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Non-Abelian bosonic currents in cosmic strings

Marc Lilley,[1,2,*] Fabrizio Di Marco,[3,†] Jérôme Martin,[1,‡] and Patrick Peter[1,§]

[1]*GReCO—Institut d'Astrophysique de Paris, UMR7095 CNRS, Université Pierre and Marie Curie, 98 bis boulevard Arago, 75014 Paris, France*

[2]*Theoretical and Mathematical Physics Group, Centre for Particle Physics and Phenomenology, Louvain University, 2 Chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium*

[3]*ICRANet, Piazza della Repubblica 10, 65122 Pescara, Italy*

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A non-Abelian generalization of the neutral Witten current-carrying string model is discussed in which the bosonic current carrier belongs to a two-dimensional representation of SU(2). We find that the current-carrying solutions can be of three different kinds: either the current spans a U(1) subgroup, and in which case one is left with an Abelian current-carrying string, or the three currents are all lightlike, traveling in the same direction (only left or right movers). The third, genuinely non-Abelian situation, cannot be handled within a cylindrically symmetric framework, but can be shown to depend on all possible string Lorentz invariant quantities that can be constructed out of the phase gradients.

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I. INTRODUCTION

Topological cosmic strings or superstrings of cosmological size are one-dimensional extended objects which are believed to have been formed in the early phases of cosmological evolution. They are of considerable interest because they may offer an observable window on the high energy physics of the primordial universe, i.e., at grand unified scales.

Topological strings are produced in phase transitions associated with spontaneous symmetry breaking. This is the standard Kibble mechanism [1,2]. Almost all supersymmetric grand unified theories in which hybrid inflation [3–5] can be realized lead to the formation of topological strings [6–9]. Besides, most classes of superstring compactification lead to a spontaneous breaking of a pseudoanomalous U(1) gauge symmetry producing local cosmic strings [10]. Such strings also form in the case where the Higgs field has a nonminimal kinetic term [11].

The simplest kind of topological string is the Nambu-Goto string which is described by the Nambu-Goto action [12,13]. The Nambu-Goto action is the world-sheet formulation counterpart of a field theory description in which the string arises as a solitonic solution of the Abelian Higgs model [14]. Such a string has no internal structure and is described entirely in terms of a world-sheet Lagrangian and the tension per unit length of the string.

Most observational signatures in the gravitational sector expected from topological strings have been derived and simulated numerically for Nambu-Goto strings. There are five main possible observational effects (see [15,16] and references therein): beamed gravitational wave bursts from kinks and particle acceleration; deflection, gravitational lensing effects, and multiple image effects; Doppler shifting effects; background gravitational radiation from string loops; and string effects in the cosmic microwave background. The existence of kinks along the strings has been shown to occur also for current-carrying strings [17] and the electromagnetic effects of such strings, which are absent in the simpler Nambu-Goto string, have been investigated. An especially interesting observational consequence of the presence of cosmic string networks in the early universe potentially because it is susceptible to being detected in the cosmic microwave background is the Gott-Kaiser-Stebbins effect [18,19]. This effect consists of a temperature shift that is due to the gravitational lensing of photons passing near a moving source.

Cosmic superstrings are formed by tachyon condensation at the end of brane inflation [20,21]. The tachyons are complex scalars [with a local U(1) gauge symmetry] identifiable with the ground state open string modes of the Neveu-Schwarz sector that end on coincident non-BPS branes and antibranes [22–25]. There exist associated gauge fields living on the brane and antibrane so that there exists a $U(1) \times U(1)$ symmetry on the brane-antibrane configuration. A first linear combination of the U(1)'s is Higgsed [26,27] leading to the appearance of a first kind of cosmic superstrings that are D p-branes with $p - 1$ dimensions compactified [28]. In type IIB superstring theory, and given a spacetime manifold $\mathcal{M}$, such stable p-branes, can, for example, be obtained by considering a $p + 2$ brane-antibrane pair stretching over a submanifold $\mathbb{R}^{p+3} \subset \mathcal{M}$. The $p + 2$ brane-antibrane pair will annihilate unless a topological obstruction exists. This obstruction can be obtained from K-theory [29–31]. A second linear combination of the U(1)'s leads to the formation of F-strings [26,27].

All of these types of strings have until recently been considered as structureless, so their dynamics is given by

*lilley@iap.fr
†Fabrizio.DiMarco@icranet.org
‡jmartin@iap.fr
§peter@iap.fr

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the Nambu-Goto action. Numerical simulations of networks (see [32] and references therein) of such strings have been produced with the result of scaling, a property thanks to which the string network never comes to dominate the Universe evolution, but neither are the strings completely washed out of the Universe, so their effect, however small, is still detectable.

The Nambu-Goto string can be generalized to the case of a string with internal structure. Such a string can be obtained by including a coupling of the string-forming Higgs field to additional (bosonic or fermionic, with global or local, Abelian, or non-Abelian symmetry) fields in the theory. In part of the parameter space, these fields condense onto the string (the symmetry gets broken) leading to the appearance of currents on the world sheet in the form of Goldstone bosons propagating along string [33]. In such a case, the current-carrying string can be described using a world-sheet Lagrangian and a nontrivial equation of state relating the tension per unit length to the energy density of the string [34–37]; the actual form of this equation of state was discussed numerically [38–40] and analytically [41]. The presence of currents on the world sheet modifies only slightly the gravitational properties of the long strings [42,43], but it also halts cosmic string loop decay caused by dissipative effects, thereby yielding new equilibrium configurations [17,44] named vortons [45–50]. Those can potentially change drastically the cosmological network evolution, at the point of ruling such strings out.

Although the current-carrying property of cosmic strings is in fact fairly generic [51–53], a possibility that has, until now, been completely disregarded is that for which the string would be endowed not only with many currents [54], but also with currents of a non-Abelian kind, as is to be expected in most grand unified theories. This natural extension of the Witten idea leads to numerous new difficulties, as, in particular, the internal degrees of freedom manifold is intrinsically curved, so that a local, flat, description of the string world-sheet manifold turns out to be inappropriate [55,56]. This paper is devoted to the specific task of obtaining the equivalent microscopic structure of a non-Abelian current-carrying cosmic string.

To do so, we restrict attention to the global situation in which, in a way similar to the so-called neutral Witten model [38], we wish to capture the essential internal dynamics of the string without the undue complication of adding extra gauge vector fields. In the case of an Abelian current, it was indeed shown that these contributions, although of potential great cosmological relevance (see, e.g. Ref. [57] and references therein), can however be treated in a perturbative way, not modifying in any essential way the actual microscopic structure [39]. We therefore assume, as a toy model, a U(1) Higgs model whose breaking leads to the existence of the strings themselves, coupled to an SU(2) doublet through a scalar potential with parameters ensuring a condensate. We first describe the fields and notation, derive their dynamical equations in full generality, and then discuss the condensate configuration. After having recovered the Abelian cases as particular solutions of the general non-Abelian situation, we concentrate on the strictly non-Abelian solutions. We obtain an exact configuration, called trichiral, and show how this model makes explicit the obstruction theorem first obtained by Carter [55,56]. We then derive the stress-energy tensor and its eigenvalues, namely, the energy per unit length and tension, and show that they depend on all the possible two-dimensional Lorentz invariants that can be constructed from the phase gradients (and second derivatives) of the angular variables in the internal space. We conclude by discussing the possible cosmological consequences of this new category of objects.

II. FIELDS CONTENT

The simplest non-Abelian current-carrying string model that can be written down is that in which a U(1) symmetry is spontaneously broken by means of a scalar complex Higgs field ϕ, itself coupled to $\mathbf{\Sigma}$, a scalar field belonging to an arbitrary representation of a non-Abelian group G. The string-forming action stems from the Higgs Lagrangian

$$\mathcal{L}_{\rm S} = -D_\mu \phi^\star D^\mu \phi - \tfrac{1}{4} C_{\mu\nu} C^{\mu\nu} - V_{\rm H}(\phi), \qquad (1)$$

where

$$C_{\mu\nu} = \nabla_\mu C_\nu - \nabla_\nu C_\mu \qquad (2)$$

and the U(1) covariant derivative is expressed in terms of the U(1) gauge field C_μ as

$$D_\mu \phi = \nabla_\mu \phi + iqC_\mu \phi, \qquad (3)$$

where q is the charge. $V_{\rm H}$ can be chosen without lack of generality as the Higgs symmetry breaking potential, namely,

$$V_{\rm H} = \frac{\lambda_\phi}{4}(|\phi|^2 - \eta^2)^2, \qquad (4)$$

with λ_ϕ a coupling constant and η the Higgs vacuum expectation value (VEV) at infinity.

The current part of the Lagrangian reads

$$\mathcal{L}_{\rm C} = -(\partial_\mu \mathbf{\Sigma})^\dagger \cdot \partial^\mu \mathbf{\Sigma} - V_{\rm C}(\mathbf{\Sigma}), \qquad (5)$$

where $\mathbf{\Sigma}$ transforms according to a yet arbitrary representation of the global invariance group whose structure constants we write as $f^a{}_{bc}$; these are defined through the commutation relations for $\{T^a\}$, the algebra of G, namely,

$$[T^a, T^b] = if^c{}_{ab} T_c. \qquad (6)$$

In Eq. (6) and in the following, the group indices are denoted by Latin small cap letters $a, b, \ldots = 1, \ldots, N$ which run to N, the group dimension. The potential appearing in the current action is the self-interacting potential chosen as

$$V_C(\mathbf{\Sigma}) = \pm m_\sigma^2 \mathbf{\Sigma}^\dagger \cdot \mathbf{\Sigma} + \lambda_\sigma (\mathbf{\Sigma}^\dagger \cdot \mathbf{\Sigma})^2, \qquad (7)$$

thus introducing the vacuum mass and self-interaction constants m_σ and λ_σ. In Eq. (7), we have introduced a sign parameter which accounts for the possibility that SU(2) is broken ($-$) or unbroken ($+$) far from the string core. The first possibility is usually not taken into account when one considers the Witten model since in that case, one has in mind that the condensate depicts electromagnetism, which is obviously unbroken far from the string. In the non-Abelian case however, it is reasonable to assume a broken symmetry far from the string as well, in particular, if one is to identify this symmetry with that of the electroweak phenomenology.

The total action of the system can be written as

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_C - V_{\text{int}}, \qquad (8)$$

where the interaction term couples the two scalar fields $V_{\text{int}}(\phi, \mathbf{\Sigma})$. This potential, again for illustrative purposes below, shall be taken as the most general renormalizable one, namely,

$$V_{\text{int}}(\phi, \mathbf{\Sigma}) = f(|\phi|^2 - \eta^2)\mathbf{\Sigma}^\dagger \cdot \mathbf{\Sigma}, \qquad (9)$$

with a positive coupling constant f to ensure vacuum stability. The vacuum far from the string therefore depends on the representation $\mathbf{\Sigma}$ belongs to. The microscopic parameters that allow for a condensate to form are similar to those of the Abelian current case; they have been discussed, in particular, in Ref. [38].

III. FIELD EQUATIONS

Having specified the field content and the action of the system, one can now derive the corresponding equations of motion. The equations of motion of the system consisting of the string-forming fields ϕ and C_μ and the current carrier $\mathbf{\Sigma}$ are

$$\nabla_\mu \nabla^\mu \phi + 2iqC^\mu \nabla_\mu \phi + iq\phi \nabla_\mu C^\mu - q^2 C^\mu C_\mu \phi - \frac{\partial V}{\partial \phi^*} = 0 \qquad (10)$$

for the string-forming Higgs field,

$$\nabla_\mu C^{\mu\nu} - iq(\phi \nabla^\nu \phi^* - \phi^* \nabla^\nu \phi) - 2q^2 C^\nu |\phi|^2 = 0 \qquad (11)$$

for the associated U(1) gauge field, and

$$\Box \mathbf{\Sigma} = \frac{\partial V}{\partial \mathbf{\Sigma}^\dagger} \qquad (12)$$

for the current carrier.

The energy-momentum tensor of the system is given by the usual relation

$$T_{\mu\nu} \equiv g_{\mu\nu}\mathcal{L} - 2\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}, \qquad (13)$$

and can be decomposed into a scalar and a vector part, namely,

$$T_{\mu\nu} = T^{\text{s}}_{\mu\nu} + T^{\text{v}}_{\mu\nu}, \qquad (14)$$

where

$$T^{\text{s}}_{\mu\nu} = D_{(\mu}\phi^* D_{\nu)}\phi - g_{\mu\nu} D_\gamma \phi^* D^\gamma \phi + \partial_{(\mu} \mathbf{\Sigma}^\dagger \cdot \partial_{\nu)} \mathbf{\Sigma} - g_{\mu\nu}(\partial_\gamma \mathbf{\Sigma})^\dagger \cdot \partial^\gamma \mathbf{\Sigma} - g_{\mu\nu} V(\phi, \mathbf{\Sigma}), \qquad (15)$$

with parentheses denoting symmetrization of the indices, i.e., $S_{(\alpha\beta)} \equiv S_{\alpha\beta} + S_{\beta\alpha}$, and

$$T^{\text{v}}_{\mu\nu} = -(C_{\mu\alpha} C^\alpha{}_\nu + \tfrac{1}{4} g_{\mu\nu} C_{\alpha\beta} C^{\alpha\beta}). \qquad (16)$$

From this stress-energy tensor and the field equations, we shall now derive the full microscopic structure of the system.

IV. THE CONDENSATE

Having derived the most general form of the equations of motion, we now turn to the specific situation where a straight, infinitely long, cosmic string is present. A typical vortex solution aligned along the z axis in polar coordinates r and θ is then given by the Nielsen-Olesen ansatz

$$\phi = \varphi(r) e^{in\theta} \quad \text{and} \quad C_\mu = C_\theta(r) \delta^\theta_\mu, \qquad (17)$$

where $n \in \mathbb{Z}$. Although the specific form of the potential is irrelevant for most of what follows, the shape (4), being the most general renormalizable function satisfying this constraint, is used in the numerical illustrations below. Inserting the above ansatz into the equations of motion, Eq. (10) takes the form

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} = \frac{Q^2}{r^2}\varphi + \frac{\partial V}{\partial \varphi}, \qquad (18)$$

while Eq. (11) becomes

$$\frac{d^2 Q}{dr^2} - \frac{1}{r}\frac{dQ}{dr} = 2q^2 Q \varphi^2, \qquad (19)$$

where we have defined $Q \equiv n + qC_\theta$. In Eq. (18), the last term on the right-hand side (rhs) involves not only the derivative of the self-interaction potential V_H, but also that of the coupling term V_{int}, so that this equation also depends on the SU(2) doublet amplitude. It is through this "backreaction" term that the string itself is affected by the presence of the current.

Let us now discuss in more detail the form of the current-carrier scalar field $\mathbf{\Sigma}$. Our goal is to find the most general ansatz for $\mathbf{\Sigma}$ in cylindrical coordinates. The case where $G = \text{U}(1)$ represents the usual so-called superconducting string model originally introduced by Witten [58]. In this particular case, Σ is a complex field vanishing in vacuum, i.e. far from the string. Its coupling with the string-forming Higgs field yields an instability in the vortex core leading to a condensate: far from the string, in vacuum, where the Higgs field is equal to its VEV $|\phi| = \eta$, the interaction term $V_{\rm int}$ vanishes so that Σ must vanish. The string location, defined as the set of points where $\phi = 0$, however, is no longer vacuumlike from the point of view of Σ, and indeed the parameters of the potential (9) can be chosen [38,39] such that Σ does not vanish inside the vortex.

One can pick a specific gauge in which Σ is real, $\Sigma = \sigma(r)$ say, depending only on the distance to the string, with $\sigma \in \mathbb{R}$ and $\lim_{r\to\infty}\sigma(r) = 0$, and generate all the solutions by applying a gauge transformation, in this case a phase. The full solution then reads

$$\Sigma = \mathrm{e}^{i\psi(z,t)T}\sigma(r), \tag{20}$$

where the phase transformation can now depend on the world-sheet internal coordinates and we did not take into account a possible dependence in the external coordinates. In Eq. (20), we have written explicitly the generator of the U(1) translation as T, even though it is not necessary in this simplifying case for which the scalar field is a mere singlet under this extra U(1); note that this could be different if Σ were belonging to the representation of a larger group containing this U(1).

Written in the form (20) with the generator, the solution is easily generalizable to the non-Abelian case. We again choose a gauge in which $\boldsymbol{\Sigma} = \boldsymbol{\sigma}(x^\perp)$, with $\boldsymbol{\sigma}$ in the desired representation but depending only on the external coordinates $x^\perp$ (in practice the radial distance r), and produce the full solution by exponentiation of the generators T_a as

$$\boldsymbol{\Sigma} = \mathrm{e}^{i\psi^a(\xi)T_a}\boldsymbol{\sigma}(x^\perp), \tag{21}$$

where the functions ψ_a *a priori* depend on the internal coordinates ξ only. As it turns out however [55,56], in the more general case of a non-Abelian symmetry, the fields ψ_a live on a curved manifold which cannot, in general, be smoothly projected on the flat manifold describing the string world sheet. As a result, one must assume that the fields ψ_a depend on all embedding coordinates.

The form (21) is not, unfortunately, directly usable, as the derivative of the group element is not easy to handle. Indeed, for a noncommuting algebra, one has

$$\partial_\mu U = i\partial_\mu \boldsymbol{\psi}\cdot\int_0^1 U(1-p)\boldsymbol{T}U(p)\mathrm{d}p \neq i\partial_\mu\boldsymbol{\psi}\cdot\boldsymbol{T}U, \tag{22}$$

where $U(p) \equiv \exp(ip\boldsymbol{\psi}\cdot\boldsymbol{T})$ and $U \equiv \exp(i\boldsymbol{\psi}\cdot\boldsymbol{T}) = \mathrm{U}(1)$, and the last relation becomes an equality in the Abelian case. Restricting attention to SU(2) however, allows simple calculations to be carried out completely since one then has the useful relation

$$\mathrm{e}^{i\alpha\boldsymbol{n}\cdot\boldsymbol{\tau}} = \cos\alpha\mathbb{1} + i\boldsymbol{n}\cdot\boldsymbol{\tau}\sin\alpha, \quad \text{with} \quad n_a n^a = 1, \tag{23}$$

between the Pauli matrices τ^a, generators of SU(2), and their exponentiated form. We therefore restrict attention to a scalar field belonging to the representation **2** of SU(2), i.e. a doublet, and thus assume in what follows that the current carrier takes the form

$$\boldsymbol{\Sigma} = (\cos\alpha\mathbb{1} + i\boldsymbol{n}\cdot\boldsymbol{\tau}\sin\alpha)\sigma g, \quad \text{with} \quad g \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}. \tag{24}$$

Notice that Eq. (18), together with the assumption of a potential depending only on the amplitude $\boldsymbol{\Sigma}^\dagger\cdot\boldsymbol{\Sigma} = \frac{1}{2}\sigma^2$, shows that $\sigma = \sigma(r)$ only. But, as already mentioned above, the angle α and the normalized vector n^a *a priori* depend on all the coordinates.

With the form (24) for the scalar field, the variation of the potential is

$$\frac{\partial V}{\partial\boldsymbol{\Sigma}^\dagger} = \frac{1}{2}\frac{\partial V}{\partial\sigma}(\cos\alpha\mathbb{1} + i\boldsymbol{n}\cdot\boldsymbol{\tau}\sin\alpha)g, \tag{25}$$

which provides the equation of motion through Eq. (12). Indeed, projecting this equation of motion on the identity of SU(2) yields

$$\Delta\sigma - [(\partial\alpha)^2 + \tan\alpha\Box\alpha]\sigma - 2\tan\alpha\partial\alpha\cdot\partial\sigma = \frac{1}{2}\frac{\partial V}{\partial\sigma}, \tag{26}$$

while the projection on the Pauli matrices τ^a leads to

$$n^a\left\{\Delta\sigma + \left[\frac{\Box\alpha}{\tan\alpha} - (\partial\alpha)^2\right]\sigma + 2\frac{\partial\alpha\cdot\partial\sigma}{\tan\alpha}\right\} + 2\left(\partial\sigma + \frac{\sigma\partial\alpha}{\tan\alpha}\right)\cdot\partial n^a + \sigma\Box n^a = \frac{1}{2}\frac{\partial V}{\partial\sigma}n^a, \tag{27}$$

which in turn implies, upon projection on n_a, recalling this vector to be normalized to unity, that

$$\Delta\sigma - \left[(\partial\alpha)^2 - \frac{\Box\alpha}{\tan\alpha} - n_a\Box n^a\right]\sigma + 2\frac{\partial\alpha\cdot\partial\sigma}{\tan\alpha} = \frac{1}{2}\frac{\partial V}{\partial\sigma}. \tag{28}$$

This last equation can be used in order to simplify Eq. (27). Indeed, inserting Eq. (28) into Eq. (27), one obtains

$$\Box n^a + 2\left(\frac{\partial\sigma}{\sigma} + \frac{\partial\alpha}{\tan\alpha}\right)\cdot\partial n^a - (n_b\Box n^b)n^a = 0, \tag{29}$$

which provides a clean equation for the evolution of the vector n^a. Note also that Eqs. (26) and (28) can be combined to provide a dynamical equation for the angle α, namely,

$$\Box\alpha + \frac{2}{\sigma}\partial\sigma\cdot\partial\alpha + \sin\alpha\cos\alpha(n_a\Box n^a) = 0, \tag{30}$$

and the profile of the condensate then satisfies

$$\Delta\sigma - [(\partial\alpha)^2 - (n_a\Box n^a)\sin^2\alpha]\sigma = \frac{1}{2}\frac{\partial V}{\partial\sigma}, \tag{31}$$

which generalizes the Abelian case by inclusion of the nonlinear term. At this stage, Eqs. (29)–(31) are the equations that one needs to solve in order to determine σ, α, and n^a.

In fact, they can still be further simplified. Indeed, let us now expand the vector components in such a way as to implement its normalization, i.e. by projecting these components on the sphere on which it evolves in terms of angular variables $\beta(t, r, z, \theta)$ and $\gamma(t, r, z, \theta)$. This gives

$$n^1 = \sin\beta\sin\gamma, \qquad n^2 = \sin\beta\cos\gamma, \qquad n^3 = \cos\beta, \tag{32}$$

and therefore

$$n_a \Box n^a = -(\partial\beta)^2 - \sin^2\beta(\partial\gamma)^2, \tag{33}$$

which shows that Eq. (30) is indeed a dynamical equation for the variable α only. Using the expansion (32), one can transform Eq. (29) into

$$\Box\beta + 2\left(\frac{\partial\sigma}{\sigma} + \frac{\partial\alpha}{\tan\alpha}\right)\cdot\partial\beta = \cos\beta\sin\beta(\partial\gamma)^2, \tag{34}$$

and

$$\Box\gamma + 2\left(\frac{\partial\sigma}{\sigma} + \frac{\partial\alpha}{\tan\alpha} + \frac{\partial\beta}{\tan\beta}\right)\partial\gamma = 0, \tag{35}$$

that completes a new set of dynamical equations, namely, Eqs. (30), (31), (34), and (35), for the 4 independent functions σ, α, β, and γ. A particular solution for constant angles and gradients (lowest energy state) is exemplified in Fig. 1 for the cases for which SU(2) is unbroken or broken far from the string, derived using typical values for the parameters.

V. ABELIAN CASES

Since the group SU(2) contains invariant U(1)'s, it can be used, restricting to special cases, to recover the Abelian Witten model [33] as well as the bi-Abelian case [54]. The purpose of this section is precisely to establish the correspondences.

A. Witten Abelian model

The form (24) for the scalar doublet can be rewritten in terms of the angles α, β, and γ as

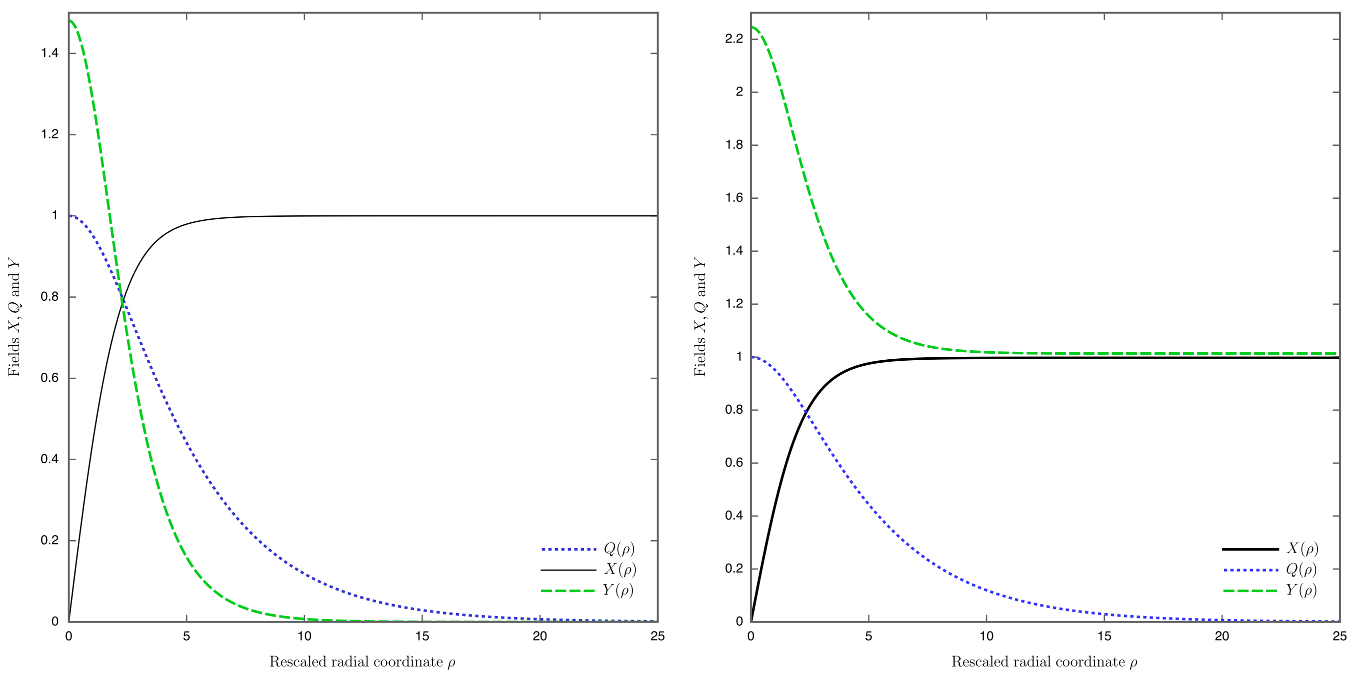

FIG. 1 (color online). Typical numerical solution of the system (18), (19), and (31) with constant phases (α, β, and γ constant) for the dimensionless fields $X(\rho) = \varphi/\eta$, $Y(\rho) = \sigma\sqrt{\lambda_\sigma}/m_\sigma$, and Q as a function of the rescaled distance to the string core $\rho = \sqrt{\lambda_\phi}\eta r$ for parameters fixed to $\tilde{q}^2 = 0.1$, $\alpha_1 = 3.37 \times 10^{-3}$, $\alpha_2 = 2.63 \times 10^{-3}$, and $\alpha_3 = 5.26 \times 10^{-4}$. The relevant free parameters are defined in a way reminiscent of Ref. [38], i.e. $\tilde{q}^2 = 2q^2/\lambda_\phi$, $\alpha_1 = m_\sigma^2/(2\lambda_\sigma\eta^2)$, $\alpha_2 = f m_\sigma^2/(2\lambda_\phi\lambda_\sigma\eta^2)$, and $\alpha_3 = m_\sigma^4/(2\lambda_\phi\lambda_\sigma\eta^4)$ (the α_i's have of course nothing to do with the angle α introduced before). The solutions are calculated by means of successive overrelaxation [69] for both cases for which either the SU(2) field does not condense in vacuum, i.e. for the plus sign in front of the massive term in Eq. (7) (left panel), or that for which the SU(2) field does condense in vacuum, i.e. for the minus sign (right panel). The fact that the three curves for the Higgs field [$X(\rho)$, full line], the U(1) gauge field [$Q(\rho)$, dashed line], and the SU(2) scalar condensate [$\sigma(\rho)$, dotted line] seem to cross at a single point for the noncondensing case of the left panel is purely coincidental and merely due to the specific choice of the parameters. The normalization of Y with respect to that of σ implies that in the large distance limit $\rho \to \infty$, one has $Y \to \frac{1}{2}(1 \mp 1)$.

$$\boldsymbol{\Sigma} = \frac{\sigma}{\sqrt{2}}\begin{pmatrix} e^{i\gamma}\sin\alpha\sin\beta \\ \cos\alpha - i\sin\alpha\cos\beta \end{pmatrix}, \tag{36}$$

from which one would like to single out a phase representing the U(1) situation. In other words, one wants to identify real functions ψ, f, and g such that

$$\boldsymbol{\Sigma} = e^{i\psi}\begin{pmatrix} f \\ g \end{pmatrix}. \tag{37}$$

Through identification of (37) with (36), one can easily convince oneself that there are only two possibilities, namely,

$$\alpha = \beta = \frac{\pi}{2}, \qquad \psi = \gamma, \qquad f = \frac{\sigma}{\sqrt{2}}, \qquad g = 0, \tag{38}$$

and

$$\psi = -\alpha, \qquad \beta = 0, \qquad \gamma \in \mathbb{R}, \qquad f = 0, \qquad g = \frac{\sigma}{\sqrt{2}}. \tag{39}$$

The first case, Eq. (38), leads to $n_a \Box n^a = -(\partial\psi)^2$, and the field equations become

$$\Delta\sigma - (\partial\psi)^2\sigma = \frac{1}{2}\frac{\partial V}{\partial\sigma}, \tag{40}$$

and

$$\Box\psi + \frac{2}{\sigma}\frac{d\sigma}{dr}\partial_r\psi = 0. \tag{41}$$

In the Abelian case, the phase does not depend on the radial distance and, hence, the last equation simply becomes $\Box\psi = 0$. This relation, together with Eq. (40), are exactly the equations of motion in the Abelian case [33–40]. The fact that we recover them from the most general framework discussed here is a consistency check of Eqs. (30), (31), (34), and (35). In the same manner, one can also check that the ansatz (39) also leads to the Abelian equations of motion.

At this point, a clarification concerning the Abelian situation is useful. With the set of equations above, one in principle assumes the phase to vary only along the world-sheet directions, i.e., $\psi = \psi(z, t)$; see above. However, this is not merely an assumption, but rather a fact that can be demonstrated through separation of variables: since the scalar field amplitude σ depends only on the radial distance r, setting $\psi = R(r) + T(\theta) + W(z, t)$, Eq. (40) tells us that

$$(\partial\psi)^2 = \left(\frac{dR}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dT}{d\theta}\right)^2 + (\partial_z W)^2 - (\partial_t W)^2 \tag{42}$$

is a yet unknown function of r only, which we write temporarily as $f(r)$. This implies that $T = T_0 + p\theta$, and hence

$$\left(\frac{dR}{dr}\right)^2 + \frac{p^2}{r^2} - f(r) = -(\partial_z W)^2 + (\partial_t W)^2 \equiv -w, \tag{43}$$

where w is a separation constant, to be later identified with the state parameter of the Abelian current-carrying cosmic string. The equation $(\partial_z W)^2 - (\partial_t W)^2 = w$ can also be solved through separation of variables. Indeed, writing $W(z, t)$ as the sum of a function of z and of a function of t, one can show that these two functions are in fact linear in z and t, respectively.

Similarly separating variables in Eq. (41) then leads to

$$\frac{d^2R}{dr^2} + \left(\frac{1}{r} + \frac{2}{\sigma}\right)\frac{d\sigma}{dr}\frac{dR}{dr} = \partial_t^2 W - \partial_z^2 W = 0, \tag{44}$$

since we have just seen that W is the sum of two linear functions (and, therefore, its second order derivatives vanish). This can be integrated to yield

$$\frac{dR}{dr} = \frac{A}{r\sigma^2}, \tag{45}$$

where A is a constant. If we insert this expression into Eq. (43), this leads to an explicit expression for the function $f(r)$, namely,

$$f(r) = w + \frac{p^2}{r^2} + \frac{A^2}{r^2\sigma^4} = (\partial\psi)^2. \tag{46}$$

This function must be plugged back into Eq. (40) in order to obtain the full profile. Since $dR/dr \propto r^{-1}\sigma^{-2}$, there is no way to obtain a regular solution for σ unless the constants p and A are made to vanish, i.e. unless $R(r)$ is in fact a constant. One recovers the possibility to concentrate on pure world-sheet phase excitations, and the dynamics of the world sheet merely depends on the phase gradients, the state parameter. It is important to notice at this stage that the second derivatives of the phase contribute neither at the level of the field equations, nor at that of the stress tensor: this is why one usually disregards them and sets, without loss of generality, the phase as $\psi = kz - \omega t$, with the state parameter being $w = k^2 - \omega^2$.

B. The bi-Abelian case

One step further in the direction of a full non-Abelian situation is that of two Abelian currents, dubbed the bi-Abelian current-carrying string, as was, in particular, studied in Ref. [54]. In this case, one identifies a U(1) × U(1) piece in SU(2) through the requirement

$$\boldsymbol{\Sigma} \equiv \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_1 e^{i\psi_1} \\ \sigma_2 e^{i\psi_2} \end{pmatrix}. \tag{47}$$

There is no direct identification that can be done here for which the phases, contrary to the actual bi-Abelian one, would depend only on the world-sheet coordinates: this is due to the fact that SU(2) is topologically equivalent to a 3-sphere, whereas the U(1) × U(1) we consider consists of two independent circles at the surface of this 3-sphere. As the phases vary, in principle independently, around the circles, they cannot describe an actual trajectory along the 3-sphere, hence the problem.

Thus, there cannot be a simply defined global solution in this case. It turns out that, in order to recover the actual $U(1) \times U(1)$, one must apply a trick, which we shall also use afterward in the full non-Abelian case. It consists of first identifying the phases as

$$\psi_1 = \gamma, \qquad \psi_2 = -\tan^{-1}(\cos\beta\tan\alpha), \tag{48}$$

so that the amplitudes are given by

$$\begin{aligned} \sigma_1^2 &= \sigma^2 \sin^2\alpha \sin^2\beta, \\ \sigma_2^2 &= \sigma^2(\cos^2\alpha + \sin^2\alpha\cos^2\beta). \end{aligned} \tag{49}$$

We immediately see where the problem originates, because in principle one expects the phases to depend on z and t, while the amplitude should be functions of the string radial distance r. But in the case of Eqs. (48) and (49), one phase, namely, $\psi_1 = \gamma$, enters independently of the rest and can therefore safely be assumed to vary along z and t, but the second phase and the amplitudes involve the *same* functions in a essentially nonlinear way.

The way to recover the previous case is to assume an ultralocal hypothesis, which consists of saying that the fields are to be evaluated at only one point of the world sheet, which we set, for simplicity, to be at $z = t = 0$, while we keep the gradients at this very point. This means in practice that we consider the angles as functions of the radial distance and set their gradients along the string to

$$\partial_z\alpha \to k_\alpha \quad \text{and} \quad \partial_t\alpha \to -\omega_\alpha, \tag{50}$$

and similar relations for β and γ.

The kinetic term $K = |\partial\Sigma_1|^2 + |\partial\Sigma_2|^2$ in the action then becomes

$$K = \tfrac{1}{2}\{\sigma'^2 + \sigma^2[\alpha'^2 + w_\alpha + \sin^2\alpha(w_\beta + w_\gamma)]\}, \tag{51}$$

where a prime denotes a derivative with respect to r and we have set $w_i \equiv k_i^2 - \omega_i^2$ for each angle $i \in \{\alpha, \beta, \gamma\}$. Taking into account the identifications (48) and (49), we see that provided we write $w_1 = w_\alpha + w_\beta + w_\gamma$ and $w_2 = w_\alpha$, it takes the canonical form for two scalar current carriers, namely,

$$K = \tfrac{1}{2}(\sigma_1'^2 + \sigma'^2 + w_1\sigma_1^2 + w_2\sigma_2^2). \tag{52}$$

In the final section, devoted to the stress-energy tensor of the string, we shall discuss the conditions on the parameters, for it can easily be seen right away that at this stage, the model contains 6 independent parameters (the phase gradients), whereas we know that the actual $U(1) \times U(1)$ case can be fully described with only 3, which are the world-sheet Lorentz invariants that can be built out of the two phase gradients. The fact that the string stress tensor can only depend on Lorentz invariant quantities must be implemented by hand at this stage, and it gives precisely the exact values for the eigenvalues that are the energy per unit length and the tension. The ultralocal procedure described below is thus validated in this case.

VI. THE NON-ABELIAN PART

Let us first build on the second solution of Sec. VA [Eq. (39)] and assume that α depends on the external coordinates and is a function of z and t only. We will show that this implies that β and γ also depend only on z and t; this would be the most natural generalization of the Witten model for which the phase excitation only moves along the world sheet. However, we find that there is only one such globally defined solution, containing three chiral propagation modes. Let us see how this happens.

A. An exact solution: The trichiral case

Let us start with seeking solutions for the angle α. Looking at Eq. (24), one notices that the term $\cos\alpha$ represents a natural Abelian part of the solution since only this term remains if one requires $n^a = 0$. In other words, α again identifies a subgroup U(1) of the original SU(2) along which the condensate behaves as a usual Abelian current-carrying cosmic string. In this situation, one also recovers the previously discussed Abelian solution. As a consequence, it seems natural to assume that α is a function of z and t, so that

$$\partial\alpha \cdot \partial\sigma = 0. \tag{53}$$

Moreover, as σ depends only on r, it is immediately clear from Eq. (26) that

$$(\partial\alpha)^2 + \tan\alpha\Box\alpha = w, \tag{54}$$

where w is a constant, again to be later identified with the state parameter of the Abelian current-carrying string. Plugging the relation (54) back into Eq. (28) now gives the constraint

$$n_a\Box n^a = -\frac{2\Box\alpha}{\sin 2\alpha} = \frac{(\partial\alpha)^2 - w}{\sin^2\alpha}. \tag{55}$$

Equation (54) can be solved setting $u = \cos\alpha$ as it then transforms into the linear Klein-Gordon equation

$$(\Box - w)u = 0, \tag{56}$$

whose general solution is easily obtained. It reads

$$u = \cos(\omega t - kz - \alpha_0) + \int [s_+(E)e^{i(Et+\sqrt{E^2-w}z)} + s_-(E)e^{i(Et-\sqrt{E^2-w}z)}]dE, \tag{57}$$

with $s_\pm(E)$ two arbitrary (unknown) functions of E and $w \equiv k^2 - \omega^2$. This general solution is made of two pieces. The first one,

$$\alpha = \alpha_0 + kz - \omega t, \tag{58}$$

is the exact equivalent of the U(1) conducting string phase.

Note that this was to be expected since, as mentioned above, α picks a special U(1) direction of the original SU(2) [59]. At this point however, it is worth mentioning that contrary to the U(1) case, there is no simple way to cancel out the constant α_0 appearing: since a simple SU(2) transformation can never be expressed as a shift in α, one cannot simply set $\alpha_0 \to 0$, so that this quantity is actually endowed with a physical (measurable) meaning. The second part of the solution represents massive particles moving along the world sheet when one considers usually normalized distribution functions $s_\pm$. We are however interested in collective modes along the string, and therefore restrict attention to the special case for which $s_\pm = 0$. Let us also notice that, if $w = 0$, then u becomes an arbitrary function of $t + z$ and $t - z$. Inserting this solution back into Eq. (54), we see that α becomes an arbitrary function of $t + z$ or $t - z$,

$$\alpha_{\rm chiral} = \alpha(t + \varepsilon z), \quad \text{with} \quad \varepsilon = \pm 1. \tag{59}$$

To summarize, we have two possible situations: either $w \neq 0$ and one must consider the solution (58) or $w = 0$ and one must work with the chiral solution given by (59).

Finally, we notice that, for the two above mentioned cases, one has $\Box\alpha = 0$ which in turn, thanks to Eq. (55), means

$$n_a \Box n^a = 0. \tag{60}$$

We then look for a nontrivial solution for the vector n^a whose dynamics is given by Eq. (29). Once one takes into account that α is a function on z and t only, see Eqs. (58) or (59), this relation reduces to

$$\Box n^a + 2\frac{\mathrm{d}\ln\sigma}{\mathrm{d}r}\partial_r n^a + \frac{2}{\tan\alpha}[(\partial_z\alpha)\partial_z - (\partial_t\alpha)\partial_t]n^a = 0. \tag{61}$$

Therefore, one must solve this equation together with the constraint (60), $n_a\Box n^a = 0$.

We first rewrite Eq. (61) as dynamical equations for the world-sheet functions β and γ. We find

$$\frac{\partial^2\beta}{\partial r^2} + \left(\frac{1}{r} + 2\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right)\frac{\partial\beta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\beta}{\partial\theta^2} + \frac{\partial^2\beta}{\partial z^2} - \frac{\partial^2\beta}{\partial t^2} + \frac{2}{\tan\alpha}\left[(\partial_z\alpha)\frac{\partial\beta}{\partial z} - (\partial_t\alpha)\frac{\partial\beta}{\partial t}\right] = 0, \tag{62}$$

and

$$\frac{\partial^2\gamma}{\partial r^2} + \left(\frac{1}{r} + 2\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right)\frac{\partial\gamma}{\partial r} + \frac{1}{r^2}\frac{\partial^2\gamma}{\partial\theta^2} + \frac{\partial^2\gamma}{\partial z^2} - \frac{\partial^2\gamma}{\partial t^2} + \frac{2}{\tan\alpha}\left[(\partial_z\alpha)\frac{\partial\gamma}{\partial z} - (\partial_t\alpha)\frac{\partial\gamma}{\partial t}\right] = 0, \tag{63}$$

showing that β and γ are subject to the same dynamics, so that their potentially different behaviors merely rely on their initial conditions. One the other hand, the constraint (60) reads

$$(\partial\beta)^2 + \sin^2\beta(\partial\gamma)^2 = 0, \tag{64}$$

showing that, in the four-dimensional embedding spacetime, the phase gradients $\partial_\mu\beta$ and $\partial_\mu\gamma$ are lightlike. However, this is not the end of the discussion, for the fields β and γ actually live in the embedding four-dimensional spacetime. They could therefore vary, in a lightlike way, in all directions around the vortex, and after integration over the transverse degrees of freedom, leave the appearance of a spacelike or timelike variation. This, in fact, is to be expected on general geometrical considerations [55,56], leading to many equation of state parameters. We shall see below that it is not what happens in the case at hand. Concretely, Eq. (64) amounts to

$$\left(\frac{\partial\beta}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\beta}{\partial\theta}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2 - \left(\frac{\partial\beta}{\partial t}\right)^2 = 0, \tag{65}$$

$$\left(\frac{\partial\gamma}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\gamma}{\partial\theta}\right)^2 + \left(\frac{\partial\gamma}{\partial z}\right)^2 - \left(\frac{\partial\gamma}{\partial t}\right)^2 = 0. \tag{66}$$

These equations have the form of two gravitational Hamilton-Jacobi equations; that is to say $g^{ik}(\partial S/\partial x^i)\times(\partial S/\partial x^k) = 0$. Consequently, they can be explicitly solved by means of separation of variables. Setting $\beta = R_\beta(r)T_\beta(\theta)b_\beta(z,t)$ and $\gamma = R_\gamma(r)T_\gamma(\theta)b_\gamma(z,t)$, the complete system of equations reads

$$\left(\frac{\mathrm{d}T_i}{\mathrm{d}\theta}\right)^2 - \nu_i^2 T_i^2 = 0, \tag{67}$$

$$\left(\frac{\mathrm{d}R_i}{\mathrm{d}r}\right)^2 - \frac{1}{r_i^2}\left(1 + \nu_i^2\frac{r_i^2}{r^2}\right)R_i^2 = 0, \tag{68}$$

$$\left(\frac{\partial b_i}{\partial z}\right)^2 - \left(\frac{\partial b_i}{\partial t}\right)^2 + \frac{b_i^2}{r_i^2} = 0, \tag{69}$$

for both $i = \beta$ and γ, where r_i and ν_i are separation constants. Of course, the solution must also satisfy the dynamical equations (62) and (63). Straightforward manipulations show that this amounts to

$$\frac{\mathrm{d}^2T_i}{\mathrm{d}\theta^2} - \mu_i^2 T_i = 0, \tag{70}$$

$$\frac{\mathrm{d}^2R_i}{\mathrm{d}r^2} + \left(\frac{1}{r} + \frac{2}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right)\frac{\mathrm{d}R_i}{\mathrm{d}r} - \left(w_i + \frac{\mu_i^2}{r^2}\right)R_i = 0, \tag{71}$$

$$\frac{\partial^2 b_i}{\partial z^2} - \frac{\partial^2 b_i}{\partial t^2} + \frac{2}{\tan\alpha}\left(\partial_z\alpha\frac{\partial b_i}{\partial z} - \partial_t\alpha\frac{\partial b_i}{\partial t}\right) + w_i b_i = 0, \tag{72}$$

where w_i and μ_i are two new constants of separation. The main question is now whether the solutions obtained from the dynamical equation are compatible with the ones derived from the constraint.

The two equations (68) and (71) controlling the behavior of R_i can only be compatible if the function R_i is a constant since one of these equations, Eq. (71), contains σ while the other, Eq. (68), does not. This immediately implies $\nu_i = 1/r_i = \mu_i = w_i = 0$ and we are left with

$$\left(\frac{\partial b_i}{\partial z}\right)^2 - \left(\frac{\partial b_i}{\partial t}\right)^2 = 0, \tag{73}$$

and

$$\frac{\partial^2 b_i}{\partial z^2} - \frac{\partial^2 b_i}{\partial t^2} + \frac{2}{\tan\alpha}\left[(\partial_z\alpha)\frac{\partial b_i}{\partial z} - (\partial_t\alpha)\frac{\partial b_i}{\partial t}\right] = 0. \tag{74}$$

Of course, one possibility is to take b_i as constant. However, this means that the vector n^a is fixed and this just corresponds to the Abelian case. In fact the general solution of the first equation above is $b_i = b_i(t + \varepsilon_i z)$, with $\varepsilon_i = \pm 1$. Inserting this solution into the second relation, one obtains

$$\varepsilon_i(\partial_z\alpha) - (\partial_t\alpha) = 0. \tag{75}$$

If α is given by Eq. (58), then the above equation becomes $\omega = -\varepsilon_i k$ which implies $w = 0$. But, if $w = 0$, then one must consider the chiral solution (59). In this case, the dynamical solution reduces to $\varepsilon\varepsilon_i = 1$. This means that one also obtains chiral solutions for these angles, namely,

$$\beta_{\rm chiral} = \beta(t + \varepsilon z), \quad \text{with} \quad \varepsilon = \pm 1, \tag{76}$$

and

$$\gamma_{\rm chiral} = \gamma(t + \varepsilon z), \quad \text{with} \quad \varepsilon = \pm 1. \tag{77}$$

We see that this solution contains three chiral-like functions, hence its name. It is of course very important to notice that the relative sign in the argument of α, β, and γ needs to be the same for these three functions. This implies that all the angles must propagate in the same direction, i.e. the string currents consist of right or left movers only. The situation is thus the same as that first discussed in Ref. [60], but with three independent copies of the currents and the additional constraint that they all move in the same direction.

Constructing a surface action over the word sheet (with coordinates ξ_i)

$$\mathcal{S} = \int d^2\xi\sqrt{-h}\mathcal{L}^{(2)}(\xi_i), \tag{78}$$

for such a trichiral string is a straightforward generalization of [60]: if one assumes a two-dimensional Lagrangian of the form

$$\mathcal{L}^{(2)} = -m^2 - \tfrac{1}{2}\mathcal{M}^{AB}h^{ij}\partial_i\psi_A\partial_j\psi_B, \tag{79}$$

where m is a constant describing the Nambu-Goto string background and $\mathcal{M}^{AB}$ is a matrix Lagrange multiplier with no kinematic term in the action, h^{ij} is the world-sheet induced metric and the ψ_A stand for our angular functions α, β, and γ. Varying with respect to this matrix immediately provides the null conditions for all the fields, namely,

$$h^{ij}\partial_i\psi_A\partial_j\psi_B = 0, \tag{80}$$

showing that not only are the fields all lightlike, but also, if the matrix $\mathcal{M}$ is nondiagonal, that all the solutions do move in the same direction, i.e. that they are all either right or left movers.

B. A no-go theorem for exact separable solutions

In fact, one can show that the trichiral solution is the only exact separable solution. Indeed, Eq. (68) can be easily solved. Its solution reads

$$\frac{R_i}{R_i^0} = \exp\left(\pm\frac{r}{r_i}\sqrt{1 + \nu_i^2\frac{r_i^2}{r^2}}\right)\left(\nu_i\frac{r_i}{r} + \sqrt{1 + \nu_i^2\frac{r_i^2}{r^2}}\right)^{\mp\nu_i}, \tag{81}$$

where R_i^0 is an integration constant. However, inserting this expression into Eq. (71) shows that it is a solution only if R_i is a constant. This is of course due to the presence of the term $(d\sigma/dr)/\sigma$ which cannot be canceled by any other term. But if R_i is a constant, then $\nu_i = 0$ which in turn implies that T is also a constant. In other words, we are back to the trichiral solution of the previous section.

This shows that there is no other exact and separable solution. Although this, of course, does not, in principle, prevent the existence of solutions which do not obey separation of variables, there exists a general argument, due to Carter [55,56], showing that one should not expect a global solution to exist. The argument relies on the fact that the generators of the currents form a manifold whose curvature is nonzero, while the cylindrically symmetric string configuration assumes vanishing extrinsic and intrinsic curvatures, thus leading to an incompatibility.

VII. ULTRALOCAL CROOKED STRING

The SU(2) condensate does not have any regular nontrivial solution expect for the trichiral: does this mean that only Abelian or chiral-like current-carrying cosmic strings can be formed?

The answer to this question involves two different perspectives. First, one must remember that when the current builds up along the string, it does so through a random process through which phases take uncorrelated values on distances larger than the correlation length. There is therefore no reason to assume the current would be, all along the world sheet, always following one particular U(1) direction. Moreover, all the above discussion heavily relies on a straight and static string whose fundamental tensor is merely the two-dimensional Minkowski metric. The string manifold, therefore, is described as flat, and this is the cause for the discrepancy: SU(2) having a nonvanishing curvature, it is normal that it cannot be projected onto the string world sheet, so only a flat subspace of it, the U(1) we identified, remains once this operation is performed.

The way to reconcile both perspectives is by considering an actual string, which, as simulations reveal, is in fact crooked, and definitely not flat. Locally, one can always approximate the string by a straight line, and assume cylindrical symmetry. However, this is only a rough approximation which, although valid in the Abelian case, is severely limited in the non-Abelian case. In order to take into account the possible variations of the phases without having a solution satisfying the requirement of cylindrical symmetry, we introduce a so-called ultralocal approximation, by which we restrict attention to one particular point on the world sheet, which we take for simplicity (and without lack of generality), to be at $z = t = 0$, but keep the phase gradients along the world sheet as parameters. This procedure, applied to the Abelian and bi-Abelian cases, gives the correct result.

In practice, the ultralocal approximation for the crooked non-Abelian current-carrying cosmic string consists of assuming the phases to depend on the radial distances, while their gradients are numbers. In other words, we set

$$\alpha \to \alpha(r) + k_\alpha z - \omega_\alpha t + \tfrac{1}{2}(\alpha^0_{,zz} z^2 + \alpha^0_{,tt} t^2) + \cdots \quad (82)$$

(and similar expressions for β and γ) and let $z, t \to 0$ in the final expressions we obtain. Note that this procedure only applies in the very final equations, and for instance it is not possible to apply it for the action itself, as the field equations derived from the approximated action would not be equivalent to the approximated field equations derived from the exact action, lacking, in particular, the squared gradients and second derivatives with respect to the world-sheet coordinates.

Using the approach described above, it is straightforward to derive the equations of motion obeyed by the three angles α, β, and γ. Since we are interested in the minimal energy configuration, we ignore a possible θ dependence. As a consequence, only equations controlling the profiles of the functions $\alpha(r)$, $\beta(r)$, and $\gamma(r)$ remain. They read

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\alpha}{\mathrm{d}r} + 2\frac{\mathrm{d}\sigma}{\mathrm{d}r}\frac{\mathrm{d}\alpha}{\mathrm{d}r} + \alpha^0_{,zz} - \alpha^0_{,tt} - \sin\alpha\cos\alpha\left[\left(\frac{\mathrm{d}\beta}{\mathrm{d}r}\right)^2 + k_\beta^2 - \omega_\beta^2\right] - \sin\alpha\cos\alpha\sin^2\beta\left[\left(\frac{\mathrm{d}\gamma}{\mathrm{d}r}\right)^2 + k_\gamma^2 - \omega_\gamma^2\right] = 0, \quad (83)$$

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\beta}{\mathrm{d}r} + 2\frac{\mathrm{d}\sigma}{\mathrm{d}r}\frac{\mathrm{d}\beta}{\mathrm{d}r} + \beta^0_{,zz} - \beta^0_{,tt} + \frac{2}{\tan\alpha}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}r}\frac{\mathrm{d}\beta}{\mathrm{d}r} + k_\alpha k_\beta - \omega_\alpha\omega_\beta\right) - \sin\beta\cos\beta\left[\left(\frac{\mathrm{d}\gamma}{\mathrm{d}r}\right)^2 + k_\gamma^2 - \omega_\gamma^2\right] = 0, \quad (84)$$

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\gamma}{\mathrm{d}r} + 2\frac{\mathrm{d}\sigma}{\mathrm{d}r}\frac{\mathrm{d}\gamma}{\mathrm{d}r} + \gamma^0_{,zz} - \gamma^0_{,tt} + \frac{2}{\tan\alpha}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}r}\frac{\mathrm{d}\gamma}{\mathrm{d}r} + k_\alpha k_\gamma - \omega_\alpha\omega_\gamma\right) + \frac{2}{\tan\beta}\left(\frac{\mathrm{d}\beta}{\mathrm{d}r}\frac{\mathrm{d}\gamma}{\mathrm{d}r} + k_\beta k_\gamma - \omega_\beta\omega_\gamma\right) = 0. \quad (85)$$

As expected, the profiles depend on the six parameters k_i and ω_i. However, and this is a new feature of the non-Abelian case, there is also an additional dependence in the second order derivatives which introduces three new Lorentz invariant parameters, namely, $\alpha^0_{,zz} - \alpha^0_{,tt}$, $\beta^0_{,zz} - \beta^0_{,tt}$, and $\gamma^0_{,zz} - \gamma^0_{,tt}$.

Our ultralocal approximation can only make sense provided Eqs. (83)–(85) depend on a finite set of new parameters: it is not a derivative expansion for which all orders contribute equally, for otherwise there would be no way to decide where to stop the expansion, and hence no way to know how many parameters are relevant for the microscopic description. Indeed, we expand the phases around the point of interest in the world sheet to *all* orders in the string coordinates close to the point in question, and take the limit $z, t \to 0$. Since the underlying theory is second order, not surprisingly, all that remains are all the Lorentz invariant first (squared) and second order derivatives of the phases.

One then wonders why not simply set these parameters to zero? This would restrict unduly to the static situation, as can be understood by the following argument: the mechanism thanks to which the current condenses along the string is akin to a phase transition, the SU(2) doublet acquiring a nonvanishing value along the world sheet only. At the transition itself, there will necessarily be a correlation length ξ—which in practice will be of the order of the Compton wavelength of the doublet—above which the phases α, β, and γ will be uncorrelated. Therefore, for two points on the string separated by a distance larger than ξ, one expects variations in the phases: the coefficients k_i, ω_i and the second order derivatives in Eq. (82) are thus expected to be given by ξ^{-1}, i.e. by the doublet mass, up to numerical coefficients. This is like in the original Witten Abelian model in which the order of magnitude of the state parameter is also of the order of the condensate's mass.

The shape of the profiles will be very similar to what one encounters in the Abelian case as a simple study of the behavior of the above equations in the limit $r \to 0$ and $r \to +\infty$ reveals. The precise form of the profiles does not bring much insight into the problem at hand and, therefore, we now turn to the calculation of the stress-energy tensor.

VIII. WORLD-SHEET STRESS-ENERGY TENSOR

Our aim is to describe the string world sheet by itself, i.e. to integrate over the transverse degrees of freedom in order to identify the stress-energy tensor eigenvalues, namely, the string tension and its energy per unit length. Let us first recall how this is done for the Witten U(1) case by reproducing the argument of Ref. [61].

In the U(1) situation, there is only one phase present, namely, α, and its general solution is the same as in our case. In fact, as discussed above, this solution is equivalent to saying that in a small but finite neighborhood of any point (z_0, t_0) on the string, the phase can be approximated as a Taylor series $\alpha \simeq \alpha_0 + k(z - z_0) - \omega(t - t_0) + \cdots$, and since there is an invariance of the theory under global transformations $\alpha \to \alpha + \text{const}$, it is always possible, at any given point, to rescale α to the simplest solution $\alpha = kz - \omega t$, i.e. to send $\alpha_0 \to 0$.

The stress-energy tensor, again for the U(1) case, does not explicitly depend on the phase itself, but on its gradients $\partial_\mu \alpha$, which, locally, can always be taken as constants. As a result, the stress-energy tensor is a function of the radial distance only if cylindrical symmetry is assumed, and its conservation $\nabla_\mu T^{\mu\nu} = 0$ implies, for $\nu = r$,

$$\int r \mathrm{d}r (T^r_r + T^\theta_\theta) = 0.$$

This sum of terms is the same as $T^x_x + T^y_y$, and the symmetry around the vortex also implies that both of these terms are the same, as the choice of directions for the axes x and y is irrelevant, nothing depending on the angle θ. Therefore, the transverse components of the stress tensor vanish. On the other hand, the $\nu = z$ and $\nu = t$ components of the conservation equation imply that the mixed parts T_{rz} and T_{rt} both behave as r^{-1}, which is not possible if this tensor is to be finite: one must impose $T_{rz} = T_{rt} = 0$. There remain the internal components T_{ab} with $a, b = z, t$: upon integration and diagonalization, they provide the relevant functions of the state parameter $w = k^2 - \omega^2$ known as energy per unit length and tension.

Unfortunately, the above does not generalize easily to the more complicated non-Abelian situation. Indeed, for the simplest possible SU(2) case we have discussed until now, the general form of the stress-energy tensor reads

$$T_{\mu\nu} = t_{\mu\nu}(r) + \sigma^2[s_{\mu\nu}(z, t) - \tfrac{1}{2} s^\alpha{}_\alpha g_{\mu\nu}], \tag{86}$$

where

$$s_{\mu\nu} = \partial_\mu \alpha \partial_\nu \alpha + \sin^2\alpha(\partial_\mu \beta \partial_\nu \beta + \sin^2\beta \partial_\mu \gamma \partial_\nu \gamma) \tag{87}$$

shows an explicit dependence in the word-sheet coordinates and the first part $t_{\mu\nu}$ only depends on r. Let us see how the above argument fails in this case.

The conservation equation, as given above, with $\nu = r$, now transforms into

$$\int r \mathrm{d}r (T^r_r + T^\theta_\theta) = \int r^2 \mathrm{d}r (\partial_t T_{tr} - \partial_z T_{zr}), \tag{88}$$

while the z and t components, respectively, give

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) T_{rz} = \partial_t T_{tz} - \partial_z T_{zz},$$

and

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) T_{tz} = \partial_t T_{tt} - \partial_z T_{zt}.$$

Assuming the separated form $T_{rz} = Z(r)\tilde{T}_{rz}$ and $T_{rt} = T(r)\tilde{T}_{rt}$, with $\tilde{T}$ being independent of r, we find, upon integration over r of these two relations, that provided the functions Z and T decay faster than r^{-1}, the surface stress tensor

$$\tilde{T}_{ab} \equiv \int r \mathrm{d}r \mathrm{d}\theta T_{ab} \tag{89}$$

is conserved, i.e. $\nabla_a \tilde{T}^{ab} = 0$.

The tensor (89) will contain all the relevant information for the dynamics of the string world sheet provided the rhs of Eq. (88) vanishes, and this gives a necessary condition for a two-dimensional world-sheet description to be valid. Given the form (86) of the stress tensor for the non-Abelian case, it is far from obvious that the two-dimensional stress-energy tensor is automatically conserved. We shall see later that the condition that Eq. (88) vanishes provides a constraint on the second time and space derivative of the angular functions α, β, and γ.

Let us now return to the crooked string in the ultralocal regime. The surface stress-energy tensor takes the form

$$\bar{T}^a{}_b = \begin{pmatrix} T^t{}_t & T^t{}_z \\ T^z{}_t & T^z{}_z \end{pmatrix} = \begin{pmatrix} -A + B & C \\ -C & -A - B \end{pmatrix}, \tag{90}$$

where

$$A = 2\pi \int r \mathrm{d}r \left\{ \varphi'^2 + \frac{Q'^2}{2q^2 r^2} + \frac{1}{2}\sigma'^2 + \frac{Q^2\varphi^2}{r^2} + \frac{1}{2}\sigma^2[\alpha'^2 + \sin^2\alpha(\beta'^2 + \sin^2\beta\gamma'^2)] \right\}, \tag{91}$$

while

$$B = \sum_{i=\alpha,\beta,\gamma} (k_i^2 + \omega_i^2) I^i \tag{92}$$

and

$$C = 2 \sum_{i=\alpha,\beta,\gamma} k_i \omega_i I^i \tag{93}$$

are expressible in terms of the profile integrals

$$I^\alpha = \pi \int \sigma^2 r \mathrm{d}r, \tag{94}$$

$$I^\beta = \pi \int \sigma^2 \sin^2\alpha r \mathrm{d}r, \tag{95}$$

$$I^\gamma = \pi \int \sigma^2 \sin^2\alpha \sin^2\beta r \mathrm{d}r. \tag{96}$$

The energy per unit length U and the tension T are then obtained as the, respectively, timelike and spacelike eigenvalues of this stress tensor, namely,

$$U = A + \sqrt{B^2 - C^2} \quad \text{and} \quad T = A - \sqrt{B^2 - C^2}, \qquad (97)$$

where the quantity $B^2 - C^2$ can be expressed in terms of all the possible Lorentz invariant scalars made from the phase gradients, namely, the parameter matrix

$$w_{ij} = k_i k_j - \omega_i \omega_j, \qquad (98)$$

and we find

$$B^2 - C^2 = \sum_{i,j=\alpha,\beta,\gamma} I^i I^j (2w_{ij} - w_i w_j), \qquad (99)$$

which generalizes the Abelian case.

Equations (97) and (99) show that the energy per unit length and tension of the non-Abelian current-carrying string depend explicitly on all the possible two-dimensional (world-sheet) Lorentz invariant parameters that can be constructed out of the phase gradients of the angular variables α, β, and γ. Although this induces a tremendous level of complexity for the description of the dynamics of the string world sheet itself, this is however not the end of the story, for the field equations for the angle profiles actually show another dependence, implicit this time: under the assumption of ultralocality, the Euler equations for α, β, and γ, namely, Eqs. (30), (31), (34), and (35), contain the parameters $\partial_{zz}\alpha^0 - \partial_{tt}\alpha^0$, $\partial_{zz}\beta^0 - \partial_{tt}\beta^0$, and $\partial_{zz}\gamma^0 - \partial_{tt}\gamma^0$, i.e. again, all the possible string Lorentz invariant second order derivatives. This makes a difference with the Abelian case for which, as we showed in Sec. VA, these second derivatives do not enter, at any level. Here, since they enter in the profiles, the energy per unit length and tension indirectly depend on their values. Thus, going from U(1) to SU(2), one increases the number of free parameters from one to eight or nine, depending on whether one considers or not yet another constraint, which we now discuss.

In Sec. VIII, we showed that the two-dimensional stress-energy tensor is conserved only provided the rhs of Eq. (88) vanishes. This, given the form (86), can be implemented in two ways. The first possibility is to simply assume the ultralocal approximation in the stress-energy tensor itself, which amounts to saying that $s_{\mu\nu}$ in Eq. (87), in fact, depends on neither z nor t; in this case, $T_{\mu\nu}$ is merely a function of the radial variable and the analysis of [61] applies.

Another way to impose the surface stress-energy tensor to be conserved is by expliciting the condition

$$\partial_t \int r^2 \mathrm{d}r s_{tr} = \partial_z \int r^2 \mathrm{d}r s_{zr} \qquad (100)$$

using the expansion (82), and only then take the ultralocal limit. This method gives a relationship between the second derivatives of the angular variables and their gradients, hence reducing the number of free parameters by one unit.

Finally, one can use the stress-energy tensor here derived to recover the bi-Abelian situation, which will allow one to illustrate a difference between many Abelian and non-Abelian currents. The $U(1) \times U(1)$ case of Sec. V B is obtained in the ultralocal limit by writing $\alpha \to \alpha(r) + k_\alpha z - \omega_\alpha t$, $\beta \to \frac{\pi}{2} + k_\beta z - \omega_\beta t$, and $\gamma \to k_\gamma z - \omega_\gamma t$, and then assuming t, $z \to 0$. Then the stress-energy tensor above is unchanged, with now $I^\beta = I^\gamma = I_1 = \pi \int \sigma_1^2(r) r \mathrm{d}r$ and $I^\alpha = I_1 + I_2 = \pi \int [\sigma_1^2(r) + \sigma_2^2(r)] r \mathrm{d}r$, where the fields are defined above Eq. (52). Setting $k_2 = k_\alpha \equiv \partial_z \psi_2$, $\omega_2 = \omega_\alpha \equiv -\partial_t \psi_2$, $\boldsymbol{k}_1^2 = k_\alpha^2 + k_\beta^2 + k_\gamma^2 \equiv (\partial_z \psi_1)^2$, and $\boldsymbol{\omega}_1^2 = \omega_\alpha^2 + \omega_\beta^2 + \omega_\gamma^2 \equiv (\partial_t \psi_1)^2$, we diagonalize the stress-energy tensor as above [Eq. (99)] to get

$$\begin{aligned} B^2 - C^2 = {} & I_2^2 (k_2^2 - \omega_2^2)^2 + I_1^2 (\boldsymbol{k}_1 - \boldsymbol{\omega}_1)^2 (\boldsymbol{k}_1 + \boldsymbol{\omega}_1)^2 \\ & + I_1 I_2 [(k_2 - \omega_2)^2 (\boldsymbol{k}_1 + \boldsymbol{\omega}_1)^2 \\ & + (k_2 + \omega_2)^2 (\boldsymbol{k}_1 - \boldsymbol{\omega}_1)^2], \end{aligned} \qquad (101)$$

which is of the form of Eq. (48) of Ref. [54] only provided the vectors $\boldsymbol{k}_1$ and $\boldsymbol{\omega}_1$ are collinear, i.e. $\boldsymbol{k}_1 = k_1 \boldsymbol{u}$ and $\boldsymbol{\omega}_1 = \omega_1 \boldsymbol{u}$, with $\boldsymbol{u}^2 = 1$. In this case, we recover indeed

$$B^2 - C^2 = w_1 I_1^2 + w_2 I_2^2 + 2x I_1 I_2, \qquad (102)$$

where $w_i = k_i^2 - \omega_i^2$ and $x = k_1 k_2 - \omega_1 \omega_2$ is the cross product. This particular choice is that which lowers the number of arbitrary parameters to only three, as demanded by the two Abelian current cases.

The bi-Abelian current case, as discussed above, has a microscopic structure (the field profiles) that depends solely on the squared phase gradients $w_1 = (\partial \psi_1)^2$ and $w_2 = (\partial \psi_2)^2$, even though the energy per unit length and tension also depend on the cross product $x = \partial \psi_1 \cdot \partial \psi_1$. By contrast, the non-Abelian current-carrying case involves in a nontrivial way not only the gradients $(\partial \alpha)^2$, $(\partial \beta)^2$, and $(\partial \gamma)^2$, but also all the possible combinations of cross products, namely, $\partial \alpha \cdot \partial \beta$, $\partial \alpha \cdot \partial \gamma$, and $\partial \beta \cdot \partial \gamma$; this is clear from the dynamical equations (30), (31), (34), and (35) defining the profiles of these angles, again provided one takes the ultralocal limit after deriving these equations.

IX. CONCLUSION

Cosmic strings are an almost generic prediction of most high energy theories, and they can have many observational cosmological consequences. They can also be current carrying, and this property changes their dynamics drastically, as it has been argued that a network of current-carrying cosmic strings could overproduce equilibrium loop configurations which, if stable, would overclose the Universe; such strings are clearly ruled out. The last case that has not been studied yet is that for which the current carrier transforms according to some representation of a non-Abelian group, and this is what has been presented above, in the particular (simplest) example of (global) SU(2). By means of such a toy model, we have been able to derive the microscopic structure of a non-Abelian current-carrying string, and exhibit the characteristic features of its stress-energy tensor, out of which one obtains, through integration over the transverse degrees of freedom, the energy per unit length and tension. In principle, these quantities allow for a complete calculation of the dynamics of the strings, hence of the motion of a network.

We have found many differences between the Abelian and the non-Abelian situations. Where the Abelian case involves a single state parameter, the simplest non-Abelian model here developed contains far more parameters, namely, at least 8. Besides, when the Abelian current case, even with more than one current, involves only the phase gradients of the fields, the non-Abelian case at hand exhibits implicit dependencies in the second derivatives with respect to the world-sheet coordinates of these phases. Those phases also acquire a profile, i.e. they must vary between the string core and the exterior: in accordance with the general Carter argument [55,56], the path followed by the phases on the SU(2) 3-sphere could not be smoothly projected onto the world sheet itself, the latter being flat while the former being intrinsically curved. Finally, whereas in the many current case the eigenvalues of the stress-energy tensor depend only explicitly on the cross gradients, the microscopic structure—the profiles—depending only on the squared gradients, in the non-Abelian case the profiles, and hence the energy per unit length and tension, depend on all the possible two-dimensional Lorentz invariants that can be built out of the phase derivatives up to the second order.

If cosmic strings were ever formed, it is quite likely that they would be current carrying, and in this category, since the well-tested standard electroweak theory already contains a broken SU(2) with a Higgs field doublet as in our case [51], the model we developed here may be relevant, depending on the values of the unknown coupling parameters. At the cosmological level, Abelian current-carrying strings do intercommute in much the same way as nonconducting ones [62]. This is made possible because the currents in both pieces of the colliding strings can merely add up at the junction, being confined in the world sheet through a linear interaction. In the non-Abelian case, it is likely that the essentially nonlinear interaction terms would forbid such a simple readjustment of the phases: it is to be expected that the intercommutation probability is much lower than for ordinary strings. This, as is well known from the superstring case [63], can imply fundamentally different cosmological consequences. Another reason why one would expect intercommutation to be far less effective in the non-Abelian current-carrying case is also related to extra dimensions: in the simplest Kaluza-Klein framework with a circular fifth dimension, the extra angular variable plays the role of the current-carrier phase and the equation of state can be calculated to be of the self-dual fixed trace kind [41] by projecting in the 4-dimensional base space [64]; it can be conjectured that introducing many extra dimensions with a complicated structure can lead to currents sharing many of the properties of the non-Abelian ones discussed here. The intercommutation of non-Abelian current-carrying cosmic string is therefore an important open problem that deserves further investigation.

The SU(2) current involved in the cosmic string discussed in this work is of the uncoupled, global kind, as was the first complete examination of the so-called neutral Witten model [38]. In order to identify this SU(2) with that actually involved in the electroweak interactions, it would be necessary to gauge it, thus introducing 3 gauge vector fields A_μ^a. These gauge vectors then couple not only with the doublet $\mathbf{\Sigma}$ but also with each other. As a result, the simple configuration we considered in this paper always seems to lead to strong divergences of the vectors, hence of the energy density. The cure to this problem may be related to confinement in QCD: as a string interaction state must be colorless, the condensate should be associated with another SU(2) field in order to form an uncharged state; within the framework of such a configuration, the total energy density contained in the world sheet can then be made finite. The resulting macroscopic string stress tensor and other relevant quantities need be evaluated in a separate work.

Finally, inclusion of a current along the strings in a cosmologically relevant network can have many consequences that need be evaluated in more details. Among those is the possibility that the currents could induce [65,66] γ-ray bursts or high energy cosmic rays [67]. The latter was recently revisited in details in Ref. [68] provided the carrier couples linearly to the string. A similar analysis applying to our case should be carried out to decide on the observability of non-Abelian current-carrying cosmic strings.

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