# Hyperscaling relations in mass-deformed conformal gauge theories

Luigi Del Debbio[1,*] and Roman Zwicky[2,†]

[1]*School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, Scotland*
[2]*School of Physics and Astronomy, University of Southampton, Highfield, Southampton SO17 1BJ, United Kingdom*
(Received 21 May 2010; published 8 July 2010)

We present a number of analytical results which should guide the interpretation of lattice data in theories with an infrared fixed point (IRFP) deformed by a mass term $\delta\mathcal{L} = -m\bar{q}q$. From renormalization group (RG) arguments we obtain the leading scaling exponent, $F \sim m^{\eta_F}$, for all decay constants of the lowest lying states other than the ones affected by the chiral anomaly and the tensor ones. These scaling relations provide a clear cut way to distinguish a theory with an IRFP from a confining theory with heavy fermions. Moreover, we present a derivation relating the scaling of $\langle\bar{q}q\rangle \sim m^{\eta_{\bar{q}q}}$ to the scaling of the density of eigenvalues of the massless Dirac operator $\rho(\lambda) \sim \lambda^{\eta_{\bar{q}q}}$. RG arguments yield $\eta_{\bar{q}q} = (3 - \gamma_*)/(1 + \gamma_*)$ as a function of the mass anomalous dimension $\gamma_*$ at the IRFP. The arguments can be generalized to other condensates such as $\langle G^2\rangle \sim m^{4/(1+\gamma_*)}$. We describe a heuristic derivation of the result on the condensates, which provides interesting connections between different approaches. Our results are compared with existing data from numerical studies of SU(2) with two adjoint Dirac fermions.

## I. INTRODUCTION

There are numerous examples of two-dimensional field theories that are invariant under the full conformal group. In four dimensions, the beta function of $\mathcal{N} = 4$ super Yang-Mills is known to vanish to all orders in perturbation theory, for any value of the coupling, so that the theory is scale invariant. Other theories have isolated zeroes of the beta function that correspond to fixed points of the renormalization group (RG) flow. For instance, the gauge coupling $g$ in QCD flows to zero as the energy scale is increased, leading to the well-known phenomenon of asymptotic freedom; in this case $g = 0$ is commonly called an UV fixed point. On the other hand, if a theory has an IR fixed point (IRFP), the couplings will flow to such a fixed point at large distances, and the theory becomes scale invariant in the large-distance regime. Theories with an IRFP do not break chiral symmetry spontaneously, and are said to lie in the *conformal window.*

Supersymmetric examples of theories within the conformal window have been studied in detail—see e.g. Ref. [1] for a review. Recently there has been a lot of interest in identifying nonsupersymmetric gauge theories with an IRFP. The main motivation, besides intrinsic interest, comes from the fact that theories *near* the conformal window correspond to the class of theories underlying walking technicolor [2–6], which is the phenomenologically most viable offspring of technicolor theories [7–10]. A chirally broken theory near the edge of the conformal window is supposedly identified by an enhancement of the ratio $\langle\bar{q}q\rangle/f_\pi^3$ with respect to a QCD-like theory [10]. Unfortunately this quantity does not display a simple

known parametric behavior. Another strategy, which is adopted in this paper, is to first identify theories within the conformal window, and then approach the boundary of the window using the available information on the color-flavor phase diagram [2,3,6,11].

The identification of conformal theories using numerical simulations is a difficult task, since the only observable quantities would be the power-law scaling of correlators at large distances. However actual lattice simulations are performed in a finite volume, and with a nonvanishing fermion mass; both the mass and the finite size of the system are relevant operators at large distances and drive the theory away from conformal behavior. Turning a technical limitation into a tool, it has become a standard strategy to consider conformal gauge theories (CGT) candidates deformed by a mass term, and to identify them from the study of their hadronic observables. Thus, if there exists an IRFP, the lattice results should be described by a mass-deformed conformal gauge theory (mCGT), obtained by adding a bare mass to the original Lagrangian

$$\delta\mathcal{L} = -m\bar{q}q. \tag{1}$$

As a consequence of the deformation, these theories are expected to develop a mass gap and a fermion condensate and thus give rise to asymptotic states and related observables, which scale to zero as the massless limit is approached. For any observable $\mathcal{O}$ the leading exponent $\eta_{\mathcal{O}}$ of the mass deformation is defined from its scaling as $m \rightarrow 0$:

$$\mathcal{O} \sim m^{\eta_{\mathcal{O}}} + \text{higher order in } m + \text{terms analytic in } m. \tag{2}$$

These critical exponents can be measured on the lattice and

it is the aim of this work to provide predictions for them that can be tested numerically.

The paper is organized as follows. In Sec. we set the framework by discussing some characteristics of theories inside the conformal window. In Sec. II we discuss general aspects of IRFPs, and introduce the standard tools for analyzing the behavior of field correlators near a fixed point of the RG flow. Thereby we obtain the hyperscaling relations that are usually derived in the context of critical phenomena [12], and we study the information that they yield in the framework of mCGT.

Section III is devoted to the study of the chiral condensate in mCGTs. First we review the relation between the scaling of the chiral condensate with the fermion mass, and the density of eigenvalues of the massless Dirac operator in the infinite-volume limit. As stated above, the chiral condensate must vanish as the fermion mass is taken to zero at a rate that is dictated by a critical exponent $\eta_{\bar{q}q}$. The nonanalytic dependence of the fermion condensate on the fermion mass is directly related to the scaling exponent for the eigenvalue density of the massless Dirac operator. As pointed out in Ref. [13], the exponents turn out to be the same:

$$\langle \bar{q}q \rangle \sim m^{\eta_{\bar{q}q}} \Rightarrow \rho(\lambda) \sim \lambda^{\eta_{\bar{q}q}}. \tag{3}$$

The scaling exponent $\eta_{\bar{q}q}$ is determined as a function of $\gamma_*$, the anomalous dimension of the mass at the IRFP. The RG analysis, which applies to all condensates, yields

$$\eta_{\bar{q}q} = \frac{(3 - \gamma_*)}{(1 + \gamma_*)}. \tag{4}$$

We then present the determination of this coefficient from a heuristic calculation, which provides some physical insight in the dynamics of mCGT. The limitations of such a heuristic approach are highlighted, and the interpretation of IR and UV cutoffs is clarified. We conclude this section by analyzing current lattice data for the eigenvalue distribution in an SU(2) gauge theory with two flavours in the adjoint representation.

In Sec. IV we explore the consequences of hyperscaling for the decay constants of the hadronic states. Our results,

summarized in Table I, can schematically written as

$$G \sim m^{(\Delta_{\mathcal{O}}-1)/1+\gamma_*}, \qquad \langle 0|\mathcal{O}(0)|H(p)\rangle = G, \tag{5}$$

for operators with scaling dimension $\Delta$. Further informations are obtained by combining these results with the chiral Ward identities in section IV B; these scaling predictions for the decay constants are then compared with recent results from numerical simulations of potential mCGT on the lattice. Finally we discuss the implication of the scaling of the decay constants for the width of the hadronic states, and compare the scaling of the decay constants in a mCGT to the one of heavy quarkonia states in a chirally broken theory like QCD.

## Conformal window—discussion and results

It is well known that SU(N) gauge theories with $n_f$ fermions are asymptotically free as long as $n_f$ does not exceed an upper limit that depends on the number of colors $N_c$ and the fermion representation $R$. At small distances the gauge coupling decreases logarithmically, and the dynamics is successfully described by perturbation theory. In the SU(3) gauge theory minimally coupled to $n_f = 2$ light flavors in the fundamental representation, the coupling increases at large distances, and the theory undergoes confinement and spontaneous chiral symmetry breaking, exhibiting a spectrum of bound states. In the massless limit, the spectrum includes three massless Goldstone bosons, known as $(\pi^0, \pi^+, \pi^-)$, reflecting the spontaneous breaking of chiral symmetry. As a consequence, there is a gap in the spectrum between the pions and the rest of the states whose masses are parametrically of the order of some hadronic scale $\Lambda \simeq \Lambda_{\text{QCD}}$, and remain finite in the chiral limit. At low energies compared to $\Lambda$ the dynamics are successfully described by an effective theory of self-interacting pions, known as chiral perturbation theory. A small nonvanishing mass can easily be incorporated as a perturbation of the massless theory.

As the number of light fermions is increased, before asymptotic freedom is lost, the theory may develop an infrared fixed point (IRFP) due to the effect of the fermions

TABLE I. Scaling laws, $G[F] \sim m^{\eta_{G[F]}}$, for the decay constants of the lowest lying states. In regard to the $V/A$ decay constants and formula (51) note that $G_{V/A} \leftrightarrow M_{V/A}F_{V/A}$ in terms of counting scaling powers. The symbol $y_m \equiv 1 + \gamma_*$ denotes the scaling dimension of the mass (7). Recall that the lowest bound state scales as $M_H \sim m^{1/y_m}$ (14) and the nonanalytic part of the quark condensate is given by: $\langle \bar{q}q \rangle \sim m^{(3-\gamma_*)/y_m}$ (4). The symbol $a$ denotes the adjoint flavour index, and $\lambda^a$ are the generator normalized as $\text{tr}[\lambda^a \lambda^b] = \frac{1}{2}\delta^{ab}$.

| $\mathcal{O}$ | def | $\langle 0|\mathcal{O}(0)|J^{P(C)}(p)\rangle$ | $J^{P(C)}$ | $\Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}}$ | $\eta_{G[F]}$ |
|---|---|---|---|---|---|
| $S$ | $\bar{q}q$ | $G_S$ | $0^{++}$ | $3 - \gamma_*$ | $(2-\gamma_*)/y_m$ |
| $S^a$ | $\bar{q}\lambda^a q$ | $G_{S^a}$ | $0^+$ | $3 - \gamma_*$ | $(2-\gamma_*)/y_m$ |
| $P^a$ | $\bar{q}i\gamma_5 q$ | $G_{P^a}$ | $0^-$ | $3 - \gamma_*$ | $(2-\gamma_*)/y_m$ |
| $V$ | $\bar{q}\gamma_\mu q$ | $\epsilon_\mu(p)M_V F_V$ | $1^{--}$ | $3$ | $1/y_m$ |
| $V^a$ | $\bar{q}\gamma_\mu \lambda^a q$ | $\epsilon_\mu(p)M_V F_{V^a}$ | $1^-$ | $3$ | $1/y_m$ |
| $A^a$ | $\bar{q}\gamma_\mu\gamma_5 \lambda^a q$ | $\epsilon_\mu(p)M_A F_{A^a}[ip_\mu F_{P^a}]$ | $1^+[0^-]$ | $3$ | $1/y_m[1/y_m]$ |

on the running of the coupling. We shall denote by $n_{f,c}$ the number of fermions above which the theory exhibits an IRFP. In this case the theory becomes scale-invariant at large distances, while the short-distance behavior is still the one dictated by asymptotic freedom. As a consequence of the scale invariance at large distances, the theory cannot be in a confining phase and chiral symmetry remains unbroken. The long-distance dynamics is governed by the critical exponents of the IRFP, which determine the scaling laws in the vicinity of the fixed point. The Banks-Zaks theories [14], where $N_c$ and $n_f$ are arranged such that the critical coupling $g^* \ll 1$, provide one working example of a theory within the conformal window. Early studies of near conformal and IRFP theories were based on approximate solutions of the Schwinger-Dyson equations [15,16]; these analyses were extended to higher representations in Ref. [11]. Unfortunately it is very difficult to control the systematic errors due to the truncation of the 1PI vertices appearing in the Schwinger-Dyson equations. Moreover Schwinger-Dyson equations predict the anomalous dimension of the mass to be around one, whereas unitarity constraints on the conformal group [17], in principle, allow for $\gamma_* \leq 2$.

Recent results have appeared recently, that address this problem either from an RG point of view [18–20], or from a gauge/string duality perspective [21–23]. We defer the investigation of the connections between our results and these other approaches for further studies.

Recent numerical simulations of gauge theories on the lattice have triggered a renewed interest in those theories and in turn in technicolor models. Algorithmic progresses have made lattice simulations with light dynamical fermions accessible on current hardware [24–26]. This opens the possibility to obtain first principles results for technicolor, and several preliminary investigations have appeared [13,27–45]. It is important to bear in mind that recent lattice results for theories that may lie inside the conformal window are plagued by systematic errors, and their interpretation still needs to be clarified. A recent discussion of the lattice artefacts in simulations of theories with a potential IRFP can be found in Refs. [44,46]. For these theories, unlike in QCD, there are no experimental results to guide the lattice simulations.

Therefore, it is crucial to develop analytical results in order to guide the lattice studies, and help in analyzing their outcome. A wider range of analytical predictions, together with more extensive simulations, will help in finding robust evidence for the existence of IRFPs.

## II. INFRARED FIXED POINTS

Let us henceforth consider theories *inside* the conformal window, i.e. gauge theories minimally coupled to a number $n_f$ of Dirac fermions, with the number of flavors and their representation adjusted so that the theories are scale-invariant at large distances when the fermions are massless.

In general, fixed points of RG flows are identified by the zeroes of the $\beta$ functions that describe the evolution of dimensionless couplings. The typical evolution of a running coupling is sketched in Fig. 1. The running coupling flows to a constant value at small energies, which corresponds to a zero of the beta function. The value $g^*$ of the coupling at the fixed point, and the precise shape of the nonperturbative function $g(\mu)$ are scheme dependent. However the existence of the fixed point and the critical exponents are universal.

The fermion mass is a relevant coupling at the IRFP, and drives the theory away from it. In a theory with a non-vanishing fermion mass, the fermionic degrees of freedom decouple at low energies, and the theory behaves like a pure Yang–Mills theory. The running of the gauge coupling for the massive theory is given by the dashed curve at small $\mu$ in Fig. 1, where the running of the coupling below some scale $\Lambda_{IR}$ is explicitly drawn. Note that in the presence of an IRFP $\Lambda_{IR}$ goes to zero as the fermion mass vanishes.

The running of the mass is described by its anomalous dimension, which has the opposite sign of the anomalous dimension of the renormalized composite operator $\bar{q}q$,

$$\mu \frac{d}{d\mu} \bar{q}q|_\mu = \gamma_{\bar{q}q}(\mu)\bar{q}q|_\mu = \gamma(\mu)\bar{q}q|_\mu. \qquad (6)$$

We have explicitly indicated the scale dependence of the various quantities. In this paper we will use the symbol $\gamma$ to denote the anomalous dimension of the mass and quark condensate: $\gamma \equiv -\gamma_m = \gamma_{\bar{q}q}$.

Note that the anomalous dimension away from the fixed point depends on the renormalization scheme. However its value $\gamma_*$ at the IRFP is a scheme-independent quantity. A
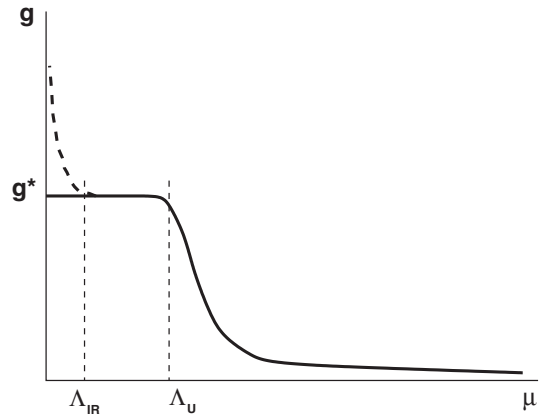


FIG. 1. Running of the coupling as a function of the energy scale for a theory with an IRFP. At low energies the coupling flows to a fixed-point value $g^*$, while the high energy behavior is the usual one expected for asymptotically free theories. The scale $\Lambda_U$ corresponds to the energy where the running starts to be dictated by asymptotic freedom. The dashed curve at low energies shows the running of the coupling when a fermionic mass term is switched on.

concise discussion of the scheme-dependent features of IRFPs can be found in Ref. [46].

Throughout this paper we will often refer to scaling dimensions of operators, denoted by $\Delta$; they are obtained as the sum of the naive mass dimension of the operator and the anomalous dimension. For example, for the operator $\bar{q}q$ we write

$$\Delta_{\bar{q}q} = d_{\bar{q}q} + \gamma_{\bar{q}q} = 3 - \gamma_*, \qquad y_m = 1 + \gamma_*, \quad (7)$$

where we have also introduced the scaling exponent $y_m$, which often appears in what follows and is widely used in the RG-literature [12]. Throughout this paper we will use these notations interchangeably.

Scaling laws are derived by assuming that the fermion mass is the only relevant operator at the IRFP. RG equations will be used below in order to derive the scaling of the chiral condensate as a function of the fermion mass. It is therefore worthwhile to briefly recall how the scaling relation for the masses in the spectrum is obtained. A recent discussion of RG flows in the vicinity of an IRFP can be found in Refs. [40,47,48].

Let us consider the zero-momentum vacuum correlator of an interpolating field $H(x)$ with the quantum numbers of a given state in the spectrum:

$$C_H(t; g, \hat{m}, \mu) = \int d^3x \langle H(t, x)H(0)^\dagger \rangle|_{g,\hat{m},\mu}, \quad (8)$$

where we have indicated explicitly the dependence on the couplings and the scale $\mu$. It is useful in this context to introduce a rescaled mass $\hat{m}(\mu) = m(\mu)/\mu$. For the specific case of lattice simulations, the scale is set by the inverse lattice spacing $\mu = a^{-1}$. The masses of the physical stable states are obtained from the Euclidean time dependence of two-point functions. At large Euclidean time $t$:

$$C_H(t; g, \hat{m}, \mu) \sim e^{-M_H t}, \quad (9)$$

where $M_H$ is the mass of the lightest state in the channel under examination. We examine the consequences of the RG equation for the two-point function.

In the vicinity of the fixed point, a RG transformation acts on the correlator according to

$$\mu = b\mu'; \qquad C_H(t; g, \hat{m}, \mu) = b^{-2\gamma_H} C_H(t; g', \hat{m}', \mu'), \quad (10)$$

where $\gamma_H$ is the anomalous dimension of the field $H$. The flow of the couplings near the RG fixed point is powerlike:

$$g' = b^{y_g} g, \qquad \hat{m}' = b^{y_m} \hat{m}. \quad (11)$$

We shall neglect henceforth the irrelevant coupling $g$ ($y_g < 0$). Multiplying all mass units by the factor $b$ we obtain

$$C_H(t; \hat{m}', \mu') = b^{-2d_H} C_H(tb^{-1}; \hat{m}', \mu), \quad (12)$$

where $d_H$ is the naive mass dimension of the operator $H$. Choosing $b$ such that $\hat{m}' = 1$, the equations above yield

$$C_H(t; \hat{m}, \mu) = \mathcal{C}_H F(t\hat{m}^{1/(1+\gamma_*)}, \mu), \quad (13)$$

where $F$ is some function that, for fixed $\mu$, depends on the rescaled variable $x = t\hat{m}^{1/(1+\gamma_*)}$ only. The detailed dependence of the prefactor $\mathcal{C}_H$ on the parameters of the theory is postponed to the next section, where it will play a prominent role. Comparing Eq. (13) with the expected behavior Eq. (9) yields

$$M_H \simeq c_H \mu \hat{m}^{1/(1+\gamma_*)} \quad \text{as } m \to 0. \quad (14)$$

Note that the scaling of the mass $M_H$ is entirely determined by the anomalous dimension $\gamma_*$ and does not depend on the specific choice of the interpolating operator $H$. Equation (14) shows that all lowest state masses scale with with same exponent $1/(1 + \gamma_*)$, while the proportionality constant $c_H$ depends on the chosen channel. While each individual mass in the spectrum vanishes, ratios of masses should remain constant as the chiral limit is approached. This scaling is consistent e.g. with the scenarios proposed in Refs. [49,50].

In the derivation above we have not considered the effects of a finite decay width. At least one channel ought to be stable and therefore not affected by the width. According to an inequality by Weingarten [51], valid for $n_F \geq 2$, this should be the mass of the lowest pseudoscalar flavor-nonsinglet, which we shall later on denote by $M_{P^a}$. For all other states one might wonder how the width interferes with the derivation above. Could the width and the mass conspire to cancel their leading mass scaling behavior in such a way as to invalidate Eq. (14)? We would like to bring forward two reasons why this should not be the case. First the difference in the large $N_c$-scaling of mass and width $[\Gamma_H/M_H \sim \mathcal{O}(1/N_c)]$ from QCD should hold in mCGT too and serve as a parametric argument against such a cancellation. Second we show in appendix C that in the approximation where the self-energy is treated as being constant such a cancellation can be excluded. This seems intuitively plausible since in Euclidian time the mass and decay width behavior are associated with exponential and oscillatory behavior, respectively.

On the contrary since mass and width do not seem to interfere in the leading large $t$- behavior Eq. (13) suggests that both the mass and the width of the resonance scale according to

$$M, \qquad \Gamma \sim m^{1/(1+\gamma^*)}, \quad (15)$$

We shall revisit the scaling of the width in Sec. IV, after discussing the scaling of the decay constants and derive $\Gamma(A \to B + C) \sim m^{1/(1+\gamma^*)}$ for a specific decay $A \to B + C$.

The behavior (14) is markedly different from what is observed in the spectrum of theories where chiral symmetry is spontaneously broken, like e.g. in QCD. In the latter theories, the Goldstone bosons become massless in the chiral limit, while the other states remain massive, with

their masses being of the order of some typical hadronic scale $\Lambda$. For theories with an IRFP, all states become massless, presumably at the same rate, which prevents a simple description of the nonperturbative low energy hadronic dynamics in terms of an effective theory like chiral perturbation theory.

Let us conclude this section by recalling how the finite-size effects can be analyzed using RG equations. We shall discuss explicitly the case of the correlator $C_H$, including the dependence on the size of the system $L$. We remind the reader that by studying finite volume effects, it is implied that the box is larger than the typical scale, $L \gg \mu^{-1}$, and therefore does not interfere with characteristic short-distance dynamics. The solution of the RG equation, including the $L$-dependence, scales as

$$C_H(t; \hat{m}, L, \mu) = b^{-2\gamma_H} C_H(t; \hat{m}', L, \mu'), \qquad (16)$$

according to a modified version of Eq. (10). Rescaling the energies by the factor $b$, and using the power-law scaling of the couplings near the IRFP yields

$$C_H(t; \hat{m}, L, \mu) = b^{-2(d_H + \gamma_H)} C_H(b^{-1}t; b^{y_m}\hat{m}, b^{-1}L, \mu). \qquad (17)$$

Choosing $b$ such that $b^{-1}L = L_0$, where $L_0$ is a reference length, yields

$$C_H(t; \hat{m}, L, \mu) = \left(\frac{L}{L_0}\right)^{-2\Delta_H} C_H\left(\frac{t}{L/L_0}; x\frac{1}{\mu L_0^{y_m}}, L_0, \mu\right), \qquad (18)$$

where we have introduced the scaling variable $x = L^{y_m} m$.

Comparing Eq. (18) with the expected asymptotic behavior in Eq. (9) we obtain

$$M_H = L^{-1} f(x), \qquad (19)$$

where $f(x)$ is some function of the scaling variable $x$, expected to vanish when $x$ goes to zero. In order to recover the correct scaling with $m$ in the thermodynamic limit

$$f(x) \sim x^{1/y_m}, \quad \text{as } x \to \infty. \qquad (20)$$

As one can see from Eq. (19), if the fermion mass is decreased at fixed $\mu$ and $L$, then the mass of the states in the spectrum will initially decrease until the Compton wavelength of the states is of the order of the linear size of the system. When this happens, the mass of the states saturates and scales with the inverse size $L^{-1}$. Results for $M_H L$ computed on different volumes should follow a universal curve when studied as a function of the scaling variable $x$.

## III. MODIFIED BANKS-CASHER RELATION

In this section we relate the scaling exponent of the chiral condensate $\eta_{\bar{q}q}$ to the scaling of the eigenvalue density of the massless Dirac operator. We then illustrate how the RG equations yield a prediction for the exponent

in terms of the anomalous dimension $\gamma_*$ introduced in Eq. (6). These results follow readily from the RG scaling of the free energy and the field correlators in the vicinity of fixed point, and were already presented in Ref. [13]. Here we discuss in detail the derivation of these results in the context of a mCGT, generalizing to other condensates such as the gluon condensate, and compare them to a more heuristic derivation.

### A. Eigenvalue density $\rho(\lambda)$ and the scaling exponent $\eta_{\bar{q}q}$

It is useful to recall the basic steps in the derivation of the Banks-Casher formula, in order to highlight the order in which the limits are taken, the divergences that may appear, and to identify the differences from the case of a conformal theory.

We closely follow the discussion in Ref. [52] and extend it at appropriate places to mCGT. The fermion propagator can be written as

$$\langle q(x)\bar{q}(y)\rangle = \sum_n \frac{u_n(x)u_n^\dagger(y)}{m - i\lambda_n}, \qquad (21)$$

where the eigenmodes of the massless Euclidean operator $D \equiv \gamma_\mu D^\mu$ have been introduced:

$$D u_n(x) = \lambda_n u_n(x). \qquad (22)$$

Since the eigenfunctions occur in pairs with opposite eigenvalues, the chiral condensate in a finite volume $V$ is given by

$$\langle \bar{q}q\rangle_V = \frac{1}{V}\int dx \langle \bar{q}(x)q(x)\rangle = -\frac{2m}{V}\sum_{\lambda_n>0}\frac{1}{m^2 + \lambda_n^2}. \qquad (23)$$

Taking the infinite volume limit at fixed mass, the sum over positive eigenvalues can be replaced by

$$\langle \bar{q}q\rangle = \lim_{V\to\infty}\langle \bar{q}q\rangle_V = -2m\int_0^\infty d\lambda\frac{\rho(\lambda)}{m^2 + \lambda^2}, \qquad (24)$$

where $\rho(\lambda)$ denotes the number density of eigenvalues per unit volume. Equation (24) is purely formal at this stage in the sense that a UV-regularization is needed on both sides. In four dimension the divergences are logarithmic and quadratic, respectively.[1] The divergences can be isolated via a twice-subtracted spectral representation:

$$\langle \bar{q}q\rangle = -2m\int_0^\mu d\lambda\frac{\rho(\lambda)}{m^2 + \lambda^2} - 2m^5\int_\mu^\infty\frac{d\lambda}{\lambda^4}\frac{\rho(\lambda)}{m^2 + \lambda^2}$$
$$+ \gamma_1 m + \gamma_2 m^3. \qquad (25)$$

The subtraction constants $\gamma_1$ and $\gamma_2$ contain the UV-divergences. Their respective behaviours are $\gamma_1 \sim \Lambda_{\text{UV}}^2$,

---

[1]Note that if the regulated theory breaks chiral symmetry explicitly, as is the case with lattice Wilson fermions, then a cubic divergence appears that survives in the chiral limit [53].

and $\gamma_2 \sim \log[\Lambda_{\text{UV}}^2]$, and their actual values depend on two physical renormalization conditions used to define the finite condensate on the left-hand side (LHS) of Eq. (24). We shall investigate the limiting behavior when $m \to 0$. The second integral and the subtraction terms in Eq. (25) vanish in the chiral limit ($m \to 0$). Therefore only the first integral, sensitive to the IR region, can result in a non-analytic term and has to be investigated further. A simple change of variable yields

$$\langle \bar{q}q \rangle = -2 \int_0^{\mu/m} dx \frac{\rho(mx)}{1 + x^2} + \mathcal{A}(m), \qquad (26)$$

where $\mathcal{A}(m)$ stands for an analytic function of $m$. From Eq. (26), following the same arguments used in QCD, one can readily obtain

$$\langle \bar{q}q \rangle \sim^{m \to 0} m^{\eta_{\bar{q}q}} \Leftrightarrow \rho(\lambda) \sim^{\lambda \to 0} \lambda^{\eta_{\bar{q}q}}. \qquad (27)$$

This in turn implies

$$\eta_{\bar{q}q}|_{\text{QCD-like}} = 0, \qquad \eta_{\bar{q}q}|_{\text{mCGT}} > 0, \qquad (28)$$

since in QCD the condensate remains finite in the chiral limit, while it vanishes in mCGT.

Let us derive the same scaling coefficient $\eta_{\bar{q}q}$ (4) from a RG analysis. The starting point is the two-point function $C_{\bar{q}q}(t; \hat{m}, \mu)$, as in Eq. (8), where the hadronic field $H = \bar{q}q$, and the explicit dependence on the coupling $g$ is suppressed. The solution of the RG equations for this specific case is

$$C_{\bar{q}q}(t; \hat{m}, \mu) = b^{-2\Delta_{\bar{q}q}} C_{\bar{q}q}(tb^{-1}; b^{y_m}\hat{m}, \mu). \qquad (29)$$

Imposing again $b^{y_m}\hat{m} = 1$, finally leads to

$$C_{\bar{q}q}(t; \hat{m}, \mu) = \hat{m}^{2\Delta_{\bar{q}q}/y_m} C_{\bar{q}q}(t\hat{m}^{1/y_m}; 1, \mu). \qquad (30)$$

Inserting a complete set of states the exponential decrease of any state other than the vacuum for large $t$ results in

$$C_{\bar{q}q}(t; \hat{m}, \mu) \sim^{t \to \infty} m^{2\eta_{\bar{q}q}}, \qquad (31)$$

whence the scaling exponent (27) follows:

$$\eta_{\bar{q}q} = \frac{\Delta_{\bar{q}q}}{y_m} = \frac{3 - \gamma_*}{1 + \gamma_*}. \qquad (32)$$

The eigenvalue density then scales as

$$\rho(\lambda) \sim \lambda^{(3 - \gamma_*)/(1 + \gamma_*)}; \qquad (33)$$

this result generalizes the Banks-Casher relation for QCD [54]:

$$\langle \bar{q}q \rangle|_{m=0} \neq 0 \Rightarrow \rho(0) = -\pi \langle \bar{q}q \rangle|_{m=0} \qquad (34)$$

to mCGT. It is interesting to remark that Refs. [13,45,55] state different predictions for the scaling exponent. Our determination of this critical exponent agrees with Ref. [13].

Surely this derivation generalizes to any other operator, for example, the gluon condensate for which one gets

$$\eta_{G^2} = \frac{\Delta_{G^2}}{y_m} = \frac{4}{1 + \gamma_*}. \qquad (35)$$

The scaling dimension of the gluon condensate is four since it appears in the Lagrangian density of a four dimensional scale-invariant theory.

### B. Alternative and heuristic derivation of $\eta_{\bar{q}q}$

Let us now present an alternative derivation of the scaling exponents $\eta_{\bar{q}q}$ and $\eta_{G^2}$ in Eqs. (32) and (35), which is of a heuristic nature but might provide some physical insight. The discussion for $\langle \bar{q}q \rangle$, which we shall adopt here before generalizing it to $\langle G^2 \rangle$ closely follows Ref. [56].[23] In this work we refine the discussion and interpretation of IR and UV terms by making use of the scaling of the hadronic masses in Eq. (14) and the interpretation of subtraction terms in Eq. (25).

In a low energy effective theory describing the dynamics of the operator $\bar{q}q$,[4] the mass deformation in Eq. (1) corresponds to a tadpole term and demands a reminimization of the potential to find the stable vacuum. The potential for $\bar{q}q$ is not known but the scaling of the two-point function is governed by the anomalous dimension. It has been proposed in Ref. [58] to mimic the continuous spectrum of such an operator by introducing a tower of scalar fields with suitably adjusted masses and couplings:

$$\bar{q}q(x) \sim \sum_n f_n \varphi_n(x); \qquad \langle \varphi_n | \bar{q}q | 0 \rangle \sim f_n,$$
$$f_n^2 = \delta^2 (M_n^2)^{\Delta_{\bar{q}q} - 2} \qquad M_n^2 = n\delta^2, \qquad (36)$$

where the quantity $\delta$ describes the mass spacing between the $\varphi_n$-modes. The decomposition (36) reproduces the two-point function of a conformal theory in Minkowski space in the limit $\delta \to 0$ [58], up to potential subtraction ambiguities. Note that in Eq. (36) we have not tried to keep track of the overall mass dimension and normalization since they are irrelevant for scaling properties. The potential part of the Lagrangian $\mathcal{L} = -m\sum_n f_n \varphi_n - 1/2\sum_n M_n^2 \varphi_n^2$ then leads to the equation of motion for $\varphi_n$ of the form

$$mf_n + M_n^2 \varphi_n = 0 \Rightarrow \langle \varphi_n \rangle = -mf_n/M_n^2, \qquad (37)$$

with solution as indicated on the right. Thus leading to a VEV,

---

[2]The computation in Ref. [56] differs by in an additional term $\delta \mathcal{L} \sim (\bar{q}q)^2$ which is not relevant here.

[3]The calculation is similar to an analysis of a scale-invariant theory with a scalar operator and tadpole term Ref. [57] in the context of the unparticle scenario, where $2 \leq \gamma_* \leq 1$ ($\Delta_{\mathcal{U}} = 3 - \gamma_*$) was assumed and made it necessary to introduce (various) IR regularizations.

[4]We refrain to change to a notation $\bar{q}q \to \mathcal{O}_U$ since we are not interested in parametrizing an effective theory for $\mathcal{O}_U$ as in Ref. [56].

$$\langle \bar{q}q \rangle \sim \sum_n f_n \langle \varphi_n \rangle = -m \sum_n \frac{f_n^2}{M_n^2} \overset{\delta \to 0}{\longrightarrow} -m \int_{\Lambda_{IR}^2}^{\Lambda_{UV}^2} d(s)ds$$

$$d(s) = \begin{cases} s^{\Delta_{\bar{q}q}-3} & \Lambda_{IR}^2 \le s \le \Lambda_{\mathcal{U}}^2 \\ (\Lambda_{\mathcal{U}}^2)^{\Delta_{\bar{q}q}-3} & \Lambda_{\mathcal{U}}^2 \le s \le \Lambda_{UV}^2 \end{cases} \qquad (38)$$

where IR and UV cutoffs, to be discussed below, were introduced. We have taken into account the running of the gauge coupling as indicated in Fig. 1, though showing a somewhat cavalier attitude towards the treatment of the transition region to be justified later on. The nonanalytic part in $m$, if present, is hidden in the IR cutoff. It seems natural that the latter is governed by the typical hadronic mass scale, i.e. $\Lambda_{IR} \simeq cM_H$, where $c$ is a constant irrelevant to our investigations. The integral can be computed and yields

$$\langle \bar{q}q \rangle \sim -m(M_H^2)^{\Delta_{\bar{q}q}-2} + m(\Lambda_{\mathcal{U}}^2)^{\Delta_{\bar{q}q}-3}\Lambda_{UV}^2. \qquad (39)$$

Thus using the scaling of the hadronic masses (14), (7), and (39) becomes

$$\langle \bar{q}q \rangle \sim m^{(3-\gamma_*)/1+\gamma_*} + \mathcal{A}(m) \Rightarrow \eta_{\bar{q}q} = \frac{3-\gamma_*}{1+\gamma_*}, \qquad (40)$$

where as previously $\mathcal{A}(m) \sim \mathcal{O}(m)$ denotes an analytic function in $m$. We have therefore derived the exponent $\eta_{\bar{q}q}$ in Eq. (3). The UV-divergent term in Eq. (39) corresponds to the quadratic divergence discussed in the previous section and is irrelevant for the nonanalytic part and the scaling exponent $\eta_{\bar{q}q}$. The nonappearance of the logarithmic divergence might be related to the fact that we do not consider the back reaction of the mass perturbation on the spectrum such as taking into consideration power correction in $m$ in the couplings $f_n$.

Surely this procedure generalizes to any gauge invariant term in the Largangian $\delta \mathcal{L} = m^{(4-\Delta_{\mathcal{O}})/y_m}\mathcal{O}$ in which case the condensate (40) assumes the form:

$$\langle \mathcal{O} \rangle|_{IR} \sim m^{(4-\Delta_{\mathcal{O}})/y_m}(M_H^2)^{\Delta_{\mathcal{O}}-2} \sim m^{\Delta_{\mathcal{O}}/y_m}, \qquad (41)$$

in accordance with Eq. (32) which is absolutely general. The subscript IR indicates that UV terms have been omitted.

We consider it worthwhile to discuss the term $\delta \mathcal{L} \sim G^2$, resulting in the gluon condensate. From Eq. (41), paying attention to the UV terms in addition one gets

$$\langle G^2 \rangle \sim (M_H^2)^{\Delta_{G^2}-2} + (\Lambda_{UV}^2)^{\Delta_{G^2}-2} \qquad (42)$$

and with (14) and $\Delta_{G^2} = 4$ one gets

$$\langle G^2 \rangle \sim m^{2(\Delta_{G^2}-2)/y_m} + \mathcal{A}(m^0)$$

$$\Rightarrow \eta_{G^2} = \frac{2(\Delta_{G^2}-2)}{y_m} = \frac{4}{1+\gamma_*}, \qquad (43)$$

the same scaling as in Eq. (35). The $\mathcal{A}(m^0)$ refers to the term proportional to $\Lambda_{UV}^4$. It originates from the region of asymptotic freedom and is interpreted as a mixing with the

identity. Such contributions, sometimes called renormalons, have hitherto prevented an extraction or a proper definition of a gluon condensate in lattice QCD.

### C. Critical discussion

Although the derivation above reproduces the correct result there remain some points that deserve further clarification. For this discussion we shall first think of the theory when $m = 0$. Assuming that this effective field theory approach, the deconstruction (36), can be extended to higher orders one would need to introduce higher order terms, starting from cubic ones, into the effective Lagrangian in order to reproduce higher correlation functions. There are two important effects, due to higher order terms.

First, are they going to modify the leading order extraction of the $\langle \bar{q}q \rangle$ directly? The answer appears to be no, since the initial potential does not know about the small parameter $m$ and perturbing the system by the term in Eq. (1) does not lead to major correction to any order but in the linear one. That this is self-consistent can be seen more explicitly by plugging in the VEV $\langle \varphi_n \rangle \sim m$ as in Eq. (37) into a fictitious higher order term. Second, does the quadratic order need to be modified? Since the higher order terms are going to modify the two-point function, the answer appears to be yes. As long as these modifications are of the form $M_n^2|_{higher\ order} \sim n^\alpha$ with $\alpha > 1$, they will give rise to subleading effects in $m$. Although this appears likely we have not tried to resolve this issue in this paper but obviously this question deserves further study.[5]

### D. Lattice data

Note that the anomalous dimension $\gamma_*$ is related to the running of the fermion mass, and has been computed by Schrödinger Functional methods in Ref. [46]. On the other hand, as discussed above, the exponent $\eta_{\bar{q}q}$ characterizes the behavior of the eigenvalue density around zero, so that in principle it can be extracted from the eigenvalue density.

First results for the eigenvalue spectrum of the Dirac operator have been presented in Refs. [32,47] for an SU(2) gauge theory with two Dirac fermions in the adjoint representation. An extensive study of the 200 lowest eigenvalues is available only for the $16^3$ lattice studied in the references above. As argued in Ref. [59], the mode number of the massive Dirac operator:

$$\nu(M, m) = \int_{-\Lambda}^{+\Lambda} d\lambda \rho(\lambda), \qquad (44)$$

---

[5]One might also be concerned that the mixing of different $\varphi_n$ modes might reshuffle hierarchies. That this is not the case for the analytic part follows from the fact that the modes with low $n$ are responsible for the nonanalytic part and that the higher modes are numerically suppressed with respect to the lower modes by $\langle \varphi_n \rangle = \langle \varphi_1 \rangle / n^{1+\gamma_*}$ [56].
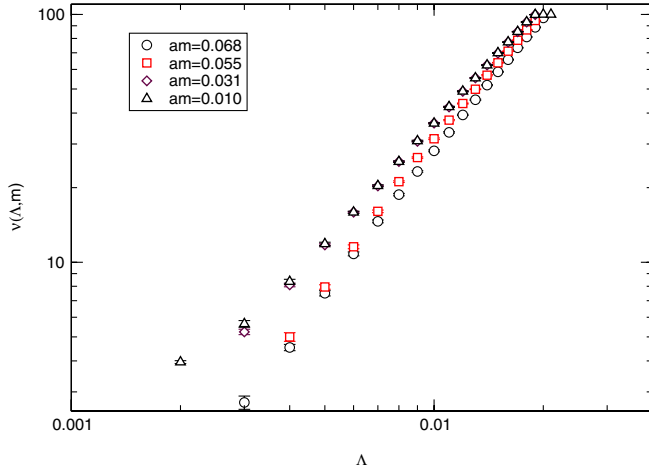
FIG. 2 (color online). The mode number $\nu(M, m)$ for the SU(2) gauge theory with two fermions in the adjoint representation. The data show the dependence of the mode number on the scale $\Lambda$ for several values of the quark mass.

where $\Lambda = \sqrt{M^2 - m^2}$, carries the same information as the density itself. The mode number can be renormalized and yields a RG invariant, universal quantity that describes the physical properties of the Dirac spectrum independent of the regularization used.

The mode number can easily be computed from the available eigenvalue distributions. Results obtained at $\beta = 2.25$ on a $32 \times 16^3$ lattice are reported in Fig. 2 for different values of the fermion mass $am$. The quantity of interest is the extrapolation of the mode number to the thermodynamic and massless limit:

$$\lim_{m \to 0} \lim_{V \to \infty} \nu(M, m). \qquad (45)$$

Using the current data at a single value of the volume, where a reasonable amount of data is available Ref. [47], the first extrapolation cannot be performed. We defer this

analysis for further studies as larger lattices become available, and concentrate instead on the extrapolation to the chiral limit. The data in Fig. 2 show that there is a dependence of the mode number on the PCAC mass. The data at the two lightest masses are compatible within the statistical errors, and we shall take the data at the lightest mass for our analysis.

In the chiral limit, the mode number is expected to scale as

$$\nu(\Lambda, 0) = C(\Lambda - g)^{\eta_{\bar{q}q}+1}, \qquad (46)$$

where the possibility of a nonvanishing spectral gap $g$, as suggested for instance in Ref. [60], is taken into account in the functional form of the fitting function. we therefore end up with fitting the data to three parameters, namely $C$, $g$, and the exponent. The data are consistent with such a power-law behavior, however the value of the exponent depends critically on the range used for the fit. The plot in the left panel of Fig. 3 shows the result of a fit that has a fit range $\Lambda \in [0.002, 0.014]$, in lattice units. The result for the critical exponent is $1 + \eta_{\bar{q}q} = 1.65(4)$, which can be translated into a fitted value for the mass anomalous dimension, yielding $\gamma_* = 1.42(5)$. Note that this value is quite different from the one found in previous studies [46,47]. However, if the fit range is reduced to $\Lambda \in [0.003, 0.008]$, then the fitted exponent is $\gamma_* = 1.1(3)$. The latter value is still larger than the one obtained form the Schrodinger functional studies, it clearly shows that the fitted exponent depends critically on the fit range. More extensive data on the eigenvalue distributions are needed in order to be able to extract the critical exponent in a reliable manner.

Note that the error induced by varying the fitting range turns out to be larger than the statistical error, and that this result is obtained at one value of the lattice volume, and could be affected by finite size effects. A more comprehensive analysis of the volume dependence of the eigen-
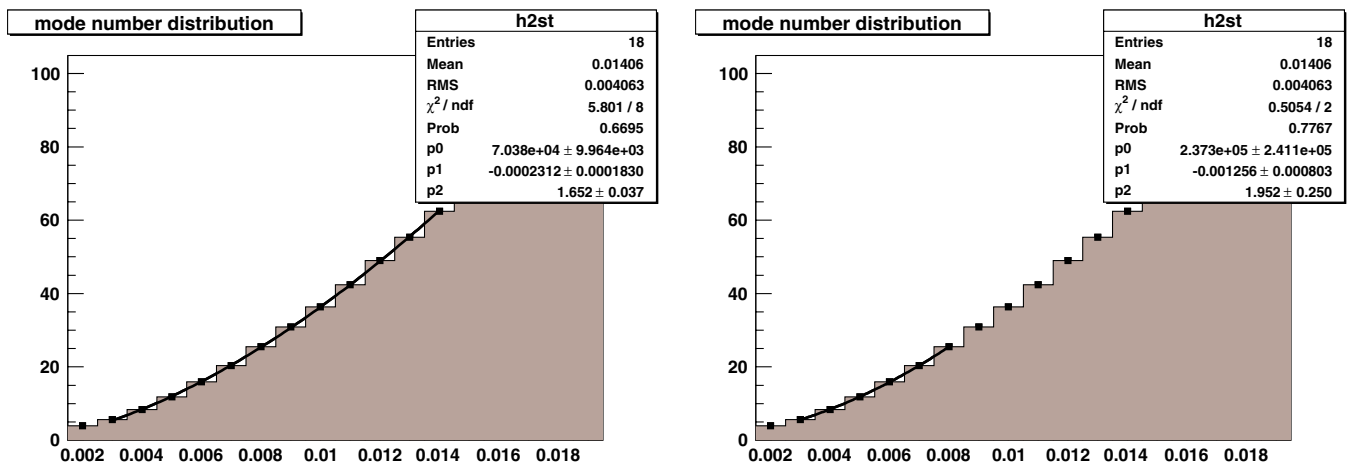


FIG. 3 (color online). Fit of the mode number $\nu(M, m)$ to a power-law behavior. The two plots represent two fits to the same data over two different ranges.

value distribution is needed. More extensive lattice data should become available in the near future.

## IV. DECAY CONSTANTS

This section explores the information that can be gathered from the RG scaling of matrix elements of given operators. In Sec. IV A we show that the scaling of decay constants of the lowest lying states directly follow from its anomalous dimension through the Callan-Symanzik equations. In Sec. IV B we deduce consequences from spectral representations of the Ward identities and low energy theorems for pseudoscalar and scalar states evaluated at zero momentum. In Sec. IV C we compare our theoretical predictions with recent lattice data. Finally, miscellaneous matters of interest are presented in Sec. IV D. In Appendix A the pseudoscalar WI at large momentum transfer is used to deduce further information.

### A. Hyperscaling and decay constants

Let us consider an operator $\mathcal{O}$ with scaling dimension $\Delta_{\mathcal{O}}$ and quantum numbers such that it couples to a state $|H(p)\rangle$ with strength $G_H$ for scalar operators and $F_H$ for vector operators. We shall choose $G_H$ to exemplify the equations below:

$$\langle 0|\mathcal{O}(0)|H(p)\rangle = G_H, \qquad \Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}}. \tag{47}$$

Information on the lowest lying state can be gained from the large time behavior of the Euclidian two-point function $C_{\mathcal{O}}(t; g, m, \mu)$ defined in Eq. (8):

$$C_{\mathcal{O}}(t; g, m, \mu) \overset{t \to \infty}{\to} e^{-M_H t} \frac{\langle 0|\mathcal{O}(0)|H(p)\rangle\langle H(p)|\mathcal{O}(0)|0\rangle}{2M_H V}$$

$$= e^{-M_H t} \frac{|G_H|^2}{2M_H V} \tag{48}$$

The scaling of $|G_H|$ can be inferred by applying a renormalization group transformation $\mu = b\mu'$ and imposing $b^{y_m}\hat{m}(\mu) = 1$ as in Sec. III A. The LHS becomes

$$C_{\mathcal{O}}(t; \hat{m}, \mu) = \hat{m}^{2\Delta_{\mathcal{O}}/y_m} C_{\mathcal{O}}(t\hat{m}^{1/y_m}; 1, \mu), \tag{49}$$

whereas the right-hand side (RHS) scales as

$$\frac{|G_H|^2}{2M_H V} \sim \hat{m}^{2\eta_{G_H} - 1/y_m + 3/y_m}. \tag{50}$$

Combining Eqs. (49) and (50) we obtain

$$|G_H| \sim \hat{m}^{\Delta_{\mathcal{O}} - 1/y_m}. \tag{51}$$

The definitions of the decay constants, their anomalous dimensions and resulting scaling coefficients are summarized in Table I.

We would like to draw the reader's attention to the fact that the pseudoscalar decay constant, as defined in Table I, is related through the PCAC relation as

$$\partial \cdot A^a = 2mP^a, \Rightarrow 2mG_{P^a} = M_{P^a}^2 F_{P^a}. \tag{52}$$

The scaling is consistent with our findings from Eq. (51) since

$$1 + (2 - \gamma_*)/y_m = 2/y_m + 1/y_m. \tag{53}$$

Let us briefly discuss the scaling dimensions of the operators given in Table I. The currents $V$, $V^a$ have vanishing anomalous dimension since they are conserved currents that are associated with global symmetries. The axial current $A^a$ is only partially conserved, see Eq. (52). It is broken by a soft term whose renormalization does not affect the divergence $\partial \cdot A^a$ and therefore $A^a$ has vanishing anomalous dimension. Moreover this implies that $mP^a$ is a renormalization group invariant and thus $\Delta_{P^a} = 3 - \gamma_*$. The scaling dimension of $S$ was already discussed in Sec. II. In the case where there are no masses $S^a$ and $P^a$ have the same renormalization constant. This is explicit to all orders in perturbation theory and should also hold nonperturbatively. Neglecting effects of the mass on $\gamma_*$ one concludes $\Delta_{S^a} = 3 - \gamma_*$. The flavor-singlet axial vector identity is anomalous. The topological charge density mixes with the axial vector, which therefore does not renormalize multiplicatively. This is further discussed in Appendix B 2

### B. Low energy theorems from ward identities and alike

In the previous subsection we have inferred the scaling laws of the decay constants of the lowest lying states from the anomalous dimensions. Further information can be obtained by analyzing WIs.

In appendix B we recall the derivation of two standard WIs, Eqs. (B2) and (B9), and a low energy theorem, Eq. (B10):

$$(2m)^2 \Pi_{P_a P_b}(0) = -2m\delta_{ab}\langle \bar{q}q \rangle$$

$$(2m)^2 \Pi_{PP}(0) - \Pi_{\tilde{Q}\tilde{Q}}(0) = -4m\langle \bar{q}q \rangle$$

$$\Pi_{SS}(0) = -\frac{\partial}{\partial m}\langle \bar{q}q \rangle, \tag{54}$$

where $\Pi_{XY}(q^2)$ is the time-ordered two-point function, cf. (B3). Information on the decay constants can be gained by investigating the dispersion representation of the two-point functions:

$$\Pi_{P_a P_b}(q^2) = \frac{1}{\pi} \int_{\text{cut}} ds \frac{\text{Im}[\Pi_{P_a P_b}(s)]}{s - q^2 - i0} + c + dq^2 + \dots, \tag{55}$$

where we have chosen $\Pi_{P_a P_b}$ as representative for definiteness. The symbols $c$ and $d$ denote subtraction constants due to UV-divergences of which only $c$ is relevant since $q^2 = 0$ in the equations above.[6] At $q^2 = 0$ Eq. (55) writes,

---

[6]Note $c$ vanishes for $\Pi_{\tilde{Q}\tilde{Q}}(0)$ since the latter vanishes to all orders in perturbation because $\tilde{Q}$ can be written as a total derivative.

$$\Pi_{P_a P_b}(0) = \frac{1}{\pi} \int_{\text{cut}} ds \frac{\text{Im}[\Pi_{P_a P_b}(s)]}{s - i0} + c_1 + c_2 m^2, \quad (56)$$

where $c = c_1 + c_2 m^2$ are the subtraction constants due to a quadratic and logarithmic divergence of which only $c_1$ is relevant for our discussion.

We would like to make a comment of speculative nature. Assuming that the lowest state,

$$\Pi_{P_a P_b}(0) = \delta_{ab} \frac{G_{P^a}^2}{M_{P^a}^2} + \dots, \quad (57)$$

contributes to the leading scaling of the RHS of Eqs. (54)

$$\frac{G_{P^a}^2}{M_{P^a}^2} + \dots + c_1 = -\frac{2}{m} \langle \bar{q} q \rangle, \quad (58)$$

then, using the results for $M_H$ and setting aside the issue of the subtraction constant $c_1 \sim \mathcal{O}(m^0)$ for the moment, the scaling laws in Table I are reproduced for $P^a$ and $S$ and

$$\frac{G_P^2 - (\tilde{G}_P / 2m)^2}{M_P^2} \sim m^{\eta_{\bar{q}q} - 1} = m^{(2(1 - \gamma_*))/(1 + \gamma_*)}, \quad (59)$$

would follow from the pseudoscalar WI, where the decay constants $G_P$ and $\tilde{G}_P$ are defined as follows.

$$\langle 0|P(0)|P(p)\rangle = G_P, \qquad \langle 0|\tilde{Q}(0)|P(p)\rangle = \tilde{G}_P. \quad (60)$$

The result for $P$, if correct, is new but on the other hand not very practical as it is the difference of two positive terms. Another possibility, or rather an extension of the scenario above, is that the hadron mass scaling is universal and that the width does not upset the parametric effects,[7] which would result in all decay constants scaling in the same way. Clearly these statements above are of a speculative nature driven by the knowledge of the lowest lying decay constants from the Callan-Symmazik equations. It is also amusing to see in what way the scaling laws for those two channels turn out to be the same. One might wonder about the influence and the origin of the subtraction terms in the WI (54). They match the divergences of the quark condensate on dimensional grounds. In fact, it can be seen that they match exactly. Following Ref. [56] we can fix the coupling

$$\langle 0|\bar{q}q|\varphi_n\rangle = \sqrt{\frac{B_{\Delta_{\bar{q}q}}}{2\pi}} f_n \quad (61)$$

(36) at $\Delta_{\bar{q}q} = 3$ by demanding that the deconstructed version matches $\Pi(0)_{SS}$ in the region where the theory is asymptotically free. At vanishing quark mass and $O(g^0)$ the time dependent scalar correlator, $\Pi(q^2)_{SS}$, is given by

$$\Pi(q^2)_{SS} = \frac{B_3}{2\pi} \int \frac{s \, ds}{s - q^2}, \qquad B_3 = \frac{N_c n_f}{4\pi}. \quad (62)$$

We have factored out the coefficient $B_3$, which matches the deconstructed version. The condensate (38) in the asymptotically free region, with the normalization (61), becomes

$$\langle \bar{q}q \rangle = -m \frac{B_3}{2\pi} \int ds \cdot 1. \quad (63)$$

Whence an exact matching of the divergences in the correlation function (54) on the LHS, and the quark condensate on the RHS is found. Thus the UV divergences of the quark condensate and the ambiguity in defining the $\Pi(q^2)$ functions from a dispersion relation do match exactly. The deconstructed version (36) of the condensate is therefore consistent with the WI. Note the extra factor of 1 half on the RHS of the nonsinglet pseudoscalar WI is due to the normalization $\text{tr}[\lambda^a \lambda^b] = \frac{1}{2} \delta^{ab}$ of the flavor generators.[8]

### C. Comparison with data

The scaling predictions obtained above can be compared to the recent lattice data presented in Ref. [44] for the SU(2) gauge theory with two flavors in the adjoint representation. The dependence of the pseudoscalar decay constant on the fermion mass is reported in Fig. 4. It is clear from the plot that the nonanalytic dependence of $F_{P^a}$ on the fermion mass can not be determined from current lattice data, where no curvature is visible. This is confirmed quantitatively by trying to fit the data to a power-law dependence on the fermion mass. Rather than trying to determine the exponent from the fit, we keep the exponent fixed, and fit the proportionality coefficient only. As shown in the figure, good fits to the data at the smaller masses can be obtained for different values of $\eta_{F_{P_a}}$ by adjusting the constant of proportionality $c$.

A preliminary estimate for the mass anomalous dimension was obtained from numerical simulations using the Schrödinger functional in Ref. [46]. A value of $\gamma_* \simeq 0.5$ is compatible with the results in Ref. [46], and leads to $\eta_{G_{P_a}} \simeq 1.0$ and $\eta_{F_{P_a}} \simeq 0.33$. This is represented by the blue line in the figure. As discussed above, the value $c$ can be adjusted to yield a good description of the data at the smaller masses. Note that we only expect the scaling to hold in the limit where the fermion mass goes to zero, therefore it seems natural to exclude the heavier points from this analysis. However, further systematic uncertainty is introduced by the choice of the fitting range.

The situation improves only slightly when looking at the dependence of the coupling $G_{P^a}$ on the fermion mass, which is presented in Fig. 5. A two-variable fit of the data can be performed in this case, and yields a scaling exponent $\eta_{G_{P^a}} = 1.3(2)$. Note that, using the scaling for-

---

[7]This should be true in the large $N_c$-limit where the width is suppressed by $1/N_c$ as compared to the mass.

[8]We have only focused on the leading quadratic divergence of the quark condensate; it would be interesting to investigate the logarithmic divergence as well in which case one could possibly learn something about mass correction to the deconstructed version presented in Sec. III B.
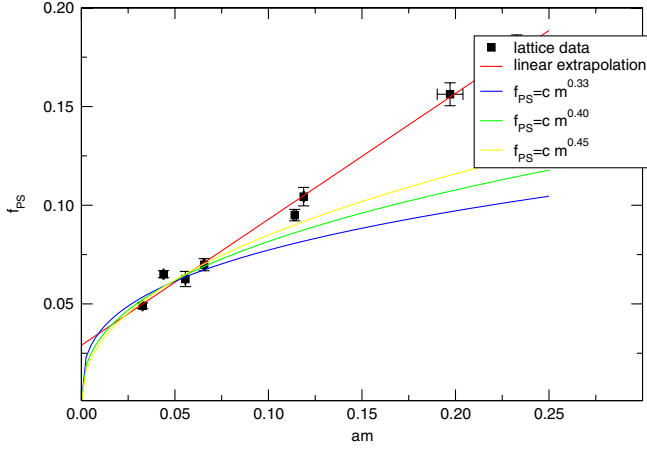
FIG. 4 (color online). The pseudoscalar decay constant $F_{P_a}$ as a function of the fermion PCAC mass $am$. All quantities are expressed in units of the lattice spacing, which is kept constant as $m$ is varied. The lines represent fits of the data to power-law scaling for different values of the critical exponents $\eta_{F_{P_a}}$.

mula in Table I, the result of the fit implies $\gamma_* = 0.30(5)$, which broadly agrees with the result of Refs. [46–48].

Better control over systematic errors is required in order to extract robust information from the scaling of the pseudoscalar decay constant. Current data can only be used to check the consistency with the scaling we presented above; Fig. 4 clearly shows that lighter masses are needed in order to actually determine the scaling exponents from lattice data.

### D. Further remarks

#### 1. Scaling of the decay width

Let us focus on a generic decay process $A \rightarrow BC$, mediated by some effective Lagrangian:
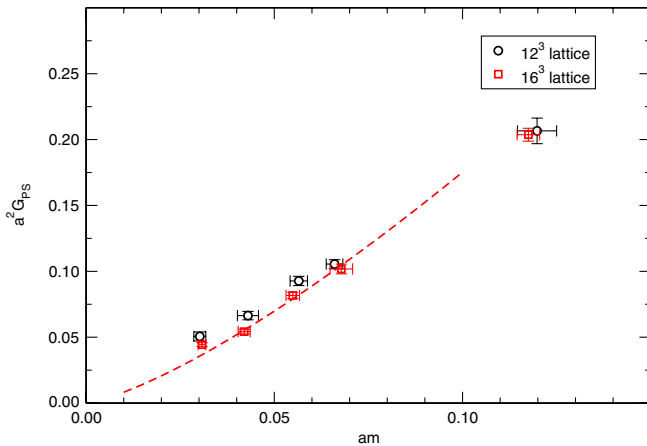


FIG. 5 (color online). The pseudoscalar coupling $G_{P_a}$ as a function of the fermion PCAC mass $am$. All quantities are expressed in units of the lattice spacing, which is kept constant as $m$ is varied. The line represents a fit to the data assuming the power-law behavior described above.

$$\mathcal{L}_{\text{eff}} = \int d^4 x G_{ABC} A(x)B(x)C(x), \qquad (64)$$

where $G_{ABC}$ is the $ABC$ coupling, and $A(x)$, $B(x)$, $C(x)$ are the fields creating and annihilating the states $A$, $B$, $C$. The fields are normalized as $\langle 0|A(0)|A(p)\rangle = 1$, etc. Let us now introduce three interpolating fields $J_A$, $J_B$, and $J_C$; these are composite fields that have an overlap with the single particle states, e.g. like quark bilinears for the simplest type of mesons. At lowest order in $G_{ABC}$ the correlator is given by

$$\langle J_A(x)J_B(y)J_C(0)\rangle \sim \int d^4 z G_{ABC}\langle J_A(x)J_B(y)J_C(0)\mathcal{L}_{\text{eff}}(z)\rangle, \qquad (65)$$

and inserting a complete set of states becomes

$$\sim \frac{G_A G_B G_C}{M_A M_B M_C}\left[\frac{VT}{V^3}\right]G_{ABC}e^{-M_A(t_x-t_z)-M_B(t_y-t_z)-M_C(-t_z)}. \qquad (66)$$

Using the scaling laws discussed above, we find for the LHS of Eq. (66):

$$\langle J_A J_B J_C\rangle \sim m^{(\Delta_A+\Delta_B+\Delta_C)/y_m}, \qquad (67)$$

while from the RHS we obtain

$$\langle J_A J_B J_C\rangle$$
$$\sim m^{(\Delta_A-1)/y_m}m^{(\Delta_B-1)/y_m}m^{(\Delta_C-1)/y_m}m^{-3/y_m}m^{5/y_m}G_{ABC}, \qquad (68)$$

and therefore

$$G_{ABC} \sim m^{1/y_m}. \qquad (69)$$

The scaling of $G_{ABC}$ determines the scaling of the decay width for this specific channel:

$$\Gamma(A \rightarrow B + C) \sim \frac{|G_{ABC}|^2}{M_A} \sim m^{1/y_m}, \qquad (70)$$

which corresponds to the same scaling we have argued for in section II Eq. (15).

#### 2. Heavy quarks or mass-deformed conformal?

It has been pointed out [42] that a confining theory with chiral symmetry breaking, large mass term and small volume could mimic a conformal theory with mass perturbation. Thus the question: how to distinguish a conformal theory with small mass perturbation from a heavy quark regime? We would like to emphasize that the scaling laws of the pseudoscalar decay constant are a major help in this respect. The decay constant of pseudoscalar meson of two heavy quarks, which we denote by $\bar{b}b$ in analogy with QCD, are expected to have the following scaling behavior:

$$F_{\bar{b}b} \sim m^{-1/2}, \qquad (\Rightarrow G_{\bar{b}b} \sim m^{1/2}). \qquad (71)$$

This follows from the two heavy-quark state behaving like a quantum mechanical bound state, and therefore being treated using a quark model; Eq. (71) expresses a result found a long time ago by van Royen and Weisskopf [61]. This is clearly different from the behavior of the decay constant of a pseudoscalar state, or any other state, in a mCGT, which vanishes in the limit $m \to 0$, cf. Table I.

## V. SUMMARY AND CONCLUSIONS

In this paper we have accumulated a number of analytical results for mass scaling exponents $\eta$ of the type (2) for lowest state observables from Callan-Symmanzik equations, which should help to identify the conformal window of four dimensional nonsupersymmetric gauge theories. Possibly the clearest evidence of the conformal window so far comes from the vanishing of the hadron masses and decay constants in the chiral limit. We have identified the scaling of $F_{\bar{b}b} \sim m^{-1/2}$ as a criterion to distinguish mCGT from a heavy quark regime at small volume which has been pointed out as a potential pitfall in identifying the conformal window [42].

In section III A mass scaling coefficients for condensates are determined from Callan-Symmanzik equations of which we mention $\eta_{\bar{q}q} = (3 - \gamma_*)/(1 + \gamma_*)$ and $\eta_{G^2} = 4/(1 + \gamma_*)$. In Sec. III B we provide a more physical, but somewhat more heuristic, derivations of those results and discuss the nature of IR- and UV-regularizations within this framework. By generalizing the Banks-Casher relation from QCD it is shown, in section III A, that the exponent of the eigenvalue density of the Dirac operator is $\eta_{\bar{q}q}$, providing an alternative method for extracting $\gamma_*$. As our discussion in Sec. III D shows, it is too early to draw conclusions on the extraction of $\gamma_*$ by this method. In Sec. IV we derived scaling laws for all lowest state decay constants, summarized in Table I, other than those affected by the chiral anomaly and of tensorial structure. Fitting to current data for $F_{P_a}$ and $G_{P_a}$ we find results for $\gamma_*$ compatible with earlier derivations of $\gamma_* \simeq 0.5$. Summarizing, the derivations indicate that lowest state observables $\mathcal{O}$ scale as

$$\eta_{\mathcal{O}} = \frac{\Delta_{\mathcal{O}}}{1 + \gamma_*}, \tag{72}$$

where the one-particle state, $|H(p)\rangle$, scaling dimension turns out to be $\eta_{|H(p)\rangle} = -1/(1 + \gamma_*)$. Whether or not this is true for higher states as well, remains unclear and is more of academic interest as higher states are difficult to assess on the lattice. The real distinction of higher states is presumably the decay width and the associated continuum thresholds. It is therefore tempting to think that in the large $N_c$-limit, where the width is supposed to vanish, all decay constants and masses in a specific channel scale in the same way as the lowest one.

To this end we would like to add a few comments on tentative conclusion on the scaling of the $S$-parameter. The electroweak $S$-parameter is proportional to the $V$-$A$ correlator evaluated at zero momentum, $\Pi_{V-A}(0) \sim \int \rho_{V-A}(s)/s\,ds$, with the pion pole subtracted. A lattice computation of the $S$-parameter for walking technicolour (WTC) theories would be phenomenologically important. Thus it is crucial to distinguish, in a parametric way, the regimes of WTC and mCGT.[9] Assuming that the correlator is saturated by the lowest resonances[10] $\Pi_{V-A}(0) \sim F_V^2/M_V^2 - F_A^2/M_A^2 - F_\pi^2/M_\pi^2$, it is just the pion pole that serves as an indicator since $(M_\pi^2)_{WTC} \sim \mathcal{O}(m)$ leads to $\Pi_{V-A}^{WTC}(0) \sim \mathcal{O}(m^{-1})$ and the results in Table I imply $\Pi_{V-A}^{mCGT}(0) \sim \mathcal{O}(1)$. To this end let us note that, since the pion mass is the lowest one in the spectrum [51], a determination of $F_V^2/F_\pi^2 \leq 1$ would imply $\Pi_{V-A}^{mCGT}(0) < 0$.

## ACKNOWLEDGMENTS

## APPENDIX A: OPERATOR PRODUCT EXPANSION IN THE DEEP EUCLIDIAN

In Ref. [62] an OPE relation is obtained which we shall derive here in a slightly modified form. Taking the Fourier transform of Eq. (B1) one arrives at the expression

$$2mq^\mu \int e^{iq\cdot x} \langle TA_\mu^a(x)P^b(0)\rangle_0 = (2m)^2 \Pi_{P_aP_b}(q^2) + 2m\langle\bar{q}q\rangle. \tag{A1}$$

Inserting a complete set of states on the LHS and using Eq. (52) yields

$$-\sum_P \frac{F_{PS}^2 M_{PS}^2}{M_{PS}^2 + Q^2} + d_1 + d_2 Q^2$$

$$= \frac{(2m)^2}{Q^2}\Pi_{P_aP_b}(-Q^2) + \frac{2m\langle\bar{q}q\rangle}{Q^2}, \tag{A2}$$

where $Q^2 \equiv -q^2$, and $d_1$, $d_2$ are subtraction constants, which can also depend on $m$. Neither of these are of relevance to us since we may simply differentiate this expression twice with respect to $Q^2$. Following Ref. [62], an expansion in one inverse power of $Q^2$ yields Eq. (58), by assuming that $\Pi_{P_aP_b}(-Q^2)$ does not vanish as $Q^2 \to \infty$ and $m \to 0$. By expanding in one more inverse power in $Q^2$

---

[9]Of course this does not solve the problem of distinguishing the WTC from TC regime *per se*.

[10]This is reasonably satisfied in QCD. Adding one more triplets of states $P$, $V$, $A$ would not really alter the conclusions above.

one observes that the scaling of $F_{PS}^2 M_{PS}^4$ has to be larger than $m^2$,

$$2\eta_{F_{PS}} + 4\eta_{M_{PS}} \geq 2. \tag{A3}$$

In the domain where $1 \leq \gamma_*$, Eq. (A3) leads to $\gamma_* \leq 2$, which corresponds exactly to the unitarity bound $\Delta_{\bar{q}q} \equiv 3 - \gamma \geq 1$ for a scalar field [17].

## APPENDIX B: WARD IDENTITIES AND A LOW ENERGY THEOREM

In this appendix sketch the derivation of two standard WI and one low energy theorem.

### 1. Ward identity for pseudoscalar non–flavor-singlet

Starting with the identity,

$$\partial^\mu \langle 0|TA_\mu^a(x)P^b(0)|0\rangle = \langle 0|T\partial \cdot A^a(x)P^b(0)|0\rangle$$
$$+ \delta(x_0)\langle 0|[A_0^a(x), P^b(0)]|0\rangle, \tag{B1}$$

and integrating the equation over $d^4x$ one arrives at the pseudoscalar Ward identity (WI)[11]:

$$(2m)^2 \Pi_{P_a P_b}(0) = -2m\delta_{ab}\langle \bar{q}q\rangle, \tag{B2}$$

where

$$\Pi_{P_a P_b}(q) = i \int_x e^{iq\cdot x}\langle 0|TP_a(x)P_b(0)|0\rangle. \tag{B3}$$

Throughout the paper $\Pi_{AB}(q)$ denotes the time-ordered two-point correlator of operators $A$ and $B$. The RHS of Eq. (B2) is obtained by evaluating the commutator

$$\langle 0|[Q_5^a|_{x_0=0}, \partial \cdot A^b(0)]|0\rangle = 2i\delta^{ab}m\langle \bar{q}q\rangle. \tag{B4}$$

The charge is defined as usual: $Q_5^a|_{x_0=0} = \int d^3x A^b(0, \vec{x})$.

### 2. (Anomalous) Ward identity for pseudoscalar flavor-singlet

The flavor-singlet sector can be analyzed by reviewing the Ward identities [63] used in discussing the $\eta'$-mass/U(1)$_A$-problem in QCD. Let us define the following flavor-singlet quantities:

$$P(x) = \sum_{j=1}^{n_f} \bar{q}_j i\gamma_5 q_j(x), \qquad A_\mu(x) = \sum_{j=1}^{n_f} \bar{q}_j \gamma_\mu \gamma_5 q_j(x),$$

$$\tilde{Q}(x) = \frac{g^2}{16\pi^2} n_f \tilde{G}_{\alpha\beta} G^{\alpha\beta}(x), \tag{B5}$$

where $\tilde{G}_{\alpha\beta} = 1/2\epsilon_{\alpha\beta\gamma\delta}G^{\gamma\delta}$. The anomaly equation is given by

$$\partial \cdot A = 2mP + \tilde{Q} \tag{B6}$$

---

[11]We are using the fact that there are no Goldstone bosons since $SU(n_f)_A$ is explicitly broken.

The integrated anomalous Ward identity is readily obtained from Eq. (B1) and reads

$$(2m)^2 \Pi_{PP}(0) + (2m)\Pi_{\tilde{Q}P}(0) = -4m\langle \bar{q}q\rangle. \tag{B7}$$

By observing that

$$0 = i \int_x \partial_\mu \langle 0|T\tilde{Q}(0)A^\mu(-x)|0\rangle$$
$$= (2m)\Pi_{\tilde{Q}P}(0) + \Pi_{\tilde{Q}\tilde{Q}}(0), \tag{B8}$$

the WI (B7) can be written in a more symmetric form:

$$(2m)^2 \Pi_{PP}(0) - \Pi_{\tilde{Q}\tilde{Q}}(0) = -4m\langle \bar{q}q\rangle. \tag{B9}$$

### 3. Low energy theorem for scalar flavor-singlet

A simple and useful relation follows from the fact that the operator $S(x) = \bar{q}q(x)$ appears in the Lagrangian. Equation (1) implies

$$\Pi_{SS}(0) = -\frac{\partial}{\partial m}\langle \bar{q}q\rangle, \tag{B10}$$

where $\Pi_{SS}(q^2)$ is the time-ordered two-point function (B3).

## APPENDIX C: THE DECAY WIDTH

In this appendix we investigate whether the large $tM_H$ behavior of a correlation function can be influenced by the width. According to [64] the Euclidian time behavior of a two-point function in the rest frame of the decaying particle can be written as a spectral integral of the type

$$C_H(t; g, m, \mu) = \frac{1}{\pi} \int dE e^{-Et}\rho(E), \tag{C1}$$

where

$$\rho(E) = \frac{\text{Im}(\Sigma(E))}{|m^2 - E^2 - \Sigma(E)^2|} \tag{C2}$$

and $\Sigma(E)$ is the self-energy and $m$ the bare mass. We shall work in the approximation where $\Sigma(E)$ does not vary appreciably around the peak $E = |M|$ and we neglect the far away singularity $E = -|M|$. The symbol $M$ denotes the renormalized mass: $M = m^2 - \text{Re}(\Sigma(M))$. In this case $\rho(E)$ assumes the form:

$$\rho(E) = \frac{\gamma(E)}{(M - E)^2 + \gamma(E)^2}, \tag{C3}$$

where $\gamma = \Gamma/2 = \text{Im}(\Sigma(E))/2M$. Then the two-point function takes the following form [64]:

$$C_H(t; g, m, \mu) = \frac{e^{-Mt}}{2M}\text{Ei}(\gamma t, (M - \lambda)t), \tag{C4}$$

where $\lambda$ is the onset of the cut, omitted in Eq. (C1), and

$$\text{Ei}(\alpha, \beta) = \int_{-\beta}^\infty \frac{\alpha e^{-x}}{x^2 + \alpha^2}dx > 0. \tag{C5}$$

If there are cancellations between the $\sim e^{-Mt}$ behavior of the mass and the width then the following must be true:

$$\text{Ei}\,(\alpha, \beta)^{\sim^{t \gg M}} \alpha^n e^{|b|\alpha}, \qquad (C6)$$

where $n$ a real number and $b$ is a number that would have to be fine-tuned. It can be shown that this cannot be the case. Consider

$$\alpha \text{Ei}(\alpha, \beta) = \int_{-\beta}^{\infty} \frac{\alpha^2 e^{-x}}{x^2 + \alpha^2} dx \leq \int_{-\beta}^{\infty} e^{-x} dx < \infty, \quad (C7)$$

but then from (C6):

$$\infty > \alpha \text{Ei}(\alpha, \beta)^{t \gg M}_{\sim} \alpha^{n+1} e^{|b|\alpha} \overset{\alpha \to \infty}{\longrightarrow} \text{divergent} \qquad (C8)$$

one gets an immediate contradiction to the hypothesis (C6). Thus we have shown that, in the approximations mentioned above, that a the large Euclidian time behavior of the mass and the width do not conspire to cancel each other.

[1] Kenneth A. Intriligator and N. Seiberg, Nucl. Phys. B, Proc. Suppl. **45BC**, 1 (1996).

[2] Bob Holdom, Phys. Lett. **143B**, 227 (1984).

[3] Koichi Yamawaki, Masako Bando, and Ken-iti Matumoto, Phys. Rev. Lett. **56**, 1335 (1986).

[4] T. Akiba and T. Yanagida, Phys. Lett. **169B**, 432 (1986).

[5] Francesco Sannino, arXiv:0911.0931.

[6] A. Markus Luty and Takemichi Okui, J. High Energy Phys. 09 (2006) 070.

[7] Steven Weinberg, Phys. Rev. D **13**, 974 (1976).

[8] Leonard Susskind, Phys. Rev. D **20**, 2619 (1979).

[9] Estia Eichten and Kenneth D. Lane, Phys. Lett. **90B**, 125 (1980).

[10] Christopher T. Hill and H. Elizabeth, Phys. Rep. **381**, 235 (2003).

[11] Dennis D. Dietrich and Francesco Sannino, Phys. Rev. D **75**, 085018 (2007).

[12] J. L. Cardy, *Cambridge Lecture Notes in Physics: 3* (Cambridge University Press, Cambridge, England, 1996), p. 238.

[13] Thomas DeGrand, arXiv:0910.3072.

[14] Tom Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982).

[15] Andrew G. Cohen and Howard Georgi, Nucl. Phys. **B314**, 7 (1989).

[16] Thomas Appelquist and Francesco Sannino, Phys. Rev. D **59**, 067702 (1999).

[17] G. Mack, Commun. Math. Phys. **55**, 1 (1977).

[18] Jens Braun and Holger Gies, J. High Energy Phys. 06 (2006) 024.

[19] Jens Braun and Holger Gies, arXiv:0912.4168.

[20] Holger Gies and Joerg Jaeckel, Eur. Phys. J. C **46**, 433 (2006).

[21] Carlos Nunez, Ioannis Papadimitriou, and Maurizio Piai, arXiv:0812.3655.

[22] Daniel Elander, Carlos Nunez, and Maurizio Piai, Phys. Lett. B **686**, 64 (2010).

[23] Maurizio Piai, arXiv:1004.0176.

[24] M. Hasenbusch and K. Jansen, Nucl. Phys. **B659**, 299 (2003).

[25] Martin Luscher, Comput. Phys. Commun. **165**, 199 (2005).

[26] M. A. Clark and A. D. Kennedy, Phys. Rev. Lett. **98**, 051601 (2007).

[27] Simon Catterall and Francesco Sannino, Phys. Rev. D **76**, 034504 (2007).

[28] Thomas Appelquist, George T. Fleming, and Ethan T. Neil, Phys. Rev. Lett. **100**, 171607 (2008).

[29] Luigi Del Debbio, Mads T. Frandsen, Haralambos Panagopoulos, and Francesco Sannino, arXiv:0802.0891.

[30] Yigal Shamir, Benjamin Svetitsky, and Thomas DeGrand, Phys. Rev. D **78**, 031502 (2008).

[31] Albert Deuzeman, Maria Paola Lombardo, and Elisabetta Pallante, Proc. Sci. LATTICE2008 (2008) 060 [arXiv:0810.1719].

[32] Luigi Del Debbio, Agostino Patella, and Claudio Pica, arXiv:0805.2058.

[33] Simon Catterall, Joel Giedt, Francesco Sannino, and Joe Schneible, arXiv:0807.0792.

[34] Thomas DeGrand, Yigal Shamir, and Benjamin Svetitsky, Phys. Rev. D **79**, 034501 (2009).

[35] Ari J. Hietanen, Jarno Rantaharju, Kari Rummukainen, and Kimmo Tuominen, J. High Energy Phys. 05 (2009) 025.

[36] Thomas Appelquist, George T. Fleming, and Ethan T. Neil, Phys. Rev. D **79**, 076010 (2009).

[37] Ari J. Hietanen, Kari Rummukainen, and Kimmo Tuominen, Phys. Rev. D **80**, 094504 (2009).

[38] A. Deuzeman, M. P. Lombardo, and E. Pallante, arXiv:0904.4662.

[39] Zoltan Fodor, Kieran Holland, Julius Kuti, Daniel Nogradi, and Chris Schroeder, J. High Energy Phys. 08 (2009) 084.

[40] Thomas DeGrand and Anna Hasenfratz, Phys. Rev. D **80**, 034506 (2009).

[41] Anna Hasenfratz, Phys. Rev. D **80**, 034505 (2009).

[42] L. Del Debbio, B. Lucini, A. Patella, C. Pica, and A. Rago, Phys. Rev. D **80**, 074507 (2009).

[43] Zoltan Fodor, Kieran Holland, Julius Kuti, Daniel Nogradi, and Chris Schroeder, Phys. Lett. B **681**, 353 (2009).

[44] C. Pica, L. Del Debbio, B. Lucini, A. Patella, and A. Rago, arXiv:0909.3178.

[45] Zoltan Fodor, Kieran Holland, Julius Kuti, Daniel Nogradi, and Chris Schroeder, J. High Energy Phys. 11 (2009) 103.

[46] Francis Bursa, Luigi Del Debbio, Liam Keegan, Claudio

Pica, and Thomas Pickup, Phys. Rev. D **81**, 014505 (2010).

[47] Luigi Del Debbio, Biagio Lucini, Agostino Patella, Claudio Pica, and Antonio Rago, arXiv:1004.3197.

[48] Luigi Del Debbio, Biagio Lucini, Agostino Patella, Claudio Pica, and Antonio Rago, arXiv:1004.3206.

[49] V. A. Miransky, Phys. Rev. D **59**, 105003 (1999).

[50] Markus A. Luty, J. High Energy Phys. 04 (2009) 050.

[51] Don Weingarten, Phys. Rev. Lett. **51**, 1830 (1983).

[52] H. Leutwyler and A. Smilga, Phys. Rev. D **46**, 5607 (1992).

[53] Peter Hasenfratz, Nucl. Phys. **B525**, 401 (1998).

[54] Tom Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).

[55] Zoltan Fodor, Kieran Holland, Julius Kuti, Daniel Nogradi, and Chris Schroeder, Proc. Sci., LATTICE2008 (2008) 058.

[56] Francesco Sannino and Roman Zwicky, Phys. Rev. D **79**, 015016 (2009).

[57] Antonio Delgado, Jose R. Espinosa, and Mariano Quiros, J. High Energy Phys. 10 (2007) 094.

[58] M. A. Stephanov, Phys. Rev. D **76**, 035008 (2007).

[59] Leonardo Giusti and Martin Luscher, J. High Energy Phys. 03 (2009) 013.

[60] Tamas G. Kovacs, Phys. Rev. Lett. **104**, 031601 (2010).

[61] R. Van Royen and V. F. Weisskopf, Nuovo Cimento A **50**, 617 (1967).

[62] Mikhail A. Shifman, A. I. Vainshtein, and Valentin I. Zakharov, Nucl. Phys. **B147**, 448 (1979).

[63] R. J. Crewther, Phys. Lett. **70B**, 349 (1977).

[64] Christopher Michael, Nucl. Phys. **B327**, 515 (1989).