

Conformal transformations of the S matrix; the β function identifies the change of spacetime

Steffen Pottel and Klaus Sibold

Institut für Theoretische Physik, Universität Leipzig, Postfach 100920, D-04009 Leipzig Germany

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First conformal transformations of the S matrix are derived in massless ϕ^4 theory. Then it is shown that the anomalous transformations can be rewritten as a symmetry once one has introduced a local coupling and interprets the charge of the symmetry accordingly. By introducing a suitable effective coupling on which the S matrix depends, one is able to identify via the β function an underlying new spacetime with a nontrivial conformal (flat) metric.

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I. INTRODUCTION

It is well known that conformal transformations realized in quantum field theories in four-dimensional spacetime as a rule are beset with anomalies. Formulated in terms of Green functions (which are off-shell quantities) they are parametrized by β functions, associated with the anomalous behavior of interaction vertices and by γ functions associated with anomalous dimensions of the fields. Exceptions were discovered mainly in the context of supersymmetric quantum field theories, notably the famous $N = 4$ super-Yang-Mills theory. The question arises, Which effects of these anomalies survive in physical quantities like S -matrix or Green functions of physical operators? A first answer has been given quite some time ago by Zimmermann [1] for the dilatations in massless ϕ^4 . In an axiomatic setting he has shown that the S operator scales with the β function

$$\kappa^2 \partial_{\kappa^2} S = \beta_{\lambda} \partial_{\lambda} S. \quad (1)$$

This can be understood as the renormalization group equation for the S operator. There is no contribution by γ . This result depends crucially on the fact that the propagator can be shown to have a pole at vanishing momentum with finite residue and therefore a field operator exists whose propagator has a pole at zero momentum with residue one. Hence the appearance of γ is a consequence of normalizing the field operator unphysically. A first attempt to extend this result to the special conformal transformations for a massive ϕ^4 theory has been undertaken in [2]. But the massless limit was not considered there, since the calculation suffered from divergences, which were too difficult to control. In the present paper we work directly in the massless limit by using the Bogoliubov-Parašiuk-Hepp-Zimmermann-Lowenstein (BPHZL) subtraction scheme. Assuming that the same normalization conditions hold as the ones Zimmermann used, we find that the special conformal transformations change the S operator according to

$$i[K_{\mu}, S] = \lim \int dx 2x_{\mu} \beta_{\lambda} \frac{\delta}{\delta \lambda(x)} S. \quad (2)$$

The term “lim” refers to *constant* coupling λ , because in

the course of deriving this result the coupling λ had been generalized to be a function of spacetime, i.e. to vary with x . Here, too, no effect of γ shows up. By putting in (2) the right-hand side to the left one can interpret the new equation as the expression for a symmetry, where the notion of charge is extended to include the external field λ . Yet another possibility of using (2) will lead to the identification of an underlying spacetime which is still flat, but has a nontrivial conformal metric. Hence one has an interacting theory which lives on a spacetime which is not Minkowski and satisfies the axioms in the sense of perturbation theory.

The paper is organized as follows. In Sec. II we recapitulate a few facts on conformal transformations for constant and for x -dependent coupling λ . In particular, we present the Ward identity (WI) for dilatations and special conformal transformations of the Green functions after the anomalies have been absorbed. In Sec. III we derive the transformation laws for the S operator. In Sec. IV these transformation laws will be reinterpreted on a transformed space (by dilatations and special conformal transformations, respectively). Section V contains discussion and conclusions.

II. DILATATIONS AND SPECIAL CONFORMAL TRANSFORMATIONS OF THE GREEN FUNCTIONS**A. The classical approximation**

The classical action

$$\Gamma_{\text{cl}} = \int d^4x \left(\frac{1}{2} \partial \phi \partial \phi - \frac{1}{4!} \lambda \phi^4 \right) \quad (3)$$

is invariant under dilatations and special conformal transformations

$$\delta^D(x, d=1)\phi = (1 + x^{\mu} \partial_{\mu})\phi, \quad (4)$$

$$\delta_{\mu}^K(x, d=1)\phi = (2x_{\mu} x^{\lambda} - \eta_{\mu}^{\lambda} x^2) \partial_{\lambda} \phi + 2x_{\mu} \phi. \quad (5)$$

Interpreting the classical action as the tree graph approximation of the generating functional Γ for vertex functions (one-particle-irreducible Green functions)

$$\Gamma_{\text{cl}} = \Gamma^{(0)} \quad (6)$$

(loop number = zero), one can express this invariance as a WI

$$W^D \Gamma^{(0)} = 0, \quad (7)$$

$$W_\mu^K \Gamma^{(0)} = 0, \quad (8)$$

where

$$W^D \equiv -i \int d^4x \delta^D \phi \frac{\delta}{\delta \phi}, \quad (9)$$

$$W_\mu^K \equiv -i \int d^4x \delta^K \phi \frac{\delta}{\delta \phi}. \quad (10)$$

In higher orders these WI's will be broken by anomalies. In order to deal with those, it is convenient to promote the coupling constant λ to an external field $\lambda(x)$, for then one can generate nonintegrated vertices by differentiation with respect to $\lambda(x)$, a process which in the BPHZL renormalization scheme is under control by the action principle. It is also very useful to construct respective currents as x moments of the improved energy-momentum tensor. This can be realized by starting from suitable WI operators (“contact terms”).

$$\tilde{w}_\nu^T \equiv \partial_\nu \phi - \frac{1}{4} \partial_\nu \left(\phi \frac{\delta}{\delta \phi} \right) + \partial_\nu \lambda \frac{\delta}{\delta \lambda}, \quad (11)$$

$$W^D[\phi, \lambda] \equiv -i \int dx x^\mu \tilde{w}_\mu^T, \quad (12)$$

$$W_\mu^K[\phi, \lambda] \equiv -i \int (2x_\mu x^\nu - \eta_\mu^\nu x^2) \tilde{w}_\nu^T. \quad (13)$$

Here the external field $\lambda(x)$ has been assigned vanishing canonical dimension and assumed to be $\lambda(x) \in \mathcal{S}(\mathbb{R}^4)$.

B. Higher orders

For quantization and calculation of higher orders we employ the BPHZL renormalization scheme which first of all means introducing an auxiliary mass term

$$\Gamma^{\text{mass}} = \left[\int d^4x \left(-\frac{1}{2} M^2 (s-1)^2 \phi^2 \right) \right]_4, \quad (14)$$

which results in a free propagator Δ_c

$$\Delta_c(x) = \frac{i}{(2\pi)^4} \int d^4p \frac{e^{-ipx}}{p^2 - M^2(s-1)^2 + i\varepsilon_Z}. \quad (15)$$

$$\varepsilon_Z = \varepsilon(M^2(s-1)^2 + \mathbf{p}^2)$$

is Zimmermann's epsilon which yields Euclidean minors and majorants for the momentum space integrals of Feynman diagrams. The variables s and $s-1$ participate in the subtractions like the external momenta, and

Zimmermann's epsilon leads to absolute convergence once subtractions have been properly performed. Nontrivial quantum corrections can show up when one wants to go to the massless limit $s=1$ which is possible only in expressions where $s-1$ appears outside of normal products and thus no longer participates in the subtractions. The relation between such normal products differing only in the position of the $s-1$ factors is given by an identity (due to Zimmermann) to which all non-naive deviations from, say, symmetry in the quantum theory can be traced back. The next ingredient is Γ_{eff} from which Feynman diagrams follow. In the BPHZL renormalization scheme, where $\Gamma_{\text{eff}} = \Gamma_{\text{free}} + \int \mathcal{L}_{\text{int}}$, Γ_{eff} is to be understood as a normal product with infrared and ultraviolet subtraction degree four. In the case of local coupling $\lambda(x)$, external field $h_{\mu\nu}(x)$ (to which the energy-momentum tensor couples), and quantum field ϕ it is given by [3,4]

$$\begin{aligned} \Gamma_{\text{eff}}(\phi, \lambda, h) = & \sum_{n=0}^{\infty} \int \left(-\frac{1}{2} z^{(n)} I_l^{(n)} - \frac{1}{2} M^2 (s-1)^2 I_M \delta_{n,0} \right. \\ & - \frac{1}{4!} \rho^{(n)} I_4^{(n+1)} + \frac{1}{2} \hat{c}^{(n)} I_c^{(n)} + \hat{z}^{(n)} I_1^{(n)} \\ & \left. + z_\lambda^{(n)} I_\lambda^{(n)} \right) \end{aligned} \quad (16)$$

with the basis of (4, 4) insertions

$$I_M = (-g)^{1/4} \phi, \quad (17)$$

$$I_l^{(n)} = (-g)^{3/8} g^{\mu\nu} \lambda^n \phi (\partial_\mu \partial_\nu - \Gamma_{\mu\nu}^{\nu'} \partial_{\nu'}) (-g)^{-1/8} \phi, \quad (18)$$

$$I_4^{(n)} = \lambda^n \phi^4, \quad (19)$$

$$I_c^{(n)} = (-g)^{1/4} \lambda^n R \phi^2, \quad (20)$$

$$I_1^{(n)} = (-g)^{1/2} g^{\mu\nu} \lambda^{n-1} \partial_\mu \lambda \partial_\nu ((-g)^{-1/4} \phi^2), \quad (21)$$

$$I_\lambda^{(n)} = (-g)^{1/4} g^{\mu\nu} \lambda^{n-2} \partial_\mu \lambda \partial_\nu \lambda \phi^2, \quad (22)$$

$$\hat{I}_k^{(n)} = (-g)^{1/2} g^{\mu\nu} \frac{1}{2} (\partial_\mu \partial_\nu - \Gamma_{\mu\nu}^{\nu'} \partial_{\nu'}) ((-g)^{-1/4} \lambda^n \phi^2), \quad (23)$$

$$\hat{I}_2^{(n)} = (-g)^{1/2} g^{\mu\nu} \frac{1}{n} (\delta_\mu^{\nu'} \partial_\nu - \Gamma_{\mu\nu}^{\nu'}) ((-g)^{-1/4} \phi^2 \partial_\nu \lambda^n). \quad (24)$$

Here $g = \det(g_{\mu\nu})$, R is the curvature scalar, and $\Gamma_{\mu\nu}^\rho$ is the Christoffel symbol in the usual conventions. In the following this curved background spacetime will not be needed explicitly—all curvature dependent terms will vanish and the Christoffel symbols are constant in the case of dilations and special conformal transformations. For the deri-

vation of the subsequent WI's they are, however, crucial and similarly for considerations of other observables than the S matrix studied in the sequel.

As far as normalization conditions are concerned, we spell out explicitly only those relevant for $h_{\mu\nu} = 0$ and $\lambda = \text{const}$

$$\partial_{p^2} \Gamma_{\phi\phi} \Big|_{p^2 = -\kappa^2} = 1, \quad \Gamma_{\phi\phi\phi\phi} \Big|_{p=p(\text{symm})} = -\lambda.$$

The subtraction scheme implies that $\Gamma_{\phi\phi}$ vanishes at $p = 0$, $s = 1$. (For the complete list we refer to [4].)

Going through some algebraic analysis which is based on the presence of $h_{\mu\nu}$ and using consistency conditions, one finds (see [4]) that in all higher orders of perturbation theory the following broken WI's hold at $s = 1$:

$$W^D \Gamma[\phi, \lambda] = -i \sum_{k=1}^{\infty} \left(\int dx \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} - \int dx \hat{\gamma}^{(k)} \lambda^k(x) \phi \frac{\delta}{\delta \phi} \right) \Gamma[\phi, \lambda], \quad (25)$$

$$W_{\mu}^K \Gamma[\phi, \lambda] = -i \sum_{k=1}^{\infty} \left(\int dx 2x_{\mu} \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} - \int dx 2x_{\mu} \hat{\gamma}^{(k)} \lambda^k(x) \phi \frac{\delta}{\delta \phi} \right) \Gamma[\phi, \lambda], \quad (26)$$

$$\beta_{\lambda}(x) \equiv \sum_{k=1}^{\infty} \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x), \quad (27)$$

$$\gamma(x) \equiv \sum_{k=1}^{\infty} \hat{\gamma}^{(k)} \lambda^k(x). \quad (28)$$

It is then tempting to absorb the right-hand sides of the WI's into new WI operators and to generate homogeneous WI's

$$\hat{W}^D[\phi, \lambda] \equiv W^D[\phi, \lambda] + i \sum_{k=1}^{\infty} \left(\int dx \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} - \int dx \hat{\gamma}^{(k)} \lambda^k(x) \phi \frac{\delta}{\delta \phi} \right), \quad (29)$$

$$\hat{W}^D \Gamma[\phi, \lambda] = 0, \quad (30)$$

$$\hat{W}_{\mu}^K[\phi, \lambda] \equiv W_{\mu}^K[\phi, \lambda] + i \sum_{k=1}^{\infty} \left(\int dx 2x_{\mu} \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \times \frac{\delta}{\delta \lambda(x)} - \int dx 2x_{\mu} \hat{\gamma}^{(k)} \lambda^k(x) \phi \frac{\delta}{\delta \phi} \right), \quad (31)$$

$$\hat{W}_{\mu}^K \Gamma[\phi, \lambda] = 0. \quad (32)$$

A good reason to proceed into this direction stems from the

remarkable fact, already stated in [4], that these new WI operators fit to the moment construction and satisfy the conformal algebra. The derivatives with respect to $\lambda(x)$ generate insertions; hence one can in the limit of constant λ understand the terms added to the original WI operators as nonlinear field transformations which complete the linear transformations to formally true symmetries of the system. We may thus expect conserved currents and associated charges operating on the Hilbert space of the theory, involving, however, the external field λ .

To pave the way to the charge operators, one needs the WI's formulated on the generating functional of the general Green functions.

In a first step one goes over to the connected Green functions by Legendre transformation

$$Z_c(J) = \Gamma(\phi) + \int J\phi, \quad (33)$$

$$-J = \frac{\delta \Gamma}{\delta \phi}, \quad (34)$$

and then in a second step to general Green functions

$$Z = e^{iZ_c}.$$

The WI's become

$$\hat{W}^D[J, \lambda] \equiv - \int dx J(x) \delta^D(x, d=1) \frac{\delta}{\delta J(x)} + \int dx \delta^D(x, d=0) \lambda(x) \frac{\delta}{\delta \lambda(x)} \quad (35)$$

$$- \sum_{k=1}^{\infty} \left(\int dx \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} + \int dx \hat{\gamma}^{(k)} \lambda^k(x) J(x) \frac{\delta}{\delta J(x)} \right), \quad (36)$$

$$\hat{W}^D Z[J, \lambda] = 0, \quad (37)$$

$$\hat{W}_{\mu}^K[J, \lambda] \equiv - \int dx J(x) \delta_{\mu}^K(x, d=1) \frac{\delta}{\delta J(x)} + \int dx \delta_{\mu}^K(x, d=0) \lambda(x) \frac{\delta}{\delta \lambda(x)} \quad (38)$$

$$- \sum_{k=1}^{\infty} \left(\int dx 2x_{\mu} \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} + \int dx 2x_{\mu} \hat{\gamma}^{(k)} \lambda^k(x) J(x) \frac{\delta}{\delta J(x)} \right), \quad (39)$$

$$\hat{W}_{\mu}^K Z[J, \lambda] = 0. \quad (40)$$

As above for the one-particle-irreducible Green functions, they express the formal invariance of general Green functions under these generalized, nonlinear transformations.

The WI's hold at $s = 1$, the massless theory. The respective Green functions exist as Lorentz invariant distributions for nonexceptional momenta, the vertex functions as Lorentz invariant functions for nonexceptional momenta. At values $s \neq 1$ the WI's are broken by soft mass contributions vanishing in the deep asymptotic Euclidean region. As long as the coupling is local there is no infrared problem anyway.

III. TRANSFORMATION LAWS FOR THE S OPERATOR

Starting point for the subsequent analysis is the S operator as defined by

$$S =: \Sigma : Z|_{J=0}, \quad (41)$$

$$\Sigma = \exp\left\{\int dx dy \phi_{\text{in}}(x) r^{-1} K(x-y) \frac{\delta}{\delta J(y)}\right\}, \quad (42)$$

$$Z[J] = \frac{\exp\{i \int \mathcal{L}_{\text{int}}(\frac{\delta}{i\delta J})\} \exp\{\frac{1}{2} \int dx dy i J(x) \Delta_c(x-y) i J(y)\}}{[\exp\{i \int \mathcal{L}_{\text{int}}(\frac{\delta}{i\delta J})\} \exp\{\frac{1}{2} \int dx dy i J(x) \Delta_c(x-y) i J(y)\}]|_{J=0}}. \quad (43)$$

Here, as above, $Z[J]$ denotes the generating functional for general Green functions, $\phi_{\text{in}}(x)$ the asymptotic field (which is free), r the wave function renormalization constant, and $K(x-y) = \square + M^2(s-1)^2$ the inverse wave operator [5]. In the massless $\lambda\phi^4$ theory we have

$$\Sigma = \exp\left\{\int dx \phi_{\text{in}}(x) r^{-1} \square_x \frac{\delta}{\delta J(x)}\right\}. \quad (44)$$

Σ amputates the external legs and when evaluating the integral, ϕ_{in} puts the external momenta on the mass shell.

Suppose now that a WI operator W^A generates a symmetry $W^A Z(J) = 0$, then one can apply a standard method (see [6,7]) to find a symmetry of the S operator in Hilbert space by establishing the following identity via integration by parts:

$$[W^A, : \Sigma :] Z|_{J=0} = [Q^A, : \Sigma :] Z|_{J=0}. \quad (45)$$

Here the left-hand side is a calculation in the functional space, whereas the right-hand side is a calculation in Hilbert space with Q^A representing the charge operator generating the transformation in question

$$i[Q^A, \phi_{\text{in}}(x)] = \delta^A \phi_{\text{in}}(x). \quad (46)$$

In simple cases the left-hand side of (45) vanishes because in the first contribution to the commutator one uses the WI, and in the second contribution one uses that at $J = 0$ the WI operator vanishes. The right-hand side represents the commutator of the charge with the S operator which is thus zero; hence the S operator is symmetric with respect to the transformation generated by A . In the present case this holds e.g. true for the Lorentz transformations and the translations. For the dilatations and special conformal transformations the situation is, however, more complicated: the $M^2(s-1)^2$ terms in Σ disturb the relation (45), the reason for working with a Σ at vanishing mass, i.e. at $s = 1$ within Σ and taking the propagators of the external legs also at $s = 1$.

A. Dilatations

Taking into account the remark at the end of the preceding paragraph, we calculate the commutator of Σ at $s = 1$ with \hat{W}^D and put also in the external legs of the Green functions $s = 1$. We obtain (Wick dots omitted)

$$\hat{W}^D(\Sigma Z)|_{J=0, s=1} = \left(\int dx \delta^D(x, d=0) \lambda(x) \frac{\delta}{\delta \lambda(x)} - \sum_{k=1}^{\infty} \int dx \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} \right) S, \quad (47)$$

where we performed as indicated the limit $J = 0$ in the WI operator. This is the nontrivial contribution of the commutator in the functional space and represents eventually the breaking of the dilational symmetry.

The commutator

$$[\hat{W}^D, \Sigma] Z|_{J=0, s=1} = \left\{ \int dx \phi_{\text{in}}(x) r^{-1} \square_x \delta^D(x, d=1) \frac{\delta}{\delta J(x)} \right. \quad (48)$$

$$\left. - \int dy \delta^D(y, d=0) \lambda(y) \times \int dx \phi_{\text{in}}(x) r^{-1} \left(\frac{\delta}{\delta \lambda(y)} \ln(r_x) \right) \square_x \frac{\delta}{\delta J(x)} \right. \quad (49)$$

$$\left. + \sum_{k=1}^{\infty} \int dy \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(y) \times \int dx \phi_{\text{in}}(x) r^{-1} \left(\frac{\delta}{\delta \lambda(y)} \ln(r_x) \right) \square_x \frac{\delta}{\delta J(x)} \right. \quad (50)$$

$$\left. + \sum_{k=1}^{\infty} \int dx \phi_{\text{in}}(x) r^{-1} \square_x \left(\hat{\gamma}^{(k)} \lambda^k(x) \frac{\delta}{\delta J(x)} \right) \right\} \Sigma Z|_{J=0, s=1} \quad (51)$$

is to stand for the commutator of the charge operator D^{op} with the S operator. Because of the special $s = 1$ prescrip-

tion (48) becomes

$$\begin{aligned} & \int dx \phi_{\text{in}}(x) r^{-1} \square_x \delta^D(x, d=1) \frac{\delta}{\delta J(x)} \\ &= - \int dx \delta^D(x, d=1) \phi_{\text{in}}(x) r^{-1} \square_x \frac{\delta}{\delta J(x)}. \end{aligned} \quad (52)$$

If we now perform the limit $\lambda(x) \rightarrow \lambda$ at least the terms with derivatives of the coupling will vanish, especially $\delta^D(x, d=1)\lambda(x)$. Hence (49) will not contribute. But similarly

$$\hat{W}^D[J, \lambda](\Sigma Z)_{|J=0, s=1} = - \sum_{k=1}^{\infty} \int dx \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} S. \quad (53)$$

In addition, we use a statement to be found in [1]. Zimmermann proved there that in a specific domain of the coupling constant, i.e. between two stationary points, it is possible to normalize on-shell even for the massless theory and the S -matrix elements exist. We choose the lowest possible regime (smallest coupling) in order to fulfill the requirements of perturbation theory and with the on-shell normalization conditions for the propagator and the collinear point normalization for the coupling we are able to set the wave function renormalization constant (power series in λ) equal to one and find

$$\ln r_x = \ln 1 = 0. \quad (54)$$

Hence (49) and (50) vanish. (The transition to the collinear point normalization for the coupling is only a finite renormalization, compatible with the subtraction scheme, maintaining the WI; the propagator normalizations of [1] and ours agree anyway.)

Zimmermann also stated that the anomalous dimension $\gamma = 0$ after normalizing on-shell. We show this in an explicit calculation for every order. Inserting

$$\phi_{\text{in}}(x) = \int \frac{d^3 q}{(2\pi)^{3/2} \sqrt{2\omega_q}} (a_q e^{-iqx} + a_q^\dagger e^{iqx}), \quad (55)$$

$$\lambda^k(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \Lambda(p), \quad (56)$$

$$\Delta_c(x-y) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2 + i\epsilon_Z} \quad (57)$$

into (51), translating \square_x into Fourier space and performing the coordinate space integration we get

$$\int dx \phi_{\text{in}}(x) \square_x (\lambda^k(x) \Delta_c(x-y)) \quad (58)$$

$$\begin{aligned} &= \int \frac{d^3 q}{(2\pi)^{3/2} \sqrt{2\omega_q}} \frac{d^4 p d^4 k}{(2\pi)^4} e^{iky} (a_q \delta^{(4)}(q+k+p) \\ &+ a_q^\dagger \delta^{(4)}(k+p-q)) \frac{-(k+p)^2}{k^2 + i\epsilon_Z} \Lambda(p) \end{aligned} \quad (59)$$

$$\begin{aligned} &= \int \frac{d^3 q}{(2\pi)^{3/2} \sqrt{2\omega_q}} \frac{d^4 p}{(2\pi)^4} \left(a_q e^{-i(p+q)y} \frac{-q^2}{(q+p)^2 + i\epsilon_Z} \right. \\ &+ \left. a_q^\dagger e^{i(p-q)y} \frac{-q^2}{(q-p)^2 + i\epsilon_Z} \right) \Lambda(p) \end{aligned} \quad (60)$$

$$= 0 \quad (61)$$

since $\phi_{\text{in}}(x)$ is the asymptotic (free) field of the theory and sits on-mass-shell, i.e. $q^2 = 0$. In addition, we know that the denominators lead to well-defined distributions and do not contain any ‘‘hard’’ singularities:

$$\lambda \in \mathcal{S}(\mathbb{R}^4) \Rightarrow \lambda^k \in \mathcal{S}(\mathbb{R}^4) \Rightarrow \Lambda \in \mathcal{S}(\mathbb{R}^4). \quad (62)$$

Collecting all arguments we obtain

$$\begin{aligned} & \int dx \delta^D(x, d=1) \phi_{\text{in}}(x) r^{-1} \square_x \frac{\delta}{\delta J(x)} \Sigma Z_{|J=0, s=1} \\ &= \sum_{k=1}^{\infty} \int dx \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} S. \end{aligned} \quad (63)$$

Recalling (46) and (48) we have

$$i[D^{\text{op}}, S] = \lim_{\lambda \rightarrow \text{const}} \sum_{k=1}^{\infty} \int dx \hat{\beta}_{\lambda}^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} S, \quad (64)$$

where D^{op} is the same charge of the dilatations as in the case of constant coupling. Because of

$$\kappa^2 \partial_{\kappa^2} \Gamma = iW^D \Gamma \quad (65)$$

the commutator (64) is nothing but the renormalization group equation, (1) for the S operator. Zimmermann derived this renormalization group equation axiomatically under the assumption that perturbation theory and the non-perturbative theory are related to each other in the sense of an asymptotic series where all derivatives on, say, Green functions etc., with respect to the coupling in the non-perturbative setting go over to the perturbative regime for small enough coupling. For the present perturbative derivation of the same relation this implies that after one has arranged the on-shell normalization conditions for the propagator (residue = 1, pole at $p = 0$) and collinear normalization for the coupling, maintaining the scheme independent WI's cancellations of potential infrared divergences occur in sums of diagrams which might not be easily seen. In the one-loop approximation, where only one diagram contributes there must not occur an infrared divergence. This is indeed true as we show in the Appendix.

B. Special conformal transformations

For the special conformal transformations we can follow the derivation of the preceding subsection line by line.

$$[\hat{W}_\mu^K, \Sigma]Z|_{J=0, s=1} = \hat{W}_\mu^K(\Sigma Z)|_{J=0, s=1} - \Sigma(\hat{W}_\mu^K Z)|_{J=0, s=1}, \quad (66)$$

$$\begin{aligned} \hat{W}_\mu^K(\Sigma Z)|_{J=0, s=1} &= \left(\int dx \delta_\mu^K(x, d=0) \lambda(x) \frac{\delta}{\delta \lambda(x)} \right. \\ &\quad \left. - \sum_{k=1}^{\infty} \int dx 2x_\mu \hat{\beta}_\lambda^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} \right) S, \end{aligned} \quad (67)$$

$$[\hat{W}_\mu^K, \Sigma]Z|_{J=0, s=1} = \left\{ \int dx \phi_{\text{in}}(x) r^{-1} \square_x \delta_\mu^K(x, d=1) \frac{\delta}{\delta J(x)} \right. \quad (68)$$

$$\begin{aligned} &- \int dy \delta_\mu^K(y, d=0) \lambda(y) \int dx \phi_{\text{in}}(x) r^{-1} \left(\frac{\delta}{\delta \lambda(y)} \ln(r_x) \right) \square_x \\ &\times \frac{\delta}{\delta J(x)} \end{aligned} \quad (69)$$

$$\begin{aligned} &+ \sum_{k=1}^{\infty} \int dy 2x_\mu \hat{\beta}_\lambda^{(k)} \lambda^{k+1}(y) \int dx \phi_{\text{in}}(x) r^{-1} \\ &\times \left(\frac{\delta}{\delta \lambda(y)} \ln(r_x) \right) \square_x \frac{\delta}{\delta J(x)} \end{aligned} \quad (70)$$

$$\left. + \sum_{k=1}^{\infty} \int dx \phi_{\text{in}}(x) r^{-1} \square_x \left(2x_\mu \hat{\gamma}^{(k)} \lambda^k(x) \frac{\delta}{\delta J(x)} \right) \right\} \Sigma Z|_{J=0, s=1}. \quad (71)$$

Then (68) becomes

$$\begin{aligned} &\int dx \phi_{\text{in}}(x) r^{-1} \square_x \delta_\mu^K(x, d=1) \frac{\delta}{\delta J(x)} \\ &= - \int dx \delta_\mu^K(x, d=1) \phi_{\text{in}}(x) r^{-1} \square_x \frac{\delta}{\delta J(x)}, \end{aligned} \quad (72)$$

and we arrive also at

$$\begin{aligned} \hat{W}_\mu^K[J, \lambda](\Sigma Z)|_{J=0, s=1} &\rightarrow \hat{W}_\mu^K[0, \lambda](\Sigma Z)|_{J=0, s=1} \\ &= \hat{W}_\mu^K[0, \lambda]S. \end{aligned} \quad (73)$$

In the limit of constant λ , $\lambda(x) \rightarrow \lambda$, at least the terms with derivatives of the coupling vanish, in particular, $\delta_\mu^K(x, d=1) \lambda(x)$. Hence (69) will not contribute and also

$$\begin{aligned} \hat{W}_\mu^K[J, \lambda](\Sigma Z)|_{J=0, s=1} &= - \sum_{k=1}^{\infty} \int dx 2x_\mu \hat{\beta}_\lambda^{(k)} \lambda^{k+1}(x) \\ &\times \frac{\delta}{\delta \lambda(x)} S. \end{aligned} \quad (74)$$

With the on-shell normalization conditions and in the limit of constant λ , as above

$$\ln r_x = \ln 1 = 0. \quad (75)$$

Finally we obtain

$$\begin{aligned} &\int dx \delta_\mu^K(x, d=1) \phi_{\text{in}}(x) r^{-1} \square_x \frac{\delta}{\delta J(x)} \Sigma Z|_{J=0, s=1} \\ &= \sum_{k=1}^{\infty} \int dx 2x_\mu \hat{\beta}_\lambda^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} S, \end{aligned} \quad (76)$$

which is nothing but

$$i[K_\mu, S] = \sum_{k=1}^{\infty} \int dx 2x_\mu \hat{\beta}_\lambda^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} S, \quad (77)$$

where K_μ is the same charge of the special conformal transformations as in the case of constant coupling.

Here, in the conformal case we cannot dispose over a renormalization group argument if we wish to relate our perturbative result to a possible nonperturbative one; there is, however, an algebraic argument available. The WI operator \hat{W}_μ^K is via the moment construction related to \hat{W}^D and satisfies with it the usual commutator relation

$$[\hat{W}^D, \hat{W}_\mu^K] = i\hat{W}_\mu^K$$

to all orders of perturbation theory. Therefore we expect that this relation is also valid in the nonperturbative theory and maintained in the sense of an asymptotic expansion. Hence we expect here the same cancellations of potential infrared divergences to take place as in the case of dilatations.

In any case the off-shell relations (37) and (40) for the Green functions hold.

C. Conserved transformations

In this subsection we round up the discussion ‘‘symmetry versus anomalies’’ by giving an interpretation of (64) and (77) which parallels (37) and (40).

A superficial look at the WI's (37) and (40) says that the Green functions are invariant under dilatational and special conformal transformations once those are modified to include transformations of the external field λ . In which sense can this be reconciled with (64) and (77) telling that the S matrix of ϕ^4 theory is anomalous; i.e. the respective charges do not commute with the S operator? The statement on the charges is obviously correct. We are, however, allowed to rewrite (64) and (77) as

$$\left\{ i[D, \bullet] - \lim_{\lambda \rightarrow \text{const}} \sum_{k=1}^{\infty} \int dx \hat{\beta}_\lambda^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} \right\} S = 0, \quad (78)$$

$$\left\{ i[K_\mu, \bullet] - \lim_{\lambda \rightarrow \text{const}} \sum_{k=1}^{\infty} \int dx 2x_\mu \hat{\beta}_\lambda^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} \right\} S = 0, \quad (79)$$

where we may interpret the terms containing the β function as a contribution to the respective charge. The first term acts as it should as an operator in Hilbert space, the second term acts as it can on the operator S . This is perfectly legitimate in a quantum field theory which depends on an external field; there it is standard that an S operator depends not only on quantum fields, which propagate, but also on classical fields, which do not propagate. Hence also a charge operator may depend on classical fields. Without the external field $\lambda(x)$ the charges D and K_μ could only change via one-particle singularities in the WI's. However, terms nonlinear in the quantum field do not cause such singularities in perturbation theory. And this is why the homogeneous WI version of the situation seduces to talk about symmetry which is, however, only correct after noticing that the changes for λ can indeed be understood as being changes of the charges, which can easily be read off from (78) and (79). The operator within curly brackets may be interpreted as a derivation, where the two terms act on their respective spaces, the whole space being Hilbert \otimes {external fields}.

Taking into account that both terms of the left-hand sides of (78) and (79) are changes, we may write the complete symmetry transformations for the S operator to first order in the transformation parameters as

$$\left\{ I + \epsilon \left(i[D, \bullet] - \lim_{\lambda \rightarrow \text{const}} \sum_{k=1}^{\infty} \int dx \hat{\beta}_\lambda^{(k)} \lambda^{k+1}(x) \frac{\delta}{\delta \lambda(x)} \right) \right\} S = S, \quad (80)$$

$$\left\{ I + \alpha^\mu \left(i[K_\mu, \bullet] - \lim_{\lambda \rightarrow \text{const}} \sum_{k=1}^{\infty} \int dx 2x_\mu \hat{\beta}_\lambda^{(k)} \lambda^{k+1}(x) \times \frac{\delta}{\delta \lambda(x)} \right) \right\} S = S. \quad (81)$$

The charges D and K_μ are not multiplicatively renormalized when going from classical approximation (no loop, $\beta = 0$) to nontrivial loop order, but acquire an additive change via the β -function terms.

IV. β FUNCTION IDENTIFIES CHANGES OF SPACETIME

In this section we would like to show that the transformation laws for the S operator under dilatations and special conformal transformations, respectively, admit the definition of an S operator on a spacetime which is obtained from standard Minkowski by performing the (infinitesimal) dilatation, resp. special conformal transformation with parameters ϵ , resp. α_μ .

For dilatations we have found the relation

$$i\epsilon[D, S(\lambda)] = \epsilon \sum_{k=1}^{\infty} \int dx \hat{\beta}_\lambda^{(k)} \lambda^{k+1} \frac{\delta}{\delta \lambda(x)} S(\lambda). \quad (82)$$

Let us recall how one finds the transformation law of a scalar field when performing an infinitesimal translation

$$x_\mu \rightarrow x_\mu - a_\mu.$$

Requiring that the transformed field at the new coordinate be the same as the old field at the old coordinate one finds

$$\phi(x) \rightarrow \phi(x + a) = \phi(x) + a^\mu \partial_\mu \phi(x).$$

In the same sense we write the S matrix in a transformed space, dilated Minkowski, with an effective coupling

$$\lambda_{\text{eff}} = \lambda + \epsilon \sum_k \hat{\beta}^{(k)} \lambda^{k+1}, \quad (83)$$

obtaining

$$S(\lambda) \rightarrow S\left(\lambda - \epsilon \sum_k \hat{\beta}^{(k)} \lambda^{k+1}\right) \quad (84)$$

$$= \sum_n S^{(n)}\left(\lambda - \epsilon \sum_k \hat{\beta}^{(k)} \lambda^{k+1}\right). \quad (85)$$

Expanding this order by order in \hbar we get

$$S^{(0)} = S^{(0)}(\lambda), \quad (86)$$

$$S^{(1)} = S^{(1)}(\lambda) - \epsilon \int dx \hat{\beta}^{(1)} \lambda^2 \frac{\delta}{\delta \lambda(x)} S^{(0)}(\lambda), \quad (87)$$

$$S^{(2)} = S^{(2)}(\lambda) - \epsilon \int dx \hat{\beta}^{(1)} \lambda^2 \frac{\delta}{\delta \lambda(x)} S^{(1)}(\lambda) - \epsilon \int dx \hat{\beta}^{(2)} \lambda^3 \frac{\delta}{\delta \lambda(x)} S^{(0)}(\lambda), \quad (88)$$

$$S^{(3)} = S^{(3)}(\lambda) - \dots, \quad (89)$$

⋮

Expanding the S matrix in (82) and omitting terms $\propto \epsilon^2$, i.e. $S(\lambda - \epsilon \sum_k \hat{\beta}^{(k)} \lambda^{k+1}) \rightarrow S(\lambda)$, we get

$$i\epsilon \left[D, S\left(\lambda - \epsilon \sum_k \hat{\beta}^{(k)} \lambda^{k+1}\right) \right] = i\epsilon \left[D, \sum_n S^{(n)}(\lambda) \right] \quad (90)$$

$$= \epsilon \sum_k \int dx \hat{\beta}^{(k)} \lambda^{k+1} \frac{\delta}{\delta \lambda(x)} \sum_n S^{(n)}(\lambda). \quad (91)$$

Performing the transformation we have

$$S\left(\lambda - \epsilon \sum_k \hat{\beta}^{(k)} \lambda^{k+1}\right) + i\epsilon \left[D, S\left(\lambda - \epsilon \sum_k \hat{\beta}^{(k)} \lambda^{k+1}\right) \right] = S(\lambda), \quad (92)$$

or by a shift

$$S(\lambda) + i\epsilon [D, S(\lambda)] = S(\lambda_{\text{eff}}), \quad (93)$$

where the right-hand side can be understood as an S operator on a flat space with metric $g_{\mu\nu} = (1 - 2\epsilon)\eta_{\mu\nu}$. Here ϵ is the parameter determining the dilatation in question. The nonvanishing β function triggers the transition to this space. If it were to vanish, dilatations would be realizable on the original Minkowski space as a true symmetry. We could not detect the transformation. Via the $S(\lambda_{\text{eff}})$ and its nontrivial dependence on ϵ we can, however, spot it. So, clearly one cannot identify the underlying spacetime *per se*, out of nothing, but the *change* from a four-dimensional Minkowski space to a four-dimensional dilated space is characterized by the β function.

The case of special conformal transformations can be dealt with in complete analogy,

$$S(\lambda) + i[\alpha^\mu K_\mu, S(\lambda)] = S(\lambda_{\text{eff}}), \quad (94)$$

where now λ_{eff} is given by

$$\lambda_{\text{eff}}^{\text{conf}} = \lambda + 2x^\mu \alpha_\mu \sum_k \hat{\beta}^{(k)} \lambda^{k+1}. \quad (95)$$

As above α is the infinitesimal parameter of the respective special conformal transformation.

The metric of the transformed space reads to first order in α

$$g_{\mu\nu} = (1 - 4x^\rho \alpha_\rho) \eta_{\mu\nu}. \quad (96)$$

The x dependence of effective coupling and metric have an interesting effect: ‘‘constant’’ coupling is now to be understood as ‘‘conformally constant,’’ i.e. relative to the new metric. Again the same comment applies as in the case of dilatations: due to the nonvanishing β function we can identify the (conformally) transformed spacetime departing from ordinary Minkowski space.

V. DISCUSSION AND CONCLUSIONS

In the present paper we treat two topics. The first one concerns the change of the S matrix under dilatations and special conformal transformations of the massless ϕ^4 theory in the context of perturbation theory. For dilatations Zimmermann [1] has obtained the respective result (1) in an axiomatic setting. We arrive perturbatively at the same, (64), by rendering the coupling local and using the fact that with the help of local λ all dilatational anomalies can be absorbed into a homogeneous WI (30) [4]. As long as λ is local there is at every vertex a nonvanishing external momentum; hence one can go on-shell without meeting

an infrared divergence. Following Zimmermann we assume that the nonperturbative theory is linked to the perturbative version in the sense of an asymptotic expansion. We know therefore from [1] that no infrared divergences can arise in the limit of constant coupling, once we realize perturbatively the same normalization conditions as employed by Zimmermann, which we do. The amputated on-shell Green functions are thus finite. Single diagrams may be infrared divergent, but these divergences cancel in the sum which represents the Green function. And, indeed, in the one-loop approximation which is given by just one diagram, no infrared divergence shows up.

For special conformal transformations we find the analogous result, (77). By the moment construction the special conformal transformations are closely related to the dilatations, hence the same cancellation of infrared divergences takes place, as the one-loop approximation shows explicitly.

The learned reader might object that the ϕ^4 theory is trivial. However, rigorous proofs of triviality exist only for dimensions strictly smaller or strictly larger than four. For exactly four dimensions no such proof seems to be available.

As the second topic we discuss an application of these results for the S matrix. We show that upon introduction of a suitable effective coupling an S matrix can be defined which signals the underlying transformed spacetime which is obtained from Minkowski space by dilatation, special conformal transformation, respectively (93) and (94). Crucial here is the fact that the β function does not vanish. In the case of dilatations this effective coupling agrees with the standard running coupling obtained from the renormalization group equation. The effective coupling for the special conformal case (95) is particularly interesting because it depends explicitly on x_μ . Constant coupling then means conformally constant. In our understanding we contribute with this result to a presently ongoing general program: the construction of quantum field theories on nontrivial spacetimes [8–10]. Beginning perhaps with Wald [11] it gradually became clear how to define S matrices on spacetimes which are globally hyperbolic and asymptotically flat. Here we present an explicit and nontrivial example: the perturbatively nontrivial S matrix of ϕ^4 theory in Minkowski space gets translated to a dilated, resp. conformally (flat) spacetime via the effective coupling. If it is possible to go to finite transformations one will obtain (in the conformal case) a space which is curved, conformally and asymptotically flat. If one succeeds in finding the corresponding expression for the S matrix (i.e. one integrates the effective coupling), one has a nontrivial S matrix on such a space.

The axioms which hold for $S(\lambda)$ on Minkowski space get via $S(\lambda_{\text{eff}}^{\text{conf}})$ translated on the transformed space, wherefrom its explicit realization can be read off, notably that of locality, i.e. causality.

β functions arise from Zimmermann identities among normal products. The Zimmermann coefficients encode the information on anomalies, hence when interpreted as we propose also on spacetimes.

Which extensions of these results can one expect in other theories? Of utmost interest are gauge theories. There, however, one has to cope with the fact that gauge fixing is not conformally invariant. Therefore the identification of the physical subspace of the entire Fock space is technically nontrivial. If the S matrix exists, i.e. in models with complete breakdown of the gauge symmetry, one has to separate the soft breaking of conformal invariance from hard breaking. But then one should find essentially similar results as presented here. This expectation is based on the fact that the β functions can be constructed as gauge independent quantities. If the S matrix does not exist (e.g. in pure QCD) one has to consider other observables, the energy-momentum tensor being a prime candidate. Here one has to check whether other anomalies than those related to the β function come into play. As far as the identification of an underlying spacetime is concerned, we expect in any case an analogous result to the above: anomalies of geometric symmetries identify respective spacetimes.

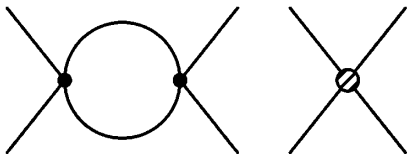
What about anomalies of internal symmetries? Here we expect a change of geometry of the internal space. The relevant anomaly coefficients are, if properly constructed, also gauge independent, hence physical. The geometry of the internal space might get a nontrivial physical meaning. These questions certainly deserve further investigation.

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APPENDIX: ONE-LOOP APPROXIMATION

In this Appendix we calculate explicitly the one-loop contribution to the scattering amplitude of two particles going into two particles and the conformal transformation of this process. There are two diagrams contributing to this S -matrix element $S_{2,2}^{(1)}$: they involve four fields and the outcome is equivalent to the four-point vertex function $\Gamma_4^{(1)}$ at the same loop order.



We expect a contribution to the breaking of conformal invariance as shown in [4], clearly for constant coupling. Since the counterterm behaves trivially under special conformal transformations, we can omit its treatment. Then the relevant relation for $S_{2,2}^{(1)}$ is

$$S_{2,2}^{(1)} = \Sigma Z_{2,2}^{(1)}[J]_{|J=0,s=1}, \quad (\text{A1})$$

$$\Sigma = \exp\{X\} = \exp\left\{\int dx \phi_{\text{in}}(x) r^{-1} \square_x \frac{\delta}{\delta J(x)}\right\}, \quad (\text{A2})$$

$$\begin{aligned} Z_{2,2}^{(1)}[J] &= \left(-i \frac{\lambda}{4!}\right)^2 \cdot 36 \cdot \int dz_1 dz_2 (\Delta_c(z_1 - z_2))^2 \\ &\quad \times \left(\int d\xi \Delta_c(z_1 - \xi) iJ(\xi)\right)^2 \\ &\quad \times \left(\int d\xi \Delta_c(z_2 - \xi) iJ(\xi)\right)^2 \cdot Z_0. \end{aligned} \quad (\text{A3})$$

By applying the Ward-Operator W_μ^K we first regain (45). (Wick dots are omitted.)

$$[\Sigma, W_\mu^K] = [X, W_\mu^K] \Sigma \quad (\text{A4})$$

$$= \int dx \phi_{\text{in}}(x) r^{-1} \square_x \delta_\mu^K(x, d=1) \frac{\delta}{\delta J(x)} \Sigma \quad (\text{A5})$$

$$= - \int dx \delta_\mu^K(x, d=1) \phi_{\text{in}}(x) r^{-1} \square_x \frac{\delta}{\delta J(x)} \Sigma \quad (\text{A6})$$

$$= -[K_\mu, \Sigma]. \quad (\text{A7})$$

In the one-loop approximation the wave function renormalization constant $r = 1$; hence the full expression of the transformed matrix element reads

$$\begin{aligned} [W_\mu^K, S_{2,2}^{(1)}] &= \frac{\lambda^2}{24} \int dz_1 dz_2 (\phi_{\text{in}}(z_1) \delta_\mu^K(z_1, d=1) \phi_{\text{in}}(z_1) \\ &\quad \times \phi_{\text{in}}^2(z_2) + \phi_{\text{in}}^2(z_1) \phi_{\text{in}}(z_2) \delta_\mu^K(z_2, d=1) \\ &\quad \times \phi_{\text{in}}(z_2)) \Delta_c^2(z_1 - z_2). \end{aligned} \quad (\text{A8})$$

In order to simplify the calculation we use the following relations,

$$\phi_{\text{in}}(x) \delta_\mu^K(x, d=1) \phi_{\text{in}}(x) = \frac{1}{2} \delta_\mu^K(x, d=2) \phi_{\text{in}}^2(x), \quad (\text{A9})$$

$$\begin{aligned} \int \Delta_c^2(x) \delta_\mu^K(x, d=2) \phi_{\text{in}}^2(x) &= - \int \phi_{\text{in}}^2(x) \delta_\mu^K(x, d=2) \\ &\quad \times \Delta_c^2(x), \end{aligned} \quad (\text{A10})$$

and obtain

$$\begin{aligned} [W_\mu^K, S_{2,2}^{(1)}] &= - \frac{\lambda^2}{48} \int dz_1 dz_2 \phi_{\text{in}}^2(z_1) \phi_{\text{in}}^2(z_2) (\delta_\mu^K(z_1, d=2) \\ &\quad + \delta_\mu^K(z_2, d=2)) \Delta_c^2(z_1 - z_2). \end{aligned} \quad (\text{A11})$$

Here Δ_c^2 is identical with $\Gamma_4^{(1)}$ and becomes well defined only after we have specified a renormalization scheme, because it is logarithmically divergent by power counting. We choose the BPHZL scheme with the propagator as given in (15) and one subtraction at $p = 0$ and $s = 0$, leading to

$$\begin{aligned} \Delta_c^2(z_1 - z_2) &= \int \frac{dp}{(2\pi)^4} e^{-ip(z_1 - z_2)} \int \frac{dk}{(2\pi)^4} \\ &\times \left(\frac{i}{k^2 - M^2(s-1)^2 + i\epsilon_Z} \right. \\ &\times \frac{i}{(p-k)^2 - M^2(s-1)^2 + i\epsilon_Z} \\ &\left. - \frac{i^2}{(k^2 - M^2 + i\epsilon_Z)^2} \right). \end{aligned} \quad (\text{A12})$$

With the Zimmermann ϵ_Z this expression is absolutely convergent. We translate $\delta_\mu^K(., d=2)$ via $\exp\{ip(z_1 - z_2)\}$ into momentum space, where monomials of dimension -1 result, i.e. derivatives with respect to the external momentum p . Hence the subtraction term vanishes and the first term becomes convergent in its own right.

We introduce Feynman parameters, which help evaluating the integrals, and obtain

$$[W_\mu^K, S_{2,2}^{(1)}] = -\frac{i\lambda^2}{24} \frac{\beta_\lambda^{(1)}}{3} \int dz_1 2z_{1,\mu} \phi_{\text{in}}^4(z_1). \quad (\text{A13})$$

Until this stage of the calculation we did not consider symmetries of the specific diagram with respect to the external lines. Noticing that the s, t, u channels give the same contribution we have a factor of 3 and the final result is

$$[W_\mu^K, S_{2,2}^{(1)}] = -\frac{i\lambda^2}{4!} \int dx 2x_\mu \beta_\lambda^{(1)} : \phi_{\text{in}}^4(x) :. \quad (\text{A14})$$

Here we reintroduced the Wick dots. We note that we did not come across any infrared divergence. This is the result for the S operator used in the main text.

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