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Scalar field theory on noncommutative Snyder spacetime

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We construct a scalar field theory on the Snyder noncommutative space-time. The symmetry underlying the Snyder geometry is deformed at the co-algebraic level only, while its Poincaré algebra is undeformed. The Lorentz sector is undeformed at both the algebraic and co-algebraic level, but the coproduct for momenta (defining the star product) is non-coassociative. The Snyder-deformed Poincaré group is described by a non-coassociative Hopf algebra. The definition of the interacting theory in terms of a nonassociative star product is thus questionable. We avoid the nonassociativity by the use of a space-time picture based on the concept of the realization of a noncommutative geometry. The two main results we obtain are (i) the generic (namely, for any realization) construction of the co-algebraic sector underlying the Snyder geometry and (ii) the definition of a nonambiguous self-interacting scalar field theory on this space-time. The first-order correction terms of the corresponding Lagrangian are explicitly computed. The possibility to derive Noether charges for the Snyder space-time is also discussed.

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I. INTRODUCTION

Snyder space-time has been the first proposal of noncommutative geometry to tame the UV divergences of quantum field theory [1,2]. The preliminary idea to solve this problem was to use a lattice structure instead of the space-time continuum [3]. However, a lattice breaks the Lorentz invariance, posing serious doubts for accepting the theory. A Lorentz invariant discrete space-time has been formulated only by Snyder. The price to pay is a noncommutative structure of space-time. Because of the success of the renormalization theory, the Snyder program has been abandoned since its rediscovery 40 years later by mathematicians [4,5]. Now, the analysis of field theories on noncommutative space-times has become a fundamental area in theoretical physics (for reviews see [6,7]).

A quantum field theory on the Snyder space-time has however not yet been constructed and thus the removal of divergences by means of noncommutativity effects has not yet been proved. In this paper we construct a self-interacting classical scalar field theory on this space-time. This model can be considered as the starting point for the quantum analysis.

The noncommutativity of the Snyder space-time is encoded in the commutator between the coordinates, which is proportional to the (undeformed) Lorentz generators. The Poincaré symmetry underlying this space-time is undeformed at the algebraic level, while the co-algebraic sector is (highly) nontrivial. In a previous paper [8] we have shown that, by using the concept of realizations, there exists infinitely many deformed Heisenberg algebras all compatible with this geometry. This freedom can be understood as the freedom in choosing momentum coordinates.

We here complete the previous analysis by studying the coproduct and star-product structures underlying the model. Equipped with this technology, we construct a scalar field theory on this noncommutative space-time. Our main goal is to define the theory without ambiguities and without needing supplementary structures (as a deformed measure) necessary in extra dimensional approaches. The momentum space of the Snyder-deformed Poincaré group does not have a Lie group structure since it is given by the coset $SO(4,1)/SO(3,1)$, i.e. the de Sitter space. The coproduct and the induced star product turn out to be *nonassociative*. This feature represents the main obstacle in studying field theories on the Snyder space-time. Such a kind of deformation of the Poincaré group cannot be recovered within the classification [9], because only deformations preserving the co-associativity are considered. The language of Hopf algebras [10] does not apply straightforwardly to the Snyder space-time geometry.

The nonassociativity propriety obstructed the analysis of the Snyder geometry with respect to other noncommutative space-times. For example κ Minkowski, a particular case of Lie algebra-type space-time, has been developed at different levels specifying star products [11,12], differential calculus [13,14], scalar field theory [15–18], and conserved charges [19–21]. The key difference between Snyder and κ Minkowski is that in the latter the momentum space has the structure of a non-Abelian Lie group and thus the coproduct is noncommutative, but still associative. In particular, the Snyder space-time is not a special case of κ Minkowski [22,23]. In fact, as clarified in [24], κ spaces are based on Lie algebra while Snyder space is grounded on trilinear commutations relations.

Our approach is based on the framework of realizations by which we bypass the nonassociativity and clearly define the self-interacting theory. The theory we construct lives on the noncommutative space-time and its dual has the

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momentum space given by a coset. Our analysis deals with the four-dimensional Lorentzian model and no extra dimensional structures are invoked. Moreover, our theory is general as we consider all realizations of the geometry, differently to the previous approaches (for other attempts to define a scalar field theory on Snyder space-time see [25–28]). The frameworks usually adopted are recovered as particular cases of our construction.

The Snyder space-time is linked to doubly special relativity models [29,30], loop quantum gravity [31], and two-time physics [32]. In particular in [33], we have shown that a Snyder-deformed quantum cosmology predicts a big-bounce phenomenology as in loop quantum cosmology [34] (for other comparisons between deformed and loop cosmologies see [35–37]).

The paper is organized as follows: In Sec. II, we describe the algebraic structure of the Snyder space-time. In Sec. III the co-algebraic sector underlying the noncommutative geometry is analyzed in detail. Section IV is devoted to the formulation of the scalar field theory on this space-time. Finally, in Sec. V the first-order corrections are computed. Concluding remarks follow.

We adopt units such that $\hbar = c = 1$, the signature given by $\eta_{\mu\nu} = \mathrm{diag}(-, +, \ldots, +)$ and the index convention $\{\mu, \nu, \ldots\} \in \{0, \ldots, n\}$.

II. SNYDER SPACE-TIME

In this section we describe the noncommutative Snyder space-time geometry. We discuss the realizations of such a geometry as well as the dispersion relation underlying the model.

A. Deformed Heisenberg algebras

Let us consider a $(n+1)$-dimensional Minkowski space-time such that the commutator between the coordinates has the nontrivial structure

$$[\tilde{x}_\mu, \tilde{x}_\nu] = sM_{\mu\nu}, \tag{1}$$

where $\tilde{x}_\mu$ denote the noncommutative coordinates, and $s \in \mathbb{R}$ is the deformation parameter with dimension of a squared length. We demand that the symmetries of such a space are described by an undeformed Poincaré algebra. This means that both Lorentz generators $M_{\mu\nu} = -M_{\nu\mu} = i(x_\mu p_\nu - x_\nu p_\mu)$ and translation generators p_μ satisfy the standard commutation relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho}$$
$$[p_\mu, p_\nu] = 0. \tag{2}$$

We also assume that momenta and noncommutative coordinates transform as undeformed vectors under the Lorentz algebra, i.e. the commutators

$$[M_{\mu\nu}, p_\rho] = \eta_{\nu\rho}p_\mu - \eta_{\mu\rho}p_\nu, \tag{3}$$

$$[M_{\mu\nu}, \tilde{x}_\rho] = \eta_{\nu\rho}\tilde{x}_\mu - \eta_{\mu\rho}\tilde{x}_\nu, \tag{4}$$

hold. The quantity $p^2 = \eta^{\mu\nu}p_\mu p_\nu$ is then a Lorentz invariant.

Relations (1)–(4) define the Snyder space-time geometry. However, they do not fix the commutator between $\tilde{x}_\mu$ and p_ν. In particular, as it was shown in [8], there exists infinitely many possible commutators that are all compatible, in the sense that the algebra closes in virtue of the Jacobi identities, with the above requirements. This feature is understood by means of the concept of realization [11,18,38–41] (for a similar framework see [42,43]). A realization on a noncommutative space is defined as a rescaling of the deformed coordinates $\tilde{x}_\mu$ in terms of ordinary phase space variables (x_μ, p_ν) as

$$\tilde{x}_\mu = \Phi_{\mu\nu}(p)x_\nu. \tag{5}$$

The most general $SO(n,1)$-covariant realization of the Snyder geometry reads [8]

$$\tilde{x}_\mu = x_\mu\varphi_1(A) + s(xp)p_\mu\varphi_2(A), \tag{6}$$

in which φ_1 and φ_2 are two (dependent) functions of the dimensionless quantity $A = sp^2$ (hereafter, the convention $(ab) = \eta^{\mu\nu}a_\mu b_\nu$ is adopted). The function φ_2 depends on φ_1 by the relation

$$\varphi_2 = \frac{1 + 2\dot{\varphi}_1\varphi_1}{\varphi_1 - 2A\dot{\varphi}_1}, \tag{7}$$

where the dot denotes differentiation with respect to A. The generic realization (6) is completely specified by the function φ_1. There are thus infinitely many ways to express, via φ_1, the noncommutative coordinates (1) in terms of the ordinary ones without deforming the original symmetry. The boundary condition $\varphi_1(0) = 1$ ensures that the ordinary commutative framework is recovered as soon as $s = 0$. The commutator between $\tilde{x}_\mu$ and p_ν immediately follows from (6) and reads

$$[\tilde{x}_\mu, p_\nu] = i(\eta_{\mu\nu}\varphi_1 + sp_\mu p_\nu\varphi_2). \tag{8}$$

This relation describes a deformed Heisenberg algebra.

It is also interesting to give the inverse of the realization (6), which reads

$$x_\mu = \frac{1}{\varphi_1}\left(\tilde{x}_\mu - \frac{1}{\varphi_1 + A\varphi_2}s(\tilde{x}p)p_\mu\varphi_2\right). \tag{9}$$

This relation allows us to construct invariants for the noncommutative framework from those arising in the commutative one. We only have to demand that the invariants in (x_μ, p_ν) coordinates will be sent into the invariants in $(\tilde{x}_\mu, p_\nu)$ coordinates by means of (9).

The use of realizations allow us to give a phase space interpretation of the Snyder space-time. Consider the *noncanonical* transformation $x_\mu \to \Phi_{\mu\nu}(p)x_\nu$, $p_\nu \to p_\nu$ in an ordinary phase space coordinatized by (x_μ, p_ν). The

Snyder noncommutative geometry results from such a map. This transformation can be a generic function of momenta, but linear in coordinates (for discussions on noncommutative classical mechanics see e.g. [44]).

B. Particular realizations

The noncommutative Snyder geometry has been analyzed in the literature from different points of view [25–33,45,46], but only two particular realizations of its algebra are usually adopted. These are the Snyder [1] and the Maggiore [47,48] types of realizations, which are particular cases of (6).

The first realization is the one originally suggested by Snyder. It is recovered from (6) if

$$\varphi_1 = 1, \tag{10}$$

which, because of (7), implies that $\varphi_2 = 1$. The second realization has been proposed by Maggiore, and it appears as soon as

$$\varphi_1 = \sqrt{1 - sp^2}, \tag{11}$$

and thus, from (7), $\varphi_2 = 0$. The momentum p_μ is bounded or unbounded, depending on the sign of s. If $s > 0$, the constraint $|p| < 1/\sqrt{s}$ holds.

Beside these types of realization, the one that realizes the Weyl symmetric ordering is the third interesting one. The Weyl ordering is obtained by the condition

$$\Phi_{\mu\nu} p_\nu = (\varphi_1 \eta_{\mu\nu} + s p_\mu p_\nu \varphi_2) p_\nu = p_\mu, \tag{12}$$

which, considering the relation (7), implies that

$$\varphi_1 = \sqrt{sp^2} \cot \sqrt{sp^2}. \tag{13}$$

As we said, there are however infinitely many possible realizations of the Snyder space-time geometry.

C. Dispersion relation

Let us discuss the fate of the standard dispersion relation $p^2 = m^2$ in the Snyder space-time. In particular, we are interested in how different realizations modify this constraint. Consider two momenta $\tilde{p}_\mu$ and p_μ in two distinct realizations. Since momenta transform as vectors under the Lorentz symmetry, see (3), the relation

$$\tilde{p}_\mu = p_\mu f(A) \tag{14}$$

holds. The function $f(A)$, such that $f(0) = 1$, depends on the realization φ_1 and can be obtained as follows. Let for example $\tilde{p}_\mu$ be a momentum in the Maggiore realization (11), i.e. the commutator $[\tilde{x}_\mu, \tilde{p}_\nu] = i\eta_{\mu\nu}\sqrt{1 - s\tilde{p}^2}$ holds. The function f is obtained by inserting (6) and (14) in this relation and reads

$$f = \frac{1}{\sqrt{\varphi_1^2 + A}}. \tag{15}$$

Notice that $f = 1$ as the realization (11) is taken into account and that, because of (10), the Maggiore momentum p_μ^M is related to the Snyder one p_μ^S by

$$p_\mu^S = p_\mu^M \sqrt{1 + s(p^S)^2}. \tag{16}$$

As expected from (3), the dispersion relation for $\tilde{p}^2$ is undeformed, but an effective mass $m_e = m_e(m)$ has to be taken into account. From (14) and (15), the Snyder-dispersion relation reads

$$\tilde{p}^2 = \frac{m^2}{\varphi_1^2(sm^2) + sm^2} \equiv m_e^2. \tag{17}$$

Let us discuss such a formula in the Snyder realization ($\varphi_1 = 1$). In the low-deformed case ($m^2 \ll 1/s$) the effective mass is given by $m_e^2 \simeq m^2(1 - sm^2)$. On the other hand, in the ultradeformed case ($m^2 \gg 1/s$ with $s > 0$) we have $m_e^2 \simeq 1/s$ (for $s < 0$, m is bounded as $m^2 < 1/|s|$).

III. CO-ALGEBRAIC SECTOR

The co-algebraic sector of the Snyder geometry is here analyzed. We first focus on a generic framework, i.e. by considering the arbitrary realization (6). The two particular realizations (10) and (11) are investigated below. A discussion about the nonassociativity follows.

A. General framework

Deformations of symmetries underlying Snyder space-time (1) are contained in the co-algebraic sector of a (nontrivial) quantum group. Generators $(\tilde{x}_\mu, p_\mu, M_{\mu\nu})$ form an algebra defined by the commutators (1)–(4) and (8). This is not a Hopf algebra. However, $(p_\mu, M_{\mu\nu})$ generate the Snyder-deformed Poincaré group $\mathcal{P}_S$ whose algebra is a generalization of the Hopf algebra.

As understood from commutators (2) and (3), the Snyder algebraic sector is the one of an undeformed Poincaré algebra. On the other hand, the co-algebraic sector, defined by the action of Poincaré generators on the Snyder coordinates $\tilde{x}_\mu$, is deformed. The action of Lorentz generators is still the standard one because of (4), but the action of momenta is modified as in (8). The Leibniz rule is thus deformed and depends on realizations. As we will see, the coproduct for momenta is no longer commutative and neither associative.

The coproduct and star-product structures can obtained from realizations as follows: Let $\mathbb{I}$ be the unit element of the space of commutative functions $\psi(x)$. By means of (6) the action of a noncommutative function $\tilde{\psi}(\tilde{x})$ on $\mathbb{I}$ gives [24,49]

$$\tilde{\psi}(\tilde{x}) \triangleright \mathbb{I} = \psi'(x). \tag{18}$$

This relation provides a map from the noncommutative space of functions to the commutative one. Notice that the commutative function $\psi'(x)$ will be in general different from $\psi(x)$. Consider now a noncommutative plane wave $e^{i(k\tilde{x})}$, in which $\tilde{x}_\mu$ refers to a given realization (6), and k_μ are the eigenvalues of $p_\mu = -i\partial/\partial x^\mu$. It is then possible to show that [24,49]

$$e^{i(k\tilde{x})} \triangleright \mathbb{I} = e^{i(Kx)}, \tag{19}$$

where $K_\mu = K_\mu(k)$ is a deformed momentum (defined below) depending on realizations. The commutative limit $s \to 0$ leads to the standard framework in which $K_\mu = k_\mu$. Consequently, given the inverse transformation $K_\mu^{-1} = K_\mu^{-1}(k)$, we have

$$e^{i(K^{-1}\tilde{x})} \triangleright \mathbb{I} = e^{i(kx)}. \tag{20}$$

It is worth noting that, in the Weyl realization (13), we have $e^{i(k\tilde{x})} \triangleright \mathbb{I} = e^{i(kx)}$ and the plane waves are undeformed.

Let us now consider two plane waves labeled by momenta k_μ and q_μ, respectively. Their action on the unit element $\mathbb{I}$ gives

$$e^{i(k\tilde{x})}(e^{i(qx)}) = e^{i(F(k,q)x)}. \tag{21}$$

The deformed momentum K_μ is thus determined by $K_\mu = F_\mu(k, 0)$, where the function $F_\mu(k, q)$ specifies the coproduct as well as the star product. It can be obtained by a straightforward implementation of the Campbell-Baker-Hausdorff formula or by the more elegant method developed in [24,49].

The star product between two plane waves is defined by $F_\mu(k, q)$ as

$$\begin{aligned} e^{i(kx)} \star e^{i(qx)} &\equiv e^{i(K^{-1}(k)\tilde{x})} e^{i(K^{-1}(q)\tilde{x})} \triangleright \mathbb{I} = e^{i(K^{-1}(k)\tilde{x})}(e^{i(qx)}) \\ &= e^{i(\mathcal{D}(k,q)x)}, \end{aligned} \tag{22}$$

in which

$$\mathcal{D}_\mu(k, q) = F_\mu(K^{-1}(k), q). \tag{23}$$

The star product defines, by means of (19) and (20), a Weyl mapping from the commutative to the noncommutative spaces provided by a one-to-one correspondence between $e^{i(k\tilde{x})}$ and $e^{i(Kx)}$. The coproduct for momenta Δp_μ (and the corresponding Leibniz rule) is obtained from $\mathcal{D}_\mu(k, q)$ as

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p). \tag{24}$$

In particular, the function $\mathcal{D}_\mu$ describes the non-Abelian sum of momenta in the Snyder noncommutative space-time, i.e.

$$\mathcal{D}_\mu(k, q) = k_\mu \oplus q_\mu \neq k_\mu + q_\mu. \tag{25}$$

As soon as the noncommutativity effects (in our case the parameter s) are switched off, the ordinary Abelian rule $\mathcal{D}_\mu(k, q) = k_\mu + q_\mu$ is recovered. By means of (22), it is possible to obtain the star product between two generic functions f and g of commuting coordinates (see, for example [11,39,49]). Adopting the plane waves relation (22), the general result for the star product stands as

$$(f \star g)(x) = \lim_{\substack{y\to x\\ z\to x}} e^{ix_\mu(\mathcal{D}^\mu(p_y,p_z) - p_y^\mu - p_z^\mu)} f(y)g(z). \tag{26}$$

Star product is a binary operation acting on the algebra of functions defined on the ordinary commutative space, and it encodes features reflecting the noncommutative nature of Snyder space-time (1). The star product is uniquely defined, but its concrete form is related to a particular realization and vice versa. For any realization the star product (22), and then (26), is nonassociative. The corresponding coproduct (24) is non-coassociative. Such a result has been confirmed by the recent analysis [28] also.

This construction is well defined and allows us to obtain, from realizations (6), both the coproduct and star-product structures underlying the Snyder space-time. The inverse path is also meaningful: starting from a star product (or a coproduct) it is always possible to recover information about the realization we are working in. However, as we shall see, to construct a scalar field theory on Snyder space-time it is more suitable to deal with realizations instead of star products. The nonassociativity of the star product in fact poses severe challenges in defining interaction terms.

Let us now compute the coproduct Δp_μ, at the first order in s, for a generic realization (6). Expanding the realization function φ_1 as $\varphi_1 = 1 + c_1 s p^2 + \mathcal{O}(s^2)$ and considering (15), we obtain

$$\begin{aligned} \Delta p_\mu &= \Delta_0 p_\mu + s\Delta_1 p_\mu + \mathcal{O}(s^2), \\ \Delta_0 p_\mu &= p_\mu \otimes 1 + 1 \otimes p_\mu, \\ \Delta_1 p_\mu &= (c - \tfrac{1}{2}) p_\mu \otimes p^2 + (2c - \tfrac{1}{2}) p_\mu p_\nu \otimes p^\nu \\ &\quad + c(p^2 \otimes p_\mu + 2p_\nu \otimes p^\nu p_\mu), \end{aligned} \tag{27}$$

where $c = (2c_1 + 1)/2$. Here $\Delta_0 p_\mu$ and $\Delta_1 p_\mu$ denote the coproduct at the zero and first order is s, respectively. The Maggiore type of realization (11) is defined by $c_1 = -1/2$ and thus it is recovered as $c = 0$. The Snyder one (10) appears for $c_1 = 0$ and thus $c = 1/2$, while the Weyl one (13) for $c = 1/6$. The coproduct (27) defines the Snyder non-Abelian sum in a generic realization. The star product is obtained from (22).

To complete the analysis of the co-algebraic sector we need to specify the coproduct $\Delta M_{\mu\nu}$ of the Lorentz generators, as well as the antipode $S(g)$ and the co-unit $\varepsilon(g)$ for any element g of $\mathcal{P}_S$. Because of relations (2)–(4), the coproduct $\Delta M_{\mu\nu}$ is trivial, i.e.

$$\Delta M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}. \tag{28}$$

The antipode $S(g)$ is defined by the equation

$$\mathcal{D}(g, S(g)) = g \oplus S(g) = 0. \tag{29}$$

From (27) and (28), we immediately realize that the antipode is not deformed for any $g = (p_\mu, M_{\mu\nu})$, that is

$$S(p_\mu) = -p_\mu \qquad S(M_{\mu\nu}) = -M_{\mu\nu}. \tag{30}$$

Because different momenta are related by (14), the antipode $S(p_\mu)$ is exactly (not only at the first order) trivial in all realizations. On the other hand, the co-unit $\varepsilon(g)$ is also trivial for any $g \in \mathcal{P}_S$. Finally, we observe that coproduct for momenta (24) is covariant because of (28), i.e. the relation

$$[\Delta M_{\mu\nu}, \Delta p_\rho] = \eta_{\nu\rho}\Delta p_\mu - \eta_{\mu\rho}\Delta p_\nu \tag{31}$$

holds. This expression is the co-algebraic counter term of (3).

Summarizing, the Snyder-deformed Poincaré group $\mathcal{P}_S$ is characterized as follows. The Lorentz symmetry is undeformed at both algebraic and co-algebraic level. The deformations are encoded in the coproduct (24) only, which, in particular, is non-coassociative. The corresponding star product (22) is nonassociative and a homomorphism relates these structures. The algebraic sector is then compatible with the co-algebraic one. Therefore, the generators $(p_\mu, M_{\mu\nu})$ of $\mathcal{P}_S$ form a generalized Hopf algebra, which we shall denote as a *non-coassociative Hopf algebra*.

B. Particular realizations

We now study the coproduct structure underlying the two particular realizations (10) and (11). In both cases a closed form of the coproduct arises.

Let us first consider the Maggiore realization (11). The basic function $F_\mu(k, q)$ in (21) is given by

$$F_\mu = q_\mu + k_\mu\sqrt{1 - A_q}\frac{\sin\sqrt{A_k}}{\sqrt{A_k}} - sk_\mu(kq)\frac{1 - \cos\sqrt{A_k}}{\sqrt{A_k}}, \tag{32}$$

where $A_p = sp^2$. The ordinary function $F_\mu = k_\mu + q_\mu$ is recovered in the $s = 0$ case. From (32) one immediately obtains the deformed momentum $K_\mu(k)$, which reads

$$K_\mu = F_\mu(k, 0) = k_\mu\frac{\sin\sqrt{A_k}}{\sqrt{A_k}}. \tag{33}$$

The coproduct Δp_μ follows from (23), and it is given, in terms of the realization function $\varphi_1 = \sqrt{1 - sp^2}$, by

$$\Delta p_\mu = p_\mu \otimes \varphi_1 - \frac{s}{1 + \varphi_1}p_\mu p_\nu \otimes p^\nu + 1 \otimes p_\mu. \tag{34}$$

Such a coproduct [namely, the addition rule (25)] is non-co-associative. The order by which we sum the momenta becomes important. As $s = 0$, we have $\varphi_1 = 1$, and the trivial coproduct $\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu$ is recovered. The first-order term coincides with (27) for $c = 0$.

Let us now analyze the Snyder realization (10). In this case the function $F_\mu(k, q)$ in (21) reads

$$F_\mu = g(hk_\mu + q_\mu)g = \left(\cos\sqrt{A_k} - \sqrt{\tfrac{s}{k^2}}(kq)\sin\sqrt{A_k}\right)^{-1} h$$
$$= \frac{1}{k^2}\left(\sqrt{\frac{k^2}{s}}\sin\sqrt{A_k} + (kq)(\cos\sqrt{A_k} - 1)\right). \tag{35}$$

The deformed momentum $K_\mu(k)$ is then given by

$$K_\mu = F_\mu(k, 0) = k_\mu\frac{\tan\sqrt{A_k}}{\sqrt{A_k}}. \tag{36}$$

The ordinary framework is restored as $s = 0$. The coproduct directly follows from (23) and reads

$$\Delta p_\mu = \frac{1}{1 - sp_\nu \otimes p^\nu}\left(p_\mu \otimes 1 - \frac{s}{1 + \sqrt{1 + A_p}}p_\mu p_\nu \otimes p^\nu + \sqrt{1 + A_p} \otimes p_\mu\right). \tag{37}$$

Also in this case the coproduct is non-coassociative. The first-order term coincides with the $c = 1/2$ case of (27).

C. On the nonassociativity

The relation between special relativity and the Snyder geometry allows us to better understand the physical meaning of the nonassociativity.

Special relativity can be analyzed (and derived) from a noncommutative point of view [50]. Consider the Galileo group $ISO(3) = SO(3) \cdot \mathbb{R}^3$. Speeds generate translations and the speed space $\mathbb{R}^3$ can be identified as $\mathbb{R}^3 \sim ISO(3)/SO(3)$. A manifold of this type is a coset space. In this case it has the (Lie) group structure. Special relativity can be viewed as arising from the deformation of $\mathbb{R}^3$ into the curved space $\mathcal{C} = SO(3, 1)/SO(3)$. This operation sends the Galileo group into the Lorentz one $SO(3, 1) = SO(3) \cdot \mathcal{C}$ (this is the Cartan decomposition of the Lorentz group [51]). The coset $SO(3, 1)/SO(3)$ is nothing but the (hyperbolic) boosts space, but it is not a Lie group. In fact the product between two boosts is not longer a boost, but an element of the full Lorentz group $SO(3, 1)$. The composition of speeds can be extracted from a coproduct structure. It turns out that the composition of (noncollinear) speeds is no longer commutative and neither associative. A physical manifestation of nonassociativity is the well-known Thomas precession [52]. From a mathematical point of view, the nonassociativity is a consequence of the fact that the coset space is not a group manifold.

The Snyder space-time geometry can be viewed from the same perspective. Consider the Poincaré group $\mathcal{P} = SO(3, 1) \cdot \mathbb{R}^4$. As above, the momentum space $\mathbb{R}^4$ can be viewed as the coset $\mathbb{R}^4 \sim \mathcal{P}/SO(3, 1)$ and of course it is a group manifold. Deforming the momentum space into the de Sitter space $dS = SO(4, 1)/SO(3, 1)$, we recover the Snyder noncommutative geometry. This is the original formulation made by Snyder himself [1]. The Snyder-deformed Poincaré group $\mathcal{P}_S$ is then factorized as

$$\mathcal{P}_S = SO(3,1) \cdot d\mathcal{S}, \tag{38}$$

showing that the Lorentz symmetry is undeformed. On the other hand, the translation sector of this (quantum) group is deformed consistently to (1). As in the previous case, the coset $d\mathcal{S}$ is not a Lie group. The non-coassociativity of Snyder coproduct can be traced back to this feature.

IV. SCALAR FIELD THEORY

In this section we construct the scalar field theory on the four-dimensional Snyder noncommutative space-time. We first consider the Fourier transformation and define the Snyder scalar field and then we write down the action for the theory. A comparison with other approaches follows.

A. Preliminaries

We define a scalar field $\tilde{\phi}(\tilde{x})$ on the Snyder noncommutative space-time by means of the Fourier transformation as

$$\tilde{\phi}(\tilde{x}) = \int [dk]\hat{\phi}(k) e^{i(K^{-1}\tilde{x})}. \tag{39}$$

The integration measure $[dk]$ is *a priori* deformed, depending on the antipode $S(k_\mu)$. However, as we have previously seen, it is trivial in any realizations. The measure in (39) is thus the ordinary one

$$[dk] = \frac{d^4k}{(2\pi)^4}. \tag{40}$$

Let us now consider the action of the Snyder scalar field (39) on the identity $\mathbb{I}$. By means of (20), this operation gives

$$\tilde{\phi}(\tilde{x}) \triangleright \mathbb{I} = \phi(x), \tag{41}$$

which ensures the Lorentz scalar behavior of the model. As a further step we consider the quadratic term $\tilde{\phi}^2(\tilde{x}) \triangleright \mathbb{I}$. Given the definition (39) and remembering (21)–(23), we obtain

$$\tilde{\phi}^2(\tilde{x}) \triangleright \mathbb{I} = (\phi \star \phi)(x). \tag{42}$$

We have thus recovered the star-product structure.

Let us now discuss the notion of a real and complex Snyder scalar field. First, because of the triviality of the antipode, the conjugation is also an ordinary one. Second, the noncommutative coordinates $\tilde{x}_\mu$ have to be Hermitian operators in any given realization. All the commutators given above are invariant under the formal antilinear involution "$\dagger$"

$$\tilde{x}_\mu^\dagger = \tilde{x}_\mu, \qquad p_\mu^\dagger = p_\mu, \qquad M_{\mu\nu}^\dagger = -M_{\mu\nu}, \tag{43}$$

where the order of elements is inverted under the involution. On the other hand, the realization (6) is in general not Hermitian. The Hermiticity condition can be immediately implemented as soon as the expression

$$\tilde{x}_\mu = \tfrac{1}{2}(x_\mu \varphi_1 + s(xp)p_\mu \varphi_2 + \varphi_1^\dagger x_\mu^\dagger + s\varphi_2^\dagger p_\mu^\dagger (xp)^\dagger) \tag{44}$$

is taken into account. However, the physical results do not depend on the choice of the representation as long as there exists a smooth limit $\tilde{x}_\mu \to x_\mu$ as $s \to 0$. We can thus restrict our attention to non-Hermitian realization only. Consequently, we focus on the real Snyder scalar field theory, while the complex one can be straightforwardly defined.

B. Action for scalar field theory

We are now able to construct a Lagrangian for the noncommutative scalar field (39). Let us start by analyzing how the ordinary kinematic term $(\partial_\mu\phi)(\partial^\mu\phi)$ is changed in the Snyder space-time. Following the previous reasoning, the corresponding term in the noncommutative framework is given by $(\partial_\mu\tilde{\phi})(\partial^\mu\tilde{\phi})\triangleright\mathbb{I}$ (notice that the derivative is still with respect to the commutative coordinates, i.e. $\partial_\mu = \partial/\partial x^\mu$). Such a term, expressed by means of the Fourier transformation (39), is uniquely defined. In fact, in order for the differentiation to make sense, we have to first project the plane waves on $\mathbb{I}$ and then act on these by differentiation. By using (21) and (30), the relation

$$(\partial_\mu e^{i(K^{-1}\tilde{x})})e^{i(qx)} = i(\mathcal{D}_\mu - q_\mu)e^{i(\mathcal{D}(k,q)x)} = ik_\mu e^{i(\mathcal{D}(k,q)x)} \tag{45}$$

follows. The kinematic part, considering (20) and (45), is then given by

$$\begin{aligned}(\partial_\mu\tilde{\phi})(\partial^\mu\tilde{\phi})\triangleright\mathbb{I} &= \int [d^2k]\hat{\phi}_{k_1}\hat{\phi}_{k_2}(\partial_\mu e^{i(K_1^{-1}\tilde{x})})\partial^\mu(e^{i(K_2^{-1}\tilde{x})}\triangleright\mathbb{I}) \\ &= -\int [d^2k]\hat{\phi}_{k_1}\hat{\phi}_{k_2}(k_1k_2)e^{i(\mathcal{D}(k_1,k_2)x)},\end{aligned} \tag{46}$$

where $[d^nk] = [dk_1]\ldots[dk_n]$ and $\phi_k = \phi(k)$. This expression leads to the correct ordinary result as $s = 0$. As in (42), the star-product prescription leads to the same result with respect to our construction:

$$(\partial_\mu\tilde{\phi})(\partial^\mu\tilde{\phi})\triangleright\mathbb{I} = (\partial_\mu\phi)\star(\partial^\mu\phi). \tag{47}$$

The action for a noninteracting massive scalar field on Snyder space-time then reads

$$\begin{aligned}I &= \int d^4x(\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + m^2\tilde{\phi}^2)\triangleright\mathbb{I} \\ &= \int d^4x[(\partial_\mu\phi)\star(\partial^\mu\phi) + m^2(\phi\star\phi)].\end{aligned} \tag{48}$$

Because of the antipode (30), the action in the momentum space can be trivially written.

The noncommutativity effects are thus summarized within the coproduct (27), i.e. within the non-Abelian sum $\mathcal{D}_\mu(k_1, k_2)$. The noncommutative corrections to the ordinary theory depend on realizations. For each type of realization different actions appear.

Finally, we investigate the role of self-interactions. In particular, we consider the cubic $\tilde{\phi}^3(\tilde{x})\triangleright\mathbb{I}$ and quartic $\tilde{\phi}^4(\tilde{x})\triangleright\mathbb{I}$ interaction terms. These terms can be immediately obtained. The generalization of (21) to three plane waves, considering also (22), reads

$$e^{i(K_3^{-1}\tilde{x})}(e^{i(K_2^{-1}\tilde{x})}(e^{i(K_1^{-1}\tilde{x})}\triangleright\mathbb{I})) = e^{i(\mathcal{D}_3(k_3,k_2,k_1)x)}, \tag{49}$$

in which $(\mathcal{D}_3)_\mu(k_3, k_2, k_1) = \mathcal{D}_\mu(k_3, \mathcal{D}(k_2, k_1))$. This defines the cubic term

$$\tilde{\phi}^{\,3}(\tilde{x})\triangleright\mathbb{I} = (\phi \star (\phi \star \phi))(x). \tag{50}$$

The quartic term $\tilde{\phi}^4(\tilde{x})\triangleright\mathbb{I}$ is determined in the same way. Given four plane waves we have

$$e^{i(K_4^{-1}\tilde{x})}(e^{i(K_3^{-1}\tilde{x})}(e^{i(K_2^{-1}\tilde{x})}(e^{i(K_1^{-1}\tilde{x})}\triangleright\mathbb{I}))) = e^{i(\mathcal{D}_4 x)}, \tag{51}$$

and therefore

$$\tilde{\phi}^{\,4}(\tilde{x})\triangleright\mathbb{I} = (\phi \star (\phi \star (\phi \star \phi)))(x), \tag{52}$$

where $(\mathcal{D}_4)_\mu = \mathcal{D}_\mu(k_4, \mathcal{D}_3(k_3, k_2, k_1))$.

Summarizing, we have defined a Lagrangian density for a self-interacting scalar field on the Snyder noncommutative space-time geometry. Our framework, which is based on realizations, uniquely fixes the theory. This is relevant because the coproduct is non-coassociative (the corresponding star product is nonassociative). This feature would lead, *a priori*, to a nonunique definition of the model. Such a shortcoming is bypassed in our construction.

C. Relation with other approaches

Our construction differs with respect to the usual ones on two main points: the dimensions of the structure underlying the theory and the adopted algebra.

The scalar field theory on Snyder space-time is usually formulated by considering a five-dimensional structure [25–28]. The same happens for the field theories on κ Minkowski [15–17]. In particular, the momentum space is the de Sitter section in a five-dimensional flat space, and a deformed Fourier measure is thus needed to ensure the Lorentz invariance [25–28]. In κ Minkowski, a five-dimensional differential structure predicts some unphysical ghost modes [15] (to overcome this feature a twist deformation of the symmetry has been proposed [53,54]). On the other hand, our theory is defined on a four-dimensional space-time. No extra measures are needed, and the theory has the same field structure of the commutative framework. The Snyder-deformed symmetry algebra is the original undeformed one, and only the coproduct structure changes. Interesting, this is exactly the framework arising from the twist formulation of noncommutative field theories [18,53,54].

The second difference with respect to other approaches is that our theory is generic. All the possible realizations of the algebra are taken into account. The other attempts to construct a scalar field theory on the Snyder space-time are in fact based on a particular realization only. Our theory in the Snyder type of realization (10) corresponds, up to the momentum-space duality, to the previous proposals [25–28].

V. FIRST-ORDER CORRECTIONS

In this section we explicitly compute the generic noncommutative corrections, up to the first order in s, to the commutative theory.

As we have seen, all the noncommutative information is summarized in the non-Abelian sum (25), namely, in the coproduct (27). We are thus interested in the function $(\mathcal{D}_4)_\mu = (\mathcal{D}_4)_\mu(k_4, k_3, k_2, k_1)$ defined in (51). The functions $(\mathcal{D}_3)_\mu(k_3, k_2, k_1)$ and $\mathcal{D}_\mu(k_2, k_1)$, which define the cubic and quadratic terms, are clearly recovered from this one as soon as $k_4 = 0$ and $k_4 = k_3 = 0$, respectively. The $(\mathcal{D}_4)_\mu$ function can be expanded in the deformation parameter s as

$$\begin{aligned}(\mathcal{D}_4)_\mu &= (\mathcal{D}_4)^0_\mu + s(\mathcal{D}_4)^1_\mu + \mathcal{O}(s^2),\\ (\mathcal{D}_4)^0_\mu &= (k_1)_\mu + (k_2)_\mu + (k_3)_\mu + (k_4)_\mu,\\ (\mathcal{D}_4)^1_\mu &= \alpha(k_1)_\mu + \beta(k_2)_\mu + \gamma(k_3)_\mu + \delta(k_4)_\mu,\end{aligned} \tag{53}$$

where the superscript denotes the order in s.

The correction term $(\mathcal{D}_4)^1_\mu$ depends on realizations through $\alpha = \alpha(\varphi_1)$, $\beta = \beta(\varphi_1)$, $\gamma = \gamma(\varphi_1)$ and $\delta = \delta(\varphi_1)$. These functions are given by

$$\begin{aligned}\alpha = c[&k_2^2 + k_3^2 + k_4^2 + 2(k_1k_2 + k_3k_2 + k_3k_1 + k_4k_3\\ &+ k_4k_2 + k_4k_1)],\end{aligned} \tag{54}$$

$$\begin{aligned}\beta = (c - \tfrac{1}{2})k_1^2 &+ (2c - \tfrac{1}{2})(k_1k_2) + c[k_3^2 + k_4^2\\ &+ 2(k_3k_2 + k_3k_1 + k_4k_3 + k_4k_2 + k_4k_1)],\end{aligned} \tag{55}$$

$$\begin{aligned}\gamma = (c - \tfrac{1}{2})&(k_1 + k_2)^2 + (2c - \tfrac{1}{2})(k_3k_2 + k_3k_1)\\ &+ c[k_4^2 + 2(k_4k_3 + k_4k_2 + k_4k_1)],\end{aligned} \tag{56}$$

$$\begin{aligned}\delta = (c - \tfrac{1}{2})&(k_1 + k_2 + k_3)^2\\ &+ (2c - \tfrac{1}{2})(k_4k_3 + k_4k_2 + k_4k_1).\end{aligned} \tag{57}$$

The value of the constant c determines the realization in which we are working. The Snyder (10), the Maggiore (11), and the Weyl (13) types of realization are, respectively, recovered for $c = 1/2, 0, 1/6$.

VI. CONCLUDING REMARKS

In this paper we have constructed a scalar field theory on the Snyder noncommutative space-time. The next step will be the quantization of the model in order to investigate the fate of UV divergences and thus fully analyze the Snyder proposal.

We have shown that the deformations of symmetries are all contained in the co-algebraic sector and that the coproduct is non-coassociative. By using the realizations of the Snyder algebra we have constructed a well-defined (namely, nonambiguous) self-interacting scalar field theory. The ambiguities carried out by the nonassociative sum of momenta (and thus the nonassociative star product) have been overcome by the use of realizations. By means of a map between the noncommutative functions and the commutative ones, a scalar field action has been constructed. This theory has been directly defined on the space-time and, since the Fourier space has been identified with the de Sitter space, it is dual to a field theory over the coset $SO(4,1)/SO(3,1)$. Finally, we have computed the first-order corrections in a generic realization.

As a last point, it is interesting to mention that we can construct Noether charges for the Snyder space-time. As was shown in [20,21], the key ingredient to build Noether charges in a noncommutative theory is a Poisson map between the deformed and the undeformed spaces of solutions of the Klein-Gordon equation. In our framework this kind of map is given by the projection of noncommutative functions on the "vacuum," as in (18). By using this map it is possible to induce a symplectic structure on the space of the noncommutative functions and thus obtain a conserved symplectic product defining charges. This analysis will be reported elsewhere [55].

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