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Diffusion in the general theory of relativity

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The Markovian diffusion theory in the phase space is generalized within the framework of the general theory of relativity. The introduction of moving orthonormal frame vectors both for the position as well the velocity space avoids difficulties in the general relativistic stochastic calculus. The general relativistic Kramers equation in the phase space is derived both in the parametrization of phase-space proper time and the coordinate time. The transformation of the obtained diffusion equation under hypersurface-preserving coordinate transformations is analyzed and diffusion in the expanding universe is studied. It is shown that the validity of the fluctuation-dissipation theorem ensures that in the quasisteady state regime, the result of the derived diffusion equation is consistent with the kinetic theory in thermodynamic equilibrium.

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I. INTRODUCTION

The theory of Markovian diffusion within the framework of the relativity theory has reattracted considerable interest during the last decade. Compared with the nonrelativistic case, relativistic Markovian diffusion becomes significantly more difficult due to conceptual and technical problems. Over the years, many papers have been devoted to this issue (see e.g. [1–12]); a few also studied diffusion in gravitational fields within the theory of general relativity [10–12]. However, to the best of the author's knowledge, a generally accepted formulation of the theory of Markovian diffusion within the framework of general relativity is still missing. Most investigations of statistical processes in relativistic fluids in cosmology and astrophysics used alternative physical approaches given by relativistic kinetic theory or phenomenological relativistic thermodynamics [13–19]. In particular, the role of dissipative processes in the early evolution of the Universe has been studied by using nonequilibrium thermodynamics [20,21]. The relativistic Boltzmann equation is an integral-differential equation, which is difficult to solve in the nonequilibrium case far from equilibrium. On the other hand, in the probabilistic approach, Fokker-Plack-type equations are differential equations which can be solved much more simply. Additionally, the probabilistic theory of diffusion processes within the framework of general relativity exhibits a fundamental interest and plays a significant role in astrophysical and cosmological problems (see e.g. [22–28]).

A crucial factor in relativistic diffusion is the fact that the velocity space in relativity is a hyperboloid (or a special three-dimensional Riemannian manifold) embedded into the four-dimensional velocity Minkowski space. In the derivation of the relativistic diffusion equation, this specific feature has to be taken into account in an appropriate way; otherwise, inconsistent results arise. One of the main difficulties in the derivation of a consistent Markovian diffusion equation in relativity is to handle the fundamental Wiener process in a stochastic differential equation on non-Euclidian manifolds in a rigorous way. There exists a well developed and rigorous mathematical theory of stochastic differential equations and diffusion processes in the base space on Riemannian manifolds with a definite metric signature [29,30]. However, this stochastic calculus cannot be applied for manifolds with indefinite metric. Recently, the author presented a physically motivated modification of this calculus to describe diffusion within the framework of special relativity in the phase space of position and velocity [31]. In the present paper, this formalism will be extended to a theory within the framework of general relativity in external gravitational fields. The main aim of the paper is the derivation of a generalized Kramers equation in external gravitational fields within the framework of general relativity theory, both in the parametrization of the phase-space proper time and of the observer time within the approach of space-time decomposition. The transformation property of this equation with respect to foliation-preserving coordinate transformations is analyzed and diffusion in the evolving universe is studied. It is shown that the quasisteady solution is in agreement with the kinetic theory in thermodynamic equilibrium.

The paper is organized as follows. In Sec. II the principal results pertaining to the special relativistic diffusion process are briefly described. In Sec. III the formalism of Markovian diffusion in gravitational fields within the framework of general relativity is presented both in the parametrization of the phase-space proper time and the observer time. In Sec. IV the transformation property of the derived diffusion equation is studied. As an example for the application of the formalism, diffusion processes in the expanding universe are discussed in Sec. V. The conclusions are presented in Sec. VI.

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II. SPECIAL RELATIVISTIC DIFFUSION PROCESS

Let us first review the principal results pertaining to the special relativistic case [31] used as the basis for the generalization within the frame of general relativity.

The important point in relativistic diffusion is the observation that the velocity space in special relativity is a noncompact hyperbolic three-dimensional Riemannian manifold embedded into the four-dimensional velocity Minkowski space. Using normalized velocity variables u^μ ($\mu = 0, 1, 2, 3$) the hyperbolic metric structure for a relativistic system of massive particles is described by the relation

$$(u^0)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2 = 1. \tag{1}$$

Therefore, relativistic Markovian diffusion processes can be described in a rigorous way by using the mathematical stochastic calculus on Riemannian manifolds, but adapted to the velocity space.

Stochastic differential equations in diffusion theory with continuous pathways are defined by the fundamental Wiener process $W^a(t)$. On a Riemannian manifold, the fundamental Wiener process is difficult to handle. The key idea in the mathematical concept of diffusion on general d-dimensional Riemannian manifolds with definite metric signature is to define a stochastic process on the curved manifold using the fundamental Wiener process, each component of which is a process in the Euclidian space R^d [29,30]. The tangent space of a Riemannian manifold is endowed with an Euclidian structure, and therefore we can move the manifold in the tangent space by construction of a parallel translation along the stochastic curve with the help of the orthonormal frame vectors $e_a = e^i_a(\mathbf{x})\partial_i$ $(i, a = 1 \dots d)$ and the Christoffel connection coefficients Γ^j_{ik}, $\mathbf{x} = (x_1 \dots x_d)$, $\partial_i = \partial/\partial x^i$. In local coordinates on a Riemannian manifold, the infinitesimal motion of a smooth curve $c^i(t)$ in M^d is that of $\gamma^i(t)$ in the tangent space by using a parallel transformation: $dc^i = e^i_a(\mathbf{x})d\gamma^a$ and $de^i_a(\mathbf{x}) = -\Gamma^i_{ml}e^l_a dc^m$. Therefore, a random curve in the stochastic mathematical calculus on Riemannian manifolds in the position space can be defined in the same way by using the canonical realization of a d-dimensional Wiener process (defined in the Euclidian space) and substituting $d\gamma^a \to dW^a(t)$. Thus the stochastic differential equations describing diffusion on a Riemannian manifold in the orthonormal frame bundle $O(M)$ with coordinates $O(M) = \{x^i, e^i_a\}$ are given by [29,30]

$$\begin{aligned} dx^i(\tau) &= e^i_a(\tau) \circ dW^a_\tau + b^i(\tau)d\tau, \\ de^i_a(\tau) &= -\Gamma^i_{ml}e^l_a \circ dx^m(\tau). \end{aligned} \tag{2}$$

Here $\delta^{ab}e^i_a(\mathbf{x}(\tau))e^j_b(\mathbf{x}(\tau)) = g^{ij}$, $\partial_i e^j_a = -\Gamma^j_{ik}e^k_a$, $b^i(\tau)$ is a vector field, g^{ij} is the Riemannian metric and δ^{ab} the flat Euclidian metric where δ^{ab} is the Kronecker symbol.

The general mathematical model given by Eq. (2) describes diffusion in the position space. It cannot be generalized to pseudo-Riemannian manifolds in the general relativity theory. But Eq. (2) exhibits a significant difference to the nonrelativistic Langevin equation in physics, where the stochastic force acts directly only on the change of the velocity and not on the position coordinates. Therefore, the mathematical stochastic calculus for Riemannian manifolds can be applied with an appropriate modification, adapting it to the velocity space. This requires the introduction of a moving velocity frame $E^i_a(\tau)$. A generalization of the Langevin equations within the special relativity theory can be defined in the fiber bundle space $F(M_L) = \{x^i, u^i, E^i_a\}$ by [31]

$$\begin{aligned} dx^i(\tau) &= u^i(\tau)d\tau, \\ du^i(\tau) &= E^i_a(\tau) \circ dW^a(\tau) + F^i(\tau)d\tau, \\ dE^i_a(\tau) &= -\gamma^i_{ml}(\mathbf{u})E^l_a(\tau) \circ du^m(\tau). \end{aligned} \tag{3}$$

Here τ is an evolution parameter along the world lines of the particles which can be chosen as the proper time. The laboratory time $t = \tau u^0/c$ is a function of the proper time τ and u^0, which here and below is defined by $u^0 = [1 + (u^1)^2 + (u^2)^2 + (u^3)^2]^{1/2}$. $\gamma^i_{ml}(\mathbf{u})$ are the Christoffel connection coefficients on the hyperboloid and $F^i = K^i/m$, where K^i are the spatial components of the 4-force ($i = 1, 2, 3$), m is the rest mass of the particles, and the a, b numbers are the moving frame vectors E_a in the hyperbolic velocity space ($a, b = 1, 2, 3$). The stochastic force is described by the fundamental Wiener process with $\langle W^a\rangle = 0$ and the correlator $\langle W^a(\tau)W^b(\tau + s)\rangle = Ds\delta_{ab}$; it is defined by an empirical diffusion constant D, which is independent of the velocity. Stochastic integrals related to Eq. (3) are defined in the Stratonovich calculus denoted by the symbol $\circ$. Since the manifold on the hyperboloid is embedded into the Minkowski space, metric $G_{ij}(u)$ and connection coefficients $\gamma^i_{jk}(\mathbf{u})$ in the velocity space are given by

$$G_{ij}(\mathbf{u}) = \delta_{ij} - (u^i u^j)/(u^0)^2, \qquad \gamma^i_{jk}(\mathbf{u}) = -u^i G_{jk}. \tag{4}$$

Associated with the diffusion process described by Eq. (3) is a diffusion generator $\mathbf{A}_{F(M_L)}$:

$$\mathbf{A}_{F(M_L)} = \frac{D}{2}\delta^{ab}H_a H_b + H_0, \tag{5}$$

where the fundamental horizontal vector fields H_a and H_0 on the fiber bundle $F(M_L)$ are given by

$$\begin{aligned} H_a &= E^i_a\frac{\partial}{\partial u^i} - \gamma^i_{ml}(\mathbf{u})E^l_a E^m_b\frac{\partial}{\partial E^i_b}, \\ H_0 &= u^i\frac{\partial}{\partial x^i} + F^i\frac{\partial}{\partial u^i} - \gamma^i_{ml}(\mathbf{u})E^l_a(\tau)F^m\frac{\partial}{\partial E^i_a}. \end{aligned} \tag{6}$$

We project the stochastic curve from the fiber space $F(M_L)$ with coordinates $\mathbf{r} = \{x^i, u^i, E^i_a\}$ to the phase space with coordinates $\{x^i, u^i\}$: $\mathbf{A}_{F(M_L)} f(\mathbf{x}, \mathbf{r}) = \mathbf{A}_P f(\mathbf{x}, \mathbf{u})$, where the diffusion generator in the phase space $\mathbf{A}_P$ is given by

$$\mathbf{A}_P = \frac{D}{2}\delta^{ab} E^i_a \frac{\partial}{\partial u^i} E^j_b \frac{\partial}{\partial u^j} + u^i \partial/\partial x^i + F^i \partial/\partial u^i. \quad (7)$$

The generator $\mathbf{A}_P$ describes how the expected value of any smooth function $f(\mathbf{x}, \mathbf{u})$ evolves in time and satisfies the following equation:

$$\frac{\partial}{\partial \tau}\varphi(\tau, \mathbf{x}, \mathbf{u}) = \mathbf{A}\varphi(\tau, \mathbf{x}, \mathbf{u}), \quad (8)$$

with $\varphi(0, \mathbf{x}, \mathbf{u}) = f(\mathbf{x}, \mathbf{u})$. Equation (8) is a Kolmogorov backward equation. A Fokker-Planck equation (or forward Kolmogorov equation) describes how the probability density function $\phi(\tau, \mathbf{x}, \mathbf{u})$ evolves with time and is determined by the adjoint of the diffusion operator $\mathbf{A}^*_P$. Therefore the special relativistic diffusion equation in the phase space takes the following form [31]:

$$\frac{\partial \phi}{\partial \tau} = -u^i \frac{\partial \phi}{\partial x^i} - \mathrm{div}_{\mathbf{u}}(\mathbf{F}\phi) + \frac{D}{2}\Delta_{\mathbf{u}}\phi, \quad (9)$$

where $\Delta_{\mathbf{u}}$ is the Laplace-Beltrami operator of the hyperbolic velocity space given by

$$\Delta_{\mathbf{u}} = G^{ij}\frac{\partial^2}{\partial u^i \partial u^j} - G^{ij}\gamma^k_{ij}\frac{\partial}{\partial u^k} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial u^i}\left(\sqrt{G}G^{ij}\frac{\partial}{\partial u^j}\right), \quad (10)$$

and the corresponding divergence operator is given by

$$\mathrm{div}_{\mathbf{u}}(\mathbf{F}\phi) = \frac{1}{\sqrt{G}}\frac{\partial}{\partial u^i}(\sqrt{G}F^i\phi), \quad (11)$$

with $G = \det\{G_{ij}\} = (u_0)^{-2}$ and $G^{ij} = \delta^{ij} + u^i u^j$.

III. GENERAL RELATIVISTIC DIFFUSION EQUATION

The special relativistic formalism briefly described in Sec. II can be extended to a theory within the framework of general relativity. In doing so, in addition to orthonormal frame vectors in the velocity space, the introduction of orthonormal frame vectors in the position space is required.

In the case of general relativity the four-velocity v^μ of massive particles in the presence of a gravitational field with a metric $g_{\mu\nu}(x)$ satisfies the condition

$$g_{\mu\nu}(x) v^\mu v^\nu = 1, \quad (12)$$

depending not only on the velocities v^ν but also on the coordinates x^μ. The complication arising by this fact can be bypassed by using the orthonormal frame vectors $e_M = e^\mu_M(x)\partial_\mu$ in the position space, where the subscript M, N numbers the vectors ($M, N = 0, 1, 2, 3$), m, n their spatial components ($m, n = 1, 2, 3$), μ their components in the coordinate basis $e_\mu = \partial_\mu$ ($\mu = 0, 1, 2, 3$), i, j the spatial components in the coordinate basis ($i, j = 1, 2, 3$) and $x = \{x^0, x^1, x^2, x^3\}$. The frame vectors satisfy the condition

$$\eta^{MN} e^\mu_M(x) e^\nu_N(x) = g^{\mu\nu}, \qquad g_{\mu\nu} e^\mu_M(x) e^\nu_N(x) = \eta_{MN}, \quad (13)$$

where $\eta_{MN} = \mathrm{diag}(-1, 1, 1, 1)$ is the metric of the Minkowski space. The dual basis of the frame fields e_M are cotangent frame one-forms $\theta^M = \theta^M_\mu dx^\mu$, satisfying the orthogonality relation

$$e^\mu_M(x)\theta^M_\nu(x) = \delta^\mu_\nu. \quad (14)$$

Each vector referring to the coordinate basis $e_\mu = \partial_\mu$ is assigned a vector referring to the frame basis e_M according to the rule

$$v^\nu = e^\nu_M(x) v^M. \quad (15)$$

Correspondingly, we find that for v^M,

$$v^M = \theta^M_\nu(x) v^\nu. \quad (16)$$

By using the orthogonal moving frames of the pseudo-Riemannian manifold in the position space, the relation (12) gets the same form as in the special relativistic case in Eq. (1):

$$g_{\mu\nu}(x) v^\mu v^\nu = g_{\mu\nu}(x) e^\mu_M(x) e^\nu_N(x) v^M v^N = \eta_{MN} v^M v^N = 1. \quad (17)$$

The covariant derivative in the natural frame is defined as usual,

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho, \quad (18)$$

where $\Gamma^\nu_{\mu\rho}$ are the Christoffel connection coefficients (or coordinate connections). The covariant derivative of vectors referring to the frame basis can be written as

$$\nabla_\mu v^M = \partial_\mu v^M + \Omega^M_{\mu N} v^N, \quad (19)$$

where $\Omega^M_{\mu N}$ are called coefficients of spin connection (or frame connection). The relation between these two kinds of connections arises from the metric compatibility condition, which here can be expressed by $\nabla_\mu \theta^M_\nu = 0, \nabla_\mu e^\nu_M = 0$. From this condition, one gets [32,33]

$$\Omega^M_{\mu N}(x) = \theta^M_\nu(x)\Gamma^\nu_{\mu\rho} e^\rho_N(x) + \theta^M_\nu(x)\partial_\mu e^\nu_N(x). \quad (20)$$

The parallel transport of the components of the four-vector v^M in the moving frame e_A and the line elements dx^μ are given by

$$dv^M = -\Omega^M_{\mu N}(x) v^N dx^\mu, \qquad dx^\mu = e^\mu_M(x) v^M d\tau. \quad (21)$$

As seen in Eq. (21), the change of the spatial components of the velocity v^m of a particle is determined by the gravitational force described by the spin connection coefficients $\Omega^m_{\mu N}(x)$. In addition, one has to take into account that the velocity is changed by a stochastic force F^m_{noise} driven by the Wiener process $dW^a(t)$. In order to avoid difficulties in the description of the Wiener process on the velocity hyperbolic manifold, the mathematical stochastic calculus on Riemannian manifolds has to be applied where the Wiener process is moved along the orthonormal frames $E_a^m(u)$ $(a = 1, 2, 3)$ [31] in the three-dimensional hyperbolic velocity space defined by the relation

$$\delta^{ab} E_a^m E_b^n = G^{mn}, \tag{22}$$

or, equivalently,

$$G_{mn} E_a^m E_b^n = \delta_{ab}, \tag{23}$$

where G_{mn} is the Riemannian metric of the hyperbolic velocity space and G^{mn} is the inverse matrix of G_{mn}. The metric and the Christoffel coefficients in the velocity space are defined in Eq. (4). The infinitesimal motion of the velocity $v^m(\tau)$ in phase space can be described by that of $u^a(\tau)$ $(a = 1, 2, 3)$ in the moving velocity frame E_a^m by using the parallel transformation

$$dv^m = E_a^m du^a \qquad dE_a^m(\tau) = -\gamma_{nl}^m(\mathbf{v}) E_a^l(\tau) dv^n \tag{24}$$

A random curve in the phase space can be defined as in Sec. II by using the Wiener process, substituting $du^a \rightarrow dW^a(\tau)$. Therefore, generalizing Eq. (3) in a consistent description of Markovian diffusion in general relativity, the noise force is described by $F_{\text{noise}}^m = E_a^m(\tau) \circ dW^a$. A remarkable feature of Markovian diffusion on a Riemannian manifold is the supposition that for the diffusion coefficients, only the orthonormal frame coefficients E_a^m, which are directly related to the hyperbolic geometry of the velocity space, are admissible. In contrast, on Euclidian manifolds, a much more general class of diffusion coefficients is permitted. Therefore, generalizing Eq. (3), the stochastic differential equations describing diffusion in gravitational and external force fields (general relativistic Langevin equations) are given by

$$\begin{aligned} dx^i &= e_M^i(x) v^M d\tau, \\ dv^m &= E_a^m(\tau) \circ dW^a - \Omega_{\mu N}^m(x) e_M^\mu(x) v^N v^M d\tau + F_{\text{ex}}^m d\tau, \\ dE_a^m(\tau) &= -\gamma_{nl}^m(\mathbf{v}) E_a^l(\tau) \circ dv^n. \end{aligned} \tag{25}$$

Here an additional external force, $F_{\text{ex}}^\mu = K_{\text{ex}}^\mu / m$, is taken into account where K_{ex}^μ are the components of the external four-force in the coordinate frame and m is the rest mass of the particles. In the moving frame, this force is expressed by $F_{\text{ex}}^m = \theta_\mu^m F_{\text{ex}}^\mu$. The Christoffel connection coefficients on the hyperboloid $\gamma_{nl}^m(\mathbf{v})$ are defined as in Eq. (4), and τ is a parameter defined along the world line of the particles, which can be chosen as the phase-space proper time.

Sufficient and necessary conditions for the existence and uniqueness of the stochastic differential Eq. (25) are that the drift and diffusion coefficients satisfy the uniform Lipschitz condition and the stochastic process $\mathbf{X}(\tau) = \{\mathbf{x}(\tau), \mathbf{v}(\tau)\}$ is adapted to the Wiener process $W^a(\tau)$, that is, the output $X(\tau_2)$ is a function of $W^a(\tau_1)$ up to that time $(\tau_1 \leq \tau_2)$.

The diffusion operator $\mathbf{A}_{F(M)}$ in the fiber bundle $F(M_L)$ with coordinates $\mathbf{r} = \{x^i, v^m, E_a^m\}$ for the stochastic process described by Eq. (25) is defined as in Eq. (5), with horizontal vector fields H_a and H_0 derived analogous, as in Sec. II:

$$\begin{aligned} H_a &= E_a^m \frac{\partial}{\partial v^m} - \gamma_{nl}^m(\mathbf{v}) E_a^l E_b^n \frac{\partial}{\partial E_b^m}, \\ H_0 &= e_M^i(x) v^M \frac{\partial}{\partial x^i} - \Omega_{\mu N}^m(x) e_M^\mu(x) v^N v^M \frac{\partial}{\partial v^m} + F_{\text{ex}}^m \frac{\partial}{\partial v^m} \\ &\quad - \gamma_{nl}^m(\mathbf{v}) E_a^l F^n \frac{\partial}{\partial E_a^m}. \end{aligned} \tag{26}$$

The operator $\mathbf{A}_{F(M)}$ can be projected to the phase space with coordinates $\{x^i, v^m\}$ by $\mathbf{A}_{F(M)} f(\mathbf{x}, \mathbf{u}) = \mathbf{A}_P f(\mathbf{x}, \mathbf{u})$, where the diffusion generator in the phase space $\mathbf{A}_P$ is given by

$$\mathbf{A}_P = \frac{D}{2} \delta^{ab} E_a^m \frac{\partial}{\partial v^m} E_b^n \frac{\partial}{\partial v^n} + e_M^i(x) v^M \frac{\partial}{\partial x^i} + F^m \frac{\partial}{\partial v^m}. \tag{27}$$

Here, the first term contains the Laplace-Beltrami operator $\Delta_{\mathbf{v}} = \delta^{ab} E_a^m \frac{\partial}{\partial v^m} E_b^n \frac{\partial}{\partial v^n}$ in the hyperbolic velocity space. The force F^m is composed of the gravitational force and the external force F_{ex}^m: $F^m = -\Omega_{\mu N}^m(x) e_M^\mu(x) v^N v^M + F_{\text{ex}}^m$. The backward Kolmogorov equation for the stochastic process described by Eq. (25) is defined by

$$\frac{\partial}{\partial \tau} \varphi(\tau, \mathbf{x}, \mathbf{v}) = \mathbf{A}_P \varphi(\tau, \mathbf{x}, \mathbf{v}). \tag{28}$$

The corresponding Fokker-Planck equation in phase space (general relativistic Kramers equation) within the frame of general relativity is defined by the adjoint of the operator $\mathbf{A}_P$. Since the Laplace-Beltrami operator is self-adjoint $\Delta_{\mathbf{v}} = \Delta_{\mathbf{v}}^+$, it takes the form

$$\frac{\partial \Phi}{\partial \tau} = -v^M \text{div}_{\mathbf{x}}(\mathbf{e}_M(x)\Phi) - \text{div}_{\mathbf{v}}(\mathbf{F}\Phi) + \frac{D}{2} \Delta_{\mathbf{v}} \Phi, \tag{29}$$

where $\Delta_{\mathbf{v}}$ is the Laplace-Beltrami operator in the hyperbolic velocity space

$$\begin{aligned} \Delta_{\mathbf{v}} &= G^{mn} \frac{\partial^2}{\partial v^m \partial v^n} - G^{mn} \gamma_{mn}^l \frac{\partial}{\partial v^l} \\ &= \frac{1}{\sqrt{G}} \frac{\partial}{\partial v^m} \left(\sqrt{G} G^{mn} \frac{\partial}{\partial v^n} \right). \end{aligned} \tag{30}$$

The divergence operator in the 3D position space is given by

$$\text{div}_{\mathbf{x}}(\mathbf{e}_M(x)\Phi) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} e_M^i(x) \Phi), \tag{31}$$

and in the 3D velocity space by

$$\mathrm{div}_{\mathbf{v}}(\mathbf{F}\Phi) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial v^m}(\sqrt{G} F^m \Phi), \tag{32}$$

with $G = \det\{G_{ij}\}$, $g = \det\{g_{ij}\}$.

Equation (29) is the diffusion equation in the parametrization of the phase-space proper time within the frame of general relativity for the probability density function $\Phi = \phi(\tau; \mathbf{x}, \mathbf{v})$ or for the transition probability $\Phi(\mathbf{x}, \mathbf{v}, \tau \mid \mathbf{x}_0, \mathbf{v}_0, 0)$.

For the solution of the relativistic diffusion equation it is convenient to introduce the hyperbolic coordinate system for the four-velocity defined by $v^1 = \mathrm{sh}\,\alpha \sin\vartheta \cos\varphi$, $v^2 = \mathrm{sh}\,\alpha \sin\vartheta \sin\varphi$, $v^3 = \mathrm{sh}\,\alpha \cos\vartheta$, and $v^0 = \mathrm{ch}\,\alpha$. We denote the velocities in the non-Cartesian coordinates by $\bar{v}^1 = \alpha$, $\bar{v}^2 = \theta$, $\bar{v}^3 = \varphi$, $a = 1, 2, 3$. The metric in these coordinates are simply to calculate and are given by $G_{11} = 1$, $G_{22} = \mathrm{sh}^2\alpha$, $G_{33} = \mathrm{sh}^2\alpha \sin^2\vartheta$, and $G_{ij} = 0$ for $i \neq j, G = \mathrm{sh}^4\alpha \sin^2\vartheta$. With the given metric, the Laplace-Beltrami operator $\Delta_{\mathbf{v}}$ takes the form

$$\Delta_{\mathbf{v}} = \frac{\partial^2}{\partial\alpha^2} + 2\,\mathrm{cth}\,\alpha \frac{\partial}{\partial\alpha} - \frac{1}{(\mathrm{sh}\,\alpha)^2}\left(\frac{\partial^2}{\partial\vartheta^2} + \mathrm{ctg}\,\vartheta \frac{\partial}{\partial\vartheta} + \frac{1}{(\sin\vartheta)^2} \frac{\partial^2}{\partial\varphi^2}\right), \tag{33}$$

and

$$\begin{aligned}\mathrm{div}_{\mathbf{v}}(\mathbf{F}\Phi) = {}& (\mathrm{sh}\,\alpha)^{-2} \frac{\partial}{\partial\alpha}((\mathrm{sh}\,\alpha)^2 F^\alpha \Phi) \\ & - (\mathrm{sh}\,\alpha)^{-1}(\sin\vartheta)^{-1} \frac{\partial}{\partial\vartheta}(\sin\vartheta F^\vartheta \Phi) \\ & - (\mathrm{sh}\,\alpha)^{-1}(\sin\vartheta)^{-1} \frac{\partial}{\partial\varphi}(F^\varphi \Phi)\end{aligned} \tag{34}$$

is the divergence operator. Here, the force components in the hyperbolic coordinate system $F^\alpha, F^\vartheta, F^\varphi$ are related with F^m by $F^\alpha = (\mathrm{ch}\,\alpha)^{-1}[\sin\vartheta(\cos\varphi F^1 + \sin\varphi F^2) + \cos\vartheta F^3]$, $F^\vartheta = (\mathrm{sh}\,\alpha)^{-1}[\cos\vartheta(\cos\varphi F^1 + \sin\varphi F^2) - \sin\vartheta F^3]$, and $F^\varphi = (\mathrm{sh}\,\alpha)^{-1}(\sin\vartheta)^{-1}[-\sin\varphi F^1 + \cos\varphi F^2]$.

The relativistic diffusion Eq. (29) is parametrized in terms of the phase-space proper time τ. But it is more convenient to parametrize the stochastic process alternatively in terms of the coordinate time because the gravitational fields and the external force fields are given in this parametrization. In general relativity, the observer time with the infinitesimal element $dx^0 = e^0_M(x) v^M d\tau$ is a function of the proper time τ and the space and velocity variables. This general definition introduces difficulties in the diffusion formalism. In order to avoid such problems and to simplify the derivation we describe diffusion in a frame system in which the timelike components of the frames $e_m(x)$ vanish: $e^o_m(x) = 0$. This condition can be achieved in general if we impose three-frame subsidiary conditions and remove a part of the six-frame arbitraries. In particular, the condition $e^o_m(x) = 0$ is introduced in the 1 + 3 space-time slicing in the Arnowitt-Deser-Misner (ADM) formalism [34] of the Hamiltonian formulation of gravity. In the ADM treatment, the space-time manifold is split into a one-parameter family of spacelike hypersurfaces $\Sigma(t)$ parametrized by a timelike function t or as a foliation of the hypersurfaces $t = \mathrm{const}$.

With vanishing frame components $e^o_m(x) = 0$, the proper time is given by $dx^0 = (g_{00})^{1/2} v^0 d\tau$ where v^0 is defined by the relation (17). The diffusion equation in the parametrization of the observer time can be derived from the stochastic differential Eq. (25) using the mathematical theorem of random time change in stochastic differential equations (see e.g. [29,30]). The proper time τ is related to x^0 by the random transformation

$$\tau = \int_0^{x^0} N(s)(v^0(s))^{-1} ds, \tag{35}$$

with $N = (g_{00})^{-(1/2)}$. Since τ depends only on the stochastic events v^m earlier than x^0 this random time change is an adapted process, and therefore the time change of an Ito integral is again an Ito integral, but driven by a different Wiener process $d\tilde{W}(x^0) = dW(\tau) N^{-(1/2)} (v^0)^{1/2}$ [29,30]. This rule for a random time change is valid within the Ito calculus. Using the transformation rules of an Ito integral into a Stratonovich integral and $d\tau = (v^0)^{-1} N dx^0$ the relativistic Langevin equation (25) can be rewritten in the parametrization with x^0 as follows:

$$\begin{aligned}dx^i(x^0) &= e^i_M(x) v^M (v^0)^{-1} N dx^0, \\ dv^m(x^0) &= E^m_a (v^0)^{-1/2} N^{1/2} \circ d\tilde{W}^a(x^0) \\ &\quad - \frac{D}{2} \delta^{ab} E^m_a E^n_b N (v^0)^{-1/2} \frac{\partial}{\partial v^n} \\ &\quad \times ((v^0)^{-1/2}) dx^0 + F^m (v^0)^{-1} N dx^0, \\ dE^m_a(x^0) &= -\gamma^m_{nl}(\mathbf{v}) E^l_a \circ dv^n(x^0).\end{aligned} \tag{36}$$

Note that in Eq. (36) an additional drift term proportional to the diffusion constant D arises which originates from the transformation rule from the Ito to the Stratonovich calculus for random time changes. The diffusion operator $\mathbf{A}_{F(M)}$ in the fiber bundle $F(M_L)$ with coordinates $\mathbf{r} = \{x^i, v^m, E^m_a\}$ for the stochastic process described by Eq. (25) is defined as in Eq. (5) with horizontal vector fields H_a and H_0 now given by

$$H_a = E_a^m(v^0)^{-1/2}N^{1/2}\frac{\partial}{\partial v^m} - \gamma_{nl}^m(\mathbf{v})E_a^l E_b^n (v^0)^{-1/2}N^{1/2}\frac{\partial}{\partial E_b^m}$$

$$H_0 = \left\{-\frac{D}{2}\delta^{ab}E_a^m E_b^n N(v^0)^{-1/2}\frac{\partial}{\partial v^n}((v^0)^{-1/2}) + (v^0)^{-1}NF^m\right\}\frac{\partial}{\partial v^m} + (v^0)^{-1}Ne_M^i(x)v^M\frac{\partial}{\partial x^i} \tag{37}$$

$$- (v^0)^{-1}N\gamma_{nl}^m(\mathbf{v})E_a^l\left\{F^n + \frac{D}{2}\delta^{bc}E_b^n E_c^k\frac{\partial}{\partial v^k}((v^0)^{-1/2})\right\}\frac{\partial}{\partial E_a^m},$$

with the gravitational and the external force $F^m = -\Omega_{\mu N}^m(x)e_M^\mu(x)v^N v^M + F_{\rm ex}^m$. The operator $\mathbf{A}_{F(M)}$ can be projected to the phase space with coordinates $\{x^i, v^i\}$ by $\mathbf{A}_{F(M)}f(\mathbf{x},\mathbf{v}) = \mathbf{A}_P f(\mathbf{x},\mathbf{v})$, where the diffusion generator in the phase space $\mathbf{A}_P$ is given by

$$\mathbf{A}_P = \frac{D}{2}(v^0)^{-1}N\delta^{ab}E_a^i\frac{\partial}{\partial v^i}E_b^j\frac{\partial}{\partial v^j} + e_M^i(x)v^M(v^0)^{-1}N\frac{\partial}{\partial x^i} + F^m(v^0)^{-1}N\frac{\partial}{\partial v^m}. \tag{38}$$

The general relativistic Kramers equation in the parametrization of the observer time x^0 can be derived analogous as above and is written as follows:

$$N^{-1}v^0\frac{\partial\Phi}{\partial x^0} = -v^M{\rm div}_{\mathbf{x}}(\mathbf{e}_M(x)\Phi) - {\rm div}_{\mathbf{v}}(\mathbf{F}\Phi) + \frac{D}{2}\Delta_{\mathbf{v}}\Phi. \tag{39}$$

As seen in the limit of special relativity, the left side and the first-term on the right-hand side (rhs) of Eq. (39) can be identified with the covariant expression $v^\mu\partial/\partial x^\mu$, while the other terms are identical with corresponding terms in Eq. (29).

Equation (39) is the main result of the present paper and represents the generalization of the Kramers equation within the frame of general relativity in the parametrization of the observer time for the probability density function $\Phi = \phi(x_0;\mathbf{x},\mathbf{v})$ with the initial condition $\phi(x_0 = 0;\mathbf{x},\mathbf{v}) = \phi_0(\mathbf{x},\mathbf{v})$. The transition probability is determined by the same equation but is defined by the initial condition $\Phi(\mathbf{x},\mathbf{v},\mathbf{0}\,|\,\mathbf{x}_0,\mathbf{v}_0,0) = (Gg)^{-1/2}\delta(v^1 - v_0^1)\times\delta(v^2 - v_0^2)\delta(v^3 - v_0^3)\delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$. For an external electromagnetic field $F_{\mu\nu}$, the normalized force $F_{\rm ex}^m$ in the moving frame is $F^m = e\theta_\mu^m g^{\mu\rho}F_{\rho\nu}e_N^\nu(x)v^N$.

IV. COORDINATE TRANSFORMATIONS

In the general relativistic framework, the invariance of the physical laws with respect of general coordinate transformations is one of the most fundamental properties. In the frame of special relativity, the probability density function $\phi(\tau;\mathbf{x},\mathbf{v})$ in the phase space is Lorentz invariant; i.e. it fulfills the condition

$$\phi'(\tau',\mathbf{x}',v') = \phi(\tau,\mathbf{x},\mathbf{v}), \tag{40}$$

where the variables $\mathbf{x}'$, $\mathbf{v}'$ are related to $\mathbf{x}$, $\mathbf{v}$ by a Lorentz transformation. Note that in contrast, the particle density and the current density (i.e. the integrals over the velocities) transform like a four-vector. As shown in [31] the special relativistic Eq. (9) suffices this condition and is invariant with respect to a Lorentz transformation. In the general theory of relativity, this invariance should be valid for general coordinate transformations. However, in the derivation of Eq. (39) using the parametrization of the observer time, we have taken advantage of the freedom in the choice of the orthonormal frame components and used the condition $e_m^o(x) = 0$, or correspondingly, a hypersurface foliation. In this frame, a coordinate-independent notion of time is demanded, and therefore we have to consider a foliation-preserving diffeomorphism described by the general space coordinate transformation

$$x'^i = f^i(x^j, t), \tag{41}$$

which preserve the hypersurface geometry. The hypersurface foliation is also preserved under an arbitrary time rescaling,

$$t' = f^0(t), \tag{42}$$

which does not affect the hypersurfaces. General covariance will then become a hidden feature, similar to the Hamiltonian formulation of gravity in the ADM formalism, but the underlying invariance of general relativity is intact and general coordinate transformations still map solutions into solutions.

Let us study the transformation property of the diffusion Eq. (39) in the parametrization of the observer time. Since the moving one-forms $\theta_\nu^A(x)$ transform like a covector, we find from the relation Eq. (16) that the vector components v^M in the moving frame are invariant with respect to general coordinate transformations $v^M = v'^M$, $F^M = F'^M$. Therefore, the operators ${\rm div}_v$ and Δ_v in the last two terms on the rhs of Eq. (39) are also invariant. From the transformation described by Eq. (41), we find with $x^0 = x'^0$ the following:

$$\frac{\partial}{\partial x^i} = \frac{\partial f^j}{\partial x^i}\frac{\partial}{\partial x'^j}. \tag{43}$$

The first term on the rhs of Eq. (39), including the three-dimensional ${\rm div}_{\mathbf{x}}$ operator with respect to the 3-geometry of the hypersurface, is intrinsically defined by the hypersurface and therefore invariant under the hypersurface-preserving transformation (41). This can be simply proven by the transformation property of the intrinsic covariant differentiation ${}^{(3)}\Delta_i F^j$ on the hypersurface given by ${}^{(3)}\Delta'_j F'^i = (\partial x'^i/\partial x^k)(\partial x^l/\partial x'^j){}^{(3)}\Delta_l F^k$. For the divergence operator $\mathrm{div}_{\mathbf{x}}(\mathbf{F}\phi) = {}^{(3)}\Delta_j(F^j\phi)$, this yields with $(\partial x'^i/\partial x^k)(\partial x^l/\partial x'^i) = \delta^l_k$ the relation ${}^{(3)}\Delta'_j F'^j = {}^{(3)}\Delta_j F^j$ or $\mathrm{div}_{\mathbf{x}'}(\mathbf{F}'\phi) = \mathrm{div}_{\mathbf{x}}(\mathbf{F}\phi)$. The time derivative on the left-hand side transforms under Eq. (41) as

$$\frac{\partial \Phi}{\partial x^0} = \frac{\partial \Phi}{\partial x'^0} - \beta^k \frac{\partial}{\partial x'^k}\Phi \qquad \beta^k = \frac{\partial f^k}{\partial x^j}\left(\frac{\partial x^j}{\partial x^0}\right)_{f^k=\mathrm{const}}. \tag{44}$$

The transformed relativistic Kramers equation (39) therefore takes the following form:

$$N^{-1}v'^0\left(\frac{\partial \Phi}{\partial x'^0} - \beta^k\frac{\partial}{\partial x'^k}\Phi\right) = -v'^A \mathrm{div}_{\mathbf{x}'}(\mathbf{e}'_A(x)\Phi) - \mathrm{div}_{\mathbf{v}'}(\mathbf{F}\Phi) + \frac{D}{2}\Delta_{\mathbf{v}'}\Phi, \tag{45}$$

where

$$\mathrm{div}_{\,\mathbf{x}'}(\mathbf{e}'_A(x')\Phi) = \frac{1}{\sqrt{g'}}\frac{\partial}{\partial x'^i}(\sqrt{g'}e'^i_A(x')\Phi), \tag{46}$$

and $g' = gJ^2$ where J is the Jacobian matrix of the transformation $J = \det\{\frac{\partial f^j}{\partial x^i}\}$. On the other hand, a time rescaling as given by Eq. (42) does not change the left-hand side of Eq. (39) because the entity N is transformed like $N' = N(\partial t/\partial t')$, and therefore $(N)^{-1}\partial/\partial t = (N')^{-1}\partial/\partial t'$.

Note that transformation properties as found in Eq. (45) with the replacement of the time derivative by the left side in Eq. (45) is a general feature of the evolution formalism in the 3 + 1 space-time decomposition in the general relativity theory (see e.g. [35]).

V. DIFFUSION IN THE EXPANDING UNIVERSE

Let us discuss the above given relativistic diffusion Eq. (39) in gravitational fields for the example of the expanding universe. The metric that characterizes the expanding spatial homogenous and isotropic universe can be described by the Robertson-Walker metric, which has the form

$$ds^2 = (cdt)^2 - \frac{R(t)^2}{1-\varepsilon r^2}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2], \tag{47}$$

where ε may assume the values 0, 1, or -1 for a flat, closed, or open universe, respectively, $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and $R(t)$ is the cosmic scale factor. In the following, we restrict ourselves to the spatial flat case with $\varepsilon = 0$. With the metric coefficients $g_{00} = -1, g_{0i} = 0, g_{ij} = -R(t)^2\delta_{ij}$, we find for the Christoffel coefficients [19,32],

$$\Gamma^i_{jk} = 0, \qquad \Gamma^0_{jk} = R\frac{\partial R}{\partial x^0}\delta_{jk}, \qquad \Gamma^i_{0k} = \frac{\partial R}{\partial x^0}R^{-1}\delta_{ik}. \tag{48}$$

The moving frames related to the metric in Eq. (47) are given by

$$\begin{aligned} e^j_m &= R(x^0)^{-1}\delta^j_m, \qquad e^0_0 = 1, \\ \theta^m_j &= R(x^0)\delta^m_j, \qquad \theta^0_0 = 1. \end{aligned} \tag{49}$$

Using Eqs. (48) and (49) we find for the spin connection coefficients

$$\Omega^m_{jn} = 0, \qquad \Omega^m_{j0} = \frac{\partial R}{\partial x^0}\delta^m_j, \qquad \Omega^m_{0n} = 0. \tag{50}$$

The gravitational force $F^m_g = -\Omega^m_{\mu B}(x)e^\mu_C(x)v^B v^C$ is given by

$$F^m_g = -\frac{\partial R}{\partial x^0}R^{-1}v^m v^0. \tag{51}$$

Now we consider the diffusion of massive particles in the expanding universe. Introducing the hyperbolic coordinate system for the four-velocity, defined by $v^1 = \mathrm{sh}\,\alpha\sin\vartheta\cos\varphi$, $v^2 = \mathrm{sh}\,\alpha\sin\vartheta\sin\varphi$, $v^3 = \mathrm{sh}\,\alpha\cos\vartheta$, and $v^0 = \mathrm{ch}\,\alpha$ we find for the gravitational force $F^\alpha_g(\eta) = -H(\eta)\,\mathrm{sh}\,\alpha$, $F^\vartheta_g = F^\varphi_g = 0$, where $H(\eta) = \frac{\partial R}{\partial \eta}R^{-1}$ is the Hubble constant and $\eta = x_0 = \tau\,\mathrm{ch}\,\alpha$. The random impacts of surrounding particles generally cause two kinds of effects: they act as a random driving force leading to a random motion and they give rise to a frictional force. In the nonrelativistic theory, the friction force is given by $f^i_F = -\nu m\tilde{v}^i$, where ν is the friction coefficient and $\tilde{v}^i$ are the components of the nonrelativistic velocity. The relativistic generalization of the friction force requires the introduction of a friction tensor ν^i_α, similar to the pressure tensor in the relativity theory [4,5]. The friction force is expressed as $F^i_F = \nu^i_\alpha[v^\alpha - U^\alpha]$, where U^α is the four-velocity of the heat bath. For an isotropic homogeneous heat bath, the friction tensor is given by

$$\nu^i_\alpha = \nu(\eta^i_\alpha + v^i v_\alpha), \tag{52}$$

with ν denoting the scalar friction coefficient measured in the rest frame of the particles. In the laboratory frame, the heat bath is at rest, described by $U^\alpha = (1, 0, 0, 0)$. Therefore, the friction force is given by $F^i_F = -\nu v^i v^0$ or in the frame basis and with hyperbolic coordinates

$$F^a_F = -\nu\,\mathrm{sh}\,\alpha, \qquad F^\vartheta_F = F^\varphi_F = 0. \tag{53}$$

In the expanding universe, the coefficients $\nu(\eta)$ and $D(\eta)$ depend on time and it is convenient to use the diffusion Eq. (39) in term of the parametrization with the observer time η. In the case of a spatial homogenous and isotropic solution in Eq. (39), the spatial derivatives vanish. Substituting the ansatz $\phi = \phi^J_M(\alpha, \vartheta, \varphi, \eta) = g_J(\alpha, \eta)Y^J_M(\theta, \varphi)$ with $Y^J_M(\theta, \varphi) = P^J_M(\vartheta)e^{iM\varphi}$ as the spherical harmonics and the associated Legendre functions $P^J_M(\vartheta)$ into Eq. (39), the following equation can be derived for $g_J = g_J(\alpha, \eta)$:

$$\operatorname{ch}\alpha \frac{\partial}{\partial\eta} g_J = \frac{D(\eta)}{2}\left(\frac{\partial^2}{\partial\alpha^2} + 2\operatorname{cth}\alpha\frac{\partial}{\partial\alpha} - \frac{J(J+1)}{\operatorname{sh}^2\alpha}\right)g_J + \chi(\eta)(\operatorname{sh}\alpha)^{-2}\frac{\partial}{\partial\alpha}(g_J \operatorname{sh}^3\alpha), \tag{54}$$

with $\chi(\eta) = \frac{\partial R}{\partial\eta}R^{-1} + \nu(\eta)$. Here the discrete index J takes the values $J = 0, 1, 2, \ldots$ and $M = -J, -J + 1, \ldots 0, 1, \ldots J$. For the fundamental solution $J = 0$, we find the diffusion equation

$$\operatorname{ch}\alpha \frac{\partial}{\partial\eta} g_0 = \frac{D(\eta)}{2}\left(\frac{\partial^2}{\partial\alpha^2} + 2\operatorname{cth}\alpha\frac{\partial}{\partial\alpha}\right)g_0 + \chi(\eta)(\operatorname{sh}\alpha)^{-2}\frac{\partial}{\partial\alpha}(g_0 \operatorname{sh}^3\alpha). \tag{55}$$

Let us discuss the solution of this equation within a certain range of validity, substituting the special ansatz

$$\phi(\alpha, \eta) = C\exp\{\beta(\eta) - \gamma(\eta)\operatorname{ch}\alpha\} \tag{56}$$

into Eq. (55), which yields the relation

$$\frac{\partial\beta}{\partial\eta} - \frac{\partial\gamma}{\partial\eta}\operatorname{ch}\alpha = \chi(\eta)[3 - \gamma\operatorname{sh}\alpha\operatorname{th}\alpha] + \frac{D(\eta)}{2}(-3\gamma + \gamma^2\operatorname{sh}\alpha\operatorname{th}\alpha). \tag{57}$$

In general, there do not exist functions $\beta(\eta)$ and $\gamma(\eta)$ which solve this equation, but if we restrict ourselves to the ultrarelativistic case $\alpha \gg 1$ we find as equations for the coefficients $\beta(\eta)$ and $\gamma(\eta)$ the following, respectively:

$$\frac{\partial\gamma}{\partial\eta} = \chi(\eta)\gamma - \frac{D(\eta)}{2}\gamma^2, \qquad \frac{\partial\beta}{\partial\eta} = 3\chi(\eta) - \frac{3D(\eta)}{2}\gamma. \tag{58}$$

The constant β is included into the normalization of $\phi(\alpha, \eta)$. An analytical solution of Eq. (58) for $\gamma(\eta)$ can be found by a transformation of the variable $\gamma(\eta) = (Y(\eta))^{-1}$, which yields the solution

$$Y = Y_0\frac{R_0}{R(\eta)}e^{-h(\eta)} + \frac{1}{2}\frac{1}{R(\eta)}e^{-h(\eta)}\int_{\eta_0}^{\eta} e^{h(t')}R(t')D(t')dt', \tag{59}$$

with $h(t) = \int_{t_0}^{t}\nu(t')dt'$.

By comparing the Jüttner distribution with the solution ansatz (56) we can introduce a time-dependent temperature of the expanding universe,

$$\gamma(\eta) = mc^2(kT(\eta))^{-1}. \tag{60}$$

Since the diffusion and drift constants D and ν are determined by the scattering processes of the particles in the system described by different physical parameters; in particular, by the temperature, the solution (59) has for temperature-dependent friction and diffusion coefficients the meaning of an integral equation. However, we have to taken into account that the diffusion and friction coefficients are related each others by the fluctuation-dissipation theorem. This relation is well known in the nonrelativistic case where the viscous friction coefficient ν of a Brownian particle must be related to the diffusion constant D of the particles by the Einstein relation

$$D = \frac{2\nu kT}{mc^2}. \tag{61}$$

The nature of the random force is independent of the presence of the gravitational field. Therefore, in the relativistic case the stationary solution of the Eq. (55) for $\frac{\partial R}{\partial\eta}R^{-1} \to 0$ must coincide with the Jüttner distribution. From the recently derived special relativistic diffusion equation [32], it was shown that the Jüttner distribution arises as the stationary solution of the special relativistic diffusion Eq. (9) if the Einstein relation is not only valid in the nonrelativistic case, but also in the relativistic regime. Substituting the relation (61) with (60) into Eq. (58), one can see that the effect of diffusion is canceled by the viscosity. Then, from Eq. (58), we obtain

$$\gamma(\eta) = \gamma_0\frac{R(\eta)}{R_0}. \tag{62}$$

This relation is identical with the result in the kinetic theory for the expanding universe [32,34]. In the radiation-dominated period in a flat cosmos, we have $R(\eta) \sim \sqrt{\eta}$, and in the matter-dominated period, $R(\eta) \sim \eta^{2/3}$, and therefore we find for the ultrarelativistic case $T \sim \eta^{-1/2}$ or $T \sim \eta^{-2/3}$, respectively, for the corresponding periods.

Note that the solution (56) does not satisfy physically determined initial conditions; this solution describes the asymptotic quasistatic regime and is valid only after a certain time, when the system is already in the equilibrium. As shown above, in this case the diffusion is compensated by the drift process, and the result that follows is consistent with the equilibrium state in kinetic theory. On the other hand, in the opposite transient case up to a certain time η after the initial time η_0, one can neglect in Eq. (55) the last term proportional to $\chi(\eta)$. Then, in the ultrarelativistic case, the fundamental solution $J = 0$ is determined by the equation

$$\frac{\partial}{\partial\varsigma}\Phi = \frac{1}{2}\left(\xi\frac{\partial^2}{\partial\xi^2} + 3\frac{\partial}{\partial\xi}\right)\Phi, \tag{63}$$

where the new variables $\varsigma = \int_{\eta_0}^{\eta} D(t)dt$ and $\xi = \exp(\alpha)$ are introduced. Using the Laplace transformation $\Phi(\varsigma, \xi) = \int_0^\infty \tilde{\Phi}(\lambda, \xi)\exp(-\lambda\varsigma)d\varsigma$, we find the solution

$$\tilde{\Phi}(\lambda, \xi) = \xi^{-1}J_1(2\sqrt{\xi\lambda}), \tag{64}$$

where J_1 is the first-order Bessel function. The eigenfunctions $\tilde{\Phi}(\lambda, \xi)$ satisfy the relations of orthogonality. Therefore, the transition probability is determined by

$$\Phi(\xi, \varsigma \mid \xi_{0,} 0) = \int_0^\infty \tilde{\Phi}(\lambda, \xi)\tilde{\Phi}^*(\lambda, \xi_0) \exp(-\lambda\varsigma) d\lambda. \tag{65}$$

Substituting Eq. (64) into Eq. (65) we find

$$\Phi(\xi, \varsigma \mid \xi_{0,} 0) = C\xi^{-1}\varsigma^{-1} I_1\left(\frac{2\sqrt{\xi\xi_0}}{\varsigma}\right) \exp\left(-\frac{\xi + \xi_0}{\varsigma}\right), \tag{66}$$

where C is the normalization constant and $I_1(x) = -iJ_1(ix)$. In the general case of arbitrary time, the solution of Eq. (54) can be obtained by numerical methods. But for the study of this problem under the conditions of the earliest epoch of the universe, we need a realistic microscopic model for the viscosity in the nonequilibrium epoch in a plasma of relativistic particles, including quarks, leptons, gauge bosons, and Higgs bosons. A detailed discussion of this issue is beyond the scope of the present paper. Corresponding cosmological estimations suggest the universe may not have been in thermal equilibrium during its earliest epoch in a time range earlier than about 10^{-38} s after the big bang or temperatures greater than 10^{16} GeV [34]. Standard phenomenological inflationary cosmological models relate this epoch with symmetry breaking phase transitions.

Let us finally briefly discuss the nonrelativistic limit $\alpha \ll 1$. From Eq. (57) we find under this condition the following:

$$\begin{aligned} \frac{\partial \gamma}{\partial \eta} &= 2\chi(\eta)\gamma - D(\eta)\gamma^2, \\ \frac{\partial \beta}{\partial \eta} &= -\frac{\partial \gamma}{\partial \eta} + 3\chi(\eta) - \frac{3D(\eta)}{2}\gamma. \end{aligned} \tag{67}$$

If we use the Einstein relation (61), the diffusion term is again canceled by the friction and from Eq. (67) the solution $\gamma(\eta) = \gamma_0 R^2(\eta)/R_0^2$ follows. The same solution is obtained in the kinetic theory for a nonrelativistic gas. In the nonrelativistic case, the temperature of the equilibrium distribution therefore depends on the cosmic scale factor $R(\eta)$ like $T \sim R^{-2}(\eta)$. Thus, both in the relativistic and in the nonrelativistic case, the validity of the general fluctuation-dissipation theorem with the Einstein relation (61) ensures that the result of the kinetic theory derived from the vanishing of the collision integral in the Boltzmann equation is consistent with the here derived probabilistic general relativistic diffusion theory in the quasisteady state regime or in thermodynamic equilibrium.

VI. CONCLUSIONS

In conclusion, a theory of Markovian diffusion processes within the framework of the general theory of relativity is formulated. In the derivation of the basic relativistic diffusion equation, the mathematical calculus of stochastic differential equations on Riemannian manifolds is used, which here is modified for the description of diffusion in the phase space of pseudo-Riemanian manifolds with an indefinite metric by using orthonormal frame vectors both in the position and in the velocity space. A generalized Langevin equation in the fiber space of position, velocity, and orthonormal velocity frames is defined, and the generalized Kramers equation within the framework of general relativity is derived both in the parametrization of the phase-space proper time and the observer time. The transformation of the obtained diffusion equation under hypersurface-preserving coordinate transformations is studied, and diffusion in the expanding universe is discussed. It is shown that the validity of the fluctuation-dissipation theorem in the relativistic case ensures that in the quasisteady state regime, the result of the derived diffusion equation is consistent with the kinetic theory in thermodynamic equilibrium. Additionally, a transient analytical solution valid for small times has been derived.

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