PHYSICAL REVIEW D **82,** 013008 (2010)

Is the tribimaximal mixing accidental?

Mohammed Abbas[1,2,3,*] and A. Yu. Smirnov[3,4,†]

[1]*Ain Shams University, Faculty of Sciences, Abbassiyah 11566, Cairo, Egypt*

[2]*Center for Theoretical Physics (CTP), The British University in Egypt, BUE, El-Sherouk City, Cairo, Egypt*

[3]*The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, I-34014 Trieste, Italy*

[4]*Institute for Nuclear Research, Russian Academy of Sciences, Moscow, Russia*

(Received 7 May 2010; published 16 July 2010)

The tribimaximal (TBM) mixing is not accidental if structures of the corresponding leptonic mass matrices follow immediately from certain (residual or broken) flavor symmetry. We develop a simple formalism which allows one to analyze effects of deviations of the lepton mixing from TBM on the structure of the neutrino mass matrix and on the underlying flavor symmetry. We show that possible deviations from the TBM mixing can lead to strong modifications of the mass matrix and strong violation of the TBM-mass relations. As a result, the mass matrix may have an ''anarchical'' structure with random values of elements or it may have some symmetry that differs from the TBM symmetry. Interesting examples include matrices with texture zeros, matrices with certain ''flavor alignment'' as well as hierarchical matrices with a two-component structure, where the dominant and subdominant contributions have different symmetries. This opens up new approaches to understanding the lepton mixing.

DOI: 10.1103/PhysRevD.82.013008 PACS numbers: 14.60.Pq

I. INTRODUCTION

The lepton mixing determined from the results of neutrino experiments can be well described by the so-called tribimaximal mixing (TBM) matrix [1][1]:

$$U_{\rm TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (1)$$

In terms of the standard parameterization of the lepton mixing matrix,

$$U_{PMNS} = U_{23}(\theta_{23})\Gamma_\delta U_{13}(\theta_{13})\Gamma^*_\delta U_{12}(\theta_{12}),$$

where $\Gamma_\delta \equiv {\rm diag}(1, 1, e^{i\delta})$, the TBM matrix corresponds to the maximal 2–3 mixing, zero 1–3 mixing, and ''democratic'' 1–2 mixing:

$$\sin^2\theta_{23} = \tfrac{1}{2}, \qquad \sin\theta_{13} = 0, \qquad \sin^2\theta_{12} = \tfrac{1}{3}. \quad (2)$$

The Dirac CP phase is irrelevant.[2]

The result (1) and (2) is very suggestive of certain underlying symmetry, and this has triggered enormous activity in the model building [2]. It is assumed that TBM is a consequence of some symmetry of the neutrino mass matrix in certain (often flavor) basis. We will refer to this as to the TBM symmetry.

For the Majorana neutrinos in the flavor basis $(\nu_e, \nu_\mu, \nu_\tau)$, the mass matrix that leads to the TBM mixing equals

$$m_{\rm TBM} = U_{\rm TBM} m_\nu^{\rm diag} U_{\rm TBM}^T, \quad (3)$$

where $m_\nu^{\rm diag} \equiv {\rm diag}(m_1, m_2, m_3)$ is the matrix of neutrino mass eigenstates. In general, m_i are complex, and we can represent them as

$$m_1 = |m_1|, \qquad m_2 = |m_2|e^{i2\phi_2}, \qquad m_3 = |m_3|e^{i2\phi_3}.$$

Here ϕ_1 and ϕ_2 are the Majorana CP-violating phases. Using (3) and (1), we find explicitly

$$m_{TBM} = \begin{pmatrix} a & b & b \\ \dots & \frac{1}{2}(a+b+c) & \frac{1}{2}(a+b-c) \\ \dots & \dots & \frac{1}{2}(a+b+c) \end{pmatrix}, \quad (4)$$

where the parameters a, b, c are determined by the neutrino masses as

$$a = \tfrac{1}{3}(2m_1 + m_2), \qquad b = \tfrac{1}{3}(-m_1 + m_2), \qquad c = m_3. \quad (5)$$

Elements of the $\mu\tau$ block of the mass matrix (4) equal

$$a + b + c = \tfrac{1}{3}m_1 + \tfrac{2}{3}m_2 + m_3,$$
$$a + b - c = \tfrac{1}{3}m_1 + \tfrac{2}{3}m_2 - m_3.$$

According to (4), the elements of matrix, $||m_{\alpha\beta}||$, $\alpha, \beta = e, \mu, \tau$, which leads to the TBM mixing, satisfy the following three conditions:

$$m_{e\mu} = m_{e\tau}, \quad (6)$$

$$m_{\mu\mu} = m_{\tau\tau}, \quad (7)$$

$$m_{ee} + m_{e\mu} = m_{\mu\mu} + m_{\mu\tau}. \quad (8)$$

*mabbas@ictp.it
†smirnov@ictp.it

[1]There is an ambiguity in the form of the mixing matrix related to the sign of rotation.

[2]In (2) $U_{ij} \equiv U_{ij}(\theta_{ij})$ is the rotation in ij subspace on the angle θ_{ij}.

(The latter is equivalent to $\sum_\alpha m_{e\alpha} = \sum_\beta m_{\mu\beta}$.) Inversely, the mass matrix, which satisfies these relations, leads to the TBM mixing independently of values of neutrino masses. The form of relation (8) changes under the field rephasing: $\nu_e \to -\nu_e$, etc. Recall that in the case of the bimaximal mixing instead of the condition (8) we would have $m_{ee} = m_{\mu\mu} + m_{\mu\tau}$.

In general, fixing any specific set of values of three mixing angles would imply three relations between the elements of the mass matrix. The point is that in the TBM case, these relations are very simple: they are just equalities of certain elements and equality of sums of elements of columns, and therefore have a good chance to follow from certain symmetry.

The TBM symmetry can appear as a residual of the flavor symmetry of the Lagrangian. (In all the models the underlying flavor symmetry for TBM is broken.) Indeed, the TBM-mass matrix (4) is invariant under transformations [3,4]

$$V_i m_{\rm TBM} V_i^T = m_{\rm TBM},$$

where

$$V_1 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ \dots & -1 & 2 \\ \dots & \dots & -1 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 1 & 0 & 0 \\ \dots & 0 & 1 \\ \dots & \dots & 0 \end{pmatrix}. \tag{9}$$

At the same time, the mass matrix of charged leptons can be diagonal due to symmetry with respect to transformation $V_3 = \mathrm{diag}(1, \omega, \omega^2)$, where $\omega \equiv e^{i2\pi/3}$. The transformations V_1, v_2, v_3 are generators of the group S_4.

Some recent developments have given rise to doubts that the TBM is of fundamental character, i.e. follows from certain approximate (broken) symmetry. The TBM mixing can be accidental—just a numerical coincidence of parameters without underlying symmetry. The arguments follow:

(1) Analysis of experimental data shows deviations from the TBM mixing. According to two recent global analyses [5,6], the best fit values as well as the 1σ allowed ranges for the mixing angles deviate from the TBM values (see Table I). Notice, however, that the latest analysis of the atmospheric neutrino data only [7] gives the best fit values (and the 90% C.L. allowed regions) as $\sin\theta_{13} = 0.00$ (< 0.2) in the case of normal mass hierarchy (NH) and $\sin\theta_{13} = 0.077$ (< 0.3) for the inverted mass hierarchy (IH). So, no significant deviation of the 1–3 mixing from zero is found, but the upper bound is in agreement with the global fit results. For the 2–3 mixing, essentially no deviation from the maximal value is obtained: $\sin^2\theta_{23} = 0.50$ (NH) and $\sin^2\theta_{23} = 0.53$ (IH). At the same time, larger deviations from the maximal mixing are allowed in comparison to the global fit: $0.407 < \sin\theta_{23} < 0.583$ (90%) C.L. Comparing the results of Table I with those in (2), we find that significant deviations from the TBM values are allowed.

(2) No simple and convincing model for the TBM mixing has been proposed so far, although the simplest possibilities have been explored almost systematically. The proposed models have rather complicated structure with a large number of assumptions, new elements (fields), new parameters, *ad hoc* quantum number assignments, and yet additional auxiliary symmetries. Attempts to realize the proposal "TBM from symmetry" can be qualified as the "symmetry building" by introduction and tuning of the complicated structure of models. The mixing does not appear as an *immediate* consequence of symmetry. On the other hand, if true, this means that there is rich physics behind observed lepton mixing. One should add, however, that from the simple assumption of the existence of discrete symmetry, which has irreducible triplet representation, one gets structures that resemble the TBM mixing but often with the wrong mass spectrum.

(3) In most proposed models there is no immediate relation between the masses and mixing angles and different physics should be introduced to explain the mass hierarchies. This is still a matter of opinion, and some authors do not consider the lack of the relations as a shortcoming in spite of the existence of the Fritzsch or Gatto-Sartory-Tonin type relations in the quark sector.

(4) The quark sector has small mixing and in the first approximation it can be neglected so that the quark mixing matrix is diagonal, as a consequence of certain symmetry. This drastically differs from the lepton mixing and therefore further complications are required to include the quark sector into a model. The grand unification puts further additional requirements [8]. Of course, it is difficult to expect that quark and lepton mixings are similar: values of neutrino masses strongly differ from values of quark

TABLE I. The best fit values and 1σ intervals for the mixing angles according to global oscillation analysis of different groups. The analysis GM-I uses the solar neutrino spectrum according to the solar model with high metallicity (GS98) and a normal gallium cross section, whereas GM-II is based on the high surface metallicity (AGSS09) and modified gallium cross section; see [6] for details.

	Bari group [5]	GM-I [6]	GM-II [6]
$\sin\theta_{13}$	$0.126^{+0.053}_{-0.049}$	$0.127^{+0.036}_{-0.055}$	$0.118^{+0.038}_{-0.048}$
$\sin^2\theta_{23}$	$0.466^{+0.073}_{-0.058}$	$0.463^{+0.071}_{-0.048}$	$0.463^{+0.071}_{-0.048}$
$\sin^2\theta_{12}$	$0.312^{+0.019}_{-0.018}$	$0.319^{+0.016}_{-0.016}$	$0.321^{+0.016}_{-0.016}$

masses. And furthermore, the neutrino mass may have a different nature being of the Majorana type.

(5) The quark-lepton complementarity [9] with different underlying physics leads to mixing that is very close to the TBM mixing.

There are several possible implications of these statements:

(i) The TBM mixing is not accidental in spite of arguments (1–5), and there is a certain flavor symmetry behind this mixing. This symmetry cannot be an exact symmetry of the Lagrangian (in the proposed models it is broken spontaneously or explicitly), and therefore deviations from the TBM mixing at some level are expected anyway. The deviations can originate from (i) renormalization group effects [10], (ii) deviations from "correct" vacuum expectation value (VEV) alignment [11] [12], (iii) a soft breaking of the $\mu - \tau$ and CP symmetries [13], (iv) higher order corrections of a flavor symmetry breaking and higher dimensional mass operators [14], (v) perturbation of the TBM-mass matrix and contribution from the charged lepton sector [15], (vi) breaking of the mass degeneracy of three heavy (right-handed) Majorana neutrinos [16], etc.

(ii) The approximate TBM mixing is not accidental but is a manifestation of some other structure or other symmetry which differs from the flavor symmetries proposed so far as an explanation of TBM. Viable alternatives are the quark-lepton complementarity [9] and weak complementarity [17], when the bimaximal mixing is obtained as a result of flavor symmetry.

(iii) The approximate TBM mixing is accidental: it results from an interplay of different, and to a large extent, independent factors or/and contributions. Some other physics apart from the flavor symmetry is involved. The mixing results are from a many-step construction and fixing various parameters by introduction of additional auxiliary symmetries and structures.

The main question we address in the paper is how to disentangle these possible implications. Clearly, the conclusive way to answer the question is to check predictions of specific models that explain the TBM mixing. Unfortunately, most of the proposed models do not give new generic or strict predictions. Therefore, interpretation of results will be rather ambiguous. Furthermore, in many cases the underlying physics is at very high mass scales (grand unified theory or even higher), so that its direct tests are not possible.

The symmetry, if it exists, is realized in terms of the mass matrix and not the mixing matrix. Therefore, the step is to explore violation of the TBM symmetry of the mass matrix. If the deviations of the mass matrix from $m_{\rm TBM}$ are large (enhanced), and the symmetry is broken strongly, the symmetry explanation of the TBM is less plausible. If in the large region of parameters (which would correspond to a large variety of different structures of matrix) the mass matrix leads to the approximate TBM mixing, the TBM looks accidental.

A somewhat similar question ("is TBM hidden or accidental symmetry?") has been discussed in [18]. In a sense, the inverse problem has been considered: small ("soft") $\sim 20\%$ relative corrections (perturbations) to the TBM-mass matrix elements have been introduced, and the consequences of these perturbations for mixing angles have been studied, depending on the mass hierarchy and phases. Our approach, criteria of accidental, and conclusions differ from those obtained in [18] (see Sec. IV).

This paper is organized as follows: In Sec. II, we present a simple formalism which accounts for the effects of deviations from the TBM on the structure of the neutrino mass matrix. Using this formalism, in Sec. III, we study the properties of the neutrino mass matrices (in the presence of the deviations) for different mass spectra and values of the CP phases. In Sec. IV, we consider the implications of the obtained results for the flavor symmetries. We search for some alternative structures of the mass matrix, and correspondingly, alternative explanation of the observed mixing. Conclusions are given in Sec. V.

II. DEVIATIONS OF THE MASS MATRIX FROM THE TBM FORM

A. Deviations from the TBM mixing

Let us define the parameters that characterize the deviation of the mixing angles from the TBM values as

$$D_{12} \equiv \tfrac{1}{3} - s_{12}^2, \qquad D_{23} \equiv \tfrac{1}{2} - s_{23}^2, \qquad D_{13} \equiv s_{13}, \quad (10)$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. Using the results in Table I, we find the central values and the 1σ allowed intervals of these deviations (see Table II). For the 1–2 and 1–3 mixings, the relative deviations equal, correspondingly, $3D_{12}$ and $2D_{23}$. The central values of these deviations and the maximal allowed values at 1σ level are (3–6)% and (6–12)% for the 1–2 mixing, and (8–10)% and (18–19)% for the 2–3 mixing. Thus, the typical size of the relative deviations is about 10% for the 1–2 mixing and 20% for the 2–3 mixing. The 1–3 mixing can be compared with values of other mixings: for the central value $s_{13}/s_{12} = 0.23$ and in the 1σ interval, $s_{13}/s_{12} = 0.33$. The 1–3 mixing can be smaller but not much smaller than other mixings.

Instead of D_{12} and D_{23}, we could introduce deviations for sines:

$$d_{12} \equiv \frac{1}{\sqrt{3}} - s_{12}, \qquad d_{23} \equiv \frac{1}{\sqrt{2}} - s_{23}. \quad (11)$$

In the lowest order there are linear relations between d_{ij} and D_{ij}: $d_{23} = D_{23}/\sqrt{2}$, $d_{12} = D_{12}\sqrt{3}/2$ in contrast to s_{13},

TABLE II. Central values and 1σ allowed intervals for the TBM deviation parameters according to the global analysis of different groups (for further explanation, see the caption for Table I).

Deviation	Bari group [5]	GM-I [6]	GM-II [6]
$\sin\theta_{13}$	$0.126(0.077 \div 0.179)$	$0.127(0.071 \div 0.163)$	$0.118(0.069 \div 0.156)$
D_{23}	$0.034(-0.039 \div 0.092)$	$0.037(-0.034 \div 0.085)$	$0.037(-0.034 \div 0.085)$
D_{12}	$0.021(0.002 \div 0.040)$	$0.014(-0.0016 \div 0.027)$	$0.012(-0.0036 \div 0.028)$

which gives the deviation from zero. Furthermore, in contrast to s_{13}, the linear deviations D_{12}, D_{23} are smaller than the quadratic ones. For the linear deviations, we have $s_{13} \gg d_{12} \sim d_{23}$, and for the present best fit values,

$$s_{13}^2 \sim d_{12} \sim d_{23}. \tag{12}$$

It can be a hierarchy of the deviations.

B. Corrections to the neutrino mass matrix

To account for the effects of deviation from the TBM mixing on the structure of the mass matrix, we will perform an expansion of the matrix in powers of the deviation parameters D_{ij}. In the lowest approximation, the correction due to D_{ij} equals

$$U_{\rm TBM} m^{\rm diag} \delta U_{ij}^{(1)T} + \text{ transponent}, \tag{13}$$

where $\delta U_{ij}^{(1)}$ is the first order correction to $U_{\rm TBM}$ due to the deviation D_{ij}. Equation (13) can be also rewritten in the form $m_{\rm TBM} U_{\rm TBM} \delta U_j^T$ + transponent. Because of hierarchy (12), we compute also corrections of the order s_{13}^2, which are given by

$$U_{\rm TBM} m^{\rm diag} \delta U_{13}^{(2)T} + U_{13}^{(2)} m^{\rm diag} U_{\rm TBM}^T + \delta U_{13}^{(1)} m^{\rm diag} \delta U_{13}^{(1)T}. \tag{14}$$

Here $U_{13}^{(2)}$ is the matrix of second order in s_{13}^2. Using (13) and (14), we find the mass matrix in the lowest order approximation as

$$m_\nu = m_{\rm TBM} + s_{13}\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}e^{-i\delta}g & \frac{1}{\sqrt{2}}e^{-i\delta}g \\ \ldots & \sqrt{2}be^{i\delta} & 0 \\ \ldots & \ldots & -\sqrt{2}be^{i\delta} \end{pmatrix} + \frac{s_{13}^2}{2}\begin{pmatrix} 2e^{-2i\delta}g & -b & -b \\ \ldots & -g & g \\ \ldots & \ldots & -g \end{pmatrix} + D_{23}\begin{pmatrix} 0 & b & -b \\ \ldots & a+b-c & 0 \\ \ldots & \ldots & -(a+b-c) \end{pmatrix} + 3bD_{12}\begin{pmatrix} -1 & -\frac{1}{4} & -\frac{1}{4} \\ \ldots & \frac{1}{2} & \frac{1}{2} \\ \ldots & \ldots & \frac{1}{2} \end{pmatrix}, \tag{15}$$

where

$$g \equiv c - ae^{2i\delta}, \tag{16}$$

a, b, c are combinations of the neutrino masses defined in Eq. (5). Notice that corrections are proportional to the elements of the TBM matrix and therefore correlate with the original TBM structure. It follows from the expression (15) immediately that

(1) s_{13} as well as D_{23} corrections break all three TBM conditions (6)–(8);

(2) s_{13}^2 and D_{12} corrections violate only the third condition.

Corrections due to the nonzero 1–3 mixing depend on b and the combination g of original parameters a and c.

The expression for the mass matrix (15) can be rewritten in terms of matrices that explicitly violate the TBM conditions:

$$m_\nu = m_{\rm TBM} + m'_{\rm TBM} + x\begin{pmatrix} 0 & 1 & -1 \\ \ldots & 0 & 0 \\ \ldots & \ldots & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 & 0 \\ \ldots & 1 & 0 \\ \ldots & \ldots & -1 \end{pmatrix} + z\begin{pmatrix} 1 & 0 & 0 \\ \ldots & 0 & 0 \\ \ldots & \ldots & 0 \end{pmatrix}. \tag{17}$$

Here $m_{\rm TBM}$ is the original TBM matrix (4) for a given mass spectrum. The matrix $m'_{\rm TBM}$ has an exact TBM form with the following parameters:

$$a' = \frac{b}{2}\left(\frac{15}{2}D_{12} + s_{13}^2\right), \qquad b' = -\frac{b}{2}\left(\frac{3}{2}D_{12} + s_{13}^2\right), \qquad c' = -gs_{13}^2.$$

Notice that, all the elements of $m'_{\rm TBM}$ are suppressed in comparison to the zero order matrix $m_{\rm TBM}$ by small deviations: $D_{12} \sim s_{13}^2 \leq 0.02$. In Eq. (17), x and y are the strengths of violation of the first and second TBM conditions, and z is the correction to m_{ee}:

$$x = -\frac{s_{13}}{\sqrt{2}}ge^{-i\delta} + bD_{23}, \tag{18}$$

$$y = \sqrt{2}s_{13}be^{i\delta} + (a+b-c)D_{23} = \sqrt{2}s_{13}be^{i\delta} + 2m_{\mu\tau}^{\rm TBM}D_{23}, \tag{19}$$

$$z = -\frac{27}{4}bD_{12} + \left[ge^{-2i\delta} - \frac{b}{2}\right]s_{13}^2. \tag{20}$$

The corrections to the TBM structure have the following properties. Contributions to x and y from s_{13} and D_{23} can sum up, thus enhancing the violation of the TBM structure. All corrections to the elements $m_{e\mu}$ and $m_{e\tau}$ but those of s_{13} are proportional to b; z depends on the smallest deviation D_{12} and is second order in s_{13}. In general, parameters x and y are independent. If $b \ll a, c$ which, as we will see, is realized in many situations, then $x \propto s_{13}$, whereas $y \propto D_{23}$. If $b \sim a, c$, one can obtain $x \gg y$ or $x \ll y$ by selecting a particular value of the phase δ. In some cases correlation between corrections x and y and the structure of the original TBM-mass matrix appear.

The total correction to the ee element is

$$\Delta m_{ee} = a' + z = -3bD_{12} + e^{-2i\delta} g s_{13}^2. \tag{21}$$

Although D_{12} is small, it enters Δm_{ee} with the coefficient 3. In other places its effect is small. The correction to the $\mu\tau$ element originates from $m'_{\rm TBM}$:

$$\Delta m_{\mu\tau} = \tfrac{3}{2} b D_{12} + \tfrac{1}{2} g s_{13}^2. \tag{22}$$

It is about 2 times smaller than Δm_{ee} and has an additional phase difference between the two terms; $\Delta m_{\mu\tau} = -\frac{1}{2}\Delta m_{ee}$ at $\delta = \pi/2$. Apart from some special cases, this correction is negligible.

The exact expression for the mass matrix is simplified substantially if $D_{12} = \delta = 0$:

$$m_\nu = \begin{pmatrix} a & c_{13} b\sqrt{1+2D_{23}} - \xi_- & c_{13} b\sqrt{1-2D_{23}} + \xi_+ \\ \dots & \frac{1}{2}(a+b+c) + y & \frac{1}{2}(a+b-c)\sqrt{1-4D_{23}^2} - 2\sqrt{2} b D_{23} s_{13} \\ \dots & \dots & \frac{1}{2}(a+b+c) - y \end{pmatrix} + s_{13}^2 (c-a) \times \begin{pmatrix} 1 & 0 & 0 \\ \dots & D_{23} - \frac{1}{2} & \frac{1}{2}\sqrt{1-4D_{23}^2} \\ \dots & \dots & -D_{23} - \frac{1}{2} \end{pmatrix}. \tag{23}$$

Here

$$\xi_\pm \equiv \frac{1}{\sqrt{2}} s_{13} c_{13} \sqrt{1 \pm 2D_{23}} (c-a), \qquad y \equiv \sqrt{2} b s_{13} \sqrt{1-4D_{23}^2} + (a+b-c) D_{23}. \tag{24}$$

The next order corrections, being proportional to $s_{13}D_{23}$, appear in the off-diagonal elements: $m_{\mu\tau}$, $m_{e\mu}$ and $m_{e\tau}$. From (24), we have

$$\xi_\pm = \frac{1}{\sqrt{2}} s_{13} c_{13} (c-a) \pm \frac{1}{\sqrt{2}} s_{13} c_{13} D_{23} (c-a),$$

where the second term gives the same corrections to $m_{e\mu}$ and $m_{e\tau}$. In the lowest order, we obtain

$$\xi_+ = \xi_- = \xi \equiv \frac{1}{\sqrt{2}} s_{13} c_{13} (c-a), \qquad y = D_{23}(a+b-c) + \sqrt{2} s_{13} b,$$

so that

$$m_\nu = \begin{pmatrix} a & b\sqrt{1+2D_{23}} - \xi & b\sqrt{1-2D_{23}} + \xi \\ \dots & \frac{1}{2}(a+b+c) + y & \frac{1}{2}(a+b-c) \\ \dots & \dots & \frac{1}{2}(a+b+c) - y \end{pmatrix}.$$

As follows from the formulas obtained above, modifications of the matrix depend on the structure of the original matrix. (The latter, in turn, depends strongly on the absolute mass scale, mass hierarchy, and CP phases. For the general dependence of the mass matrices on CP phases, see [19]). According to (15), corrections are proportional to the deviations multiplied by different original matrix elements:

$$\Delta m_{\alpha\beta} = \sum_{i>j} \sum_{\gamma\delta} f^{ij}_{\gamma\delta} D_{ij} m_{\gamma\delta},$$

where $(i, j = 1, 2, 3)$, $(\alpha, \beta, \gamma, \delta = e, \mu, \tau)$, and $f^{ij}_{\gamma\delta}$ are numerical coefficients that can contain also the phase factors $e^{i\delta}$ and $e^{-i\delta}$. Inserting into (15) $a = m^0_{ee}$, $b = m^0_{e\mu}$, and $c = m^0_{\mu\mu} - m^0_{\mu\tau}$, we find that the s_{13} corrections mix the e-line and $\mu\tau$-block elements: the corrections to the e-line elements $m_{e\mu}$ and $m_{e\tau}$ are proportional to the elements of the $\mu\tau$ block as well as to m_{ee}, whereas the corrections to the $\mu\tau$ block are proportional to $m^0_{e\mu}$. The D_{23} corrections do not mix elements from different blocks: $\Delta m_{\mu\mu} = f^{23}_{\mu\mu} D_{23} m_{\mu\tau}$. The D_{12} corrections to all elements are proportional to $m^0_{e\mu}$. The s_{13}^2 corrections mix the $\mu\tau$-block elements and m_{ee}. The correction to the subdominant elements can be proportional to the element of the dominant block and be much larger than the original element. The elements of the dominant block can get relative corrections of the order (20–30)% because the corrections can be enhanced by some additional numerical factors 2–3. In turn, these factors originate from the correction itself as well as some smallness of the original element (say by factor 1/2–1/3). In the cases when the original flavor matrix has no hierarchy, the corrections of the order 30% can lead to the ‘‘anarchical’’ character of the matrix with random values of elements.

An alternative parameterization of deviations from the TBM-mass matrix is proposed in [20], in which the element m_{ee} is unchanged.

C. Basis corrections

The basis in which the symmetry is introduced may differ from the flavor basis. In the symmetry basis, the elements of the mass matrix equal $m_{\alpha\beta}^{(\rm sym)} = m_{\alpha\beta} + \Delta m_{\alpha\beta}^b$, where $\Delta m_{\alpha\beta}^b$ is the basis corrections. Taking into account mixing in the quark sector, one can assume that the symmetry basis differs from the flavor basis by the Cabibbo-Kobayashi-Maskawa type rotation. To get some idea about the possible effects, we will consider for simplicity the 1–2 rotation only, with the angle θ_b of the order of Cabibbo angle: $s_b \equiv \sin\theta_b \sim \sin\theta_C \sim 0.2$. This rotation gives the following basis corrections:

$$\begin{aligned}
\Delta m_{ee}^b &= -2s_b c_b m_{e\mu} + s_b^2(m_{\mu\mu} - m_{ee}),\\
\Delta m_{\mu\mu}^b &= -\Delta m_{ee},\\
\Delta m_{\mu e}^b &= s_b c_b(m_{\mu\mu} - m_{ee}) - 2s_b^2 m_{e\mu},\\
\Delta m_{e\tau}^b &= -s_b m_{\mu\tau} + (1 - c_b)m_{e\tau} \approx -s_b m_{\mu\tau} + \frac{s_b^2}{2} m_{e\tau},\\
\Delta m_{\mu\tau}^b &= s_b m_{\mu\tau} + (1 - c_b)m_{e\tau} \approx s_b m_{\mu\tau} + \frac{s_b^2}{2} m_{e\tau}. \quad (25)
\end{aligned}$$

Apparently certain correlations between corrections to different elements exist, especially for some original structures of the mass matrix. For instance, if $m_{\mu e}$ and $m_{e\tau}$ are small (as, e.g., in the case of strong normal mass hierarchy), then $\Delta m_{ee}^b = \tan\theta_b \Delta m_{\mu e}^b$ corrections to $\Delta m_{e\mu}^b$ and $\Delta m_{e\tau}^b$ are large, $\Delta m_{e\tau}^b = -\Delta m_{e\mu}^b m_{\mu\tau}/(m_{\mu\mu} - m_{ee})$, etc.

Alternatively, the basis corrections can be accounted for by further deviation of the mixing angles from their TBM values: $\theta_{ij} \to \theta_{ij} + \Delta\theta_{ij}^b$. Therefore, in our consideration this can be taken into account by enlarging possible intervals for D_{ij}. For instance, a change of the 1–2 mixing by θ_C leads to the interval $\theta_{12} = 20° \div 45°$. The upper value corresponds to the maximal 1–2 mixing and the quark lepton complementarity case. This interval corresponds to $D_{12} = -0.17 \div 0.22$.

We will comment on possible additional changes of structure of the mass matrix due to these corrections.

D. Violation of the TBM conditions

Violation of the TBM symmetry of the neutrino mass matrix can be characterized by parameters that describe violation of the equalities (6)–(8). For the first two equalities, we can introduce

$$\Delta_e \equiv \frac{m_{e\mu} - m_{e\tau}}{m_{e\mu}}, \quad (26)$$

$$\Delta_{\mu\tau} \equiv \frac{m_{\mu\mu} - m_{\tau\tau}}{m_{\tau\tau}}. \quad (27)$$

Since the difference $(m_{ee} + m_{e\tau}) - (m_{\mu\mu} + m_{\mu\tau})$ depends on Δ_e and $\Delta_{\mu\tau}$,[3] we define the third violation parameter in a different way to avoid the strong correlation between the parameters. The third TBM condition (8) can be rewritten using (6) and (7) as $\Sigma_L = \Sigma_R$, where

$$\Sigma_L \equiv m_{ee} + \frac{m_{e\mu} + m_{e\tau}}{2}, \Sigma_R \equiv m_{\mu\tau} + \frac{m_{\mu\mu} + m_{\tau\tau}}{2}.$$

Then the third TBM-violation parameter can be introduced as

$$\Delta_\Sigma \equiv \frac{\Sigma_L - \Sigma_R}{\Sigma_R}. \quad (28)$$

In Δ_Σ, the effects of large violations of the first and second conditions are excluded.

Specific values of the violation parameters correspond to certain features of the mass matrix. For instance, $\Delta_e = 1$ corresponds to the texture zero $m_{e\tau} = 0$, $\Delta_{\mu\tau} \to \infty$ gives condition for $m_{\tau\tau} = 0$, etc. These values, in turn, can testify for some new symmetries of the mass matrix.

In what follows we will express the TBM-breaking parameters in terms of D_{ij} and study their dependence on the absolute mass scale, type of mass spectrum, and *CP* phases. We identify situations when the TBM conditions can be strongly violated. It is convenient to present the diagonal mass matrix in (3) as

$$m^{\rm diag} = {\rm diag}(m_1, m_2, m_3) = m_1 I + {\rm diag}(0, m, M),$$

where $m \equiv m_2 - m_1$, $M \equiv m_3 - m_1$ and I is the unit matrix. For definiteness, we will take $s_{13} > 0$.

1. The parameter Δ_e

According to (17), this parameter can be written as

$$\Delta_e = 2\frac{s_{13} + \alpha}{s_{13} - \tilde{s}_{13} e^{i\tilde{\phi}}}, \quad (29)$$

where in the first approximation α and $\tilde{s}_{13}$ do not depend on s_{13}, and furthermore, $\alpha \propto D_{23}$. The factor 2 originates from the fact that $m_{e\mu} - m_{e\tau} = 2x$, whereas $m_{e\mu} = x + A$. The quantity $\tilde{s}_{13} e^{i\tilde{\phi}}$ plays a crucial role: It determines the position of the pole of Δ_e which corresponds to texture zero $m_{e\mu} = 0$. Also, it determines the values of s_{13} at which some other special features of the neutrino mass matrix can be realized. Indeed, a given value of Δ_e corresponds to

$$s_{13} = \frac{\Delta_e \tilde{s}_{13} e^{i\tilde{\phi}} + \alpha}{\Delta_e - 2}.$$

[3]In the lowest order the difference equals $x - y + z - 3bD_{23} \approx m_{e\mu}\Delta_e/2 + m_{\tau\tau}\Delta_{\mu\tau}/2 + O(D_{12}, s_{13}^2)$.

TABLE III. Special values of the violation parameter Δ_e and the corresponding relations between elements of the mass matrix. Here values of the ratio $s_{13}e^{-i\tilde{\phi}}/\tilde{s}_{13}$ are given for $\alpha = 0$.

$s_{13}e^{-i\tilde{\phi}}/\tilde{s}_{13}$	$-\frac{1}{3}$	$\frac{1}{3}$	-1	1	$\gg 1$
Δ_e	$\frac{1}{2}$	-1	1	∞	≈ 2
Mass relation	$2m_{e\tau} = m_{e\mu}$	$m_{e\tau} = 2m_{e\mu}$	$m_{e\tau} = 0$	$m_{e\mu} = 0$	$m_{e\tau} = -m_{e\mu}$

So, if α is zero or small, which is realized in many cases, $\tilde{s}_{13}e^{i\tilde{\phi}}$ determines special values of Δ_e, and correspondingly, special mass relations (see Table III). Which of the possibilities in Table III can be realized depends on the upper bound on s_{13} and the value of $\tilde{s}_{13}$, which in turn is given by the mass spectrum and CP phases. Realization of the possibilities from the left to right in Table III requires decreasing values of $\tilde{s}_{13}$.

In terms of masses and mixing angles, Δ_e has the following expression:

$$\Delta_e = \frac{ms_{12}c_{12}(c_{23} - s_{23}) - s_{13}\kappa(c_{23} + s_{23})}{ms_{12}c_{12}c_{23} - \kappa s_{13}s_{23}}, \qquad (30)$$

where

$$\kappa \equiv Me^{-i\delta} - ms_{12}^2 e^{i\delta} - 2im_1 \sin\delta.$$

Consequently, the pole value and the phase equal

$$\tilde{s}_{13} \equiv s_{12}c_{12}\cot\theta_{23}\frac{m}{\kappa} \approx s_{12}c_{12}\frac{m}{\kappa}(1 + 2D_{23}),$$

$$\tilde{\phi} \equiv \arg\left[\frac{m}{\kappa}\right].$$

The expression for Δ_e can be rewritten approximately as

$$\Delta_e \approx 2\frac{s_{13}(1 + D_{23}) - \tilde{s}_{13}D_{23}}{s_{13} - \tilde{s}_{13}e^{i\tilde{\phi}}}.$$

Then

$$\alpha \approx (s_{13} - \tilde{s}_{13})D_{23}.$$

According to (29), $\Delta_e = 1$, which corresponds to $m_{e\tau} = 0$, is realized at

$$s_{13} = -(\tilde{s}_{13}e^{i\tilde{\phi}} + 2\alpha) = -\frac{\tilde{s}_{13}(e^{i\tilde{\phi}} - 2D_{23})}{1 + 2D_{23}}. \qquad (31)$$

At

$$s_{13} = \frac{1}{3}(\tilde{s}_{13}e^{i\tilde{\phi}} - 2\alpha) = \frac{\tilde{s}_{13}(e^{i\tilde{\phi}} + 2D_{23})}{3 + 2D_{23}},$$

we obtain $m_{e\tau} = 2m_{e\mu}$.

The strongest dependence of Δ_e is on s_{13}. In the case of the maximal 2–3 mixing, $D_{23} = 0$, Eq. (31) gives

$$\tilde{s}^{\,0}_{13} \equiv \left|\frac{s_{12}c_{12}m}{Me^{-i\delta} - ms_{12}^2 e^{i\delta} - 2im_1 \sin\delta}\right|. \qquad (32)$$

Since the CP phases are unknown, in general, $\tilde{\phi}$ can take any value. Therefore, for a given mass hierarchy and s_{13} and varying CP phases, the maximal and minimal values of Δ_e are realized for $\tilde{\phi} = 0$ and π: $\Delta_e = |2s_{13}/(s_{13} \pm \tilde{s}_{13})|$.

If $\tilde{\phi} = 0$, at $s_{13} = \tilde{s}_{13}$, Δ_e has a singularity. If $\tilde{\phi} \neq 0$, the function $|\Delta_e|$ has the peak

$$|\Delta_e| = \frac{2s_{13}}{\sqrt{(s_{13} - \tilde{s}_{13}\cos\tilde{\phi})^2 + (\tilde{s}_{13}\sin\tilde{\phi})^2}}, \qquad (33)$$

see Fig. 1. The maximum is at $s_{13} \simeq \tilde{s}_{13}\cos\tilde{\phi}$. For $s_{13} \gg \tilde{s}_{13}$, Δ_e approaches the asymptotic value $\Delta_e^{as} = 2$, which corresponds to the equality $m_{e\mu} = -m_{e\tau}$.

The parameter Δ_e depends on m_1 via $\tilde{s}_{13}$. As we will see, changing m_1 one can increase or decrease $\tilde{s}_{13}$, depending on the CP phases.

According to (31), a nonzero D_{23} shifts the pole: $\tilde{s}_{13} = \tilde{s}^0_{13}(1 + 2D_{23})$. For the present best fit value of s_{23}, we obtain $\tilde{s}_{13} = 1.07\tilde{s}^0_{13}$, and for $D_{23} \simeq 0.09$, we have a $\sim$10% change of Δ_e. The asymptotic value of Δ_e for large s_{13} becomes

$$\Delta_e = 1 + \cot\theta_{23} \approx 2 + D_{23}.$$

In the limit $s_{13} \to 0$, we obtain from (30) $\Delta_e = 1 - \tan\theta_{23} \approx 2D_{23}$. Then the central and the 1σ allowed values for D_{23} ($D_{23} = 0.034$ and 0.09) give, correspondingly, $\Delta_e = (0.07, 0.18)$.

If $D_{23} > 0$, the deviation Δ_e is greater than that in the case of the maximal 2–3 mixing. For example, in the case

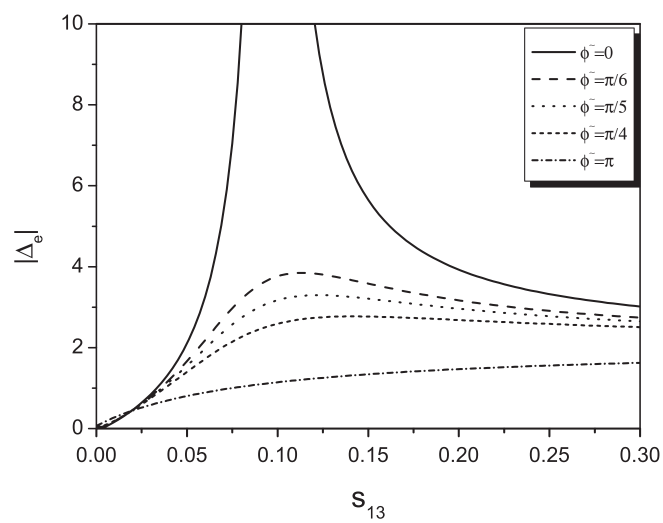

FIG. 1. $|\Delta_e|$ as a function of s_{13} for different values of $\tilde{\phi}$. We take the best fit values of θ_{23} and θ_{12}.

of strong mass hierarchy ($m_1 \simeq 0$) and for the best fit values of mixing angles, we obtain $\Delta_e \sim 12$ instead of 8.

2. The parameter $\Delta_{\mu\tau}$

Similarly to the previous case and according to Eq. (17), this violation parameter can be presented as

$$\Delta_{\mu\tau} = -2\frac{D_{23} + \beta}{D_{23} - \tilde{D}_{23}}, \tag{34}$$

where in the lowest order β and the pole value $\tilde{D}_{23}$ do not depend on D_{23}. In the limit $\beta \approx 0$, the parameter $\tilde{D}_{23}$ determines special values of $\Delta_{\mu\tau}$, and consequently, special relations between the matrix elements (see Table IV).

Explicitly, in terms of deviation parameters, we obtain

$$\tilde{D}_{23} = -\frac{1}{2\kappa_{23}}[Mc_{13}^2 + mc_{12}^2 + 2m_1 + 2m'\sqrt{1 - 4D_{23}^2} + s_{13}^2(ms_{12}^2e^{2i\delta} + m_1(e^{2i\delta} - 1))]$$

and

$$\beta = \frac{m'}{\kappa_{23}}\sqrt{1 - 4D_{23}^2} \approx \frac{m'}{\kappa_{23}},$$

where

$$m' \equiv -ms_{13}s_{12}c_{12}e^{i\delta}$$

and

$$\kappa_{23} \equiv Mc_{13}^2 - mc_{12}^2 + ms_{12}^2s_{13}^2e^{2i\delta} + m_1s_{13}^2(e^{2i\delta} - 1).$$

Neglecting the s_{13}^2 terms, we have in the first approximation

$$\beta \approx \frac{m'}{\kappa_{23}} = -\frac{(m_2 - m_1)s_{13}s_{12}c_{12}e^{i\delta}}{m_3 - m_2c_{12}^2 - m_1s_{12}^2}$$

and

$$\tilde{D}_{23} \approx -\frac{m_3 + m_2c_{12}^2 + m_1s_{12}^2 - 2(m_2 - m_1)s_{12}c_{12}s_{13}e^{i\delta}}{2(m_3 - m_2c_{12}^2 - m_1s_{12}^2)}.$$

For real values of $\tilde{D}_{23}$, this quantity determines the position of the pole of $\Delta_{\mu\tau}$ which corresponds to $m_{\tau\tau} = 0$. According to (34), the equality $\Delta_{\mu\tau} = -1$, ($m_{\mu\mu} = 0$), is realized at $D_{23} = -(\tilde{D}_{23} + 2\beta)$, and at $D_{23} = \frac{1}{3}(\tilde{D}_{23} - 2\beta)$ we obtain $m_{\mu\mu} = 2m_{\tau\tau}$ ($\Delta_{\mu\tau} = 1$). In many situations $\beta \approx 0$. Nonzero β leads to a shift of the special points from the values indicated in Table IV.

In the lowest order $\Delta_{\mu\tau}$ depends on the 1–3 mixing via m' only. Neglecting the s_{13}^2 corrections, we have $m'_1 = m_1$.

The strongest dependence of $\Delta_{\mu\tau}$ is the one on D_{23}. For $s_{13} = 0$, we have $m' = 0$, $\beta = 0$, and

$$\Delta_{\mu\tau} \approx \left|\frac{2D_{23}}{D_{23} - \tilde{D}_{23}}\right|. \tag{35}$$

In this case,

$$\tilde{D}_{23} = -\frac{m_3 + m_2c_{12}^2 + m_1s_{12}^2}{2(m_3 - m_2c_{12}^2 - m_1s_{12}^2)}. \tag{36}$$

For maximal 2–3 mixing, $D_{23} = 0$, we obtain from (34)

$$\Delta_{\mu\tau} = \frac{4m'}{m_1s_{12}^2 + m_2c_{12}^2 + m_3 + 2m'}. \tag{37}$$

According to (19), in the first approximation the corrections are proportional to the $e\mu$ element of the original TBM matrix: $\sqrt{2}s_{13}b = \sqrt{2}s_{13}m_{e\mu}^0$.

If $s_{13} \neq 0$ and $D_{23} \neq 0$ simultaneously, $\Delta_{\mu\tau}$ can be further enhanced. The dependence of $\Delta_{\mu\tau}$ on D_{23} is shown in Fig. 2.

Notice that the $\mu\tau$ block of the mass matrix in all the cases with strong enhancement of $\Delta_{\mu\tau}$ can be presented as

$$m_\nu \approx 2m_0\begin{pmatrix} D_{23} + \tilde{D}_{23} & \frac{1}{2}\sqrt{1 - 4D_{23}^2} \\ \ldots & -D_{23} + \tilde{D}_{23}\end{pmatrix}. \tag{38}$$

This shows that when violation of the second condition is strong, the off-diagonal elements are much larger (by factor $(2D_{23})^{-1} > 5$) than the diagonal elements. In other words, violation of the TBM condition is large when $m_{\mu\mu}$ and $m_{\tau\tau}$ elements are subleading. This means that the structure of the whole mass matrix does not change substantially by these corrections.

The TBM parameters can be introduced in a different way:

$$\Delta'_e \equiv \frac{m_{e\mu} - m_{e\tau}}{m_{e\mu} + m_{e\tau}}, \tag{39}$$

thus, excluding the linear dependence of the denominator on s_{13}. The two parameters are related by

$$\Delta'_e = \frac{\Delta_e}{2 - \Delta_e}.$$

So, the texture zero $m_{e\mu} = 0$ would correspond to $\Delta'_e = -1$ and the relation $m_{e\mu} = -m_{e\tau}$ is realized when $\Delta'_e \to \infty$, etc. The pole value $\tilde{s}_{13}$ is determined from the condition $\Delta'_e(\tilde{s}_{13}) = -1$.

TABLE IV. Special values of the violation parameter $\Delta_{\mu\tau}$ and the corresponding relations between the elements of the mass matrix. Values of the ratio $D_{23}/\tilde{D}_{23}$ are given for $\beta = 0$.

$D_{23}/\tilde{D}_{23}$	$-\frac{1}{3}$	$\frac{1}{3}$	-1	1	$\gg 1$
$\Delta_{\mu\nu}$	$-\frac{1}{2}$	1	-1	∞	≈ -2
Mass relation	$m_{\tau\tau} = 2m_{\mu\mu}$	$2m_{\tau\tau} = m_{\mu\mu}$	$m_{\mu\mu} = 0$	$m_{\tau\tau} = 0$	$m_{\tau\tau} = -m_{\mu\mu}$

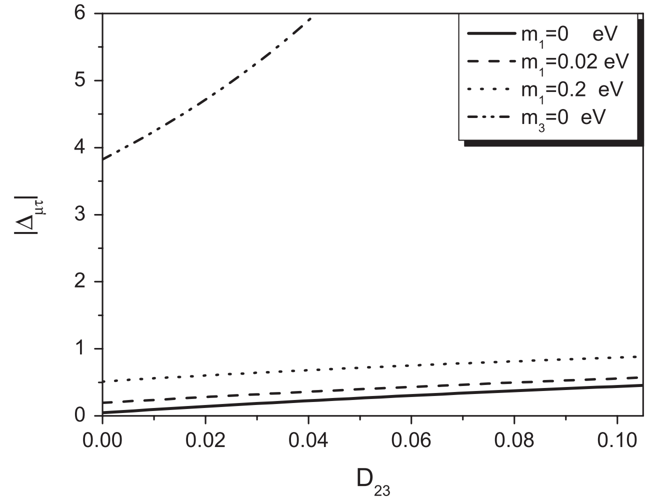

FIG. 2. Dependence of $|\Delta_{23}|$ on D_{23} for different values of the lightest neutrino mass and $\phi_2 = \frac{\pi}{2}$. We take the best fit values of θ_{13} and θ_{12}. The value $m_3 = 0$ corresponds to the inverted mass hierarchy.

3. The parameter Δ_Σ

Using (17), we find

$$\Sigma_L = a + b + \left(ce^{-2i\delta} - a - \frac{b}{2}\right)s_{13}^2 - \frac{15}{4}bD_{12},$$
$$\Sigma_R = a + b + 3bD_{12}.$$

And consequently,

$$\Delta_\Sigma \simeq \frac{s_{13}^2(ce^{-2i\delta} - a - \frac{b}{2}) - \frac{27}{4}D_{12}b}{a + b + 3bD_{12}} = \frac{s_{13}^2[m_3e^{-2i\delta} - \frac{1}{2}(m_1 + m_2)] - \frac{9}{4}D_{12}(m_2 - m_1)}{\frac{1}{3}m_1 + \frac{2}{3}m_2 + (m_2 - m_1)D_{12}}.$$

Δ_Σ reflects violation of the TBM structure by m_{ee} and $m_{\mu\tau}$. Therefore, instead of Δ_Σ we can simply use the deviation of m_{ee} from its TBM value:

$$\Delta m_{ee} \equiv m_{ee} - m_{ee}^{\mathrm{TBM}} = -(m_2 - m_1)D_{12} + [m_3e^{-2i\delta} - m_1 - (m_2 - m_1) \times (\tfrac{1}{3} - D_{12})]s_{13}^2.$$

This correction is not affected by the 2–3 mixing. Contribution of D_{12} is rather small. The larger effect can be due to s_{13}^2. If $m_1 \approx 0$, the last term can dominate: $m_{ee} \approx m_3 s_{13}^2$. Equation (40) reproduces the one in (21) when high order terms $\sim D_{12}s_{13}^2$ are neglected. In the case of strong mass hierarchy and $s_{13} = 0$, we have $m_{ee} \approx m_2(1/3 - D_{12})$.

The proposed formalism allows us immediately (and very precisely) to trace an impact of the deviations from the TBM mixing on structure of the neutrino mass matrix. The effect of future measurements of the mixing angles can be seen immediately.

III. PROPERTIES OF THE NEUTRINO MASS MATRIX

Formulas obtained in the previous section allow us to "design" neutrino mass matrices with certain required properties which agree with observations. We reconstruct the neutrino mass matrix in the cases of TBM mixing and deviations from TBM for different mass hierarchies and CP-violation phases. The results of the numerical computations are given in Tables V and VI. Tables V and VI illustrate the maximal possible modifications of structures at a certain confidence level. Apparently, any intermediate structure between the original TBM and matrices with deviations presented in Table V are possible. As the best fit values, we take $D_{12} = 0.012$, $D_{23} = 0.037$ and $s_{13} = 0.118$ and for 1σ deviations, we use $D_{12} = 0.028$, $D_{23} = 0.085$ and $s_{13} = 0.156$.

Modification of the mass matrix (for fixed values of the deviations) depends on the CP-violating phases. Table V corresponds to $\delta = 0$. For certain cases this does not correspond to the maximal deviation of the mass matrix from the TBM form. In Table VI, we show the mass matrices for $\delta = \pi$ when they lead to stronger deviations than in Table V.

Because of the hierarchy of the allowed deviations (12), the following combinations of mass matrix elements are approximately invariant under corrections:

$$m_{e\mu} + m_{e\tau} \approx \text{const}, \qquad m_{\mu\mu} + m_{\tau\tau} \approx \text{const}. \tag{40}$$

The elements m_{ee} and $m_{\mu\tau}$ receive only small corrections.

We will consider several "benchmark" spectra determined by the mass hierarchy/ordering, and CP parities. For each case, we (i) compute the parameters of the mass matrix and reconstruct the TBM matrix, (ii) find the lowest order corrections using (18)–(20) and identify conditions at which corrections are maximal, (iii) compute $\tilde{s}_{13}$, $\tilde{D}_{23}$ and the TBM-violation parameters, and (iv) discuss the properties of the mass matrix with corrections.

A. Normal mass hierarchy

In the case of strong normal mass hierarchy, we take $m_1 \approx 0$; see lines $NH(0, 0)$ and $NH(0, \frac{\pi}{2})$ in Table V.

(a) The parameters of the mass matrix

$$a = b \approx \frac{m_2}{3} \approx \frac{\sqrt{\Delta m_{21}^2}}{3},$$
$$a, b \ll c \approx m_3 \approx \sqrt{\Delta m_{31}^2}$$

give the TBM matrix

$$m_\nu \approx \begin{pmatrix} \frac{m_2}{3} & \frac{m_2}{3} & \frac{m_2}{3} \\ \cdots & \frac{m_3}{2} + \frac{m_2}{3} & -\frac{m_3}{2} + \frac{m_2}{3} \\ \cdots & \cdots & \frac{m_3}{2} + \frac{m_2}{3} \end{pmatrix}.$$

(b) The lowest order corrections equal

$$\begin{aligned} x &\approx -\frac{1}{\sqrt{2}} s_{13} m_3 e^{-i\delta}, \\ y &\approx -D_{23} m_3 + \frac{\sqrt{2}}{3} s_{13} m_2 e^{i\delta}, \\ \Delta m_{ee} &\approx -m_2 D_{12} + s_{13}^2 m_3 e^{-2i\delta}, \\ \Delta m_{\mu\tau} &\approx \frac{1}{2}(m_2 D_{12} + m_3 s_{13}^2). \end{aligned} \tag{41}$$

Notice that in y the two contributions can be of the same size and enhance each other. The same is in Δm_{ee} and $\Delta m_{\mu\tau}$. For $D_{23} > 0$, the maximal deviations are achieved if $\phi_3 = \pi/2$ and $\delta = 0$ or $\phi_3 = 0$ and $\delta = \pi$. For the best fit values of the mixing angles, the maximal deviations equal (in the units 10^{-2} eV) $|\Delta m_{ee}| \sim 0.10$, $|x| \sim 0.45$, $|y| \sim 0.25$, and the correction to the subleading elements are bigger than the original TBM elements. At the 1σ level, the corrections become $|\Delta m_{ee}| \approx 0.15$, $|x| \approx 0.65$, $|y| \approx 0.5$, and the structure of the mass matrix can substantially deviate from the TBM form.

(c) The parameters of violation of the TBM conditions: At $D_{23} = 0$, we have

TABLE V. Numerical examples of neutrino mass matrices in the cases of NH, partially degenerate spectrum (PD), IH, and degenerate spectrum (D). The numbers in brackets of the scenario definition indicate the *CP* phases (ϕ_2, ϕ_3). We show matrices for the exact TBM (left column), the best fit values of the mixing angles (central column), and the mixing angles allowed at 1σ level (right column). We take $\delta = 0$ and the elements of the matrices are in the unit 10^{-2} eV.

Scenario	Exact TBM	Best fit values	1σ deviation
NH(0,0)	$\begin{pmatrix} 0.3 & 0.3 & 0.3 \\ \cdots & 2.7 & -2.1 \\ \cdots & \cdots & 2.7 \end{pmatrix}$	$\begin{pmatrix} 0.35 & -0.06 & 0.7 \\ \cdots & 2.6 & -2.1 \\ \cdots & \cdots & 2.8 \end{pmatrix}$	$\begin{pmatrix} 0.39 & -0.15 & 0.8 \\ \cdots & 2.4 & -2.0 \\ \cdots & \cdots & 3.0 \end{pmatrix}$
NH(0, $\frac{\pi}{2}$)	$\begin{pmatrix} 0.3 & 0.3 & 0.3 \\ \cdots & -2.1 & 2.7 \\ \cdots & \cdots & -2.1 \end{pmatrix}$	$\begin{pmatrix} 0.2 & 0.7 & -0.16 \\ \cdots & -1.8 & 2.7 \\ \cdots & \cdots & -2.4 \end{pmatrix}$	$\begin{pmatrix} 0.14 & 0.8 & -0.34 \\ \cdots & -1.5 & 2.6 \\ \cdots & \cdots & -2.6 \end{pmatrix}$
PD(0,0)	$\begin{pmatrix} 2.06 & 0.06 & 0.06 \\ \cdots & 3.7 & -1.6 \\ \cdots & \cdots & 3.7 \end{pmatrix}$	$\begin{pmatrix} 2.1 & -0.19 & 0.34 \\ \cdots & 3.6 & -1.5 \\ \cdots & \cdots & 3.8 \end{pmatrix}$	$\begin{pmatrix} 2.1 & -0.25 & 0.44 \\ \cdots & 3.4 & -1.5 \\ \cdots & \cdots & 3.9 \end{pmatrix}$
PD(0, $\frac{\pi}{2}$)	$\begin{pmatrix} 2.06 & 0.06 & 0.06 \\ \cdots & -1.6 & 3.7 \\ \cdots & \cdots & -1.6 \end{pmatrix}$	$\begin{pmatrix} 1.95 & 0.65 & -0.57 \\ \cdots & -1.2 & 3.6 \\ \cdots & \cdots & -1.8 \end{pmatrix}$	$\begin{pmatrix} 1.9 & 0.79 & -0.81 \\ \cdots & -0.86 & 3.5 \\ \cdots & \cdots & -2.1 \end{pmatrix}$
PD($\frac{\pi}{2}$, 0)	$\begin{pmatrix} 0.6 & -1.4 & -1.4 \\ \cdots & 2.2 & -3.0 \\ \cdots & \cdots & 2.24 \end{pmatrix}$	$\begin{pmatrix} 0.7 & -1.8 & -0.9 \\ \cdots & 1.7 & -3.0 \\ \cdots & \cdots & 2.6 \end{pmatrix}$	$\begin{pmatrix} 0.8 & -1.9 & -0.7 \\ \cdots & 1.3 & -2.9 \\ \cdots & \cdots & 2.9 \end{pmatrix}$
PD($\frac{\pi}{2}$, $\frac{\pi}{2}$)	$\begin{pmatrix} 0.6 & -1.4 & -1.4 \\ \cdots & -3.0 & 2.2 \\ \cdots & \cdots & -3.0 \end{pmatrix}$	$\begin{pmatrix} 0.57 & -0.95 & -1.8 \\ \cdots & -3.1 & 2.2 \\ \cdots & \cdots & -3.0 \end{pmatrix}$	$\begin{pmatrix} 0.57 & -0.86 & -1.9 \\ \cdots & -2.9 & 2.1 \\ \cdots & \cdots & -3.1 \end{pmatrix}$
IH(0,0)	$\begin{pmatrix} 4.8 & -0.03 & -0.03 \\ \cdots & 2.4 & 2.4 \\ \cdots & \cdots & 2.4 \end{pmatrix}$	$\begin{pmatrix} 4.8 & 0.36 & -0.44 \\ \cdots & 2.6 & 2.4 \\ \cdots & \cdots & 2.3 \end{pmatrix}$	$\begin{pmatrix} 4.7 & 0.45 & -0.60 \\ \cdots & 2.9 & 2.3 \\ \cdots & \cdots & 2.1 \end{pmatrix}$
IH(0,0)	$\begin{pmatrix} 1.6 & -3.2 & -3.2 \\ \cdots & -0.7 & -0.8 \\ \cdots & \cdots & -0.7 \end{pmatrix}$	$\begin{pmatrix} 1.7 & -3.2 & -3.2 \\ \cdots & -1.4 & -0.8 \\ \cdots & \cdots & -0.2 \end{pmatrix}$	$\begin{pmatrix} 1.9 & -3.2 & -3.1 \\ \cdots & -1.7 & -0.8 \\ \cdots & \cdots & 0.05 \end{pmatrix}$
D(0,0)	$\begin{pmatrix} 20.0 & 0.006 & 0.006 \\ \cdots & 20.3 & -0.3 \\ \cdots & \cdots & 20.3 \end{pmatrix}$	$\begin{pmatrix} 20.0 & -0.04 & 0.06 \\ \cdots & 20.2 & -0.3 \\ \cdots & \cdots & 20.3 \end{pmatrix}$	$\begin{pmatrix} 20.0 & -0.05 & 0.07 \\ \cdots & 20.2 & -0.3 \\ \cdots & \cdots & 20.3 \end{pmatrix}$
D($\frac{\pi}{2}$, 0)	$\begin{pmatrix} 6.6 & -13.3 & -13.3 \\ \cdots & 6.9 & -13.6 \\ \cdots & \cdots & 6.9 \end{pmatrix}$	$\begin{pmatrix} 7.3 & -14.7 & -11.5 \\ \cdots & 3.4 & -13.6 \\ \cdots & \cdots & 9.8 \end{pmatrix}$	$\begin{pmatrix} 8.1 & -15.2 & -10.2 \\ \cdots & 1.02 & -13.3 \\ \cdots & \cdots & 11.4 \end{pmatrix}$
D(0, $\frac{\pi}{2}$)	$\begin{pmatrix} 20 & 0.006 & 0.006 \\ \cdots & -0.3 & 20.3 \\ \cdots & \cdots & -0.3 \end{pmatrix}$	$\begin{pmatrix} 19.4 & 3.2 & -3.4 \\ \cdots & 1.5 & 19.9 \\ \cdots & \cdots & -1.5 \end{pmatrix}$	$\begin{pmatrix} 19.0 & 4.0 & -4.8 \\ \cdots & 3.6 & 19.5 \\ \cdots & \cdots & -3.2 \end{pmatrix}$
D($\frac{\pi}{2}$, $\frac{\pi}{2}$)	$\begin{pmatrix} 6.6 & -13.3 & -13.3 \\ \cdots & -13.6 & 6.9 \\ \cdots & \cdots & -13.6 \end{pmatrix}$	$\begin{pmatrix} 6.8 & -11.4 & -15.0 \\ \cdots & -15.4 & 6.7 \\ \cdots & \cdots & -12.0 \end{pmatrix}$	$\begin{pmatrix} 7.1 & -11.1 & -15.1 \\ \cdots & -15.6 & 6.5 \\ \cdots & \cdots & -12.0 \end{pmatrix}$

TABLE VI. The same as in Table V for the Dirac phase $\delta = \pi$.

Scenario	Best fit values	1σ deviation
NH(0,0)	$\begin{pmatrix} 0.35 & 0.67 & -0.11 \\ \ldots & 2.5 & -2.1 \\ \ldots & \ldots & 2.9 \end{pmatrix}$	$\begin{pmatrix} 0.38 & 0.77 & -0.28 \\ \ldots & 2.3 & -2.0 \\ \ldots & \ldots & 3.1 \end{pmatrix}$
PD(0,0)	$\begin{pmatrix} 2.1 & 0.32 & -0.21 \\ \ldots & 3.5 & -1.55 \\ \ldots & \ldots & 3.8 \end{pmatrix}$	$\begin{pmatrix} 2.1 & 0.38 & -0.32 \\ \ldots & 3.4 & -1.5 \\ \ldots & \ldots & 3.9 \end{pmatrix}$
PD($\frac{\pi}{2}, \frac{\pi}{2}$)	$\begin{pmatrix} 0.57 & -1.9 & -0.81 \\ \ldots & -2.6 & 2.1 \\ \ldots & \ldots & -3.4 \end{pmatrix}$	$\begin{pmatrix} 0.57 & -2.05 & -0.51 \\ \ldots & -2.4 & 2.0 \\ \ldots & \ldots & -3.7 \end{pmatrix}$
IH(0,0)	$\begin{pmatrix} 4.8 & -0.41 & 0.39 \\ \ldots & 2.6 & 2.4 \\ \ldots & \ldots & 2.3 \end{pmatrix}$	$\begin{pmatrix} 4.7 & -0.51 & 0.55 \\ \ldots & 2.9 & 2.3 \\ \ldots & \ldots & 2.1 \end{pmatrix}$
IH($\frac{\pi}{2}$, 0)	$\begin{pmatrix} 1.75 & -3.44 & -2.91 \\ \ldots & -0.36 & -0.9 \\ \ldots & \ldots & -1.3 \end{pmatrix}$	$\begin{pmatrix} 1.9 & -3.6 & -2.6 \\ \ldots & -0.37 & -1.05 \\ \ldots & \ldots & -1.4 \end{pmatrix}$
D($\frac{\pi}{2}, \frac{\pi}{2}$)	$\begin{pmatrix} 6.7 & -15.8 & -10.2 \\ \ldots & -11.0 & 6.3 \\ \ldots & \ldots & -16.3 \end{pmatrix}$	$\begin{pmatrix} 7.1 & -16.7 & -8.4 \\ \ldots & -10.0 & 5.5 \\ \ldots & \ldots & -17.7 \end{pmatrix}$

$$\tilde{s}^0_{13} \approx s_{12}c_{12}\sqrt{\frac{\Delta m^2_{21}}{\Delta m^2_{31}}} \simeq 0.09,$$

and $\tilde{\phi} \approx 2\phi_2 - 2\phi_3 + \delta$. Notice that $\tilde{s}^0_{13}$ in (42) is slightly smaller than the present best fit value of s_{13} and at the 1σ level $s_{13}/\tilde{s}_{13} \leq 2$. Therefore, all the possibilities indicated in Table III can be realized. For the best fit value of the 1–3 mixing, $\Delta_e = 11.6$. For the 1σ upper bounds on the 1–3 mixing, the parameter equals $\Delta_e = 6.4$. Thus, the first TBM relation in (6) can be broken very strongly. Such a strong influence (even for small s_{13}) originates from the fact that s_{13} mixes the large and small mass scales in the mass matrix, and therefore the corrections to the subleading elements ($m_{e\mu}$, $m_{e\tau}$) are proportional to the large mass: $\sim s_{13}\sqrt{\Delta m^2_{31}}$.

From (36), we have

$$\tilde{D}_{23} = -1/2, \qquad \beta \approx s_{13}s_{12}c_{12}\frac{m}{M} \ll \tilde{D}_{23},$$

and therefore

$$\Delta_{\mu\tau} \approx \frac{4D_{23}}{1 + 2D_{23}} \approx 4D_{23}.$$

Since $D_{23}/\tilde{D}_{23} < 0.2$ (1σ), no texture zeros or special relations indicated in Table IV can be obtained. The effect of the 1–3 mixing is very small, since the element b is small. According to (37), $\Delta_{\mu\tau} \approx 2s_{13}\sin2\theta_{12}\sqrt{\Delta m^2_{21}/\Delta m^2_{31}} \sim s_{13}/3$.

The examples in Table V correspond to $\delta = 0$. For $\delta = \pi$, according to Eq. (41), the values of $m_{e\mu}$ and $m_{e\tau}$ permute, see Table V. Also, in this case m_{ee} is suppressed. Signs of corrections to the $\mu\tau$-block and e-line elements can be independently changed, varying ϕ_3 and δ. The correction to m_{ee} is then fixed.

(d) Properties of the mass matrix:

(a) The allowed corrections to the subleading $e\mu$ and $e\tau$ elements dominate the original TBM values: $x \gg b$; changes of elements of the $\mu\tau$ block can be of the order 1; m_{ee} can be suppressed by the corrections of the order s^2_{13}.

(b) Texture zeros appear: $m_{e\tau} = 0$ or $m_{e\mu} = 0$ at s_{13} determined by $\tilde{s}_{13}$.

(c) Special relations $m_{e\mu} = rm_{e\mu}$, with $r = 1/2$, 2 can be obtained.

(d) The equality $m_{ee} = -m_{e\mu}$ can be approximately realized.

(e) The sharp difference of the elements of the $\mu\tau$ block and the e line disappear. So, one may have a smooth decrease of values of the elements from $m_{\tau\tau}$ to m_{ee} with the additional smallness of $m_{e\tau}$. This structure resembles the structure of the quark mass matrices with, however, a much larger expansion parameter $\lambda \sim 0.5$–0.8.

(f) The maximal deviation of m_ν from $m_{\rm TBM}$ corresponds to $m_2 > 0$, $m_3 > 0$, and $\delta = \pi$, which leads to the strong increase of $m_{e\mu}$ and the decrease of $m_{\mu\mu}$. In this case, the correction to m_{ee} is positive. The ee element is suppressed, if $m_2 > 0$, $m_3 < 0$ and $\delta = 0$. In this case, the mass matrix has the following form:

$$m_\nu = \begin{pmatrix} 0.4 & 0.8 & 0.2 \\ \ldots & 2.3 & -2.0 \\ \ldots & \ldots & 3.0 \end{pmatrix} 10^{-2}\ \text{eV}.$$

The basis corrections can further smear difference of the e-line and $\mu\tau$-block elements. Varying S_b in the interval $-0.2 \div 0.2$, one finds $\Delta m^b_{ee} = (-0.24 \div 0.4)$, $\Delta m^b_{e\mu} = (0.32 \div -0.44)$, $\Delta m^b_{e\tau} = (0.4 \div -0.4)$ in the units 10^{-2} eV.

The total correction to m_{ee} can be as large as 0.002 eV, which is still smaller than the original $a = 0.003$ eV. However, $m_{ee} = 0$ can be realized with the increase of m_1. This can be achieved if

$$m_1 = -\frac{m_2}{2} + \frac{9}{4c^2_{13}}m_2D_{12} + \frac{3}{2}m_3\tan^2\theta_{13}.$$

Numerically, $m_{ee} = 0$ if $m_1 \approx 5.2.10^{-3}$ eV for the TBM case, $m_1 \approx 3.3.10^{-3}$ eV and $m_1 \approx 6.10^{-3}$ eV for the best fit values of the mixing angles with $\delta = 0, \frac{\pi}{2}$, respectively.

B. Partially degenerate spectrum

Suppose $|m_1| \approx |m_2| \approx \bar{m} < |m_3|$. Numerically this corresponds to $\bar{m} \sim (2 - 3) \cdot 10^{-2}$ eV and $m_3 = (5.5$–$6.0) \cdot 10^{-2}$ eV. The phase ϕ_2 becomes important.

1. The case $\phi_2 = 0$, lines $PD(0, 0)$ and $PD(0, \frac{\pi}{2})$ in Table V.

(a) The parameters of the mass matrix

$$m_1 \approx m_2 \approx \bar{m} > 0, \qquad a \approx \bar{m},$$

$$b = \epsilon \equiv \frac{\Delta m_{21}^2}{6\bar{m}} \approx 7 \cdot 10^{-4} \text{ eV},$$

give the TBM-mass matrix

$$m_\nu \approx \begin{pmatrix} \bar{m} & \epsilon & \epsilon \\ \ldots & \frac{1}{2}(m_3 + \bar{m}) & -\frac{1}{2}(m_3 - \bar{m}) \\ \ldots & \ldots & \frac{1}{2}(m_3 + \bar{m}) \end{pmatrix}.$$

The main feature of this matrix is a strong (factor of 30) suppression of the $m_{e\mu}$ and $m_{e\tau}$ elements in comparison with the other elements which are of the same order. Now $\bar{m} \sim m_3$, and consequently, a strong difference between $m_{\mu\mu}$ and $m_{\mu\tau}$ can appear.

(b) The lowest order corrections equal

$$x = -\frac{1}{\sqrt{2}} s_{13}(m_3 e^{-i\delta} - \bar{m} e^{i\delta}),$$

$$y = D_{23}(\bar{m} - m_3),$$

$$\Delta m_{ee} = s_{13}^2 (m_3 e^{-2i\delta} - \bar{m}).$$

In the case of $PD(0, 0)$, the corrections are not large: the m_3 and $\bar{m}$ terms partially cancel each other in x and y. Although $x \gg \epsilon$, the elements $m_{e\mu}$ and $m_{e\tau}$ are small, and the structure of the matrix with the dominant $\mu\tau$ block does not change.

The situation is different for $\phi_3 = \pi/2$, see line $PD(0, \frac{\pi}{2})$ of Table V. Corrections are maximal if $m_3 < 0$ and $\delta = 0$:

$$x = -\frac{1}{\sqrt{2}} s_{13}(|m_3| + \bar{m}), \qquad y = D_{23}(\bar{m} + |m_3|),$$

$$\Delta m_{ee} = -s_{13}^2(|m_3| + \bar{m}).$$

For $\delta = \pi$, the correction x changes the sign and values of $m_{e\mu}$ and $m_{e\tau}$ interchange.

(c) Violation of the TBM conditions: According to (31)

$$\tilde{s}_{13} = s_{12} c_{12} \frac{\Delta m_{21}^2}{2\bar{m}\kappa} \sim 10^{-2},$$

so that for the maximal allowed $s_{13}^{\max}$, we have $s_{13}^{\max}/\tilde{s}_{13} \sim 20$ and therefore all special mass relations in Table III can be satisfied. $\alpha = (s_{13} - \tilde{s}_{13}) D_{23}$, and since $s_{13}^{\max} \gg \tilde{s}_{13}$, the corrections of the order $s_{13} D_{23}$ become important.

From (36), we find $\tilde{D}_{23} = -(m_3 + \bar{m})/2(m_3 - \bar{m}) \sim 3/2$, which is larger than in the case of strong mass hierarchy, and correspondingly, the effect of violation of the second condition is weaker. In this case, $\beta \approx 0$.

(d) Properties of the mass matrix $PD(0, \frac{\pi}{2})$:

(a) Corrections to the subleading elements are large: about an order of magnitude larger than the TBM values. Therefore, the subleading mass matrix can be modified completely.

(b) At the 1σ level, the mass matrix has all of the elements of the same order (within a factor of 3). This can be considered as a realization of the anarchical structure.

(c) Equality $m_{e\mu} \approx -m_{e\tau}$ can be achieved. Exact zero of one of these elements is realized for very small s_{13}. Equalities $m_{e\mu} = -m_{\mu\mu}$ or $m_{ee} = -m_{\mu\mu}$ can be obtained.

Basis corrections can further "equilibrate" elements. For $s_b = 0.2$, they equal $\Delta m_{ee}^b = 0.04$, $\Delta m_{\mu e}^b = 0.31$, $\Delta m_{e\tau}^b = 0.32$, and $\Delta m_{\mu\tau}^b = -0.31$ in the units 10^{-2} eV. They are of the order of the TBM-violation corrections for the ee and $e\tau$ elements.

2. $\phi_2 = \pi/2$; see line $PD(\frac{\pi}{2}, 0)$ and $PD(\frac{\pi}{2}, \frac{\pi}{2})$.

(a) The parameters of the mass matrix

$$m_1 \approx \bar{m}, \qquad m_2 \approx -\bar{m}, \qquad a \approx \frac{\bar{m}}{3}, \qquad b \approx -\frac{2\bar{m}}{3}$$

lead to the TBM-mass matrix

$$m_\nu \approx \begin{pmatrix} \frac{1}{3}\bar{m} & -\frac{2}{3}\bar{m} & -\frac{2}{3}\bar{m} \\ \ldots & \frac{m_3}{2} - \frac{\bar{m}}{6} & -\frac{m_3}{2} - \frac{\bar{m}}{6} \\ \ldots & \ldots & \frac{m_3}{2} - \frac{\bar{m}}{6} \end{pmatrix}. \tag{42}$$

All the elements are of the same order, so that corrections do not change the structure strongly. The ee element is the smallest one.

(b) The lowest order corrections equal

$$x = -\frac{1}{\sqrt{2}} s_{13} m_3 e^{-i\delta} - \frac{2}{3}\bar{m} D_{23},$$

$$y = -\frac{2\sqrt{2}}{3} s_{13} \bar{m} e^{i\delta} - D_{23}\left(\frac{1}{3}\bar{m} + m_3\right),$$

$$\Delta m_{ee} = s_{13}^2 m_3 + 2\bar{m} D_{12},$$

$$\Delta m_{\mu\tau} = \frac{1}{2}\Delta m_{ee} - \bar{m} D_{12}.$$

For $D_{23} > 0$, the largest deviations appear when $\delta = 0$ and $m_3 > 0$ ($\phi_3 = 0$) or $\delta = \pi$ and $\phi_3 < \pi/2$. For the best fit (bf) values of the mixing parameters (and 1σ), we have $x \sim -0.5(-0.8)$, $y \sim -0.3(-0.7)$, $\Delta m_{ee} \sim 0.15(0.3)$ (in the units 10^{-2} eV). Corrections to the $e\mu$ and $e\tau$ elements are of the order 1; corrections to other elements are up to (20–30)%.

(c) The parameters of violation of the TBM conditions: The poles of Δ_e and $\Delta_{\mu\tau}$ are at

$$\tilde{s}_{13} = \frac{-\bar{m} \sin 2\theta_{12}(1 + 2D_{23})}{M e^{-i\delta} + 2\bar{m} s_{12}^2 e^{i\delta} - 2i m_1 \sin\delta},$$

$$\tilde{D}_{23} = -\frac{m_3 - \bar{m}(1 - s_{13}\tan 2\theta_{12} e^{i\delta})}{2(m_3 + \bar{m})} \gg D_{23}^{\max},$$

where $D_{23}^{\max}$ is the maximal allowed value of D_{23} at 1σ, so no texture zeros are realized in the $\mu\tau$ block.

(d) Properties of the mass matrix:

(a) Texture zero $m_{e\tau} = 0$, is realized at $s_{13} \approx (0.13, 0.19, 0.34)$ for $m_1 = (0.005, 0.01, 0.02)$ eV;

(b) In the case of $PD(\frac{\pi}{2}, 0)$, the structure is possible with all elements being of the same order and m_{ee} and $m_{e\tau}$ being the smallest ones.

The basis corrections for $s_b = 0.2$ equal $\Delta m_{ee}^b = 0.6$, $\Delta m_{\mu e}^b = 0.43$, $\Delta m_{e\tau}^b = 0.58$ and $\Delta m_{\mu\tau}^b = -0.63$ in the units 10^{-2} eV. They are of the order of the TBM-violation corrections for the $e\mu$ and $e\tau$ elements and large for the ee and $\mu\tau$ elements.

C. Inverted mass hierarchy

If $m_3 \approx 0$, we obtain $|m_1| \approx |m_2| \approx \bar{m}$, $\bar{m} \sim \sqrt{\Delta m_{31}^2} \approx 5 \cdot 10^{-2}$ eV. The structure of the mass matrix is similar to that in the partially degenerate case. Similarly, the results strongly depend on the phase ϕ_2.

1. $\phi_2 = 0$, line $IH(0, 0)$.

(a) Parameters of the mass matrix

$$m_1 \approx m_2 \approx \bar{m} > 0, \qquad a \approx \bar{m},$$
$$b = \epsilon \sim 2.7 \cdot 10^{-4} \text{ eV},$$

give the TBM-mass matrix

$$m_\nu \approx \begin{pmatrix} \bar{m} & \epsilon & \epsilon \\ \dots & \frac{1}{2}\bar{m} & \frac{1}{2}\bar{m} \\ \dots & \dots & \frac{1}{2}\bar{m} \end{pmatrix}.$$

The elements $m_{e\mu}$ and $m_{e\tau}$ are suppressed by 2 orders of magnitude in comparison to the other elements.

(b) The lowest order correction:

$$x = \frac{1}{\sqrt{2}} s_{13}\bar{m}e^{i\delta}, \qquad y = D_{23}\bar{m},$$
$$\Delta m_{ee} = -s_{13}^2\bar{m}, \qquad \Delta m_{\mu\tau} = \frac{1}{2}\Delta m_{ee}.$$

Corrections strongly correlate with the TBM structure: they suppress the ee and $\tau\tau$ elements, and enhance the $\mu\mu$ element (if $D_{23} > 0$). Corrections to $m_{e\mu}$ and $m_{e\tau}$ dominate, so that $m_{e\mu} \approx -m_{e\tau}$. The matrix with corrections can be written as

$$m_\nu \approx \bar{m}\begin{pmatrix} 1 - s_{13}^2 & \frac{1}{\sqrt{2}} s_{13}e^{i\delta} & -\frac{1}{\sqrt{2}} s_{13}e^{i\delta} \\ \dots & \frac{1}{2} + D_{23} & \frac{1}{2} \\ \dots & \dots & \frac{1}{2} - D_{23} \end{pmatrix}.$$

Corrections are small to m_{ee} and at the bf values (1σ level) of the deviation they equal approximately 10% (20%) for the elements of the $\mu\tau$ block.

(c) The parameters of violation of the TBM conditions: If $D_{23} = 0$, we obtain from (32) very small pole value

$$\tilde{s}_{13} \approx s_{12}c_{12}\frac{\Delta m_{21}^2}{2\Delta m_{31}^2} \approx 0.008.$$

Consequently, for the central values of the 1–3 mixing we have nearly maximal TBM violation, $\Delta_e \approx 2$. All the relations in Table III can be satisfied.

For $s_{13} = 0$, we have $\tilde{D}_{23} \approx 0.5$, and as can be immediately seen from Eq. (43),

$$\Delta_{\mu\tau} = 4D_{23}\frac{1}{1 - 2D_{23}} \approx 4D_{23},$$

independently of the phase ϕ_2. If $D_{23} = 0$, we obtain from (37) $\Delta_{\mu\tau} \approx s_{13}\sin 2\theta_{12}\frac{\Delta m_{21}^2}{\Delta m_{31}^2}$, which is strongly suppressed.

(d) Properties of the mass matrix, (line $IH(0, 0)$):

(a) Structure of the dominant block of the mass matrix does not change substantially in comparison to the TBM form.

(b) No texture zeros can be obtained in the $\mu\tau$ block.

(c) The matrix has no special structure apart from some trend of increase of elements from $m_{\tau\tau}$ to m_{ee}, with m_{ee} being the largest one.

(d) Equality $m_{e\mu} \approx -m_{e\tau}$ can be achieved. Texture zeros $m_{e\mu}$ or $m_{e\tau}$ are possible for very small values of s_{13}.

The basis corrections for $s_b = 0.2$ equal $\Delta m_{ee}^b = -0.08$, $\Delta m_{\mu e}^b = -0.47$, $\Delta m_{e\tau}^b = -0.48$, and $\Delta m_{\mu\tau}^b = 0.48$ in the units 10^{-2} eV. They are of the order of the TBM corrections for the $e\mu$ and $e\tau$ elements.

2. $\phi_2 = \pi/2$, line $IH(\frac{\pi}{2}, 0)$:

(a) The parameters of the mass matrix

$$m_1 \approx \bar{m}, \qquad m_2 \approx -\bar{m}, \qquad a \approx \bar{m}/3,$$
$$b \approx -\frac{2\bar{m}}{3}, \qquad c = 0$$

give the TBM matrix that equals

$$m_\nu \approx \bar{m}\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \dots & -\frac{1}{6} & -\frac{1}{6} \\ \dots & \dots & -\frac{1}{6} \end{pmatrix}.$$

Now the elements of the e row dominate.

(b) Lowest order corrections equal

$$x = \frac{\bar{m}}{3}\left(\frac{1}{\sqrt{2}} s_{13}e^{-i\delta} - 2D_{23}\right),$$
$$y = -\frac{\bar{m}}{3}(2\sqrt{2}s_{13}e^{i\delta} - D_{23}),$$
$$\Delta m_{ee} = \bar{m}\left(2D_{12} - \frac{1}{3}s_{13}^2\right),$$
$$\Delta m_{\mu\tau} = -\bar{m}\left(D_{12} + \frac{1}{6}s_{13}^2\right).$$

For the bf (and 1σ) values of the mixing parameters, we have $x \sim -0.25(-0.5)$, $y \sim 0.2(0.3)$, $\Delta m_{ee} \sim 0.2(0.3)$ in the units 10^{-2} eV. The maximal values of the corrections can be achieved for $\delta = \pi$, see Table VI.

The overall structure of the mass matrix does not change substantially. Corrections correlate with zero order structure being proportional to the same $\bar{m}$:

$$m_\nu \approx \frac{1}{3}\bar{m}\begin{pmatrix} 1 + 6D_{12} - s_{13}^2 & -2 + \frac{1}{\sqrt{2}}s_{13}e^{-i\delta} - 2D_{23} & -2 - \frac{1}{\sqrt{2}}s_{13}e^{-i\delta} + 2D_{23} \\ \dots & -\frac{1}{2} - 2\sqrt{2}s_{13}e^{i\delta} + D_{23} & -\frac{1}{2} \\ \dots & \dots & -\frac{1}{2} + 2\sqrt{2}s_{13}e^{i\delta} - D_{23} \end{pmatrix}.$$

(c) The parameters of violation of the TBM conditions: Since the $e\mu$ and $e\tau$ elements are dominant, the relative corrections are small. Indeed, the "pole" value of s_{13} equals

$$\tilde{s}_{13} \approx \frac{2s_{12}c_{12}}{(1 - 2s_{12}^2)} \sim 3,$$

and for the allowed range, $s_{13} \ll \tilde{s}_{13}$, the TBM violation is suppressed:

$$\Delta_e \approx \frac{2s_{13}}{\tilde{s}_{13}} \approx \frac{2}{3}s_{13}.$$

In contrast, since the original elements of the $\mu\tau$ block are suppressed, the relative corrections to $m_{\mu\mu}$ and $m_{\tau\tau}$ can be large, thus strongly violating the second TBM condition. For $\Delta_{\mu\tau}$, we find the pole at $\tilde{D}_{23} = \frac{1}{2} - s_{13}\tan 2\theta_{12}e^{i\delta}$. At the 1σ level, $\tilde{D}_{23} \approx 0.09$, and one can achieve $m_{\tau\tau} = 0$, as is shown in Table V.

For $D_{23} = 0$, we have

$$\Delta_{\mu\tau} \approx \frac{4s_{13}\sin 2\theta_{12}}{\cos 2\theta_{12} - 2s_{13}\sin 2\theta_{12}}.$$

This dependence has a pole at $s_{13} = 0.5\cot 2\theta_{12} \approx 0.17$–$0.20$, at the maximal allowed values of the 1–3 mixing. Thus, $\Delta_{\mu\tau} \to \infty$ and the violation of the TBM structure is strongly enhanced. According to (43), the s_{13}corrections are enhanced by the additional factor $2\sqrt{2} \sim 3$. For smaller values of s_{13}, $\Delta_{\mu\tau} \approx 4s_{13}\tan 2\theta_{12}$.

(d) Properties of the mass matrix:

(a) The matrix can show "inverted flavor hierarchy" with $m_{\tau\tau}$ being the smallest element;

(b) Depending on δ, $m_{\mu\mu} = 0$ or $m_{\tau\tau} = 0$ texture zero can be obtained at the 1σ level.

The basis corrections are $\Delta m_{ee}^b = 1.16$, $\Delta m_{\mu e}^b = -0.19$, $\Delta m_{e\tau}^b = 0.09$, and $\Delta m_{\mu\tau}^b = -0.22$ in units of 10^{-2} eV. The correction to m_{ee} is large.

D. Degenerate spectrum

In the case of degenerate spectrum, $m_1 \approx m_2 \approx m_3 \approx m_0$, the structure of the mass matrix depends strongly on values of both Majorana phases.

1. $\phi_2 = \phi_3 = 0$, line $D(0, 0)$.

(a) The parameters of the mass matrix equal

$$a \approx m_0, \qquad b \approx \epsilon_S \equiv \frac{\Delta m_{21}^2}{6m_0}, \qquad c = m_0 + \epsilon_A,$$
$$a + b + c = 2m_0 + \epsilon_A,$$
$$a + b - c \approx -\epsilon_A \equiv -\frac{\Delta m_{31}^2}{2m_0}.$$

Numerically, for $m_0 = 0.2$ eV we find $\epsilon_S = 6.7 \cdot 10^{-5}$ eV and $\epsilon_A = 6.2 \cdot 10^{-3}$ eV. The TBM-mass matrix is very close to the unit matrix:

$$m_\nu \approx \begin{pmatrix} m_0 & \epsilon_S & \epsilon_S \\ \dots & m_0 + \frac{1}{2}\epsilon_A & -\frac{1}{2}\epsilon_A \\ \dots & \dots & m_0 + \frac{1}{2}\epsilon_A \end{pmatrix} = m_0 I + \begin{pmatrix} 0 & \epsilon_S & \epsilon_S \\ \dots & \frac{1}{2}\epsilon_A & -\frac{1}{2}\epsilon_A \\ \dots & \dots & \frac{1}{2}\epsilon_A \end{pmatrix}.$$

Furthermore, there is a strong hierarchy of the subleading (off-diagonal) elements.

(b) The lowest order corrections: Neglecting terms proportional to ϵ_S, we find

$$x = -\frac{1}{\sqrt{2}}s_{13}[(m_0 + \epsilon_A)e^{-i\delta} - m_0 e^{i\delta}],$$
$$y = -\epsilon_A D_{23},$$
$$\Delta m_{ee} = s_{13}^2[(m_0 + \epsilon_A)e^{-2i\delta} - m_0],$$

and the size of corrections strongly depends on δ:

$$x \approx \begin{cases} -\frac{1}{\sqrt{2}}s_{13}\epsilon_A & \delta = 0 \\ i\sqrt{2}s_{13}m_0 & \delta = \pi/2 \end{cases}.$$

For $\delta = \pi/2$, the correction Δm_{ee} is the maximal: $\Delta m_{ee} = -2m_0 s_{13}^2$.

(c) Parameters of the violation of the TBM conditions: For $\phi_2 = 0$, the $e\mu$ and $e\tau$ elements are very small, so that they can be canceled at very small $\tilde{s}_{13}$.

Indeed,

$$\tilde{s}_{13} \approx s_{12}c_{12}\frac{\Delta m_{21}^2}{\Delta m_{31}^2} = 0.015.$$

Correspondingly, the singularity and the peak move to small values of s_{13}.

The pole value $\tilde{D}_{23} \approx -1/s_{13}^2 e^{2i\delta} \to \infty$, so that

$$\Delta_{\mu\tau} \approx D_{23}\frac{\Delta m_{31}^2}{2m_0^2}$$

turns out to be strongly suppressed as a consequence of the dominance of the $\mu\mu$ and $\tau\tau$ elements. For $s_{13} \neq 0$ and $D_{23} = 0$, $\Delta_{\mu\tau} \approx s_{13}\sin 2\theta_{12}\Delta m_{21}^2/2m_0^2$, and for $m_0 = 0.2$ eV, the breaking is very strongly suppressed due to the smallness of b: $\Delta_{\mu\tau} \sim -10^{-3}s_{13}$.

(d) Properties of the mass matrix:

(a) Corrections do not affect the dominant elements but can change completely the subdominant structure.

(b) The only significant change in the neutrino mass matrix is the violation of the equality of $m_{e\mu}$ and $m_{e\tau}$: $\Delta_e \approx 2$ can be achieved, which corresponds to $m_{e\tau} \simeq -m_{e\mu}$:

$$m_\nu \simeq m_0 I + \begin{pmatrix} \Delta m_{ee} & x & -x \\ \dots & y & -\frac{1}{2}\epsilon_A + \Delta m_{\mu\tau} \\ \dots & \dots & -y \end{pmatrix}.$$

Notice that due to corrections the elements of the second subdominant matrix in (43) can be of the same order: $|x| \sim |y| \sim \epsilon_A$, or can obey certain symmetry.

(c) Since $\tilde{s}_{13}$ is very small, all special mass relations indicated in Table III, including texture zeros, can be achieved.

The basis corrections equal $\Delta m_{ee}^b = 0.01$, $\Delta m_{\mu e}^b = 0.06$, $\Delta m_{e\tau}^b = 0.06$, and $\Delta m_{\mu\tau}^b = -0.06$ (in the units 10^{-2} eV). They are of the order of the TBM deviation corrections for the $e\mu$ and $e\tau$ elements.

2. $\phi_2 = 0$, $\phi_3 = \pi/2$; line $D(0, \frac{\pi}{2})$.

(a) The parameters of the TBM-mass matrix

$$a \approx m_0, \qquad b \approx \epsilon_S, \qquad c = -m_0 - \epsilon_A,$$
$$a + b + c \approx -\epsilon_A, \qquad a + b - c \approx 2m_0 + \epsilon_A,$$
$$a - c = 2m_0 + \epsilon_A$$

give the TBM matrix

$$m_\nu \approx \begin{pmatrix} m_0 & \epsilon_S & \epsilon_S \\ \dots & -\frac{1}{2}\epsilon_A & m_0 + \frac{1}{2}\epsilon_A \\ \dots & \dots & -\frac{1}{2}\epsilon_A \end{pmatrix} = m_0 T + \begin{pmatrix} 0 & \epsilon_S & \epsilon_S \\ \dots & -\frac{1}{2}\epsilon_A & \frac{1}{2}\epsilon_A \\ \dots & \dots & -\frac{1}{2}\epsilon_A \end{pmatrix},$$

where T is the "triangle" matrix with the only nonzero elements $m_{ee} = m_{\mu\tau} = m_{\tau\mu}$. The elements of the matrix are strongly hierarchical.

(b) The lowest order corrections equal

$$x = \frac{1}{\sqrt{2}}s_{13}[(m_0 + \epsilon_A)e^{-i\delta} + m_0 e^{i\delta}],$$
$$y = -2m_0 D_{23}, \qquad \Delta m_{ee} = 2s_{13}^2 m_0.$$

The largest deviation is for $\delta = 0$: $x \approx \sqrt{2}s_{13}m_0$.

(c) The parameters of violation of the TBM conditions:

$$\tilde{s}_{13} \approx s_{12}c_{12}\frac{\Delta m_{21}^2}{4m_0^2} = 2.5 \cdot 10^{-4}$$

for $m_0 = 0.2$ eV. The reason for this smallness is that the original elements of the e row are very small. For the allowed values of s_{13}, the maximal TBM violation, $\Delta_e \approx 2$, can be nearly achieved, and all of the special mass relations in Table III can be realized.

The $\mu\mu$ and $\tau\tau$ elements are strongly suppressed, and the corrections dominate. The pole of $\Delta_{\mu\tau}$, which corresponds to $m_{\tau\tau} = 0$, is at

$$\tilde{D}_{23} \approx -\frac{\Delta m_{31}^2}{8m_0^2} = -0.008$$

and is achieved for the negative values of D_{23} ($\theta_{23} > \pi/4$). Then from (35), we obtain

$$\Delta_{\mu\tau} \approx 2\frac{D_{23}}{D_{23} + \frac{\Delta m_{31}^2}{8m_0^2}(1 + 2D_{23})}.$$

Numerically, we have $\tilde{D}_{23} = -(0.03, 0.0075, 0.0033)$ for $m_0 = (0.1, 0.2, 0.3)$ eV, correspondingly. This pole value is well within the 1σ allowed range for D_{23}, so, all of the special mass relations in Table IV can be obtained. In particular, for positive D_{23} at $D_{23} \approx -\tilde{D}_{23}$, we have $\Delta_{\mu\tau} = 1$ which corresponds to $m_{\mu\mu} = 0$. For $|D_{23}| \gg |\tilde{D}_{23}|$, $\Delta_{\mu\tau} \to 2$ independently of the sign of D_{23}. This value of $\Delta_{\mu\tau}$ corresponds to $m_{\mu\mu} \simeq -m_{\tau\tau}$. Notice that for $m_0 = 0.2$ eV and 1σ allowed D_{23}, the ratio $D_{23}/\tilde{D}_{23} \approx 12$, so that the limit can be realized with a good accuracy.

For nonzero 1–3 mixing but $D_{23} = 0$,

$$\Delta_{\mu\tau} \approx 2s_{13}\sin 2\theta_{12}\frac{\Delta m_{21}^2}{4m_0^2 s_{13}^2 + \Delta m_{21}^2 c_{12}^2}.$$

For $s_{13} \gg c_{12}\sqrt{\Delta_{21}^2}/2m_0 \sim 10^{-2}$, we have

$$\Delta_{\mu\tau} \approx \frac{\sin 2\theta_{12}}{s_{13}}\frac{\Delta m_{21}^2}{2m_0^2}.$$

The deviation increases with a decrease of s_{13};

however, even in the maximum $\Delta_{\mu\tau}$ does not exceed 0.03.

(d) Properties of the mass matrix:

(a) With corrections the neutrino mass matrix takes the following form:

$$m_\nu \approx m_0 \begin{pmatrix} 1 & \sqrt{2}s_{13} & -\sqrt{2}s_{13} \\ \dots & 2D_{23} - \frac{1}{2m_0}\epsilon_A & 1 + \frac{1}{2m_0}\epsilon_A \\ \dots & \dots & -2D_{23} - \frac{1}{2m_0}\epsilon_A \end{pmatrix}.$$

Here two TBM conditions are maximally broken; however, $x \ll a$. Corrections have a completely different symmetry from that of the dominant block which has the triangle form.

(b) Corrections to the dominant triangle structure can all be of the same order and of the size of the Cabibbo angle with respect to the dominant structure

$$m_\nu = m_0[T + 0.2D],$$

where D is the democratic matrix or matrix with elements of the same order.

(c) Since both $\tilde{s}_{13}$ and $\tilde{D}_{23}$ are very small, special relations for elements of the e line and $\mu\tau$ block can be satisfied simultaneously.

The basis corrections are $\Delta m^b_{ee} = -0.81$, $\Delta m^b_{\mu e} = -4.0$, $\Delta m^b_{e\tau} = -4.1$, and $\Delta m^b_{\mu\tau} = 4.1$ (in the units 10^{-2} eV). The corrections to the $e\mu$ and $e\tau$ elements are of the order of the corrections due to the TBM-mixing deviations.

3. $\phi_2 = \pi/2$ and $\phi_3 = 0$; line $D(\frac{\pi}{2}, 0)$.

(a) The parameters of the mass matrix

$$m_2 \approx -m_0, \qquad a = \frac{m_0}{3}, \qquad b = -\frac{2m_0}{3},$$
$$c = m_0, \qquad a + b + c \approx \frac{2m_0}{3},$$
$$a + b - c \approx -\frac{4m_0}{3}$$

lead to TBM matrix

$$m_\nu \approx m_0 \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \dots & \frac{1}{3} & -\frac{2}{3} \\ \dots & \dots & \frac{1}{3} \end{pmatrix} = m_0 I - \frac{2}{3} m_0 D, \tag{43}$$

where D is the democratic matrix.

(b) The lowest order corrections equal

$$x = -\frac{1}{\sqrt{2}} s_{13}\left(e^{-i\delta} - \frac{1}{3}e^{i\delta}\right)m_0 - \frac{2}{3}D_{23}m_0,$$
$$y = -\frac{2}{3}m_0(2D_{23} + \sqrt{2}s_{13}e^{i\delta}),$$
$$\Delta m_{ee} = 2m_0\left(D_{12} + \frac{1}{3}s^2_{13}\right),$$
$$\Delta m_{\mu\tau} = m_0\left(-D_{12} + \frac{1}{3}s^2_{13}\right).$$

The relative corrections are enhanced because the elements of the original matrix are suppressed by numerical factors. The corrections equal 50% (100%) for y, 20% (30) for x and 10% (20%) for the ee element.

(c) The violation of the TBM conditions: The original elements $e\mu$ and $e\tau$ are large and s_{13} produces a relatively small effect. The pole value equals $\tilde{s}_{13} \approx \cot\theta_{12} > 1$; as a result, for the allowed values of s_{13} the breaking parameter is suppressed:

$$\Delta_e = \frac{2s_{13}}{\cot\theta_{12}e^{i\tilde{\phi}} - s_{13}} \approx 2s_{13}\tan\theta_{12}. \tag{44}$$

The pole of $\Delta_{\mu\tau}$ is given by

$$\tilde{D}_{23} = -\tfrac{1}{2}\tan^2\theta_{12}(1 + 2\cot\theta_{12}s_{13}e^{i\delta})$$
$$\approx -\tfrac{1}{4}(1 + 2\sqrt{2}s_{13}e^{i\delta}).$$

Consequently, $\tilde{D}_{23} \gg D^{\max}_{23}$ for $\delta = 0$. The minimal value of $\tilde{D}_{23}$ is realized at $\delta = \pi$ and the maximal possible s_{13}: $\tilde{D}_{23} \sim 0.1$ for which $\tilde{D}_{23} \approx D^{\max}_{23}$. Therefore, one can reach the pole and $m_{\tau\tau} \approx 0$. Since $\phi_2 = \pi/2$, the parameter β is not suppressed and becomes important:

$$\beta = \frac{s_{13}\sin 2\theta_{12}e^{i\delta}}{2(1 + \cos 2\theta_{12})}.$$

Texture zero $m_{\mu\mu} = 0$ is realized if $D_{23} = -(\tilde{D}_{23} + \beta)$. For $\delta = 0$, we obtain at the 1σ level $-(\tilde{D}_{23} + \beta) \approx 0.085$, which is close to the 1σ allowed value of D_{23}. Consequently, $m_{\mu\mu} \approx 0$ can be achieved at the 1σ level, as can be seen in Table V. For $s_{13} = 0$, we have

$$\Delta_{\mu\tau} \approx 4\frac{D_{23}}{\tan^2\theta_{12} + 2D_{23}}. \tag{45}$$

For negative D_{23}, the deviation can be substantially enhanced ($\sim 12D_{23}$); however, the pole is still not realized.

The 1–3 mixing effect (for $D_{23} = 0$) on violation of the second TBM condition is given by

$$\Delta_{\mu\tau} \approx 4s_{13}\frac{c_{12}}{s_{13} + 2s_{13}c_{12}} \approx 4s_{13}\cot\theta_{12}.$$

Here the correction is enhanced by the factor $4\cot\theta_{12} \sim 6$, so that for the maximal allowed 1–3 mixings, we obtain $\Delta_{\mu\tau} \approx 1$.

(d) Properties of the mass matrix:
(a) The $e\mu$ and $e\tau$ elements can differ by (50–60)%,
(b) Texture zeros $m_{\mu\mu} = 0$ or $m_{\tau\tau} = 0$ can be achieved;
(c) The equalities $m_{ee} \approx -m_{e\tau}$, $m_{\mu\tau} = m_{\tau\tau}$ are possible;
(d) The matrix may have a rather random anarchical character;
(e) At the 1σ level, the structure of the matrix can change strongly, and the TBM conditions can be strongly broken.

The basis corrections are $\Delta m^b_{ee} = 5.2$, $\Delta m^b_{\mu e} = 1.1$, $\Delta m^b_{e\tau} = -2.4$, and $\Delta m^b_{\mu\tau} = 2.98$ (in the units 10^{-2} eV). They are of the order of the TBM corrections for $e\mu$ and $e\tau$ elements and large for ee elements.

4. $\phi_2 = \phi_3 = \frac{\pi}{2}$, line $(D(\frac{\pi}{2}, \frac{\pi}{2}))$:
(a) The parameters of the mass matrix

$$m_1 = m_0, \qquad m_2 \approx -m_0 - \epsilon_S,$$
$$m_3 \approx -m_0 - \epsilon_A, \qquad a = \frac{m_0}{3}, \qquad b = -\frac{2m_0}{3},$$
$$c = -m_0, \qquad a + b + c \approx -\frac{4m_0}{3},$$
$$a + b - c \approx \frac{2m_0}{3}$$

give the TBM-mass matrix

$$m_\nu \approx m_0 \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \dots & -\frac{2}{3} & \frac{1}{3} \\ \dots & \dots & \frac{2}{3} \end{pmatrix} = -m_0 V_1 = m_0 T - \frac{2}{3} m_0 D,$$

where D is the democratic matrix. This matrix differs from the one in the previous case by permutation in the $\mu\tau$ block. It is proportional to the symmetry matrix V_1.

(b) The lowest order corrections equal

$$x = \frac{1}{\sqrt{2}} s_{13}\left(e^{-i\delta} + \frac{1}{3} e^{i\delta}\right) m_0 - \frac{2}{3} D_{23} m_0,$$
$$y = \frac{2}{3} m_0 (D_{23} + \sqrt{2} s_{13} e^{i\delta}),$$
$$\Delta m_{ee} = 2m_0\left(D_{12} - \frac{2}{3} s_{13}^2\right).$$

The deviation x is enhanced if $\delta = \pi$:

$$x = -\tfrac{2}{3} m_0 (\sqrt{2} s_{13} + D_{23}).$$

In this case, $y = \frac{2}{3} m_0 (D_{23} - \sqrt{2} s_{13})$. All the elements of the TBM matrix are of the same order and just differ by factor 2. The elements of the e raw and $\mu\tau$ block affected by the corrections are large, and therefore the effect of the corrections is relatively small: for the bf values and 1σ, we have 12% (25%) for y ($\mu\tau$ block), and 25% (45%) for x and 8% (10%) for the ee element.

(c) The parameters of violation of the TBM conditions: Now $\tilde{s}_{13} \approx \tan\theta_{12}$, so that

$$\Delta_e = \frac{2s_{13}}{\tan\theta_{12} e^{i\tilde{\phi}} - s_{13}} \approx 2s_{13}\cot\theta_{12}. \qquad (47)$$

The violation parameter $\Delta_{\mu\tau}$ equals

$$\Delta_{\mu\tau} \approx 4D_{23} \frac{D_{23}}{\cot^2\theta_{12} + 2D_{23}}.$$

Here enhancement is weaker than in the previous case. For $D_{23} = 0$, we have

$$\Delta_{\mu\tau} \approx 4s_{13} \frac{s_{12}}{c_{12} - 2s_{13}s_{12}} \approx 4s_{13}\tan\theta_{12}.$$

Since

$$\tilde{D}_{23} = -\frac{c_{12}^2 - \sin\theta_{12} s_{13} e^{i\delta}}{2s_{12}^2}, \qquad |\tilde{D}_{23}| \gg D_{23}^{\max},$$

no zeros can be obtained in the $\mu\tau$ block.

(d) Properties of the mass matrix: It may have the form

$$m_\nu \approx \begin{pmatrix} a & y & z \\ y & z & a \\ z & a & y \end{pmatrix} + \delta m_\nu = m_0 \begin{pmatrix} \frac{1}{3} & -\frac{2}{3}(1 + D_{23}) + \frac{\sqrt{2}}{3}\sin 2\theta_{13} & -\frac{2}{3}(1 - D_{23}) - \frac{\sqrt{2}}{3}\sin 2\theta_{13} \\ \dots & -\frac{2}{3} + \frac{y}{m_0} & \frac{1}{3} \\ \dots & \dots & -\frac{2}{3} - \frac{y}{m_0} \end{pmatrix}.$$

This matrix has approximate cyclic symmetry and the elements of the second diagonal are equal (see also the line $D(\frac{\pi}{2}, \frac{\pi}{2})$ in Table V).

If $\delta = 0$, then y is enhanced: $y = \frac{2}{3} m_0 (D_{23} + \sqrt{2} s_{13})$, and the two contributions sum up. At the same time, in x the two contributions partially cancel each other. The elements of the mass matrix have a rather random spread within factor 3, without a clear structure. The TBM conditions are broken by $O(1)$ factors.

The basis corrections for $s_b = 0.2$ equal $\Delta m^b_{ee} = 4.4$, $\Delta m^b_{\mu e} = -2.9$, $\Delta m^b_{e\tau} = -1.6$, and $\Delta m^b_{\mu\tau} = 1.1$ (in the units 10^{-2} eV). They are large for the ee element and significant for the $e\mu$ and $e\tau$ elements.

IV. DEVIATIONS FROM THE TBM AND FLAVOR SYMMETRY

Using results of the previous sections, we will consider implications of the mass matrices with deviations from the TBM structure for the flavor symmetries.

Recall that the TBM as well as other flavor structures could be an immediate consequence of symmetry, if, e.g., (i) a single mechanism of neutrino mass generation dominates and various corrections are negligible; (ii) Higgses are flavorless, so that the problem of VEV alignment does not exist. In this case, one needs to adjust the Yukawa coupling constants only. It can be shown, however, that flavor symmetry, should be broken to explain TBM. The flavor structures that can be obtained in this scenario do not reproduce TBM but they can serve as the dominant structures of the mass matrix.

The operator responsible for the Majorana neutrino masses has the form

$$L = h_{ij} L_i L_j X,$$

where, in general, the ''Higgs factor'' X is some combination of the Higgs fields.

A. Deviations from TBM and new flavor symmetries?

Do neutrino mass matrices with deviations from TBM have some new symmetry which differs from the TBM symmetry? Here we briefly note some possibilities; their detailed realizations will be presented elsewhere [21].

As we have established in the previous sections, the corrections can lead to new equalities between the matrix elements. In particular,

$$m_{e\mu} \approx -m_{e\tau}, \tag{48}$$

as well as $m_{\mu\mu} = -m_{\tau\tau}$, $m_{e\mu} = m_{\mu\mu}$, $m_{e\mu} = n m_{e\tau}$ with, e.g., $n = 2$, $1/2$, etc. Let us consider implications of the equality (48).

If $b = m^0_{e\mu}$ and the deviation D_{23} are very small, then the correction x dominates, y is negligible, and furthermore, the corrections of the order $s_{13}D_{23}$, which contribute to $m_{e\mu}$ and $m_{e\tau}$ equally, are also small. In this case, the mass matrix has the following approximate form:

$$\begin{pmatrix} a & -x & x \\ \dots & \frac{1}{2}(a+c) & \frac{1}{2}(a-c) \\ \dots & \dots & \frac{1}{2}(a+c) \end{pmatrix}.$$

The conditions for this form of matrix are realized in the cases of spectra with quasidegenerate first and second states: $m_1 \approx m_2$ and $\phi_2 = 0$: $PD(0,0)$, $PD(0,\frac{\pi}{2})$, $IH(0,0)$, $D(0,0)$, $D(0,\frac{\pi}{2})$. [Notice that in Table V the examples of matrices correspond to the maximal allowed value of D_{23}, so that the correction $s_{13}D_{23}$ leads to the violation of equality (48)].

In the case of inverted mass hierarchy, $IH(0,0)$, also $c \simeq 0$. The matrix (49) is invariant under the transformation

$$V_2' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

which is one the generators of S_4. This is new residual symmetry, since the first TBM condition is broken by the 1–3 mixing. As we have mentioned before, the equality (48) and symmetry with respect to V_2 (9) can be restored by redefinition: $\nu_\mu \to -\nu_\mu$. In this case, the $\mu\tau$ element changes the sign: $\frac{1}{2}(a-c) \to -\frac{1}{2}(a-c)$ and the third TBM condition turns out to be broken: $\Sigma_L - \Sigma_R = a - c + x$. Now the matrix is invariant with respect to V_2 but not V_1.

In contrast to the TBM matrix, the matrices with deviations can contain texture zeros [22] and agree with observed neutrino masses. Interesting examples, which can testify for certain symmetries, follow:

(1) The texture zeros $m_{e\mu} = 0$ or $m_{e\tau} = 0$ can be achieved in the cases of normal mass hierarchy: $NH(0,0)$, $NH(0,\frac{\pi}{2})$, partial degeneracy: $PD(0,0)$, $PD(0,\frac{\pi}{2})$, inverted hierarchy: $IH(0,0)$, degenerate spectrum: $D(0,0)$, $D(0,\frac{\pi}{2})$. In all these cases, the original elements of the e line are small.

(2) The texture zeros $m_{\mu\mu} = 0$ or $m_{\tau\tau} = 0$ can be obtained in the cases of inverted mass hierarchy $IH(\frac{\pi}{2},0)$, and degenerate spectrum $D(0,\frac{\pi}{2})$. The condition for that is $m^0_{\mu\mu} = m^0_{\tau\tau} \ll m^0_{\mu\tau}$.

(3) Matrices with two texture zeros become allowed: various combinations of zeros in the e line and $\mu\tau$ block (indicated above) can be obtained for the degenerate spectrum $D(0,\frac{\pi}{2})$. In particular, in the case of very small 1–3 mixing ($s_{13} = 0.001$), $D_{23} \sim 0.032$, $\delta = 0$, and $m_1 = 0.09$ eV, all the elements of the second diagonal can be zero, $m_{\mu\mu} = m_{e\tau} = 0$. By changing the value of δ and the sign of D_{23}, we can change the positions of ''zeros.'' If $D_{23} \sim -0.032$, the two texture zeros $m_{\tau\tau} = m_{e\tau} = 0$ are achieved. If $\delta = \pi$, we get $m_{\mu\mu} = m_{e\mu} = 0$, so that the 1–2 mixing is induced. If $D_{23} \sim -0.032$ and $\delta = \pi$, we get $m_{\tau\tau} = m_{e\mu} = 0$.

(4) An interesting possibility is the matrix with two texture zeros: $m_{e\mu} = m_{ee} = 0$ which can be achieved in the case of normal mass hierarchy with $m_1 \sim 0.0031$ eV at the best fit values of the mixing angles and $(\phi_2, \phi_3, \delta) = (\pi/2, 0, \pi)$. This is the signature of yet another class of underlying symmetries.

(5) Also, the Fritzsch-type matrix with $m_{ee} = 0$, $m_{e\tau} = 0$ and relatively small $m_{\mu\mu}$ can be realized in the case of normal mass hierarchy and $m_1 \sim 0.0035$ eV at the best fit values of the mixing angles and $(\phi_2, \phi_3, \delta) = (\pi/2, 0, 0)$.

B. Two-component structure of the mass matrix

In a number of cases, the neutrino mass matrix has a strongly hierarchical structure with large elements forming the dominant block and small subdominant elements. This may indicate that the mass matrix has a two-component structure

$$m_\nu = M_d + \mu_s, \tag{49}$$

where M_d and μ_s are the dominant and subdominant contributions. The matrices M_d and μ_s may have different origins and different symmetries, the subdominant matrix μ_s may appear as a result of breaking of symmetry of M_d, and symmetry can be completely broken in μ_s.

As we have shown the relative corrections to the dominant block elements are of the order 30%, whereas corrections to the subdominant elements can be much larger than the original elements. Therefore, if the mass matrix, indeed, has two different contributions, the corrections can completely change the structure and possible symmetries of the subdominant matrix. There are different scenarios for (49). The dominant M_d can be a consequence of unbroken symmetry, whereas the subdominant block appears as a result of symmetry breaking.

Here we briefly consider possible symmetries which lead to various dominant structures:

(1) The $\mu - \tau$-dominant block (the case of normal mass hierarchy) has, e.g., the $U(1)$ symmetry with the charge prescriptions $L(\nu_e) = 1$, $L(\nu_\mu) = L(\nu_\tau) = 0$ [23].

(2) The matrix with the dominant block, which consists of the $\mu\mu$, $\tau\tau$, $\mu\tau$, and ee elements, is realized in the case of partially degenerate spectrum $PD(0, 0)$. It is invariant under

$$\nu_e \to \nu_e, \qquad \nu_\mu \to -\nu_\mu, \qquad \nu_\tau \to -\nu_\tau.$$

Clearly this symmetry cannot be the exact symmetry of the whole Lagrangian, but it can appear as a residual summery for neutrino Yukawa couplings.
The matrix proportional to the unit matrix, $M_d = m_0 I$, is the dominant structure for the degenerate spectrum $D(0, 0)$. It can be a consequence of various discrete and continuous symmetries. Suppose the lepton doublets, L_i, form a triplet of some symmetry group G_f: $L \sim \mathbf{3}$, and Higgses are flavorless. Then to get the invariant combination $\mathbf{3} \times \mathbf{3} \sim \mathbf{1}$, the group G_f should be $SO(3)$ or some of its subgroup. The smallest group with irreducible representation $\mathbf{3}$ is A_4, and the invariant combination $L_i L_i$ produces the required unit matrix.
Suppose the Higgs factor X is a singlet of the symmetry group but not invariant, e.g., $X \sim \mathbf{1}'$ or $X \sim \mathbf{1}''$ of A_4, then $L_i L_j$ should transform as $\mathbf{1}''$ and $\mathbf{1}'$, correspondingly. These combinations produce either zero mass (because of the antisymmetric nature of couplings) or the matrix proportional to the diagonal phase matrix: $m_\nu = m_0 \mathrm{diag}(1, e^{2i\pi/3}, e^{4i\pi/3})$.

(3) The triangle matrix $M_d = m_0 T$ is the dominant structure in the case of degenerate spectrum $D(0, \frac{\pi}{2})$. This structure can be a consequence of discrete or continuous symmetries, as in the previous case. In particular, the A_4 model with triplet L_i, in the complex representation leads to the triangle form.

Also the triangle dominant structure with $m_{ee} \neq m_{\mu\tau}$ is possible in the case of deviation from TBM. This structure can be produced in models where ν_μ and ν_τ form a doublet of some (discrete) symmetry group: $L_1 \sim \mathbf{1}$ and $\tilde{L} = (L_2, L_3)^T \sim \mathbf{2}$. The neutrino mass matrix is diagonal for the real representation and of the triangle form for complex representation. Such a situation can be realized in the case of the S_3 group and its further embedding like S_4, etc.

If the lepton doublets transform as singlets of the symmetry group, e.g., $L_1 \sim \mathbf{1}$, $L_2 \sim \mathbf{1}'$, $L_3 \sim \mathbf{1}''$ (and $X \sim \mathbf{1}$), the neutrino mass matrix is of the triangle form

$$m_\nu = v \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & 0 & h_{23} \\ 0 & h_{23} & 0 \end{pmatrix},$$

where h_{11} and h_{23} can be of the same order.

A possibility to get some flavor structures immediately from symmetry is to use a single Yukawa coupling, but with X having nontrivial flavor structure. If $L \sim 3$ and $X \sim \mathbf{3}$, the neutrino mass matrix equals

$$m_\nu = h \begin{pmatrix} v_1 & v_3 & v_2 \\ v_3 & v_2 & v_1 \\ v_2 & v_1 & v_3 \end{pmatrix}. \tag{50}$$

The TBM form can be achieved if $v_2 = v_3$ but in this case $|m_1| = |m_2|$. With possible deviations from TBM, we can easily reproduce the required structure of (50) with $m_1 \neq m_2$. It appears in the case of the degenerate spectrum $D(\frac{\pi}{2}, \frac{\pi}{2})$ [see Eq. (48)]. The problem is reduced now to VEV alignment: $v_1 \approx 1/3$, $v_2 \approx -2/3 - x$, $v_2 \approx -2/3 + x$, and $x = -y$.

C. No-symmetry case

1. *Anarchical matrix* with random values of elements [24] is an extreme case. Matrix of this type appears for certain intervals of CP phases in the cases of partial degeneracy or degenerate spectra: $PD(\frac{\pi}{2}, 0)$, $D(\frac{\pi}{2}, \frac{\pi}{2})$, $D(\frac{\pi}{2}, 0)$ when the original TBM-mass matrix has no or weak hierarchy of elements. In these, the "random" mass matrix leads accidentally to the strong degeneracy of mass eigenstates. This implies fine-tuning, unless certain new symmetry is introduced. Alternatively, this can imply that the mixing comes from the charged lepton sector whereas the neutrino mass matrix has a diagonal quasidegenerate form and obeys certain symmetry.

There are various possible origins of the anarchical structure, for instance, the seesaw mechanism with many ($n \gg 3$) right-handed neutrinos. Another possibility is when two different and independent mechanisms give comparable contributions to the mass matrix. Each of these contributions separately may have rather regular structure.

D. Matrices with flavor alignment

There are two possibilities:

(1) Normal flavor alignment. In the case of normal mass hierarchy with $m_1 \neq 0$ the corrections due to deviations from TBM as well as basis corrections can wash out sharp differences between the elements of the $\mu\tau$ block and e line. As a result, one obtains a gradual decrease in the size of the elements from $m_{\tau\tau}$ to m_{ee}.

(2) In the case of inverted mass hierarchy (see $IH(\frac{\pi}{2}, 0)$), the corrections can produce an inverse flavor hierarchy when the values of the matrix elements increase , while moving from τ to μ flavors.

These possibilities may indicate some perturbative origins and a kind of the Froggatt-Nielsen mechanism [25] with a large expansion parameter.

V. CONCLUSION

Is the TBM mixing accidental? The question is reduced, essentially, to the question of whether this mixing immediately follows from some (broken) symmetry or other principle, or if it appears as a result of a many-step construction, and fixing various parameters by introduction of additional symmetries and structures.

The symmetry is formulated at the level of the mass matrix. Therefore, if the data imply a very specific mass matrix with small deviations from the TBM form, we can say that TBM is not accidental. We find the opposite: very strong deviations of m_ν from $m_{\rm TBM}$ and strong violations of the TBM conditions (immediate manifestation of the symmetry) are allowed. This can be considered as an indication that TBM is accidental. We find that a large variety of the mass matrices with deviations from TBM explain experimental data.

Strong deviations of m_ν from $m_{\rm TBM}$ open up a possibility of the some alternative approaches to explain the data. Namely, some other symmetry (which differs from the TBM symmetry) or other principle can be involved. For instance, matrices with texture zeros are allowed which indicates, e.g., $U(1)$ underlying symmetry. Also, matrices with different relations between the elements are possible, which testify for yet another class of symmetries.

We show that the mass matrix may show no trace of symmetry having random values of elements. However, this corresponds to the quasidegenerate spectrum which implies another way to explain the data. In some cases, the matrix has certain flavor alignment: gradual change of values of matrix elements from m_{ee} to $m_{\tau\tau}$.

For certain ranges of masses and CP phases, the mass matrix has a structure with a strong hierarchy between matrix elements: dominant and subdominant ones. We find that corrections can change the dominant elements by factors $O(1)$ and be much larger than the subdominant elements. This may support the idea of the two-component structure of the mass matrix when the dominant block has certain (unbroken) flavor symmetry and appears at the lowest renormalizable level, whereas the subdominant structures can be result of symmetry breaking by, e.g., high order operators with flavon fields.

If it turns out that these new approaches lead to a simpler and more straightforward explanation of the data, the TBM-symmetry approach will be disfavored.

The 1–3 mixing leads to the most strong corrections. So, forthcoming measurements of this mixing will play a crucial role in the understanding of the underlying physics [26]. Corrections to other angles produce a next order effect (as s_{13}^2), although in some cases they can be enhanced by additional numerical factors.

ACKNOWLEDGMENTS

The authors are grateful to M. Frigerio and S. Khalil for useful discussions.

[1] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B **458**, 79 (1999); Phys. Lett. B **530**, 167 (2002).

[2] A selective list of publications includes: E. Ma and G. Rajasekaran, Phys. Rev. D **64**, 113012 (2001); W. Grimus, A. S. Joshipura, S. Kaneko, L. Lavoura, and M. Tanimoto, J. High Energy Phys. 07 (2004) 078; J. Kubo, A. Mondragon, M. Mondragon, and E. Rodriguez-Jauregui, Prog. Theor. Phys. **109**, 795 (2003); **114**, 287(E) (2005); R. N. Mohapatra, M. K. Parida, and G. Rajasekaran, Phys. Rev. D **69**, 053007 (2004); C. Hagedorn, M. Lindner, and R. N. Mohapatra, J. High Energy Phys. 06 (2006) 042; I. de Medeiros Varzielas, S. F. King, and G. G. Ross, Phys. Lett. B **648**, 201 (2007); E. Ma, Phys. Lett. B **660**, 505 (2008); C. Luhn, S. Nasri, and P. Ramond, Phys. Lett. B **652**, 27 (2007); E. Ma and G. Rajasekaran, Phys. Rev. D **64**, 113012 (2001); E. Ma, Mod. Phys. Lett. A **17**, 627 (2002); K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. B **552**, 207 (2003); for recent publication and complete list of references see for instance, C. Hagedorn, S. F. King, and C. Luhn, arXiv:1003.4249; and the review G. Altarelli and F. Feruglio, arXiv:1002.0211.

[3] C. S. Lam, Phys. Lett. B **656**, 193 (2007); Phys. Rev. Lett.

101, 121602 (2008); Phys. Rev. D **78**, 073015 (2008); arXiv:0907.2206.

[4] W. Grimus, L. Lavoura, and P. O. Ludl, J. Phys. G **36**, 115007 (2009).

[5] G. L. Fogli, E. Lisi, A. Marrone, A. Palazzo, and A. M. Rottunno, Phys. Rev. Lett. **101**, 141801 (2008).

[6] M. C. Gonzalez-Garcia, M. Maltoni, and J. Salvado, arXiv:1001.4524.

[7] R. Wendell *et al.*, arXiv:1002.3471.

[8] R. d. A. Toorop, F. Bazzocchi, and L. Merlo, arXiv:1003.4502.

[9] A. Y. Smirnov, in *The 2nd International Workshop on Neutrino Oscillations in Venice (NO-VE 2003) 3–5 December 2003, Venice, Italy*, edited by M. Baldo Ceolin (Papergraf, Padua, Italy, 2003); M. Raidal, Phys. Rev. Lett. **93**, 161801 (2004); H. Minakata and A. Y. Smirnov, Phys. Rev. D **70**, 073009 (2004).

[10] S. Antusch, J. Kersten, and M. Lindner, Nucl. Phys. **B674**, 401 (2003); A. Dighe, S. Goswami, and S. Ray, Phys. Rev. D **79**, 076006 (2009).

[11] J. Barry and W. Rodejohann, Phys. Rev. D **81**, 093002 (2010).

[12] M. Honda and M. Tanimoto, Prog. Theor. Phys. **119**, 583 (2008).

[13] S. F. Ge, H. J. He, and F. R. Yin, J. Cosmol. Astropart. Phys. 05 (2010) 017.

[14] A. Hayakawa, H. Ishimori, Y. Shimizu, and M. Tanimoto, Phys. Lett. B **680**, 334 (2009).

[15] F. Plentinger and W. Rodejohann, Phys. Lett. B **625**, 264 (2005).

[16] S. K. Kang, Z.-z. Xing, and S. Zhou, Phys. Rev. D **73**, 013001 (2006).

[17] L. Merlo, Acta Phys. Pol. B **40**, 3179 (2009).

[18] C. H. Albright and W. Rodejohann, Phys. Lett. B **665**, 378 (2008); C. H. Albright and W. Rodejohann, Eur. Phys. J. C **62**, 599 (2009).

[19] M. Frigerio and A. Y. Smirnov, Nucl. Phys. **B640**, 233 (2002).

[20] C. D. Carone and R. F. Lebed, Phys. Rev. D **80**, 117301 (2009).

[21] M. Abbas and A. Y. Smirnov (work in progress).

[22] For references, see the recent paper R. Verma, G. Ahuja, N. Mahajan, M. Randhawa, and M. Gupta, arXiv:1004.5452.

[23] A. Palcu, Mod. Phys. Lett. A**22**, 939 (2007); G. Altarelli and R. Franceschini, J. High Energy Phys. 03 (2006) 047; W. Grimus and L. Lavoura, J. Phys. G **31**, 683 (2005); S. T. Petcov and W. Rodejohann, Phys. Rev. D **71**, 073002 (2005); K. S. Babu and R. N. Mohapatra, Phys. Lett. B **522**, 287 (2001).

[24] L. J. Hall, H. Murayama, and N. Weiner, Phys. Rev. Lett. **84**, 2572 (2000); A. de Gouvea and H. Murayama, Phys. Lett. B **573**, 94 (2003); G. Altarelli, F. Feruglio, and I. Masina, J. High Energy Phys. 01 (2003) 035; Y. E. Antebi, Y. Nir, and T. Volansky, Phys. Rev. D **73**, 075009 (2006).

[25] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. **B147**, 277 (1979).

[26] M. Mezzetto and T. Schwetz, arXiv:1003.5800.