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Quasiblack holes with pressure: General exact results

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A quasiblack hole is an object in which its boundary is situated at a surface called the quasihorizon, defined by its own gravitational radius. We elucidate under which conditions a quasiblack hole can form under the presence of matter with nonzero pressure. It is supposed that in the outer region an extremal quasihorizon forms, whereas inside, the quasihorizon can be either nonextremal or extremal. It is shown that in both cases, nonextremal or extremal inside, a well-defined quasiblack hole always admits a continuous pressure at its own quasihorizon. Both the nonextremal and extremal cases inside can be divided into two situations, one in which there is no electromagnetic field, and the other in which there is an electromagnetic field. The situation with no electromagnetic field requires a negative matter pressure (tension) on the boundary. On the other hand, the situation with an electromagnetic field demands zero matter pressure on the boundary. So in this situation an electrified quasiblack hole can be obtained by the gradual compactification of a relativistic star with the usual zero pressure boundary condition. For the nonextremal case inside the density necessarily acquires a jump on the boundary, a fact with no harmful consequences whatsoever, whereas for the extremal case the density is continuous at the boundary. For the extremal case inside we also state and prove the proposition that such a quasiblack hole cannot be made from phantom matter at the quasihorizon. The regularity condition for the extremal case, but not for the nonextremal one, can be obtained from the known regularity condition for usual black holes.

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I. INTRODUCTION

In recent years, the taxonomy of relativistic objects has increased to include the so-called quasiblack holes. The general definition and description of the general properties of these objects can be found in [1]. Here, we recall that a quasiblack hole is, roughly speaking, an object on the verge of forming a horizon but without collapsing, so the system remains static even when the boundary approaches its own gravitational radius surface, or the quasihorizon, as nearly as one likes. It turns out that nonextremal quasiblack holes are connected with the appearance of diverging surface stresses when the boundary approaches the quasihorizon, so only extremal quasiblack holes are free from infinite surface stresses.

The significance of quasiblack holes is twofold. First, it is a useful methodical tool for better understanding the general features of black holes such like the relation to black hole mimickers [2], the mass formula [3,4], and entropy [5,6]. In doing so, one should not bother about the physical realization of such construction and even admit infinite surface stresses to obtain finite final formulas for physical quantities (see [3]). Second, quasiblack holes can be of interest by themselves, as real physical objects. There are several examples of objects that exhibit quasiblack hole behavior. Simple systems, which can be treated analytically, like Bonnor stars, made of Majumdar-Papapetrou matter, i.e., extremal dust where the density of matter is equal to that of the charge so that the matter pressure is zero, matched to an extreme Reissner-Nordström vacuum, admit quasiblack holes [7–9]. Continuous Majumdar-Papapetrou systems made purely from extremal dust also admit quasiblack holes [10]. More complex structures like self-gravitating Yang-Mills–Higgs magnetic monopoles also possess quasiblack holes, as found previously in [11,12].

In [13] exact relativistic charged sphere solutions with pressure were found. Drawing upon this work on exact solutions [13] and upon previous work on charged systems with pressure [14], it was shown in [15] that there are electrically charged quasiblack holes with pressure which are obtained as limiting cases of the relativistic charged spheres of [13], namely, these quasiblack holes can be thought of as being formed when a star made of charged matter with pressure is sufficiently compressed. In the

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study [15], the corresponding models have the attractive feature that in some range of parameters the speed of sound is real and less than that of light. In [16,17], numerical work was performed on a different but similar type of relativistic charged spheres which degenerates into quasiblack holes with pressure when the spheres are sufficiently compact. The study of pressure charged systems not only extends the class of electrically charged quasiblack holes but also brings an important feature connected with the issue of stability to those systems. The point is that quasiblack holes made purely from extremal dust, are unstable with respect to a dynamic perturbation having kinetic energy. With the presence of pressure, there is the possibility of finding stable configurations. Indeed, in [17] it was found that there were instances in which the systems are stable against radial perturbations, and this might indicate that the quasiblack holes found in [15] are also stable. The self-gravitating Yang-Mills–Higgs magnetic monopole quasiblack holes studied in [11,12] can be considered as quasiblack holes with pressure since an intrinsic inbuilt effective pressure is present in the Yang-Mills–Higgs equations, and thus, might also be stable systems.

Following our previous works [1–6], we want to put forward a general model-independent approach and find the conditions under which quasiblack holes, extremal to the outside, with pressure are possible. We work with quasiblack holes that are extremal from the outside because only these are regular and free from infinite surface stresses, nonextremal quasiblack holes having diverging surface stresses [3]. The study is quite general, in the sense that the outside extremality condition can be of any type, it can be due to a specific mass to charge relation, or to a specific mass to cosmological constant relation, to name two cases among others. If, for instance, the external region is described by the Reissner-Nordström metric, its charge q is equal to mass m, $q = m$. On the other hand, from inside we allow that the quasihorizon can be either nonextremal or extremal. Nonextremal quasihorizons from the inside with matter pressure were found in [15]. Extremal quasihorizons with pressure for self-gravitating magnetic monopoles were studied in [11,12]. Our analysis includes all these systems and extends to pressure systems the pressureless cases treated in [1]. Moreover, we treat the cases in which from the outside the quasihorizon is always extremal whereas from the inside the quasihorizon can be either nonextremal or extremal.

This paper is organized as follows: In Sec. II, we write the basic formulas for a generic spherically symmetric system and for the system when it is in a state of transition to a quasiblack hole. In Sec. III, we make a deep analysis of the conditions on the radial pressure the quasihorizon of a quasiblack hole must obey in the cases where there is an nonextremal quasihorizon from the inside and an extremal quasihorizon from the inside. We also study the conditions on the energy density and make some comments related to the null energy condition. In Sec. IV, we conclude.

II. BASIC FORMULAS AND LIMITING TRANSITION

A. Basic formulas

Consider a metric $g_{\mu\nu}$ with line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ for a spherically symmetric spacetime containing matter, i.e.,

$$ds^2 = -U(r)dt^2 + V(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

The stress-energy tensor of the matter has the form

$$T_\mu{}^\nu = \mathrm{diag}(-\rho, p_r, p_\perp, p_\perp), \tag{2}$$

where ρ, p_r, and $p_\perp$ are the energy density, the radial pressure, and the tangential pressure, respectively. The Einstein equations are $G_{\mu\nu} = 8\pi T_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor and $G = 1$, $c = 1$ here. The two equations of interest are the tt and rr components. If we put

$$U(r) = V(r)\exp(2\psi(r)), \tag{3}$$

then it follows from the Einstein equations that

$$2\psi(r) = \int^r d\bar{r}\frac{\sigma(\bar{r})}{V(\bar{r})}, \tag{4}$$

where we have defined the quantity $\sigma(r)$ as

$$\sigma(r) = 8\pi r(p_r(r) + \rho(r)). \tag{5}$$

And if we put

$$V(r) = 1 - \frac{2m(r)}{r}, \tag{6}$$

then it follows that

$$m(r) = 4\pi\int_0^r d\bar{r}\bar{r}^2\rho(\bar{r}). \tag{7}$$

Here, we assume that the center $r = 0$ is a regular one, and there is no horizon *a priori*.

Let us consider a compact body situated in the inside region such that $r \leq r_0$. The radius $r = r_0$ defines the boundary which divides the inside region from the outside one. We do not specify the metric outside, for $r > r_0$. In particular, it can be the Reissner-Nordström metric. In what follows we will use subscripts "in" and "out" to distinguish quantities in each of the two regions. To match the two metrics, i.e., the first quadratic forms, at the boundary $r = r_0$, we need the condition

$$U_{\mathrm{in}}(r_0) = U_{\mathrm{out}}(r_0). \tag{8}$$

We assume that there is no massive shell on the boundary, which entails the continuity of the metric potential V,

$$V_{\mathrm{in}}(r_0) = V_{\mathrm{out}}(r_0). \tag{9}$$

In addition, without essential loss of generality, we deal with metrics for which $U_{\mathrm{out}}(r) = V_{\mathrm{out}}(r)$, since this simplifies the formulas. In particular, the Reissner-Nordström metric belongs to this class, in which case $U_{\rm out}(r) = 1 - \frac{2m(r)}{r}$, with $m(r) = m - q^2/2r$, and in the extremal case $m = q$, we are interested in one that has $m(r) = m - m^2/2r$, so that $U_{\rm out}(r) = (1 - m/r)^2$. Then, after simple manipulations, we obtain that

$$U_{\rm in}(r) = V_{\rm in}(r)\exp(2\psi(r_0, r)), \tag{10}$$

with

$$2\psi(r_0, r) = \int_{r_0}^{r} d\bar{r}\frac{\sigma(\bar{r})}{V(\bar{r})}. \tag{11}$$

We do not specify further properties beforehand, in particular, that the presence of transverse surface stresses is allowed.

B. Limiting transition

Now we make the next assumption, namely, there is a limiting transition in the course of which a horizon almost forms. From (8), one can then write

$$U_{\rm in}(r_0) = U_{\rm out}(r_0) \equiv U(r_0) = \varepsilon, \tag{12}$$

where ε is any number that can be made as small as one wants, $\varepsilon \ll 1$. Since we are interested in the limit $\varepsilon \to 0$, this means that the quantity $U(r_0) = \varepsilon$ becomes a small parameter and the areal radius r_0 approaches the radius of a would-be horizon r_+. We want to examine whether and under which condition a quasiblack hole can appear. By itself, the proximity of r_0 to r_+ is insufficient. It is also required that in the whole inner region $r \leq r_0$ the lapse function $U_{\rm in}(r) \to 0$ in such a way that

$$U_{\rm in}(r) = \varepsilon f(r), \tag{13}$$

where $f(r)$ is some bounded function. Furthermore, $f(r_+) \neq 0$. The latter condition is needed to distinguish a quasiblack hole from a true black hole. More exactly, this function must obey the condition $f(r_+) = 1$, as is seen from (12) and (13). Formally, we can also admit a nonmonotonic $f(r)$ which inside, in some subregion, is of the order $\varepsilon^{-\gamma}$ with $0 < \gamma < 1$. Then $U \to 0$ everywhere inside. However, for the most physically interesting cases of quasiblack holes, $U(r)$ is a monotonically decreasing function of r, see Appendix B of [1].

From an outside perspective, the supposed quasihorizon can be, in principle, nonextremal or extremal. From a physical viewpoint, the latter case is more important since it is the extremal quasiblack hole case which is indeed regular [1], whereas the nonextremal quasiblack hole case leads to infinite surface stresses [3]. Thus, we assume that to the outside the quasiblack hole is extremal. The study is valid for any extremal type of outside horizon. In the situation where there is an extremal electrically charged horizon, then the charge equals the mass, $q = m$.

Now, even being extremal to the outside, the quasiblack hole can have a horizon which, from the inside, is either nonextremal or extremal. Indeed, an extremal horizon for outside observers implies that the metric potential $V(r)$ has in the limit a double root when considered from outside. However, as shown in a concrete example in [15], from inside, the horizon can be either nonextremal or extremal. Therefore, we will consider the two cases separately, i.e., we will consider first quasiblack holes with a nonextremal horizon from the inside, and second quasiblack holes with an extremal horizon from the inside. Both are extremal quasiblack holes from the outside.

III. QUASIBLACK HOLES WITH PRESSURE

A. Quasiblack holes with pressure, nonextremal from the inside

1. General considerations

In the nonextremal from the inside case, near the gravitational radius of the configuration, the asymptotic form of the metric potential V inside should be

$$V_{\rm in} = \varepsilon + k(r_0 - r) + \ldots, \tag{14}$$

with $\varepsilon \ll 1$, $k > 0$, k being some quantity with units of inverse length. See [15] for concrete examples of this case of quasiblack holes with pressure, nonextremal from the inside. We want to elucidate the conditions on the parameters of the system, when the quantity U is uniformly bounded everywhere inside, i.e., is of the form (13). We analyze first the behavior of the functions in the bulk of the matter $r < r_0$, and second at the boundary r_0, and in both cases we assume that the quasiblack hole is being formed, $r_0 \to r_+$.

Region in the bulk of the matter, $r < r_0$. To this end, let us rewrite Eqs. (10) and (11) in the form

$$U_{\rm in} = V_{\rm in}P_1P_2. \tag{15}$$

Here

$$P_1 = \exp(2\psi_1), \qquad 2\psi_1 = \int_{r_0}^{r} d\bar{r}\frac{\sigma_0}{V_{\rm in}(\bar{r})}, \tag{16}$$

$$P_2 = \exp(2\psi_2), \qquad 2\psi_2 = \int_{r_0}^{r} d\bar{r}\frac{\sigma(\bar{r}) - \sigma_0}{V_{\rm in}(\bar{r})}, \tag{17}$$

where $\sigma(r)$, defined in Eq. (5), is a quantity with units of surface density (i.e., inverse length) and $\sigma_0 \equiv \sigma(r_0)$ in an obvious notation. It is also useful to define $\sigma_+ \equiv \sigma(r_+)$, i.e.,

$$\sigma_+ = 8\pi r_+(p_r(r_+) + \rho(r_+)). \tag{18}$$

Taking into account (14), we see that $\lim_{\varepsilon\to 0}P_2$ is a well-defined nonzero quantity that remains everywhere bounded, including the boundary $r = r_0 = r_+$. Let us focus attention on P_1. Then, one can write ψ_1 in the form

$$2\psi_1(r) = \frac{\sigma_+}{k}(\ln\varepsilon + 2\psi_{1\epsilon}(r)) + 2\psi_{11}(r), \tag{19}$$

where $2\psi_{1\epsilon}(r) = -\ln(\varepsilon + k(r_0 - r))$. It follows from (18) and the asymptotic behavior (14) that in the limit when $r_0 \to r_+$ (that entails $\varepsilon \to 0$) the quantity $2\psi_{11}(r)$ is finite everywhere inside, including the limit $\varepsilon = 0$, $2\psi_{11}(r_0) = 0$. Making the rescaling of time according to $T = t(\frac{\varepsilon}{k})^{(\sigma_+/2k)}$, we obtain inside the metric

$$ds^2 = -\frac{V(r)}{(\varepsilon + k(r_0 - r))^{\sigma_+/k}} g(r) dT^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{20}$$

where $g(r) \equiv \exp(2\psi_{11} + 2\psi_2)$ is everywhere finite and does not vanish.

Now, we want to impose that the metric (20) be free of curvature singularities by requiring that in an orthonormal frame the components of the Riemann tensor be finite. There is only one such potentially divergent term for the metric (20). It is the component

$$R_{0r}{}^{0r} = -\tfrac{1}{4}V'(\ln U)' - \tfrac{1}{4}V(2(\ln U)'' + (\ln U)'^2), \tag{21}$$

where U is the potential of dT^2 in (20), and a $'$ denotes a derivative with respect to the argument, in this case r. A simple, but nontrivial, analysis shows that there are only two ways to achieve finiteness of (21). Indeed, using Eqs. (6), (10), and (11) in (21) one finds,

$$R_{0r}{}^{0r} = K - Q, \tag{22}$$

where

$$K = -\left(\frac{m}{r^2} + 4\pi r p_r\right)', \tag{23}$$

and

$$Q = \frac{\sigma(\sigma + V')}{4V}. \tag{24}$$

We want to exclude the presence of a shell, so we want that the pressure be continuous. Then, p_r' is finite and so is the quantity K. The potential divergences can be connected with the term Q only. It follows from Eq. (14) that in the limit under discussion

$$Q \approx \frac{\sigma_+(\sigma_+ - k)}{4V}. \tag{25}$$

There are thus two possibilities: either $\sigma_+ = 0$, which as we will see yields the regular black hole, or $\sigma_+ = k$, which yields the quasiblack hole.

The first way is to put $\sigma_+ = 0$. Then, we get from (19) $2\psi_1(r) = 2\psi_{11}(r)$, so that, as P_2 (see above), P_1 is finite. Since $U = VP_1P_2$, it follows from (14) that $U \sim r_0 - r$, and thus $U \sim V$. So, instead of a quasihorizon, in the limit of $\varepsilon \to 0$, $r_0 \to r_+$, we obtain a regular event horizon (see, e.g., [18]), of the type found in the Schwarzschild, Reissner-Nordström or generic regular black holes discussed in [19] (see also [20,21]). In doing so, the metric coefficient $U_{\text{in}}(r)$ does not have the form (13). Thus, as we want to ensure the existence of a quasiblack hole, we reject the choice $\sigma_+ = 0$.

The second way to achieve finiteness is to put $\sigma_+ = k$. Then, in the limit $\varepsilon \to 0$ one has $2\psi_1(r) = \ln\varepsilon +$ finite terms, so that $P_1 \sim \varepsilon$, and so also $U \sim \varepsilon$, i.e., we obtain the metric function U in the form (13), the form appropriate for a quasiblack hole. Thus, we choose $\sigma_+ = k$. Using the expressions (6) and (7), the equality $r_+ = 2m(r_+)$, and neglecting the difference between r_0 and r_+, one obtains $V'(r_+) = -(8\pi\rho^{\text{in}}(r_+)r_+ - \frac{1}{r_+})$. From Eq. (14) one has $k = -V'(r_+)$, i.e., $k = 8\pi\rho^{\text{in}}(r_+)r_+ - \frac{1}{r_+}$. Then, since we are considering the case $\sigma_+ = k$, we finally get from Eq. (18) that

$$p_r^{\text{in}}(r_+) = -\frac{1}{8\pi r_+^2}, \tag{26}$$

the desired condition. The inside pressure of a quasiblack hole with pressure has to obey this condition. It cannot be obtained by the straightforward limit $\varepsilon \to 0$ from the regularity condition on the horizon of a true black hole, which as we have seen above demands $\sigma_+ = 0$ (i.e., $p_r^{\text{out}}(r_+) = -\rho^{\text{out}}(r_+)$), see [18]. This represents a remarkable result which clearly demonstrates that, although for an outside remote observer a true black hole and a quasiblack hole are undistinguishable, in the inner region the properties of a quasiblack hole can be very different from those of a black hole. Our general statement that $\sigma_+ \neq 0$ on a quasihorizon nonextremal from inside, can be checked in the particular examples given in [7,8] (see also [9]) of quasiblack holes made from pressureless matter, i.e., charged dust. Indeed, for such systems $\sigma_+ = 8\pi r_+\rho(r_+)$ where $\rho(r_+)$ is the density of matter and its matter pressure obeys $p_r = 0$ (see also [1]). Trivially, in these examples, $\rho(r_+)$ is clearly different from zero, so $\sigma_+ \neq 0$, as it must. It is worth noting that the limit discussed while checking the regularity condition can be characterized as $\lim_{r\to r_0}\lim_{\varepsilon\to 0}$.

Region at the boundary, $r = r_0$. We can also consider the immediate vicinity of the boundary by taking the opposite limit: $\lim_{\varepsilon\to 0}\lim_{r\to r_0}$. Then, it follows from (11) that for any $\varepsilon \neq 0$ we have that $\psi(r, r_0) \to 0$ when $r \to r_0$. Thus, $U_{\text{in}}(r_0) = V_{\text{in}}(r_0) = \varepsilon$ and the procedure is self-consistent.

2. Discussion: Conditions on the pressure and energy density at the boundary and more on the regularity requirement

(i) Conditions on the pressure and energy density at the boundary. We divide this discussion into two situations, when there is no electromagnetic field and when there is one.

(a) No electromagnetic field. Suppose that there is no electromagnetic field. Then, since from Eq. (26) the radial pressure p_r on the boundary is negative, we deduce that quasiblack holes with no electromagnetic field are connected with tension on the boundary. To proceed in the analysis, note that at a outside sphere with radius r, from Eq. (7) the mass $m(r)$ can be written as $m(r) = m(r_+) + 4\pi \int_{r_+}^{r} d\bar{r}\bar{r}^2 \rho$. Thus, since $r_+ = 2m(r_+)$, from Eq. (7) one can write for the outside

$$V_{\rm out}(r) = 1 - \frac{r_+}{r} - \frac{2m_{\rm out}}{r}, \qquad m_{\rm out} = 4\pi \int_{r_+}^{r} d\bar{r}\bar{r}^2 \rho_{\rm out}(\bar{r}), \tag{27}$$

where the difference between a horizon and a quasihorizon has been neglected. So, $V'_{\rm out}(r)$ at r_+ is given by

$$V'_{\rm out}(r_+) = \frac{1}{r_+}(1 - 8\pi\rho_{\rm out}(r_+)r_+^2). \tag{28}$$

We recall that we are dealing with extremal quasiblack holes from outside, since it is this kind of quasiblack holes which is free of curvature singularities or infinite surface stresses [1,3]. Therefore, $V'_{\rm out}(r_+) = 0$, and from Eq. (28) we find

$$\rho_{\rm out}(r_+) = \frac{1}{8\pi r_+^2}. \tag{29}$$

From the regularity condition on the horizon of a black hole (see, e.g., a detailed discussion in [18]) it also follows that

$$p_r^{\rm out}(r_+) = -\rho^{\rm out}(r_+), \tag{30}$$

and so

$$p_r^{\rm out}(r_+) = -\frac{1}{8\pi r_+^2}. \tag{31}$$

Thus, from Eq. (26) one always has

$$p^{\rm in}(r_+) = p^{\rm out}(r_+). \tag{32}$$

This means we automatically have obtained a quasiblack hole with continuous pressure on the boundary. So there is no need for a shell, certainly an elegant result, since thin shells always imply in some type of primary, albeit mild, discontinuity in the metric fields. On the other hand, we are considering the case in which the matter inside is not extremal in the sense that $V'_{\rm in}(r_+) \neq 0$ by construction. This means that a jump in density is mandatory. Jumps in density are well handled in gravitational systems, so this means that there is no problem. It is also important to pay attention to the following point. In principle, quasiblack holes which are extremal from outside, admit nonzero surface stresses and hence jumps in the radial pressure. This conclusion was obtained in [1,3] from a general form of the metric of extremal quasiblack holes. However, if, additionally, we take into account Einstein equations, it turns out that for configurations which are extremal outside and nonextremal inside, these surface stresses vanish.

(b) Electromagnetic field. Suppose now that there is an electromagnetic field. Now, the pressure receives contribution from two fields, the electromagnetic field and the matter field, so that the radial pressure can be written as $p_r = p_r^{\rm matter} + p_r^{\rm em}$. The electromagnetic pressure has the form $p_r^{\rm em} = -\frac{q^2(r)}{8\pi r^4}$ where $q(r)$ is the charge enclosed inside a sphere of radius r. Bearing in mind that we are interested in configurations which are (or tend to) extremal when viewed from outside, we have in the limit under discussion, $q(r_+) = r_+$, in accordance with the properties of an extremal Reissner-Nordström metric. Thus, $p_r^{\rm em} = -\frac{1}{8\pi r_+^2}$. Then, it follows from Eq. (26) that

$$p_r^{\rm matter}(r_+) = 0. \tag{33}$$

This situation, of existence of an electromagnetic field, is physically preferable since it means that we can build a quasiblack hole by considering a relativistic star with pressure obeying $p_r^{\rm matter}(r_0) = 0$ on the boundary and then taking the quasihorizon limit, as was done in [15]. In doing so, the configuration outside either represents an extremal Reissner-Nordström quasiblack hole or tends to it as shown in [1].

(ii) More on the regularity requirement. We now want to emphasize the role of the regularity requirement, i.e., regularity in the components of the Riemann tensor and so a spacetime free of curvature singularities. In principle, a metric in which Eq. (13) holds can occur without this requirement. For example, if we take $p_r = 0$ and $\rho = \rho_0 = {\rm const}$ everywhere for $r \le r_0$, and vacuum outside, an exact solution can be obtained [22,23] for which $V = 1 - \frac{8\pi\rho_0 r^2}{3}$ and $U = (1 - \frac{8\pi\rho_0 r_0^2}{3})^{3/2}(1 - \frac{8\pi\rho_0 r^2}{3})^{-1/2}$. Here r_0 is the surface at which this solution matches the outer Schwarzschild solution. One can try to obtain a quasiblack hole from this solution by taking the limit $r_0 \to \sqrt{\frac{3}{8\pi\rho_0}}$. Then, the metric potential U does indeed acquire the form given in Eq. (13). However, in this limit the surface $r = r_0$ becomes singular. By construction, condition (26) is not satisfied, so the absence of a regular quasiblack hole is justified. This, being an example in which the outside metric is Schwarzschild rather than extremal Reissner-Nordström, also shows that it is much harder to find nonextremal regular quasiblack holes than extremal ones.

B. Quasiblack holes with pressure, extremal from the inside

1. General considerations

In [1] we have analyzed the properties of quasiblack holes in which the matter in the inside region is extremal, i.e., matter for which the energy density is equal to the charge density. These quasiblack holes of [1] are thus quasiblack holes without pressure, with extremal matter in the inside region. Here we generalize those results by analyzing the properties of quasiblack holes with pressure extremal from the inside. Extremal from the inside means that the horizon from the inside is extremal (this is obligatory for quasiblack holes without pressure, but not for quasiblack holes with pressure). The horizon from the outside is always extremal for us.

In the case we have an extremal horizon from the inside, instead of (11) we have the asymptotic form

$$V = \varepsilon + \kappa^2(r_0 - r)^2 + \ldots, \tag{34}$$

with $\varepsilon \ll 1$, and κ being some positive quantity with units of inverse length. See [11,12] for concrete examples of this case of quasiblack holes with nonzero pressure which represent dispersed systems and have quasihorizons which are extremal both from inside and outside. Note that in (34) we can neglect the difference between r_0 and r_+. Inside we can distinguish two regions, the region in the bulk of the matter $r < r_0$, and the region at the boundary $r = r_0$. We consider now both regions separately.

Region in the bulk of the matter, $r < r_0$. In this region, $r < r_0$, the proper distance l, given by $l = \int_r^{r_0} \frac{d\bar{r}}{\sqrt{V}}$, from any point to the boundary diverges in the limit $\varepsilon \to 0$ as it is clear from (34). Indeed, defining dl as the infinitesimal proper distance, one obtains in the limit $\varepsilon \to 0$, $l \approx -\frac{1}{2\kappa} \times \ln\varepsilon$. It is useful to proceed along the same lines as in Sec. III A but now, because of the different asymptotic form of V, it is more convenient to rewrite ψ in another form,

$$U_{\rm in} = V_{\rm in} P_1 P_2 P_3. \tag{35}$$

Using the definition (5), we can rewrite the function ψ in (11), in this limit, as

$$P_1 = \exp(2\psi_1), \qquad 2\psi_1 = \int_{r_0}^{r} d\bar{r} \frac{\sigma_0}{V_{\rm in}(\bar{r})}, \tag{36}$$

$$P_2 = \exp(2\psi_2), \qquad 2\psi_2 = \int_{r_0}^{r} d\bar{r} \frac{\sigma(\bar{r}) - \sigma_0 - \bar{\sigma}_0'(\bar{r} - r_0)}{V_{\rm in}(\bar{r})}, \tag{37}$$

$$P_3 = \exp(2\psi_3), \qquad 2\psi_3 = \int_{r_0}^{r} d\bar{r} \frac{\sigma_0'(\bar{r} - r_0)}{V_{\rm in}(\bar{r})}, \tag{38}$$

where again a $'$ denotes a derivative with respect to the argument. Consider each term on (35) separately in the limit $\varepsilon \to 0$. In the first term, the integral is of the order $\varepsilon^{-1/2}$. To make the whole expression finite, we must conclude that $\sigma_0 \approx \sigma_+$ is also of the same order to compensate these divergences, namely, $\sigma_+ \lesssim O(\sqrt{\varepsilon})$, i.e., $p_r + \rho \lesssim O(\sqrt{\varepsilon})$, see [1] (Sec. II.A.d) for the analogous result for extremal charged dust. The second term remains finite since near r_0 both the numerator and denominator are proportional to $(\bar{r} - r_0)^2$ in the limit under discussion. Consider now the third term. We are discussing the region $r < r_0$. Thus, if $\sigma_+' > 0$, it is seen that in the region under discussion $\psi_3 \to +\infty$, $P_3 \to +\infty$, $U_{\rm in} \to +\infty$. Such a behavior has nothing to do with a quasiblack hole and should be rejected. Therefore, we must have $\sigma_+' \leq 0$. Because of the logarithmic behavior of the integral, we can represent ψ_3 in the form

$$2\psi_3 = \frac{\sigma_+'}{2\kappa^2}\left(\ln\left((r - r_0)^2 + \frac{\varepsilon}{\kappa^2}\right) - \ln\left(\frac{\varepsilon}{\kappa^2}\right)\right) + 2\psi_{33}, \tag{39}$$

where $2\psi_{33}$ is finite in the limit under discussion [cf. Eq. (19)]. Then, we can write the metric as [cf. Eq. (20)],

$$ds^2 = -V(r)\left((r - r_0)^2 + \frac{\varepsilon}{\kappa^2}\right)^{(\sigma_+'/2\kappa^2)} g(r) dT^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{40}$$

where $T = t(\frac{\varepsilon}{\kappa^2})^{-(\sigma_+'/4\kappa^2)}$, and $g = \exp(2\psi_{33} + 2\psi_1 + 2\psi_2)$ is finite. The concrete form of the metric potentials in the interior region is model dependent, see examples in [1] (see also [11,12] for extremal pressure systems, and [9,10] for extremal pressureless systems). We can also discuss the regularity of the Riemann tensor as we did in the nonextremal case. Using (40) and (34) in (21) and (24) gives in the limit $r \to r_0$ that

$$Q \approx \frac{\sigma_+^2}{4V}, \tag{41}$$

so the only possible choice is indeed

$$\sigma_+ = O(\sqrt{\varepsilon}) \to 0, \tag{42}$$

as already found.

Region at the boundary, $r = r_0$. This region is in the immediate vicinity of the boundary (which tends to the quasihorizon in the limit under discussion). In this region, by definition, the proper distance l remains finite since, although the double root of V is being approached, the limit of integration shrinks. We assume that the metric is well-defined, with 2ψ being finite in the vicinity of r_0. Then, bearing in mind that $\sigma_+ \lesssim O(\sqrt{\varepsilon})$ as found above, neglecting a weak dependence of σ/r on r, so that $\frac{\sigma}{r} \approx a\kappa^2\sqrt{\varepsilon}$ for some constant a, we can write near the quasihorizon r_+,

$$\sigma_+ \approx a\kappa^2 r_+ \sqrt{\varepsilon}. \tag{43}$$

Here, $a \geq 0$ since, as discussed above, near the quasihorizon we want to have $\sigma' < 0$ and $\sigma > 0$. In the limit $\varepsilon = 0$ we obtain from (43) the regularity condition for the quasihorizon, the condition being $\sigma_+ = 0$, which by Eq. (18) means

$$p_r(r_+) = -\rho(r_+). \tag{44}$$

This regularity condition is the same as for usual, i.e., true, horizons, see, e.g., [18]. Thus, if a quasihorizon is extremal from inside, the regularity condition (44) similar to that for black hole (30) is reproduced, in contrast to the situation with the quasihorizon nonextremal from inside. To obtain the metric in this limit we make the substitution $r = r_0 - \sqrt{\frac{\varepsilon}{\kappa^2}}y$. Then the metric is

$$ds^2 = -(1 + y^2)e^{-(a\kappa r_+ \arctan y)}dT^2 + \frac{1}{\kappa^2}\frac{1}{1 + y^2}dy^2 + r_0^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{45}$$

We have used $t = \frac{T}{\sqrt{\varepsilon}}$ to absorb the factor ε into g_{00}, a procedure that is typical of quasiblack holes [1]. The metric (45) is a slight generalization of the Bertotti-Robinson metric. To see it, we note that for a pure electromagnetic situation one has $\kappa^2 = \frac{1}{r_+^2}$ and $a = 0$, so that (45) coincides with the Bertotti-Robinson metric. The proper distance $l = \frac{1}{\kappa}\int_0^y \frac{d\tilde{y}}{\sqrt{1+y^2}}$ is finite for any y but it diverges in the limit $y \to \infty$, so we obtain an infinitely long tube.

2. Discussion: conditions on the pressure and energy density at the boundary and a proposition

(i) Conditions on the pressure and energy density at the boundary. Since the value of $\rho(r_+)$ is fixed by the condition $V'(r_+) = 0$ both from outside and inside, both radial pressure and density are continuous, in contrast to the nonextremal case from inside where the density is discontinuous. In the situation (a) there is no electromagnetic field then the quasihorizon is supported by matter tension, in the situation (b) there is an electromagnetic field the matter pressure is equal to zero at the quasihorizon, both results can be deduced as before.

(ii) A proposition. From the above considerations an interesting result follows. In order to have a well-defined U, and thus a well-defined metric, we need to have σ defined in Eq. (5) obeying $\sigma > 0$ in some vicinity of the quasihorizon. Since on the quasihorizon itself $\sigma = \sigma_0 = \sigma_+ \to 0$, we must have $\sigma_0' \leq 0$ as is explained above. But from (5), $\sigma = 8\pi r(p_r(r) + \rho(r))$. Thus, we can state the following proposition: (i) One cannot build an extremal quasiblack hole entirely from phantom matter, i.e., matter with the null energy condition violated everywhere inside, $p_r + \rho < 0$. (ii) In case there is phantom matter, it cannot border the quasihorizon but must lie inside the inner region only. Thus, at least in some vicinity of the quasihorizon the null energy condition is satisfied everywhere, so that $p_r + \rho \geq 0$.

For a discussion of the energy conditions within the related context of regular black holes see [24]. Alternation of regions with normal and phantom matter is discussed in [25] in another context.

IV. CONCLUSION

We have studied extremal quasiblack holes, as seen from the outside, with nonzero pressure and have shown how these objects are attainable on general grounds. From the inside these quasiblack holes can have nonextremal and extremal quasihorizons. The total pressure at the matter boundary is less or equal to zero and it is always continuous there. In the situation where there is an electric field the matter pressure is zero at that boundary. The density behaves as expected, either showing a jump at the boundary in the nonextremal case or being continuous in the extremal case. The regularity conditions for the nonextremal inside case is completely different from the regularity condition for the usual regular black holes, whereas the regularity conditions for the extremal inside case can be obtained from the known regularity conditions for the usual regular black holes. For the extremal inside case we show that the quasiblack holes cannot be made from phantom matter at the quasihorizon. Further properties that one can envisage depend on the particular model under study, see [15–17] for the nonextremal inside case and [11,12] for the extremal inside case. In our previous studies [1–6] we have shown that quasiblack holes with nonextremal and extremal quasihorizons for the outside are distinct entities and must be considered as such when one studies them. Here we have shown that the same holds for quasiblack holes with nonextremal and extremal quasihorizons for the inside. They have to be carefully considered as separate entities with distinct properties.

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