Influence of strong field vacuum polarization on gravitational-electromagnetic wave interaction

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The interaction between gravitational and electromagnetic waves in the presence of a static magnetic field is studied. The field strength of the static field is allowed to surpass the Schwinger critical field, such that the QED effects of vacuum polarization and magnetization are significant. Equations governing the interaction are derived and analyzed. It turns out that the energy conversion from gravitational to electromagnetic waves can be significantly altered due to the QED effects. The consequences of our results are discussed.

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I. INTRODUCTION

As studied by many authors [1–19] there exist numerous mechanisms for the conversion between gravitational waves (GWs) and electromagnetic (EM) waves. In particular, the propagation of GWs across an external static magnetic field gives rise to a linear coupling to the electromagnetic field (see e.g. Refs. [1–3]), which may lead to the GW excitation of ordinary EM waves in vacuum, or of magnetohydrodynamic (MHD) waves in a plasma [3–5]. Many nonlinear coupling mechanisms are also possible [4,6–10]. Cosmological aspects of GW-EM couplings have been reviewed by Ref. [11], and also studied recently by e.g. Refs. [12,13]. Conversion of energy from gravitational to electromagentic degrees of freedom has been pointed out as a means to indirect detection of gravitational waves by several authors (see e.g. Refs. [3,6,14]), since the latter is so much easier to detect. For astrophysical application (see e.g. Refs. [1,3,6,9,18,19]), naturally this requires well-developed theories to recognize the signature of the gravitational origin. Furthermore, there must be a sufficient amount of energy conversion taking place. Specifically, considering the coupling due to a static magnetic field, it has been noted that more energy can be converted from gravitational to electromagnetic degrees of freedom if the interaction region is larger, and if the magnitude B_0 of the static magnetic field is larger [3]. In the case that the interaction region is magnetized vacuum, with a size smaller than the background curvature radius, it has been found that the energy converted is linear in the background field energy density [2,3]. This result, however, does not account for QED vacuum polarization effects [20–23], which become significant when B_0 approaches the value $E_{\rm cr}/c$, where $E_{\rm cr} \equiv m_e^2 c^3/\hbar e \simeq 10^{18}$ V/m is the Schwinger critical field, m_e is the electron mass, e is the elementary charge, c is the speed of light in vacuum, and $h = 2\pi\hbar$ is the Planck constant.

In the present paper we will investigate the QED influence on gravitational-electromagnetic interaction in a static field B_0 that may be stronger than the characteristic QED scale $E_{\rm cr}/c$. It should be noted that such intense field do occur in nature, specifically close to magnetars where close to the surface the magnetic field strength may reach 10^{10}–10^{11} T [24]. Starting from Einstein's equations, together with the Heisenberg-Euler Lagrangian to describe vacuum polarization and magnetization in the electromagnetic theory, the basic equations for small amplitude wave propagation on a background with a strong static magnetic field B_0 is derived. In order to simplify the calculation, the size of the interaction region is assumed to be much smaller than the background curvature. It is found that the vacuum polarization effects lead to a saturation, such that the energy conversion (almost) stops to grow with B_0 beyond a certain value $B_{\rm sat}$. This value depend on the length L of the interaction region. For a large L, the saturation value is much smaller than the QED scale, i.e. $B_{\rm sat} \ll E_{\rm cr}/c$ (in which case the weak field QED corrections [20] of the Heisenberg-Euler theory would have sufficed), but for shorter interaction regions we may have $E_{\rm cr}/c \ll B_{\rm sat}$ in which case the full theory is required. The relevance of our model calculation to astrophysical problems is discussed at the end of the paper.

II. BASIC EQUATIONS

According to classical electrodynamics, photons does not interact, and Maxwell's equations are linear (in the absence of current and charge density sources that may depend on the field) and can be derived from the simple Lagrangian density $\mathcal{L}_c = -(1/4\mu_0)F^{\alpha\beta}F_{\alpha\beta}$, where $F^{\alpha\beta}$ is the electromagnetic field tensor, by varying the four-potential. When QED enters the picture, photons may interact also in the absence of real charged sources, due to the ubiquitous virtual electron-positron pairs. As a consequence, Maxwell's equations get new types of source terms that are nonlinear in the field strengths, and can be interpreted as vacuum polarization and vacuum magnetization, see e.g. Ref. [20]. An effective field theory capturing these effects within a Lagrangian was first put forward by Ref. [25] and was subsequently rigorously derived from QED by Ref. [26]. The Lagrangian for soft photon (i.e. photon energy much smaller than electron rest mass energy) light propagation, taking one loop corrections into account, is given by [22,23,26]

$$\mathcal{L} = -\frac{1}{\mu_0}\mathcal{F} - \frac{\alpha}{2\pi\mu_0 e^2}\int_0^{i\infty}\frac{ds}{s^3}e^{-eE_{\rm cr}s/c} \times\left[(es)^2 ab\coth(eas)\cot(ebs) - \frac{(es)^2}{3}(a^2-b^2) - 1\right] - A_\alpha j^\alpha, \tag{1}$$

where $a = [\sqrt{(\mathcal{F}^2+\mathcal{G}^2)} + \mathcal{F}]^{1/2}$, $b = [\sqrt{(\mathcal{F}^2+\mathcal{G}^2)} - \mathcal{F}]^{1/2}$, $\mathcal{F} = (1/4)F_{\alpha\beta}F^{\alpha\beta}$, $\mathcal{G} = (1/4)F_{\alpha\beta}\hat{F}^{\alpha\beta}$, $\hat{F}^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}\frac{F_{\mu\nu}}{2}$, $\epsilon^{\alpha\beta\mu\nu}$ the totally antisymmetric tensor, A_α the four-potential, j^α the four-current and α the fine structure constant. The Euler-Lagrange equations of motion for the Lagrangian (1) becomes

$$\gamma_{\mathcal{F}}F^{\mu\nu}_{;\mu} + \gamma_{\mathcal{G}}\hat{F}^{\mu\nu}_{;\mu} + \tfrac{1}{2}[\gamma_{\mathcal{F}\mathcal{F}}F^{\mu\nu}F_{\alpha\beta} + \gamma_{\mathcal{G}\mathcal{G}}\hat{F}^{\mu\nu}\hat{F}_{\alpha\beta}]F^{\alpha\beta}_{,\mu} + \gamma_{\mathcal{F}\mathcal{G}}[F^{\mu\nu}\hat{F}_{\alpha\beta} + \hat{F}^{\mu\nu}F_{\alpha\beta}]F^{\alpha\beta}_{,\mu} = -j^\nu, \tag{2}$$

where we have applied the Eq. (2) of Ref. [22] to a curved background, and introduced the quantities

$$\gamma_{\mathcal{F}} = \frac{\partial\mathcal{L}}{\partial\mathcal{F}}, \qquad \gamma_{\mathcal{G}} = \frac{\partial\mathcal{L}}{\partial\mathcal{G}}, \qquad \gamma_{\mathcal{F}\mathcal{F}} = \frac{\partial^2\mathcal{L}}{\partial\mathcal{F}^2}, \qquad \gamma_{\mathcal{G}\mathcal{G}} = \frac{\partial^2\mathcal{L}}{\partial\mathcal{G}^2}, \qquad \gamma_{\mathcal{F}\mathcal{G}} = \frac{\partial^2\mathcal{L}}{\partial\mathcal{F}\partial\mathcal{G}}. \tag{3}$$

The physics of strong field vacuum polarization and vacuum magnetization is thus encoded in the parameters introduced in Eq. (3). For the case of interest to us, i.e. no external electric field, the scalars, $\gamma_{\mathcal{F}}$, $\gamma_{\mathcal{G}}$, $\gamma_{\mathcal{F}\mathcal{F}}$, $\gamma_{\mathcal{G}\mathcal{G}}$ and $\gamma_{\mathcal{F}\mathcal{G}}$ can be computed analytically as functions of the external constant magnetic field strength B_0. This procedure which involves the solution of numerous integrals is described in Ref. [22], and the explicit expressions of the scalars can be found in Appendix A.

In the paper we will study the influence of a GW on a strong magnetic field. The metric of a linearized GW propagating in the z-direction can be written

$$ds^2 = -c^2dt^2 + (1+h_+)dx^2 + (1-h_+)dy^2 + 2h_\times dxdy + dz^2, \tag{4}$$

where the two independent polarizations h_+ and $h_\times$ depend on the coordinates as $h_{+,\times} = h_{+,\times}(z-ct)$. Furthermore, we define an orthonormal tetrad by

$$\mathbf{e}_0 = \frac{1}{c}\partial_t, \qquad \mathbf{e}_1 = \left(1 - \frac{1}{2}h_+\right)\partial_x - \frac{1}{2}h_\times\partial_y, \qquad \mathbf{e}_2 = \left(1 + \frac{1}{2}h_+\right)\partial_y - \frac{1}{2}h_\times\partial_x, \qquad \mathbf{e}_3 = \partial_z. \tag{5}$$

In linearized theory of gravity, the relevant components of the Einstein equations read:

$$(e_0^2 - \partial_z^2)h_+ = \kappa(\delta T_{11} - \delta T_{22}), \qquad (e_0^2 - \partial_z^2)h_\times = 2\kappa(\delta T_{12}), \tag{6}$$

where $\kappa = 8\pi G/c^4$, and G is the gravitational constant. The energy-momentum tensor associated with the Lagrangian (1) is written $T_{\mu\nu} = -\gamma_{\mathcal{F}}F^\alpha_\mu F_{\alpha\nu} + (\mathcal{G}\gamma_{\mathcal{G}} - \mathcal{L})g_{\mu\nu}$, see [27], and expressions for δT_{11}, δT_{22} and δT_{12}, linearized around the strong magnetic field B_0, is worked out in Appendix A.

Next we follow the covariant approach presented in Ref. [28] for splitting the EM and material fields in a 1 + 3 fashion. Suppose an observer moves with 4-velocity u^α. This observer will measure the electric and magnetic fields $E_\alpha \equiv F_{\alpha\beta}u^\beta$ and $B_\alpha \equiv \epsilon_{\alpha\beta\gamma}F^{\beta\gamma}/2$, respectively, where $F_{\alpha\beta}$ is the EM field tensor and $\epsilon_{\alpha\beta\gamma}$ is the volume element on hyper-surfaces orthogonal to u^α. We also define the spatial gradient operator as $\nabla = (e_1, e_2, e_3)$. Using the 1 + 3 split we write the Maxwell equations in the tetrad basis (5). From Eq. (2) and the Faraday equation, $F_{[ij;k]} = 0$, we obtain

$$c\nabla\cdot\mathbf{B} = \frac{\rho_B}{\epsilon_0}, \tag{7}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\left(\frac{\rho}{\gamma_F} + \rho_E\right), \tag{8}$$

$$e_0\mathbf{B} + \frac{\nabla\times\mathbf{E}}{c} = -\mu_0\mathbf{j}_B, \tag{9}$$

$$\frac{1}{c}e_0\mathbf{E} - \nabla\times\mathbf{B} = -\mu_0\left(\mathbf{j}_Q + \frac{\mathbf{j}}{\gamma_F} + \mathbf{j}_E\right), \tag{10}$$

where $\mathbf{j}_Q$ is the combined vacuum polarization and vacuum magnetization current density, which from Eq. (2) can be seen to take the form

$$j^\alpha_Q \equiv -\frac{1}{2\mu_0}\left(\frac{\gamma_{GG}}{\gamma_F}\hat{F}^{kl}\hat{F}^{i\alpha} + \frac{\gamma_{FF}}{\gamma_F}F^{kl}F^{i\alpha}\right)e_iF_{kl}, \tag{11}$$

and the effective (i.e. gravity induced) charge densities and current densities are

$$\rho_E \equiv -\epsilon_0[\gamma^\alpha_{\beta\alpha}E^\beta + \epsilon^{\alpha\beta\gamma}\gamma^0_{\alpha\beta}cB_\gamma],$$
$$\rho_B \equiv -\epsilon_0[\gamma^\alpha_{\beta\alpha}cB^\beta - \epsilon^{\alpha\beta\gamma}\gamma^0_{\alpha\beta}E_\gamma],$$
$$j^\alpha_E \equiv \frac{1}{\mu_0}\left[-(\gamma^\alpha_{0\beta} - \gamma^\alpha_{\beta0})\frac{E^\beta}{c} + \gamma^\beta_{0\beta}\frac{E^\alpha}{c} - \epsilon^{\alpha\beta\gamma}(\gamma^0_{0\beta}B_\gamma + \gamma^\delta_{\beta\gamma}B_\delta)\right],$$
$$j^\alpha_B \equiv \frac{1}{\mu_0}\left[-(\gamma^\alpha_{0\beta} - \gamma^\alpha_{\beta0})B^\beta + \gamma^\beta_{0\beta}B^\alpha + \epsilon^{\alpha\beta\gamma}\left(\gamma^0_{0\beta}\frac{E_\gamma}{c} + \gamma^\delta_{\beta\gamma}\frac{E_\delta}{c}\right)\right], \tag{12}$$

where the Greek indices takes values between 1 and 3, and the Latin indices between 0 and 3. From here on we will be concerned with a GW wave propagating across a magnetic field. Explicit expressions of the source terms for this case is obtained by substituting the QED parameters from Appendix A into Eq. (11), and the rotation coefficients for a linearized GW presented in Appendix B into Eq. (12).

III. WAVE INTERACTION

The most efficient interaction of a GW with a static magnetic field occurs if the GW propagates perpendicular to the magnetic field. As has been found by e.g. Refs. [2,3], the fact that the GW fulfills the same dispersion relation as EM waves, makes the energy conversion resonant. As a consequence, the energy conversion from a GW to copropagating EM waves is directly proportional to the background field energy density as well as the length of the interaction region, defined as the region occupied by the static magnetic field B_0. This conclusion holds as long as QED effects is negligible, and the length of the interaction region is smaller than the radius of curvature associated with the magnetic field energy density. Our aim here is to investigate to what extent the QED effects, associated with fields strengths approaching the Schwinger limit, modifies the energy conversion between GWs and EM waves. For this purpose we will still assume that the interaction region is smaller than the radius of curvature due to B_0, such that the interaction can be considered as taking place on a Minkowski background.

As we will see, in addition to an EM wave copropagating with the monochromatic GW, with metric perturbation $h_{\times,+} = \tilde{h}_{\times,+}\exp[i(kz - \omega t)]$ and $\omega = kc$, a counterpropagating wave with the same frequency will also be induced. We thus make the ansatz $\mathbf{B} = B_0\mathbf{e}_1 + \delta\mathbf{B}(z)\times\exp[-i\omega t]$ and $\mathbf{E} = \delta\mathbf{E}(z)\exp[-i\omega t]$, where $\delta\mathbf{B}$ and $\delta\mathbf{E}$ includes both positive (along z) and negative propagating waves. Taking the curl of Eq. (10) and using (9) one obtains,

$$\begin{aligned} -e_0^2\mathbf{B} - \nabla\times(\nabla\times\mathbf{B}) + \mu_0\nabla\times\mathbf{j}_Q \\ = -\mu_0\nabla\times\mathbf{j}_E + \mu_0 e_0\mathbf{j}_B, \end{aligned} \tag{13}$$

to linear order, with the components of the polarization current Eq. (11) given by

$$\begin{aligned} j_Q^1 &= -\frac{\gamma_{GG}}{\gamma_F}B_0^2\frac{1}{c\mu_0}e_0\delta E_1, \\ j_Q^2 &= -\frac{\gamma_{FF}}{\gamma_F}B_0^2\frac{1}{\mu_0}\partial_z\delta B_1, \qquad j_Q^3 = 0. \end{aligned} \tag{14}$$

From Eq. (12) and Eqs. (B1) the gravitational contribution is found to be:

$$\begin{aligned} \rho_E = \rho_B = 0, \qquad j_E^1 = \frac{B_0}{2\mu_0}\frac{\partial h_\times}{\partial z}, \\ j_E^2 = -\frac{B_0}{2\mu_0}\frac{\partial h_+}{\partial z}, \qquad j_E^3 = 0, \\ j_B^1 = -\frac{B_0}{2c\mu_0}\dot{h}_+, \qquad j_B^2 = -\frac{B_0}{2c\mu_0}\dot{h}_\times, \qquad j_B^3 = 0. \end{aligned} \tag{15}$$

Using Eqs. (9) and (13)–(15) we will next demonstrate that different EM wave polarizations couple to different GW polarizations. The result is most easily expressed in terms of the magnetic field components, and can then be written:

$$\begin{aligned} [k_E^{+2} + \partial_z^2]\delta B_1 &= k_E^{+2}B_0\tilde{h}_+\exp[ikz] \\ [k_E^{\times 2} + \partial_z^2]\delta B_2 &= \frac{1}{2}\left(\frac{\omega^2}{c^2} + k_E^{\times 2}\right)B_0\tilde{h}_\times\exp[ikz], \end{aligned} \tag{16}$$

where $k_E^{+2} = \omega^2/(c^2(1 + B_0^2\gamma_{FF}/\gamma_F))$ and $k_E^{\times 2} = \omega^2(1 - B_0^2\gamma_{GG}/\gamma_F)/c^2$. As can be seen, all effects of the QED-vacuum polarization and magnetization is encoded in the effective wave numbers k_E^+ and $k_E^\times$, that approach ω/c for $cB_0/E_{\rm cr} \ll 1$. Note that Eq. (16) agrees with Ref. [22], when the GW-coupling terms on the right hand sides are dropped [29]. The backreaction on the GW can be obtained by combining Eqs. (6) and (A11). Whether or not this effect is important depends on the ratio of the excited wave energy density compared to the (pseudo) wave energy density of the GW. Roughly the scaling is as follows: For weak background magnetic fields (i.e. negligible QED effects), the excited wave energy density is limited by $W_{\rm em} \sim B_1^2/\mu_0 \sim (kL)^2|\tilde{h}_{+,\times}|^2 B_0^2/\mu_0$, where k is the incident wave number and L is the length of the interaction region. As we will see in the next section, whenever QED effects are important, the excited wave energy is reduced compared to this scaling. Thus at most the ratio of the excited wave energy to the GW (pseudo) wave energy density becomes $W_{\rm em}/W_{\rm GW} = L^2(G/8\pi c^2)B_0^2/\mu_0$. Whenever the interaction region is smaller than the background curvature due to the unperturbed magnetic field (as we have assumed above), this ratio is much smaller than unity, and hence the backreaction on the GW can be neglected. As a consequence, the approximation of "no GW backreaction" will be employed in the next section.

IV. A SPECIFIC EXAMPLE

As a specific example we will now consider a boundary value problem, where the GW propagating in the $\mathbf{e}_3$-direction, enters the interaction region, given by $-L/2 < z < L/2$, which is the region where the external magnetic field $B_0\mathbf{e}_1$ is taken to be nonzero. The general solution to Eq. (16), for the interaction region $-L/2 < z < L/2$, is

$$\delta B_{1,2} = T_{1,2}e^{ik_E^{+,\times}z} + R_{1,2}e^{-ik_E^{+,\times}z} + C_{1,2}e^{ikz},$$

where $C_1 = k_E^{+2} B_0 \tilde{h}_+ / (k_E^{+2} - k^2)$, $C_2 = (k_E^{\times 2} + k^2) B_0 \tilde{h}_\times / 2(k_E^{\times 2} - k^2)$, and $R_{1,2}$ and $T_{1,2}$ are constants determined by the boundary conditions. This must be matched with the EM wave solutions with constant amplitudes outside the interaction region

$$\delta B_{1,2} = f_{1,2}^R e^{-ikz}, \qquad z \in (-\infty, -L/2),$$
$$\delta B_{1,2} = f_{1,2}^T e^{ikz}, \qquad z \in (L/2, \infty),$$

at $z = \pm L/2$. Furthermore, the electric fields must be matched as well. The relevant Maxwell equations are

$$\delta E_2 = \frac{i}{\omega}\left(\frac{\omega^2}{k_E^{+2}} \partial_z \delta B_1 + \frac{B_0}{2} \partial_z h_+\right), \tag{17}$$

$$\delta E_1 = -\frac{i\omega}{k_E^{\times 2}}\left(\partial_z \delta B_2 + \frac{B_0}{2} \partial_z h_\times\right), \tag{18}$$

The matching of the electric field is done in the same way as that of the magnetic field to give four equations for four quantities, for each set of coupled polarizations. Solving these equations, the resulting amplitudes of the "reflected" and "transmitted" (or strictly speaking counterpropagating and copropagating) EM waves becomes

$$f_1^R = \frac{B_0 \tilde{h}_+}{2} \eta_+ e^{-i\theta} \frac{(1+\eta_+)e^{i\eta_+\theta} + (1-\eta_+)e^{-i\eta_+\theta} - 2e^{i\theta}}{(1+\eta_+)^2 e^{-i\theta\eta_+} - (\eta_+ - 1)^2 e^{i\theta\eta_+}}, \tag{19}$$

and

$$f_1^T = \frac{B_0 \tilde{h}_+}{2} \frac{\eta_+}{(1-\eta_+)^2 e^{i\eta_+\theta} - (1+\eta_+)^2 e^{-i\eta_+\theta}} \times \left[\frac{(1-\eta_+)^2}{1+\eta_+} e^{i\eta_+\theta} + \frac{(1+\eta_+)^2}{(1-\eta_+)} e^{-i\eta_+\theta} + 2\frac{3\eta_+^2 + 1}{\eta_+^2 - 1} e^{-i\theta}\right], \tag{20}$$

for the mode that couples to the plus-polarization. For the mode that couples to the cross-polarization we similarly obtain

$$f_2^R = \frac{\tilde{h}_\times B_0}{2} (\eta_\times^2 + 1) \times \frac{e^{-i\theta}[e^{i\theta\eta_\times} - e^{-i\theta\eta_\times}]}{(\eta_\times - 1)^2 e^{i\theta\eta_\times} - (\eta_\times + 1)^2 e^{-i\theta\eta_\times}}, \tag{21}$$

and

$$f_2^T = \frac{\tilde{h}_\times B_0}{2}\left(\frac{\eta_\times^2 + 1}{\eta_\times^2 - 1}\right) \times \frac{(\eta_\times - 1)^2 e^{i\theta\eta_\times} - (\eta_\times + 1)^2 e^{-i\theta\eta_\times} + 4\eta_\times e^{-i\theta}}{(\eta_\times + 1)^2 e^{-i\theta\eta_\times} - (\eta_\times - 1)^2 e^{i\theta\eta_\times}}. \tag{22}$$

Here we have introduced the notation $\eta_{+,\times} \equiv k_E^{+,\times}/k$ and $\theta = kL$. An example of the magnetic profile (containing both the transmitted and reflected wave) is given in Fig. 1 for $kL = 40$ and $cB_0/E_{\rm cr} = 100$. The expressions (19)–(22) contains all information about the energy conversion to the different EM-modes. However, to appreciate these results and the effects due to QED, we must first evaluate some results for the low-field limit when $\eta_{+,\times} \to 1$. The squared coefficient $|f_1^T|^2$, proportional to the energy density of the transmitted wave excited by the +-polarization, then becomes

$$|f_1^T|^2 = \tfrac{1}{4}|\tilde{h}_+|^2 B_0^2 k^2 L^2, \tag{23}$$

and similarly for the mode excited by the opposite polarization,

$$|f_2^T|^2 = \tfrac{1}{4}|\tilde{h}_\times|^2 B_0^2 k^2 L^2. \tag{24}$$

Thus we see that the transmitted energy density is directly proportional to the background energy density. However, this behavior is dramatically changed when QED effects are taken into account. The main reason is that the EM wave dispersion relation is changed in the interaction region (that makes $\eta_{+,\times}$ deviate from unity) which in turn detunes the excited wave with the GW. The consequence for the transmitted wave excited by the $\times$-polarization is depicted in Fig. 2, for $kL = 20$ and $kL = 100$. The steady increase in the absence of QED is replaced by an oscillatory behavior, mainly due to the detuning of the GW and EM wave dispersion relation. Note that we here have normalized the transmission coefficient with $|\tilde{h}_{+,\times}|^2 B_0^2 k^2 L^2$, such that the coefficient without QED effects is represented by a straight line. For a longer interaction region, a smaller mismatch of dispersion relations are needed for the phase difference to accumulate, and hence the curve with the lower value of kL ($kL = 20$)

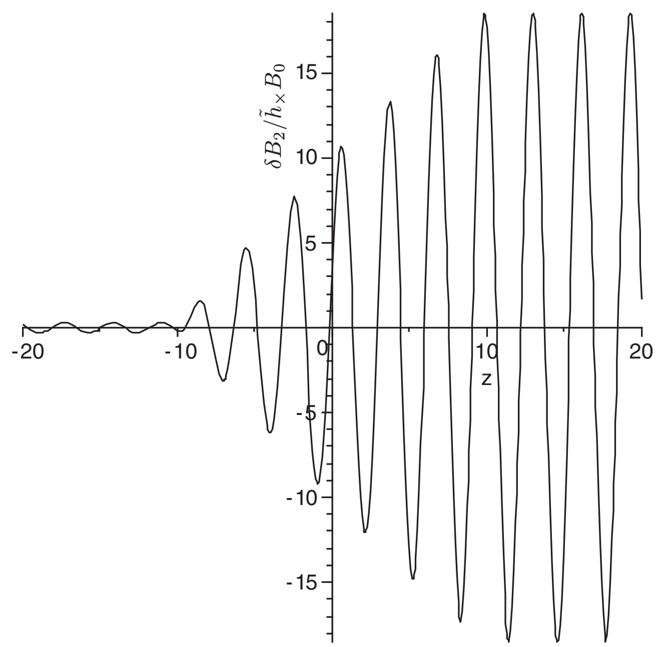

FIG. 1. The wave profile for $kL = 40$ and $cB_0/E_{\rm cr} = 100$. The magnetized region lies between $z = -10$ and $z = 10$.

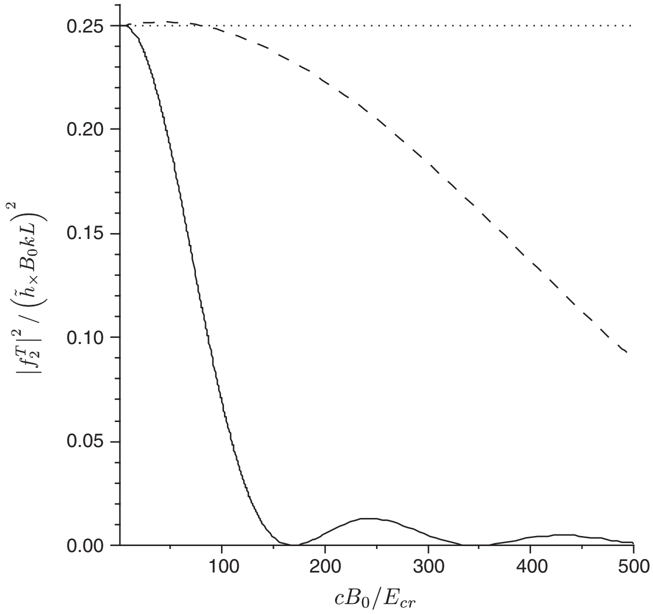

FIG. 2. Normalized energy density of the copropagating EM wave, excited by a cross-polarized GW, as a function of background magnetic field strength for $\theta = 20$ (dashed line) and $\theta = 100$ (solid line) compared to the non-QED case (dotted line).

needs a much higher field strength before significant QED effects are seen. A similar point is illustrated by Fig. 3 that depicts the energy density for the copropagating mode excited by the $+$-polarization. Note that the energy conversion to this EM-mode is much less affected by the QED effects. The reason is that the QED modification of the EM

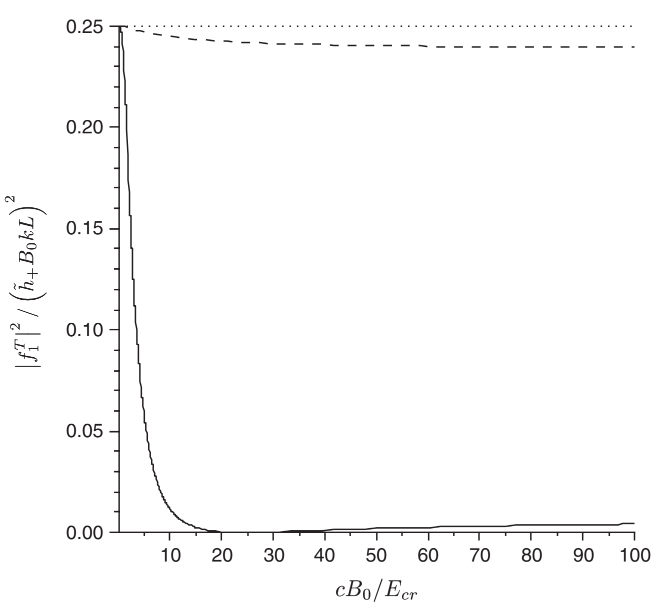

FIG. 3. Normalized energy density of the copropagating EM wave, excited by a plus-polarized GW, as a function of background magnetic field strength for $\theta = 2000$ (dashed line) and $\theta = 20000$ (solid line) compared to the non-QED case (dotted line).

dispersion relation effectively saturates at a value $cB_0/E_{\rm cr} \sim 10$. Accordingly we have chosen higher values of kL, namely $kL = 2000$ and $kL = 20\,000$, which is needed in order to see the deviation from the classical behavior induced by QED. In addition to the copropagating EM modes there are also counter-propagating EM waves. From a practical point of view, these are much less significant, since the counter-propagating modes are always nonresonant with the source GW, and hence the energy density of these modes does not systematically increase with a larger interaction region, i.e. increasing kL. From a more theoretical point of view, an interesting effect can be seen in the coefficients (19) and (21), however. Without QED effects, the $+$-polarization does not cause a back-scattered wave, independent of the value of kL, as seen by (19) when letting $\eta_+ \rightarrow 1$. However, the situation for the $\times$-polarization is different, as we find a finite but small counter-propagating mode from (21) also in the limit $\eta_\times \rightarrow 1$.

V. SUMMARY AND CONCLUSION

In this paper we have studied the interaction between GWs and EM waves in the presence of a strong static magnetic field B_0, using the Heisenberg-Euler Lagrangian in order to take QED vacuum polarization and magnetization into account. The high-frequency approximation has been applied to zeroth order, i.e. all effects of the background curvature has been neglected, which is permissible if the spatial extension of the interaction region is much smaller than the radius of curvature. The specific boundary conditions considered is an incoming GW incident on a static magnetic field with a given extent L, which give raise to an excited EM wave in the same direction as the GW, as well as one propagating in the opposite direction. The role of the QED effects is twofold: First, the coupling strength between the GWs and the electromagnetic waves are modified [as described by the coefficients of the right-hand side in Eq. (16)]. Second, the change in phase velocity ($< c$) of the EM waves induced by the vacuum polarization, as described by the expressions $k_E^\times$ and k_E^+, destroys the perfect resonance with the gravitational source wave, which gives a saturation of the possible energy conservation at a finite value of L. These effects are similar in principle for the $h_\times$- and h_+-polarizations (which couples to different EM polarizations), and the dimensionless parameter $(cB_0/E_{\rm cr})^2 kL$ need to reach $(cB_0/E_{\rm cr})^2 kL \sim 10^5$ in order for QED effects to be important in both cases. However, since the QED modification of the EM mode excited by the h_+-polarization saturates at a value $cB_0/E_{\rm cr} \sim 10$, a much higher value of kL is needed for the QED effects to be significant in this case.

The problem considered here has been highly idealized and has mainly been motivated by a theoretical interest to study GW and EM wave interaction in a strong field environment, allowing for field strengths larger than the

Schwinger critical field $E_{\rm cr}$. However, we would like to point out that there is a certain astrophysical relevance of the problem, as the effect of QED detuning is found to be of significance for field strengths $B_0 \simeq 3E_{\rm cr}/c \approx 10^{10}$ T (see e.g. Fig. 3), a value that has been observed at magnetar surfaces [24], although a high GW frequency would be required. Interesting generalizations of the present work includes the study of the back-scattering on the GW, and the possible existence of an energy conservation law. Furthermore, some of the restrictions of the present study can likely be removed, e.g. by considering propagation at an arbitrary angle to the magnetic field, and/or by relaxing the short wave approximation, i.e. including the background curvature effects.

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APPENDIX A: STRONG FIELD VACUUM POLARIZATION AND MAGNETIZATION PARAMETERS

With only a strong magnetic field present the quantities $\gamma_{\mathcal{F}}$, $\gamma_{\mathcal{G}}$, $\gamma_{\mathcal{F}\mathcal{F}}$, $\gamma_{\mathcal{G}\mathcal{G}}$, and $\gamma_{\mathcal{F}\mathcal{G}}$ can be determined analytically, see Ref. [22]. The resulting expressions for these QED parameters are

$$\begin{aligned}
\gamma_{\mathcal{G}} &= 0, \quad \gamma_{\mathcal{F}\mathcal{G}} = 0,\\
\gamma_{\mathcal{F}} &= -\frac{1}{\mu_0} - \frac{\alpha}{2\pi\mu_0}\bigg[\frac{1}{3} + 2h^2 - 8\zeta'(-1,h) + 4h\ln(\Gamma(h))\\
&\quad - 2h\ln h + \frac{2}{3}\ln h - 2h\ln 2\pi\bigg],\\
\gamma_{\mathcal{F}\mathcal{F}} &= \frac{\alpha}{2\pi\mu_0 B^2}\bigg[\frac{2}{3} + 4h^2\psi(1+h) - 2h - 4h^2 - 4h\ln\Gamma(h)\\
&\quad + 2h\ln 2\pi - 2h\ln h\bigg],\\
\gamma_{\mathcal{G}\mathcal{G}} &= \frac{\alpha}{2\pi\mu_0 B^2}\bigg[-\frac{1}{3} - \frac{2}{3}\psi(1+h) - 2h^2 + \frac{1}{3h} + 8\zeta'(-1,h)\\
&\quad - 4h\ln\Gamma(h) + 2h\ln 2\pi + 2h\ln h\bigg], \qquad \text{(A1)}
\end{aligned}$$

where $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$ is the fine structure constant, $h = \frac{E_{\rm cr}}{2cB}$, $\Gamma(h)$ the gamma function, $\psi(h)$ the digamma function and $\zeta'(-1,h)$ the first derivative of the Hurwitz zeta function with respect to its first argument.

Furthermore, in the absence of a strong electric field we can calculate the integral in the Lagrangian (1) analytically. Since there is only a strong magnetic field present we have $b = 0$. Thus, to compute the integral in Eq. (1), we expand the integrand and take the limit as $b \to 0$, thereby obtaining

$$I = \int_0^{i\infty} \frac{ds}{s^3} e^{-eE_c rs/c}\bigg[(eas)\coth(eas) - \frac{(eas)^2}{3} - 1\bigg]. \qquad \text{(A2)}$$

By changing the variables such that $eas = z$, dividing the integral into three parts, altering the integration path and using the regulator z^ϵ we obtain

$$\begin{aligned}
I = (ea)^2\bigg\{&\int_0^\infty dz e^{-E_{\rm cr}z/ca} z^{\epsilon-2}\coth(z)\\
&- \int_0^\infty dz e^{-E_{\rm cr}z/ca}\frac{z^{\epsilon-1}}{3} - \int_0^\infty dz e^{-E_{\rm cr}z/ca} z^{\epsilon-3}\bigg\} \qquad \text{(A3)}
\end{aligned}$$

Since $E_{\rm cr}/ca = 2h$ we find the first, second and third part of the integral to be

$$\begin{aligned}
I_1 &\equiv \int_0^\infty dz e^{-2hz} z^{\epsilon-2}\coth(z)\\
&= \frac{1}{\epsilon}\bigg(2h^2 + \frac{1}{3}\bigg) + (1 - C - \ln 2)\bigg(2h^2 + \frac{1}{3}\bigg)\\
&\quad - 4\zeta'(-1,h) - 2h\ln(h), \qquad \text{(A4)}
\end{aligned}$$

$$I_2 \equiv \int_0^\infty dz e^{-2hz}\frac{z^{\epsilon-1}}{3} = -\frac{1}{3\epsilon} + \frac{1}{3}C + \frac{\ln(2h)}{3}, \qquad \text{(A5)}$$

and

$$\begin{aligned}
I_3 &\equiv \int_0^\infty dz e^{-2hz} z^{\epsilon-3}\\
&= -\bigg[\frac{2h^2}{\epsilon} + h^2 - 2h^2\ln h + (1 - C - \ln 2)2h^2\bigg], \qquad \text{(A6)}
\end{aligned}$$

respectively, where C is Euler's constant. With Eqs. (A4)–(A6) we can now rewrite Eq. (A2) as

$$\begin{aligned}
I = (ea)^2\bigg\{&\frac{1}{3}[1 - \ln 2 - \ln(2h)] + h^2[2\ln h - 1]\\
&- 2h\ln h - 4\zeta'(-1,h)\bigg\}, \qquad \text{(A7)}
\end{aligned}$$

and thus the Lagrangian (1) becomes

$$\begin{aligned}
\mathcal{L} = -\frac{1}{\mu_0}\mathcal{F} - \frac{\alpha B^2}{2\pi\mu_0}\bigg\{&\frac{1}{3}[1 - \ln 2 - \ln(2h)]\\
&+ h^2[2\ln h - 1] - 2h\ln h - 4\zeta'(-1,h)\bigg\} - A_\alpha j^\alpha. \qquad \text{(A8)}
\end{aligned}$$

Since we have only a magnetic field, $\mathcal{G} = 0$ holds, and the energy-momentum tensor associated with the Lagrangian (A8) becomes

$$T_{\mu\nu} = -\gamma_{\mathcal{F}} F^{\alpha}_{\mu} F_{\alpha\nu} - \mathcal{L} g_{\mu\nu}. \tag{A9}$$

Next we proceed by expanding the energy-momentum tensor (A9). The first order contribution becomes

$$\delta T_{\mu\nu} = \delta\gamma_{\mathcal{F}} F^{\alpha}_{\mu} F_{\alpha\nu} - \gamma_{\mathcal{F}}[\delta F^{\alpha}_{\mu} F_{\alpha\nu} + F^{\alpha}_{\mu} \delta F_{\alpha\nu}] - \delta\mathcal{L} g_{\mu\nu}, \tag{A10}$$

where

$$\delta\gamma_{\mathcal{F}} = \frac{\alpha}{2\pi\mu_0}\bigg[4h_0 + 4\ln\Gamma(h_0) + 2\ln h_0 - 2\ln 2\pi - 2 - \frac{2}{3h_0} - 4h_0\Psi(h_0)\bigg]\bigg(-h_0\frac{\delta B_1}{B_0}\bigg),$$

and $h_0 = E_{\mathrm{cr}}/2cB_0$, so the relevant energy-momentum tensor terms in Eq. (6) becomes

$$\begin{aligned} \delta T_{11} - \delta T_{22} &= B_0^2 \delta\gamma_F - 2\gamma_F B_0 \delta B_1, \\ \delta T_{12} &= \gamma_F B_0 \delta B_2. \end{aligned} \tag{A11}$$

APPENDIX B: RICCI-ROTATION COEFFICIENTS

The Ricci-rotation coefficients of a Minkowski spacetime perturbed by a GW propagating in the $\mathbf{e}_3$-direction expressed in the tetrad (5) is given by

$$\begin{aligned} \gamma^0_{11} &= -\gamma^0_{22} = \gamma^1_{01} = -\gamma^2_{02} = \frac{1}{2c}\dot{h}_+, \\ \gamma^0_{12} &= \gamma^0_{21} = \gamma^1_{02} = \gamma^2_{01} = \frac{1}{2c}\dot{h}_\times, \\ \gamma^1_{31} &= -\gamma^2_{32} = -\gamma^3_{11} = \gamma^3_{22} = \frac{1}{2}\frac{\partial h_+}{\partial z}, \\ \gamma^1_{32} &= \gamma^2_{31} = -\gamma^3_{12} = -\gamma^3_{21} = \frac{1}{2}\frac{\partial h_\times}{\partial z}, \end{aligned} \tag{B1}$$

to first order in $h_{+,\times}$.

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[29] It should be noted that although the interaction Eq. (16) can be expressed solely in terms of δB_1 and δB_2, the EM-wave polarization in the presence of strong QED effects is nontrivial. For a more complete description, see e.g. Ref. [22].