrivial twists.

$$G(\{x,y\},\{x_0,y_0\},t) = \sum_{m_1,m_2} \frac{1}{(4\pi t)^3} \times e^{-(1/4t)[(x-x_0)^2+(y-y_0+m\cdot\ell)^2]} T_1^{m_1} T_2^{m_2}. \tag{B13}$$

From this solution, the associated ζ function is

$$\zeta_\Delta(s) = \frac{V^{4+2}}{(4\pi)^3\Gamma(s)}\left[\frac{t^{s-3}}{s-4}\Big|_{t=0}^{t=\infty} + \sum_{m_1,m_2\neq 0} \mathrm{Tr}(W_1^{m_1}W_2^{m_2}) \times \int_0^\infty dt t^{s-4} e^{-(1/4t)\sum_{a=1}^2 (l_a m_a)^2}\right],$$

where V^{4+2} is the $6D$ volume, Tr denotes the trace over the chosen $U(N)$ representation, and we have used Eq. (15) to write the Wilson loop.

The first term in Eq. (A14) comes from the $m_1 = m_2 = 0$ contribution and it is divergent. The zero winding numbers case corresponds, in fact, to local operator contributions and it is independent of the continuous $U(N)$ SS parameters. For m_1 and/or m_2 different from zero, the integral and the sum in the second term converge and so they can be safely interchanged. This contribution, in fact, proceeds from the Wilson loops that wrap around the noncontractible cycles of the torus at least once. Consequently the effective action is given by

$$\Gamma_{(1)}^{\mathrm{ren}} = -\frac{V^{4+2}}{\pi^3} \sum_{w_1,w_2\neq 0} \frac{\mathrm{Tr}(W_1^{w_1}W_2^{w_2})}{[(l_1w_1)^2 + (l_2w_2)^2]^3}. \tag{B14}$$

A comparison with the previous result obtained from the $4D$ spectrum shows immediately that the higher-dimensional and dimensionally reduced computations of the finite part of the effective action actually agree. Notice that this is not in contradiction with the statements of [32] since there nonlocal sectors were not considered. Conversely, here we have discarded the local UV divergent contributions studied in those works.

---

[1] L. Susskind, Phys. Rev. D **20**, 2619 (1979).

[2] C. T. Hill, S. Pokorski, and J. Wang, Phys. Rev. D **64**, 105005 (2001); N. Arkani-Hamed, A. G. Cohen, and H. Georgi, Phys. Lett. B **513**, 232 (2001).

[3] D. B. Fairlie, Phys. Lett. **82B**, 97 (1979); J. Phys. G **5**, L55 (1979); N. S. Manton, Nucl. Phys. **B158**, 141 (1979); P. Forgacs and N. S. Manton, Commun. Math. Phys. **72**, 15 (1980).

[4] M. Luscher, Nucl. Phys. **B219**, 233 (1983).

[5] Y. Hosotani, Phys. Lett. **126B**, 309 (1983); Y. Hosotani, Phys. Lett. **129B**, 193 (1983); Y. Hosotani, Ann. Phys. (N.Y.) **190**, 233 (1989); Y. Hosotani, arXiv:hep-ph/0408012; arXiv:hep-ph/0504272.

[6] J. E. Hetrick and C. L. Ho, Phys. Rev. D **40**, 4085 (1989).

[7] A. T. Davies and A. McLachlan, Phys. Lett. B **200**, 305 (1988); A. Mclachlan, Phys. Lett. B **222**, 372 (1989); **237**, 650(E) (1990); A. McLachlan, Nucl. Phys. **B338**, 188 (1990).

[8] M. Burgess and D. J. Toms, Phys. Lett. B **234**, 97 (1990).

[9] H. Hatanaka, T. Inami, and C. S. Lim, Mod. Phys. Lett. A **13**, 2601 (1998).

[10] Y. Hosotani, N. Maru, K. Takenaga, and T. Yamashita, Prog. Theor. Phys. **118**, 1053 (2007); Y. Hosotani, arXiv:hep-ph/0607064;

[11] A. Higuchi and L. Parker, Phys. Rev. D **37**, 2853 (1988); M. Kubo, C. S. Lim, and H. Yamashita, Mod. Phys. Lett. A **17**, 2249 (2002); G. Burdman and Y. Nomura, Nucl. Phys. **B656**, 3 (2003); C. Scrucca, M. Serone, and L. Silvestrini, Nucl. Phys. **B669**, 128 (2003); N. Haba, M. Harada, Y. Hosotani, and Y. Kawamura, Nucl. Phys. **B657**, 169 (2003); N. Haba, Y. Hosotani, and Y. Kawamura, Prog. Theor. Phys. **111**, 265 (2004); N. Haba, Y. Hosotani, Y. Kawamura, and T. Yamashita, Phys. Rev. D **70**, 015010 (2004); N. Haba and T. Yamashita, J. High Energy Phys. 04 (**2004**) 016; C. S. Lim and N. Maru, Phys. Lett. B **653**, 320 (2007).

[12] C. Csaki, C. Grojean, L. Pilo, and J. Terning, Phys. Rev. Lett. **92**, 101802 (2004); Y. Nomura, J. High Energy Phys. 11 (2003) 050; G. Burdman and Y. Nomura, Phys. Rev. D **69**, 115013 (2004); Y. Hosotani and M. Mabe, Phys. Lett. B **615**, 257 (2005); Y. Hosotani, K. Oda, T. Ohnuma, and Y. Sakamura, Phys. Rev. D **78**, 096002 (2008).

[13] C. Csaki, C. Grojean, and H. Murayama, Phys. Rev. D **67**, 085012 (2003); C. Scrucca, M. Serone, L. Silvestrini, and A. Wulzer, J. High Energy Phys. 02 (2004) 049; Y. Hosotani, S. Noda, and K. Takenaga, Phys. Lett. B **607**, 276 (2005); C. S. Lim, N. Maru, and K. Hasegawa, J. Phys. Soc. Jpn. **77**, 074101 (2008); C. S. Lim and N. Maru, Phys. Rev. D **75**, 115011 (2007).

[14] Y. Hosotani, S. Noda, and K. Takenaga, Phys. Rev. D **69**, 125014 (2004).

[15] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Nucl. Phys. **B261**, 678 (1985); **B274**, 285 (1986).

[16] E. Witten, "*Fermion Quantum Numbers In Kaluza-Klein Theory*," 1983 (unpublished).

[17] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **125B**, 136 (1983); C. G. Callan and J. A. Harvey, Nucl. Phys. **B250**, 427 (1985).

[18] S. Randjbar-Daemi, A. Salam, and J. Strathdee, Nucl. Phys. **B214**, 491 (1983).

[19] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, Phys. Rev. Lett. **54**, 502 (1985).

[20] G. Aldazabal, L. E. Ibanez, F. Quevedo, and A. M. Uranga, J. High Energy Phys. 08 (2000), 002; G. Aldazabal *et al.*, J. High Energy Phys. 02 (2001) 047; C. P. Burgess, D. Hoover, C. de Rham, and G. Tasinato, J. High Energy Phys. 03 (2009) 124; I. Antoniadis, A. Kumar, and B. Panda, Nucl. Phys. **B823**, 116 (2009).

[21] J. Alfaro *et al.*, J. High Energy Phys. 01 (2007) 005.
[22] M. Salvatori, J. High Energy Phys. 06 (2007) 014.
[23] J. Ambjorn and H. Flyvbjerg, Phys. Lett. **97B**, 241 (1980).
[24] N. K. Nielsen and P. Olesen, Nucl. Phys. **B144**, 376 (1978); **B79**, 304 (1978); J. Ambjorn, N. K. Nielsen, and P. Olesen, Nucl. Phys. **B152**, 75 (1979).
[25] G. 't Hooft, Nucl. Phys. **B153**, 141 (1979); Commun. Math. Phys. **81**, 267 (1981).
[26] G. von Gersdorff, Nucl. Phys. **B793**, 192 (2008); J. High Energy Phys. 08 (2008) 097.
[27] H. Abe, T. Kobayashi, and H. Ohki, J. High Energy Phys. 09 (2008) 043; H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, Nucl. Phys. **B814**, 265 (2009).
[28] J. Scherk and J. H. Schwarz, Nucl. Phys. **B153**, 61 (1979); Phys. Lett. **82B**, 60 (1979).
[29] D. R. Lebedev, M. I. Polikarpov, and A. A. Roslyi, Nucl. Phys. **B325**, 138 (1989).
[30] A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D **27**, 2397 (1983).
[31] L. Giusti *et al.*, Phys. Rev. D **65**, 074506 (2002); A. Gonzalez-Arroyo and A. Ramos, J. High Energy Phys. 07 (2004) 008.
[32] E. Alvarez and A. F. Faedo, J. High Energy Phys. 05 (2006) 046; E. Alvarez and A. F. Faedo, Phys. Rev. D **74**, 124029 (2006); V. P. Frolov, P. Sutton, and A. Zelnikov, Phys. Rev. D **61**, 024021 (1999).
[33] D. V. Vassilevich, Phys. Rep. **388**, 279 (2003).
[34] J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976).
[35] J. S. Dowker and R. Critchley, Phys. Rev. D **15**, 1484 (1977); J. S. Dowker, J. Phys. A **10**, 115 (1977).