Comment on "Semitransparency effects in the moving mirror model for Hawking radiation"

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Particle production by a semitransparent mirror accelerating on trajectories which simulate the Hawking effect was recently discussed in [3]. This author points out that some results in [1] are incorrect. We show here that, contrary to statements therein, the main results and conclusions of the last paper remain valid, only Eq. (41) there and some particular implication are not. The misunderstanding actually comes from comparing two very different parameter regions, and from the fact that, in our work, the word statistics was used in an unusual way related to the sign of the β-Bogoliubov coefficient, and not with its ordinary meaning, connected with the number of particles emitted per mode.

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In this comment we clarify and improve some issues discussed in [1,2] and compare the main results in these two works with the ones recently obtained in [3]. A careful examination will show that there is actually no contradiction between the results in the two cases, and that all the main conclusions of [1,2] hold valid. The only mistake in [1] is in the equation that gives the average number of produced particles per mode, which we correct here. Some misunderstanding came, in fact, from comparing (as was done in [3]) two separate parameter regions, and also from an unusual interpretation of the word statistics that we employed. More precisely, in both our papers we claimed that a semitransparent mirror radiates a thermal flux described by the Fermi-Dirac statistics as dictated by the value of the β-Bogoliubov coefficient (which was explicitly calculated by us for the first time ever, for a semitransparent mirror). However, as pointed out in [3], the word statistics is commonly related with the number of created particles per unit mode, and this originated confusion.

We first briefly review the results obtained in [1,2]. We studied there the particle spectrum produced by a semitransparent mirror, which is initially at rest, then accelerates during a large enough (but finite) time span, u_0, along the trajectory defined in [4,5]:

$$v = \frac{1}{k}(1 - e^{-ku}) \tag{1}$$

(in lightlike coordinates, where k is some frequency), and finally, for $u \geq u_0$, is left alone moving with constant velocity on an inertial trajectory. The main result in those papers was that in the physically realistic case of a partially transmitting mirror, where the reflection and transmission coefficients are those in the model proposed by Barton and Calogeracos in [6], [$r(\omega) = \frac{-i\alpha}{\omega+i\alpha}$ and $s(\omega) = \frac{\omega}{\omega+i\alpha}$ with $\alpha \geq 0$], by assuming that $\omega \sim k \sim \alpha \ll \omega'$ and making $u_0 \to \infty$, one gets

$$|\beta^{R,R}_{\omega,\omega'}|^2 \equiv |(\phi^{\mathrm{out}*}_{\omega,R}; \phi^{\mathrm{in}}_{\omega',R})|^2 \cong \frac{1}{2\pi\omega k}\left(\frac{\alpha}{\omega'}\right)^2 (e^{2\pi\omega/k} + 1)^{-1}, \tag{2}$$

where ω' denotes the frequency of ingoing particles (coming from the right null past infinity domain $\mathcal{J}^-_R$), and ω the frequency of the outgoing ones (particles going to the right null future infinity domain $\mathcal{J}^+_R$). Moreover, in order to obtain the radiation on the right-hand side of the mirror, we also needed to calculate the Bogoliubov coefficient: $\beta^{R,L}_{\omega,\omega'} \equiv (\phi^{\mathrm{out}*}_{\omega,R}; \phi^{\mathrm{in}}_{\omega',L})^*$, where now ω' denotes the frequency of a particle coming from the left null past infinity domain $\mathcal{J}^-_L$, and ω the frequency of a particle going to the right null future infinity domain $\mathcal{J}^+_R$. Assuming, once again, that $\omega \sim k \sim \alpha \ll \omega'$, and making $u_0 \to \infty$, we then obtained

$$|\beta^{R,L}_{\omega,\omega'}|^2 \sim \frac{1}{\omega\omega'}\mathcal{O}\left[\left(\frac{\alpha}{\omega'}\right)^2\right]. \tag{3}$$

From the results obtained from the square of the β-Bogoliubov coefficient, we then concluded that these semitransparent mirrors emit, on the right-hand side, a thermal radiation of scalar massless particles obeying Fermi-Dirac statistics. To repeat, this was based on the sign of the corresponding Bogoliubov coefficient. Once we obtained Eq. (2), as a mere application of our main results, i.e. Eqs. (2) and (3), we got Eq. (41) of [1]. Unfortunately, we applied our result in the wrong way.

In the comparison of the results it is important to realize that, in our paper the energy barrier α is *not* large, but just moderate, namely, of order ω. This last condition, although implicit from the calculations themselves, was not explicitly stated in our papers and is at the origin of the main confusion, on top of the particular use of the word *statistics*

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in both our articles, as coming from the β-Bogoliubov coefficient, as just mentioned.

On the other hand, in [3] the author studies the case α large but finite (more precisely, he considers the case $k \ll \alpha$). To repeat, this case had *not* been studied in [1,2], but indeed it can be derived straightforwardly, namely, from Eq. (27) of [2] we easily conclude that, when $\omega \ll \omega' \ll \alpha^2/k$ and $k \ll \omega' \ll \alpha^2/k$, for $u_0 \to \infty$, one has

$$|\beta^{R,R}_{\omega,\omega'}|^2 \cong \frac{1}{2\pi\omega' k}(e^{2\pi\omega/k} - 1)^{-1}, \tag{4}$$

that is, we then obtain the Bose-Einstein statistics. Moreover, when $\alpha^2/k \ll \omega'$ and $\omega \ll \omega'$, for $u_0 \to \infty$, one has

$$|\beta^{R,R}_{\omega,\omega'}|^2 \cong \frac{1}{2\pi\omega k}\left(\frac{\alpha}{\omega'}\right)^2(e^{2\pi\omega/k} + 1)^{-1}, \tag{5}$$

that is, one obtains the Fermi-Dirac one. Thus, the contradiction found in [3] disappears all at once for the β-Bogoliubov coefficient.

To make things even more explicit, we will now reproduce the result of [3], in the corresponding validity region. Let us obtain, for a partially transmitting mirror, the average number, $\mathcal{N}_\omega$, of produced particles in the mode ω at infinite times on the right-hand side of the mirror, which is $\mathcal{N}_\omega \cong \int_0^\infty d\omega' |\beta^{R,R}_{\omega,\omega'}|^2$. For large values of α (more precisely $\alpha \gg k$ and $\omega \sim k$), the domain $[0, \infty)$ can be split, for convenience, into the four disjoint sets: $[0, k/\eta)$, $[k/\eta, \eta\alpha^2/k)$, $[\eta\alpha^2/k, \alpha^2/(k\eta))$, and $[\alpha^2/(k\eta), \infty)$, where η is a dimensionless parameter satisfying $k/\alpha \ll \eta \ll 1$. From Eq. (27) of [2], it is not difficult to see that, in the first (resp. third) interval, one can use the bound $|\beta^{R,R}_{\omega,\omega'}|^2 \le K(\eta)\omega'/(k^2\omega)$ (resp$|\beta^{R,R}_{\omega,\omega'}|^2 \le C(\eta)/(\omega\omega')$), where $K(\eta)$ (resp $C(\eta)$) is a dimensionless constant independent of α. Then, using Eq. (4) in the second interval and (5) in the fourth one, we get

$$\begin{aligned} &\left|\mathcal{N}_\omega - \frac{1}{\pi k}(e^{2\pi\omega/k} - 1)^{-1}[\ln(\alpha/k) - \ln(\eta)] \right. \\ &\left. - \frac{\eta}{2\pi\omega}(e^{2\pi\omega/k} + 1)^{-1}\right| \le \frac{K(\eta)}{2\omega\eta^2} + \frac{2C(\eta)}{\omega}\ln(\eta). \end{aligned} \tag{6}$$

Finally, dividing by $T_\alpha \equiv k^{-1}\ln(\alpha/k)$, and making $\alpha \to \infty$, we have

$$\lim_{\alpha\to\infty}\frac{\mathcal{N}_\omega}{T_\alpha} = \frac{1}{\pi}(e^{2\pi\omega/k} - 1)^{-1}. \tag{7}$$

This result was obtained in a more elegant way in [3] and constitutes the main result of that paper. As we see, there is no difference here: exactly *the same* result is obtained from our model, in the appropriate region α *large*. Summing up, what is important to be stressed is that, in order to make comparisons, one should work in *the same regime*. As we have shown, for large values of α there is no contradiction between our results and those obtained in [3], contrary to the claims in this last paper.

To finish, consider Eq. (41) of [1], which is, we understand, the main argument for discrepancy. Although not explicitly stated in [1], to obtain this equation we assumed there that $k \ll \omega \sim \alpha$. In order to arrive at Eq. (41) of [1], one needs to split the interval $[0, \infty)$ into two domains, namely, $[0, k)$ and $[k, \infty)$. In the first, since $k \ll \omega$, one has $\omega' \ll \omega$ and thus the incident waves have very low frequency, which means that the mirror behaves like a perfect reflector. Then, using Eq. (13) of [1] (*not* Eq. (41) of [1]), we obtained

$$\int_0^k d\omega' |\beta^{R,R}_{\omega,\omega'}|^2 \sim \mathcal{O}\left(\frac{k^2}{\omega(\omega^2 + k^2)}\right). \tag{8}$$

And here comes our error. In the second interval we incorrectly used expression (5) [note that, one can only use this equation when $\omega' \gg \alpha^2/k$]. To do things properly, one has to split the domain $[k, \infty)$ into two disjoint intervals $[k, \alpha^2/(\eta k))$ and $[\alpha^2/(\eta k), \infty)$, where, η is a dimensionless parameter which satisfies $0 < \eta \ll 1$. It is in the second one where we can actually use (5). In [3] it has been pointed out that, in the case $k \ll \alpha$ and k comparable to ω, there exists a frequency cutoff, namely, $\omega'_{\max} = \alpha^2/\omega$, for which the contribution in $\mathcal{N}_\omega$ of the frequencies ω' larger than $\omega'_{\max}$ is practically zero. Then, although in [1] it was assumed that $k \ll \omega$, since $\omega'_{\max} \ll \alpha^2/(\eta k)$, one may conclude that the sector $[\alpha^2/(\eta k), \infty)$, where Eq. (5) is valid, does not contribute to the average number of produced particles in the mode ω, and consequently, Eq. (41) of [1] is clearly incorrect. In fact, the main contribution to the average number of produced particles per mode is in the sector $[k, \alpha^2/(\eta k))$, where the Eq. (4) is valid, and thus, the correct form for Eq. (41) of [1] is

$$\begin{aligned} \mathcal{N}_\omega &\cong \int_k^{\alpha^2/k}\frac{1}{2\pi\omega' k}(e^{2\pi\omega/k} - 1)^{-1}d\omega' \\ &= \frac{1}{\pi k}\ln(\alpha/k)(e^{2\pi\omega/k} - 1)^{-1}. \end{aligned} \tag{9}$$

Summarizing the discussion above, we briefly conclude, on the one hand, that Eq. (41) of [1] needs be modified. On the other, that anyway one can never draw from this equation the conclusion that, for large α the average number of created particles diverges like α^2, because this equation has been obtained by assuming that α is *not* large, but just moderate of the order $\alpha \sim \omega$. In other words, it corresponds to a *different* range of validity as the one considered in Ref. [3], which is what was missed in that paper.

Another conclusion is that the apparent disagreement between our papers and [3] is due to the specific meaning of the word statistics in our works. From the result (2) [explicitly calculated by us for the first time, for a semitransparent mirror], we claimed that a semitransparent mirror radiates a thermal flux obeying Fermi-Dirac statistics, based on the sign of the Bogoliubov coefficients. However, the author of [3] pointed out that the word statistics should refer to the number of produced particles per mode. Again, there is no discrepancy here, but a different use of the word statistics. It is clear that according to the functional form of the expressions for the mean particle number, the author of [3] is right.

A final remark is in order. An issue that has not been studied, either in [1,2] or in [3], is the calculation of $\mathcal{N}_\omega$ assuming that $\omega \sim k \sim \alpha$. In that case, Eq. (2) is valid in the sector $\omega' \gg \alpha$, and possibly its contribution to $\mathcal{N}_\omega$ becomes important. However, since we have not been able to obtain an analytic expression for the β-Bogoliubov coefficient in the domain $[0, \alpha]$ (maybe this is impossible to do), we cannot say anything about $\mathcal{N}_\omega$ in such case. Anyhow, this is a situation that deserves further investigation.

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