

Λ CDM universe in $f(R)$ gravityPeter K. S. Dunsby,^{1,2,3} Emilio Elizalde,⁴ Rituparno Goswami,^{1,2} Sergei Odintsov,^{4,5} and Diego Saez-Gomez⁴¹*Centre for Astrophysics, Cosmology and Gravitation, University of Cape Town, Rondebosch, 7701, South Africa*²*Department of Mathematics and Applied Mathematics, University of Cape Town, 7701 Rondebosch, Cape Town, South Africa*³*South African Astronomical Observatory, Observatory 7925, Cape Town, South Africa*⁴*Institut d'Estudis Espacials de Catalunya, ICE (IEEC-CSIC), Campus UAB, Facultat Ciències, Torre C5-Par-2a pl, E-08193 Bellaterra, Barcelona, Spain*⁵*Institució Catalana de Recerca i Estudis Avançats, Barcelona, Spain*

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Several different explicit reconstructions of $f(R)$ gravity are obtained from the background Friedmann-Laîmatre-Robertson-Walker expansion history. It is shown that the only theory whose Lagrangian is a simple function of the Ricci scalar R , that admits an exact Λ CDM expansion history, is standard general relativity with a positive cosmological constant and the only way to obtain this behavior of the scale factor for more general functions of R is to add additional degrees of freedom to the matter sector.

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I. INTRODUCTION

After more than one hundred years, general relativity (GR) is still considered to be the best fundamental theory for the description of the gravitational action. When applied to cosmology, assuming homogeneity and isotropy [encoded in the Friedmann-Laîmatre-Robertson-Walker (FLRW) metric], together with a fluid description of baryons, cold dark matter (CDM) and radiation, GR gives rise to a set of field equations which when solved produce the simplest expanding cosmology—the Friedmann model, governing the dynamics of the cosmological scale factor $a(t)$. This model has been remarkably successful, giving, for example, the correct light element abundances and explaining the origin of the cosmic microwave background radiation (CMBR). In the past two decades, however, advances in observational cosmology appears to suggest that if one wishes to retain the FLRW metric, the universe must have undergone two periods of accelerated expansion. The first period of acceleration, known as the inflationary epoch is needed to explain the flatness problem and the near-scale invariant spectrum of temperature fluctuations observed in the CMBR, while the second period explains the dimming of distant type Ia supernovae relative to Einstein-de Sitter universe model. In order to explain these periods of acceleration, the strong energy condition ($\rho + 3p \geq 0$) needs to be violated. In the case of inflation, this is achieved by introducing a dynamical scalar field, while the present day acceleration is most easily explained with the introduction of a cosmological constant. The resulting description of the Universe, in which the recent expansion history is driven by a cosmological constant and ordinary matter is dominated by a CDM component has become known as the Λ CDM or *Concordance Model* [1]. Unfortunately, this beautifully simple phenomenological model, which appears to fit all currently available observations (Supernovae Ia [2], CMBR anisotropies [3], Large Scale Structure formation [4], baryon oscillations [5] and

weak lensing [6]) is affected by significant fine-tuning problems related to the vacuum energy scale and therefore it is important to investigate alternatives to this description of the Universe.

Currently, one of the most popular alternatives to the Λ CDM model is based on modifications of the standard Einstein-Hilbert action. This is due to the fact that these changes naturally admit a phase of late-time accelerated expansion (an early Universe inflationary phase is also possible [7]). In this way Dark Energy can be thought of as have a geometrical origin, rather than be due to the vacuum energy or additional scalar fields which are added by hand to the energy momentum tensor. As a result, the cosmology and astrophysics of modified gravity is currently an extremely active area of research (see the recent reviews [8] and references therein).

One of the simplest extensions of GR is based on gravitational actions which are nonlinear in the Ricci curvature R and/or contain terms involving combinations of derivatives of R [9–15]. An important feature of these theories is that the field equations can be written in a way which makes it easy to compare with GR. This is done by moving all the higher-order corrections to the curvature onto the RHS of the field equations and defining an “effective” source term, often described as the *curvature fluid*. Once this has been done, it is easy to see that how a change of sign in the deceleration parameter of the FLRW cosmology can occur, leading to a period of late-time acceleration.

Studies of the physics of these theories is however hampered by the complexity of the field equations, making it difficult to obtain both exact and numerical solutions which can be compared with observations. These problems can be reduced somewhat by using the theory of dynamical systems [16], which provides a relatively simple method for obtaining exact solutions and a (qualitative) description of the global dynamics of these models for a given $f(R)$ theory [17].

Another useful approach is to assume that the expansion history of the universe is known exactly, and to invert the field equations to deduce what class of $f(R)$ theories give rise to this particular cosmological evolution.

This has been done recently for exact power-law solutions for the scale factor, corresponding to phases of cosmic evolution when the energy density is dominated by a perfect fluid. It was found that such expansion histories only exist for R^n gravity [18]. A more extensive analysis of *reconstruction methods* has been carried out in [19] to obtain theories which give an approximate description of deceleration-acceleration transitions in cosmology and in [20] a powerful approach to reconstruction based on standard cosmic parameters instead of a time law for the scale factor was introduced.

In this paper we perform a number of explicit reconstructions which lead to a number of interesting results. We find, for example, that the only real valued Lagrangian $f(R)$ that is able to mimic an exact Λ CDM expansion history for a universe filled with dustlike matter is the Einstein-Hilbert Lagrangian with positive cosmological constant. This does not mean that $f(R)$ gravity is incompatible with an exact Λ CDM expansion history. In fact we further show that in a universe filled with a both a minimally-coupled noninteracting massless scalar field and dustlike matter, a theory of gravity can be found which exactly mimics the Λ CDM expansion history, making it impossible to distinguish it from GR at the level of the FLRW background. Moreover, number of realistic models may also mimic late-time acceleration epoch approximately.

II. FIELD EQUATIONS FOR HOMOGENEOUS AND ISOTROPIC SPACETIMES IN $f(R)$ GRAVITY

We consider the following action within the context of 4-dimensional homogeneous and isotropic spacetimes, i.e., the FLRW universes with negligible spatial curvature:

$$\mathcal{A} = \frac{1}{2} \int d^4x \sqrt{-g} [f(R) + \mathcal{L}_m], \quad (1)$$

where R is the Ricci scalar, f is general differentiable (at least C^2) function of the Ricci scalar and \mathcal{L}_m corresponds to the matter Lagrangian. Units are chosen so that $c = 8\pi G = 1$.

The field equations for homogeneous and isotropic spacetimes are the *Raychaudhuri equation*

$$3\dot{H} + 3H^2 = -\frac{1}{2f'}[\rho + 3p + f - f'R + 3Hf''\dot{R} + 3f'''R^2 + 3f''\dot{R}], \quad (2)$$

where H is the Hubble parameter; the *Friedmann equation*

$$3H^2 = \frac{1}{f'}\left[\rho + \frac{Rf' - f}{2} - 3Hf''\dot{R}\right]; \quad (3)$$

the *trace equation*

$$3\ddot{R}f'' = \rho - 3P + f'R - 2f - 9Hf''\dot{R} - 3f'''R^2; \quad (4)$$

and the *energy conservation equation* for standard matter

$$\dot{\rho} = -\Theta(\rho + P). \quad (5)$$

Combining the Friedmann and Raychaudhuri equations, we obtain

$$R = 6\dot{H} + 12H^2, \quad (6)$$

which is the usual definition for the Ricci scalar for homogeneous and isotropic flat Friedmann-L  matre-Robertson-Walker spacetimes.

The Raychaudhuri equation can be obtained by adding the Friedmann equation with its time derivative (using the energy conservation equation and the definition of the Ricci scalar). Therefore any solution of the Friedmann equation is automatically a solution to the Raychaudhuri equation and hence it is sufficient to solve the Friedmann equation to reconstruct the theory of gravity.

III. RECONSTRUCTION OF A $f(R)$ THEORY THAT ADMITS AN EXACT Λ CDM MODEL

As we know, the present day observations suggest that the variation of the Hubble parameter with the redshift is sufficiently well described by the relation

$$H(z) = \sqrt{\frac{\rho_0}{3}(1+z)^3 + \frac{\Lambda}{3}}, \quad (7)$$

where $\rho_0 \geq 0$ is the matter density (which consists of the observed and the cold dark matter) and Λ is the cosmological constant. In what follows we try and construct theories (belonging to the class of $f(R)$ gravity), that exactly mimic the above expansion history.

From Eq. (7) we see that the time derivative of the scale-factor $a(t)$ can be given as

$$\dot{a} = \sqrt{\frac{\rho_0}{3a} + \frac{\Lambda}{3}}. \quad (8)$$

Here we have used the usual definition of the redshift $1/a = 1 + z$. From the above equation we can immediately calculate the second derivative of the scale-factor, which is given by

$$\ddot{a} = \frac{1}{2}(\dot{a}^2)_{,a} = \frac{2\Lambda a^3 - \rho_0}{6a^2}. \quad (9)$$

We know for a flat FLRW universe, the Ricci scalar is defined by

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \quad (10)$$

Using Eqs. (8) and (9) in Eq. (10) we obtain the Ricci scalar in terms of the scale-factor as

$$R(a) = \frac{4\Lambda a^3 + \rho_0}{a^3}. \quad (11)$$

We would like to emphasize here that till now we have not *a priori* assumed any specific theory of gravity. Equation (8) is obtained by observations while Eq. (10) is a purely geometrical result for flat homogeneous and isotropic spacetimes, independent of the theory of gravity.

We can now invert Eq. (10), to write the scale-factor in terms of the Ricci scalar

$$a(R) = \left(\frac{\rho_0}{R - 4\Lambda} \right)^{(1/3)}. \quad (12)$$

We note that, since the scale-factor has to be real, we considered only the real root of Eq. (10), discarding the other two complex conjugate roots. Also the positivity of the scale-factor implies that the Ricci scalar reaches the value 4Λ , asymptotically in an infinite time. From Eq. (12) we can calculate the Hubble parameter and the time derivative of Ricci scalar in terms of the Ricci scalar and these are given by

$$H(R) = \frac{1}{a(R)} \sqrt{\frac{\rho_0}{3a(R)} + \frac{\Lambda}{3}}, \quad (13)$$

$$\dot{R} = R_{,a}(a(R)) \sqrt{\frac{\rho_0}{3a(R)} + \frac{\Lambda}{3}}. \quad (14)$$

Similarly, using the energy conservation equation we can write the matter density of the universe in terms of scale-factor ‘ a ’, or alternatively $\rho(a(R))$.

To investigate which functions $f(R)$ exactly mimic the Λ CDM expansion history, we substitute all the above quantities written as a function of the Ricci scalar into the Friedmann equation, obtaining

$$-3(R - 3\Lambda)(R - 4\Lambda)f''(R) + \left(\frac{R}{2} - 3\Lambda\right)f'(R) + \frac{1}{2}f(R) - \rho(R) = 0. \quad (15)$$

Since this equation has to be satisfied for all times (which imply for all $R \geq 4\Lambda \in \Re$), this becomes a differential equation for the function $f(R)$ in R -space. It is easy to see that the above equation is an exact inhomogeneous hypergeometric equation for the variable $x \equiv -3 + R/\Lambda$, with $\rho(R)$ as the inhomogeneous term. The solution to the homogeneous part is given by

$$f(x) = C_1 F([\alpha_+, \alpha_-], -\frac{1}{2}; x) + C_2 x^{3/2} F([\beta_+, \beta_-], \frac{5}{2}; x), \quad (16)$$

where $\alpha_{\pm} = (-7 \pm \sqrt{73})/12$, $\beta_{\pm} = (-11 \pm \sqrt{73})/12$ and $C_{1,2}$ are arbitrary constants of integration.

Let us now analyze the solution carefully. The two finite poles of the hypergeometric equation are at $R = 3\Lambda$ and $R = 4\Lambda$ respectively. Since the allowed range of R is $R \geq$

4Λ , one of the pole is out of the range and the other is at the boundary. However we see that in this range $x \geq 1$. We know that the convergence of a hypergeometric function for the variable ‘ x ’ is guaranteed if $|x| < 1$, and otherwise the function is either divergent or complex valued. Indeed one can check explicitly that both the solutions of the homogeneous equation are complex valued for $R \geq 4\Lambda$. Hence to ensure a real valued function $f(R)$, we must choose both the arbitrary constants $C_{1,2}$ to be zero. This is interesting as it shows that there cannot be any real valued function of Ricci scalar that can mimic a Λ CDM expansion for a vacuum universe.

Therefore from now on we only consider the particular solution for the given inhomogeneous term $\rho(R)$. Let us now explicitly reconstruct the theories for which a given matter field would obey a Λ CDM expansion history. We would like to note here that a different reconstruction approach related with local tests in modified gravity was considered in [21].

A. Reconstruction for dustlike matter

Supposing the Universe is filled with dustlike matter ($w = 0$). From the energy conservation equation we have

$$\rho(a) = \frac{\rho_0}{a^3} \Rightarrow \rho(R) = R - 4\Lambda. \quad (17)$$

Substituting in the Friedmann Eq. (15), we get the particular solution as

$$f(R) = R - 2\Lambda, \quad (18)$$

which is the well known Lagrangian for general relativity with a cosmological constant. This result is interesting as it proves that the only real valued Lagrangian $f(R)$, that can mimic an exact Λ CDM expansion history for a universe filled with dustlike matter, is the Einstein-Hilbert Lagrangian with positive cosmological constant.

It is also important to note, however, that this is not the case, if we put $\Lambda = 0$. In that case the general solution of the Friedmann equation is

$$f(R) = R + C_1 R^{\alpha_+} + C_2 R^{\alpha_-} \quad (19)$$

We see that the function is real valued for positive Ricci scalar and hence there exist classes of real valued function $f(R)$, other than GR, that can mimic a dustlike expansion history without the cosmological constant. But even a very small value of the cosmological constant would break this degeneracy, and in that case the theory must be GR.

B. Reconstruction for perfect fluid with equation of state $p = -1/3\rho$

A perfect fluid with an equation of state $p = -\frac{1}{3}\rho$ is physically interesting as it lies in the boundary of the set of matter fields that obey the *strong energy condition*. In GR such fluids give rise to a *Milne Universe* which is a *coasting* universe, and the Ricci scalar is proportional to the

square of the Hubble parameter. However as we would see, even this kind of fluids can also mimic a Λ CDM Universe in higher-order gravity.

Using the equation of state in the energy conservation equation and supposing the present density of the fluid is ρ_f we get

$$\rho(a) = \frac{\rho_f}{a^2} \Rightarrow \rho(R) = [\rho_f(R - 4\Lambda)]^{2/3}. \quad (20)$$

Now solving the Friedmann Eq. (15), the particular solution is

$$f(R) = \mu(R - 4\Lambda)^{2/3}, \quad (21)$$

where μ is a constant depending on ρ_f .

C. Reconstruction for multifluids

Let us now consider that along with dustlike matter, a noninteracting stiff fluid is also present in the universe and their present densities are ρ_0 and ρ_s respectively. This scenario is also possible if we have noninteracting minimally-coupled massless scalar field with the dustlike matter. In this case, from the conservation equation, total matter density is

$$\rho(a) = \frac{\rho_0}{a^3} + \frac{\rho_s}{a^6} \Rightarrow \rho(R) = (R - 4\Lambda) + \frac{\rho_s}{\rho_0^2}(R - 4\Lambda)^2. \quad (22)$$

Substituting this into the Friedmann Eq. (15), we get the following particular solution:

$$f(R) = \mu_1 R + \mu_2 R^2 + \mu_3, \quad (23)$$

where $\mu_n (n = 1..3)$ are constants depending on ρ_0 , ρ_s and Λ . We therefore conclude that if the universe is filled with minimally-coupled noninteracting massless scalar field with dustlike matter, then the theory of gravity described above would exactly mimic a Λ CDM expansion history and it is impossible to distinguish this from GR with present cosmological observations for the background FLRW level.

D. Reconstruction for nonisentropic perfect fluids

Nonisentropic perfect fluids are generally described by the equation of state

$$p = h(\rho, a). \quad (24)$$

Using the energy conservation relation we get the required differential equation for ρ as

$$\rho(a)_{,a} = -\frac{3}{a}(\rho + h(\rho, a)). \quad (25)$$

In general the above equation may not admit a closed form solution. However the calculation gets much simpler if $h(\rho, a)$ is a separable function of the form

$$h(\rho, a) = w(a)\rho, \quad (26)$$

that is, the barotropic index of the fluid changes with time. In this case we can directly integrate Eq. (25) to get

$$\rho(a) = \exp\left[-3 \int \frac{1+w(a)}{a} da\right]. \quad (27)$$

We then substitute this into the Friedmann equation to get the required theory of gravity.

As a specific illustration, let's assume that the time dependent barotropic index is given by

$$w(a) = \frac{2\gamma - \nu a^3}{\gamma + \nu a^3}, \quad (28)$$

where γ and ν are constants. As we can see in early times the fluid has positive pressure and at late times this behaves like a cosmological constant. Using (27) and (12), we get ρ in terms of the Ricci scalar as

$$\rho(R) = \frac{(\nu\rho_0 + \gamma R - 4\Lambda\gamma)^3}{\rho_0^2}. \quad (29)$$

Substituting this into the Friedmann Eq. (15), we obtain the particular solution to be

$$f(R) = \mu_1 R + \mu_2 R^2 + \mu_3 R^3 + \mu_4, \quad (30)$$

where $\mu_n (n = 1..4)$ are constants depending on ρ_0 , γ , ν and Λ .

Let us consider another form of nonisentropic perfect fluids where the equation of state is given by

$$p = w\rho + h(a). \quad (31)$$

This equation of state is physically interesting, due to the presence of the tuning function $h(a)$, which can compensate the effect of the fourth order gravity at intermediate times. In this way we can have a matter dominated Friedmann like epoch (necessary for structure formations in the Universe), followed by an accelerated expanding phase.

Integrating the energy conservation Eq. (25), we obtain

$$\rho(a) = \left[- \int 3a^{(2+3w)}h(a)da + C_1\right]a^{-3(1+w)}, \quad (32)$$

where C_1 is an arbitrary constant of integration. Thus, we see that the effective density is that of an isentropic perfect fluid together with a time dependent cosmological term. We can always tune this term to obtain a fit with cosmological observations.

As a specific example let us consider $h(a) \equiv a^{-12}$ and $w = 0$. This then implies that the matter field in the Universe is dust along with a time dependent cosmological term which diverges at the *Big Bang singularity* and goes to zero sufficiently fast at the later epochs.

Integrating the energy conservation equation we obtain

$$\rho(a) = \frac{\rho_0}{a^3} + \frac{\rho_1}{a^{12}} \Rightarrow \rho(R) = R - 4\Lambda + \frac{\rho_1}{\rho_0}(R - 4\Lambda)^4. \quad (33)$$

Solving the Friedmann equation, we obtain the following particular solutions

$$f(R) = \mu_1 R + \mu_2 R^2 + \mu_3 R^3 + \mu_4 R^4 + \mu_5, \quad (34)$$

where μ_n ($n = 1..5$) are constants depending on ρ_0 , ρ_1 and Λ .

IV. RECONSTRUCTION OF APPROXIMATE ΛCDM MODELS

Let us now consider a useful technique proposed in Ref. [22], where the Friedmann equations are written as functions of the number of e-foldings instead of the time, $N = \ln \frac{a}{a_0}$. In such a case, the Hubble parameter is given in terms of N as:

$$H = g(N) = g(-\ln(1+z)). \quad (35)$$

Then, the first FLRW equation for a flat universe yields,

$$\begin{aligned} 0 = & -9G(N(R))(4G'(N(R)) + G''(N(R))) \frac{d^2 f(R)}{dR^2} \\ & + \left(3G(N(R)) + \frac{3}{2}G'(N(R)) \right) \frac{df(R)}{dR} - \frac{f(R)}{2} \\ & + \sum_i \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)N(R)}, \end{aligned} \quad (36)$$

where $G(N) \equiv g(N)^2 = H^2$, and the Ricci scalar can be related with N via $R = 12G(N) + 3G'(N)$. Then the Eq. (36) can be resolved for a given Hubble expansion rate, and the corresponding $f(R)$ theory is reconstructed.

Cosmological solutions in $f(R)$ gravity with the presence of an inhomogeneous EoS fluid

We consider now a Universe governed by some specific $f(R)$ theory in the presence of a perfect fluid, whose equation of state is given by,

$$p = w(a)\rho + \zeta(a), \quad (37)$$

where $w(a)$ and $\zeta(a)$ are functions of the scale-factor a , which could correspond to the dynamical behavior of the fluid and to its viscosity. Let us write the FLRW equations for $f(R)$ as following,

$$H^2 = \frac{1}{3}(\rho' + \rho_{f(R)}), \quad (38)$$

$$2\dot{H} + 3H^2 = -(p' + p_{f(R)}), \quad (39)$$

where $\rho' = \frac{\rho}{f'(R)}$ and $p' = \frac{p}{f'(R)}$. The pressure and energy density with the subscript $f(R)$ contains the terms corresponded to $f(R)$ and are defined as

$$\rho_{f(R)} = \frac{1}{f'} \left(\frac{Rf' - f}{2} - 3H\dot{R}f'' \right), \quad (40)$$

$$p_{f(R)} = \frac{1}{f'} \left(\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf') \right). \quad (41)$$

Then, by combining both FLRW equations, and using the equation of state defined in (37), we can write

$$\begin{aligned} \zeta(a) = & (w(a)\rho_{f(R)} - p_{f(R)} - 2\dot{H} \\ & - 3(1+w(a))H^2)f'(R(a)). \end{aligned} \quad (42)$$

As $\zeta(a)$ just depends on the Hubble parameter and its derivatives, for some specific solutions, any kind of cosmology can be reproduced. Let us consider the example,

$$\frac{3}{\kappa^2}H^2 = G_\rho a^{-c} + G_q a^d, \quad (43)$$

where G_ρ and G_q are constants. We can check that in this solution, the first term in the r.h.s. corresponds to a fluid with EoS $w_\rho = -1 + c/3 > -1$, while the second term, it has an equation of state $w_q = -1 - d/3 < -1$, which corresponds to a phantom fluid. We could consider a viable $f(R)$ function proposed in [23], which is given by

$$F(R) = \frac{\alpha R^{m+l} - \beta R^n}{1 + \gamma R^l}. \quad (44)$$

This function is known to pass the local gravity tests and could contribute to drive the Universe to an accelerated phase. Then, by introducing (43) and (44) in the expression for $\zeta(a)$ in (42), we obtain the equation of state for the inhomogeneous fluid that, together with $f(R)$, reproduces the solution (43), which for $c = 3$ and $d \geq 0$ reproduces the ΛCDM model, and probably drives the Universe evolution into a phantom phase in the near future.

V. CONCLUSION

In this paper we have extended a the reconstruction programme for $f(R)$ gravity obtaining several interesting explicit reconstructions. In particular we find that the only real valued Lagrangian $f(R)$ that is able to mimic an exact ΛCDM expansion history for a universe filled with dustlike matter is the Einstein-Hilbert Lagrangian with positive cosmological constant. This does not mean that $f(R)$ gravity is incompatible with an exact ΛCDM expansion history, only that one has to extend the theory somewhat in order that this possibility can be realized. For example in a universe filled with a both a minimally-coupled noninteracting massless scalar field and dustlike matter, a theory of gravity can be found which exactly mimics the ΛCDM expansion history, making it impossible to distinguish it from GR using measurements of the background cosmological parameters. It is then an interesting problem to probe, how the perturbations in these modified theories can break this degeneracy, by predicting different structure formations, growth factor or cosmological gravitational waves from GR, that can be experimentally verified [24].

In fact the matter power spectrum for power-law $f(R)$ theories satisfies the requirement for scale invariance and has distinct features that can be detected by combining future cosmic microwave background (CMB) and large scale surveys (LSS) data. Moreover, it remains the number of realistic modified gravities which reproduce this late-time cosmic acceleration era approximately, where essentially any degree of accuracy may be achieved. In that case, it is then required to have local constraints, like the Solar System or Post Newtonian constraints (see for e.g. [21] and

the references therein, which rules out the Lagrangians which grow indefinitely as the Ricci scalar vanishes) to have a modified gravity theory that works for both local and cosmological scales.

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- [1] J. P. Ostriker and P. J. Steinhardt, [arXiv:astro-ph/9505066](#).
- [2] S. Perlmutter *et al.*, *Astrophys. J.* **517**, 565 (1999); A. G. Riess *et al.*, *Astron. J.* **116**, 1009 (1998); J. L. Tonry *et al.*, *Astrophys. J.* **594**, 1 (2003); R. A. Knop *et al.*, *Astrophys. J.* **598**, 102 (2003); A. G. Riess *et al.*, *Astrophys. J.* **607**, 665 (2004); S. Perlmutter *et al.*, *Astrophys. J.* **517**, 565 (1999); *Astron. Astrophys.* **447**, 31 (2006).
- [3] D. N. Spergel *et al.*, *Astrophys. J. Suppl. Ser.* **148**, 175 (2003); **170**, 377 (2007).
- [4] M. Tegmark *et al.*, *Phys. Rev. D* **69**, 103501 (2004); U. Seljak *et al.*, *Phys. Rev. D* **71**, 103515 (2005); S. Cole *et al.*, *Mon. Not. R. Astron. Soc.* **362**, 505 (2005).
- [5] D. J. Eisenstein *et al.*, *Astrophys. J.* **633**, 560 (2005); C. Blake, D. Parkinson, B. Bassett, K. Glazebrook, M. Kunz, and R. C. Nichol, *Mon. Not. R. Astron. Soc.* **365**, 255 (2006).
- [6] B. Jain, A. Taylor, *Phys. Rev. Lett.* **91**, 141302 (2003).
- [7] A. A. Starobinsky, *Phys. Lett.* **91B**, 99 (1980); K. S. Stelle, *Gen. Relativ. Gravit.* **9**, 353 (1978).
- [8] S. Nojiri and S. D. Odintsov, in *Proceedings of the 42nd Karpacz Winter School of Theoretical Physics: Current Mathematical Topics in Gravitation and Cosmology, Ladek, Poland, 2006*, econf C0602061, 06 (2006); *Int. J. Geom. Methods Mod. Phys.* **4**, 115 (2007); S. Capozziello and M. Francaviglia, *Gen. Relativ. Gravit.* **40**, 357 (2008); A. de Felice and S. Tsujikawa, *Living Rev. Relativity* **13**, 3 (2010); V. Faraoni and T. Sotiriou, *Rev. Mod. Phys.* **82**, 451 (2010).
- [9] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, *Phys. Rev. D* **70**, 043528 (2004); S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **68**, 123512 (2003); *Gen. Relativ. Gravit.* **36**, 1765 (2004); *Phys. Lett. B* **576**, 5 (2003); S. Capozziello, *Int. J. Mod. Phys. D* **11**, 483 (2002); V. Faraoni, *Phys. Rev. D* **72**, 124005 (2005); M. L. Ruggiero and L. Iorio, *J. Cosmol. Astropart. Phys.* **01** (2007) 010; A. de la Cruz-Dombriz and A. Dobado, *Phys. Rev. D* **74**, 087501 (2006); N. J. Poplawski, *Phys. Rev. D* **74**, 084032 (2006); A. W. Brookfield, C. van de Bruck, and L. M. H. Hall, *Phys. Rev. D* **74**, 064028 (2006); Y. Song, W. Hu, and I. Sawicki, *Phys. Rev. D* **75**, 044004 (2007); B. Li, K. Chan, and M. Chu, *Phys. Rev. D* **76**, 024002 (2007); X. Jin, D. Liu, and X. Li, [arXiv:astro-ph/0610854](#); T. P. Sotiriou and S. Liberati, *Ann. Phys. (N.Y.)* **322**, 935 (2007); T. P. Sotiriou, *Classical Quantum Gravity* **23**, 5117 (2006); R. Bean, D. Bernat, L. Pogosian, A. Silvestri, and M. Trodden, *Phys. Rev. D* **75**, 064020 (2007); A. J. Bustelo and D. E. Barraco, *Classical Quantum Gravity* **24**, 2333 (2007); J. Ford, S. Giusto, and A. Saxena, *Nucl. Phys.* **B790**, 258 (2008); F. Briscese, E. Elizalde, S. Nojiri, and S. D. Odintsov, *Phys. Lett. B* **646**, 105 (2007); S. Baghram, M. Farhang, and S. Rahvar, *Phys. Rev. D* **75**, 044024 (2007); D. Bazeia, B. Carneiro da Cunha, R. Menezes, and A. Petrov, *Phys. Lett. B* **649**, 445 (2007); P. Zhang, *Phys. Rev. D* **76**, 024007 (2007); B. Li and J. D. Barrow, *Phys. Rev. D* **75**, 084010 (2007); L. M. Sokolowski, *Classical Quantum Gravity* **24**, 3391 (2007); V. Faraoni, *Phys. Rev. D* **75**, 067302 (2007); O. Bertolami, C. G. Boehmer, T. Harko, and F. S. N. Lobo, *Phys. Rev. D* **75**, 104016 (2007); S. K. Srivastava, *Int. J. Theor. Phys.* **47**, 1966 (2008); S. Capozziello, V. F. Cardone, and A. Troisi, *J. Cosmol. Astropart. Phys.* **08** (2006) 001; S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **657**, 238 (2007); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, *Phys. Rev. D* **77**, 046009 (2008).
- [10] S. Capozziello, V. F. Cardone, S. Carloni, and A. Troisi, *Int. J. Mod. Phys. D* **12**, 1969 (2003).
- [11] K. i. Maeda and N. Ohta, *Phys. Lett. B* **597**, 400 (2004); K. i. Maeda and N. Ohta, *Phys. Rev. D* **71**, 063520 (2005); N. Ohta, *Int. J. Mod. Phys. A* **20**, 1 (2005); K. Akune, K. i. Maeda, and N. Ohta, *Phys. Rev. D* **73**, 103506 (2006).
- [12] S. Capozziello, V. F. Cardone, and A. Troisi, *Mon. Not. R. Astron. Soc.* **375**, 1423 (2007).
- [13] R. Kerner, *Gen. Relativ. Gravit.* **14**, 453 (1982); J. P. Duruisseau and R. Kerner, *Classical Quantum Gravity* **3**, 817 (1986).
- [14] P. Teyssandier, *Classical Quantum Gravity* **6**, 219 (1989).
- [15] G. Magnano, M. Ferraris, and M. Francaviglia, *Gen. Relativ. Gravit.* **19**, 465 (1987).
- [16] *Dynamical Systems in Cosmology*, edited by J. Wainwright and G. F. R. Ellis (Cambridge University Press, Cambridge, England, 1997), ISBN 0-521-55457-8.
- [17] S. Carloni, P. K. S. Dunsby, S. Capozziello, and A. Troisi, *Classical Quantum Gravity* **22**, 4839 (2005); S. Carloni and P. K. S. Dunsby, *J. Phys. A* **40**, 6919 (2007); S. Carloni, A. Troisi, and P. K. S. Dunsby, *Gen. Relativ. Gravit.* **41**, 1757 (2009).

- [18] N. Goheer, J. Larena, and P.K.S. Dunsby, *Phys. Rev. D* **80**, 061301 (2009).
- [19] S. Nojiri, S.D. Odintsov, A. Toporensky, and P. Tretyakov, [arXiv:0912.2488](https://arxiv.org/abs/0912.2488); G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, and S. Zerbini, [arXiv:0909.2747](https://arxiv.org/abs/0909.2747); S. Nojiri and S.D. Odintsov, *J. Phys. Conf. Ser.* **66**, 012005 (2007); S. Nojiri and S.D. Odintsov, *Phys. Rev. D* **74**, 086005 (2006); S. Nojiri, S.D. Odintsov, and M. Sami, *Phys. Rev. D* **74**, 046004 (2006).
- [20] S. Carloni, R. Goswami, and Peter K.S. Dunsby, [arXiv:1005.1840](https://arxiv.org/abs/1005.1840)
- [21] G.J. Olmo, *Phys. Rev. Lett.* **95**, 261102 (2005); *Phys. Rev. D* **72**, 083505 (2005).
- [22] S. Nojiri, S.D. Odintsov, and D. Sáez-Gómez, *Phys. Lett. B* **681**, 74 (2009).
- [23] S. Nojiri and S.D. Odintsov, *Phys. Rev. D* **77**, 026007 (2008).
- [24] A. de la Cruz-Dombriz, A. Dobado, and A.L. Maroto, *Phys. Rev. D* **77**, 123515 (2008); K.N. Ananda, S. Carloni, and P.K.S. Dunsby, *Classical Quantum Gravity* **26**, 235018 (2009); S. Carloni, P.K.S. Dunsby, and A. Troisi, *Phys. Rev. D* **77**, 024024 (2008).