Non-Abelian Yang-Mills–Higgs vortices

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In this article we present new, genuinely non-Abelian vortex solutions in $SU(2)$ Yang-Mills–Higgs theory with only one *isovector* scalar field. These non-Abelian solutions branch off their Abelian counterparts (Abrikosov–Nielsen-Olesen vortices) for precise values of the Higgs potential coupling constant λ. For all values of λ, their energies lie below those of the Abelian energy profiles, the latter being logarithmically divergent as $\lambda \to \infty$. The non-Abelian branches plateau in the limit $\lambda \to \infty$ and their number increases with λ, this number becoming infinite. For each vorticity, the gaps between the plateauing energy levels become constant. In this limit the non-Abelian vortices are noninteracting and are described by the *self-dual* vortices of the $O(3)$ sigma model. In the absence of a topological lower bound, we expect these non-Abelian vortices to be *sphalerons*.

I. INTRODUCTION

Non-Abelian vortices on $\mathbb{R}^2$ have attracted interest since a very long time. Nambu [1] pointed out that vortices of finite length in $\mathbb{R}^3$ require monopoles at each end. Originally, they were proposed by Mandelstam [1] as flux tubes absorbed by non-Abelian ('t Hooft-Polyakov) monopoles at each end. In this picture the monopoles are bound, implying that in the dual picture where the duals of the monopoles are the quarks, one can describe confinement in QCD.

The 't Hooft-Polyakov monopole is a topologically stable and finite energy solution of the $SU(2)$ Yang-Mills–Higgs (YMH) system on $\mathbb{R}^3$, where the Higgs field takes its values in the algebra, i.e., that it is an isovector, $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$, under $SO(3)$ rotations. Topologically stable and finite energy vortex solutions of the gauged Higgs system on $\mathbb{R}^2$ on the other hand are supported by the Abelian Higgs model, where the Higgs field is a complex scalar, $\varphi = \phi^1 + i\phi^2$, i.e., it is an isovector $\phi^M = (\phi^1, \phi^2)$ under $SO(2)$ rotations. This is the Abrikosov–Nielsen-Olesen (ANO) vortex [2]. The field multiplets in the two models do not match.

To construct a non-Abelian vortex on $\mathbb{R}^2$, it was realized by Nielsen and Olesen that it is necessary to have a model with more than one Higgs field. They chose [2] two $SO(3)$ *isovector* Higgs fields, each with its own symmetry breaking potential and vacuum expectation value, but with the vacuum value of each oriented at different directions in isospace—in the simplest case being orthogonal to each other. In this way the $SO(3)$ gauge group is completely broken on the asymptotic circle of $\mathbb{R}^2$, which is necessary for topological stability. Subsequently, this construction was extended in models featuring N distinct Higgs fields, generalizing the $SU(2)$ vortices of [2] to $SU(N)$ in [3–9]. These vortices, described as Z_N vortices, are not genuinely non-Abelian since their flux is restricted to a single direction along the Cartan subalgebra.

More recently, this problem was considered in the context of $\mathcal{N} = 2$ supersymmetric QCD models by Hanay and Tong [10], by Auzzi *et al.* [11], and by Eto *et al.* [12]. The salient feature of these models is that they have both gauge and color symmetries that are broken by the condensate of the scalar fields in such a way that the unbroken subgroup results in orientational zero modes of the string, responsible for non-Abelian flux.

Non-Abelian vortices have been studied intensively in the context of dual confinement in QCD (see the reviews in [13,14]). In addition to this physical application, they present important examples of cosmic strings [15,16], relevant to cosmological phase transitions.

In this article, we have constructed non-Abelian vortices of a $SU(2)$ YMH model with only *one* algebra valued, i.e., isovector, Higgs field. (Non-Abelian vortices in the Weinberg-Salam model were constructed in [17].) This model features exactly the same field multiplets, on $\mathbb{R}^2$, as the YMH system supporting the 't Hooft-Polyakov monopole on $\mathbb{R}^3$, differing from the two-Higgs models of [2] and those supporting Z_N vortices, and obviously from the SQCD models of [10,11].

II. YMH MODEL AND NON-ABELIAN ANSATZ

Our model on $\mathbb{R}^2$ is described by the static Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}\text{Tr}F_{ij}^2 - \text{Tr}(D_i\Phi)^2 + (4\lambda)^2\,\text{Tr}(\tfrac{1}{4}v^2 + \Phi^2)^2, \quad (1)$$

where $\Phi = -\frac{i}{2}\vec{\phi} \cdot \vec{\sigma}$ is the anti-Hermitian isovector Higgs field, and $A_j = -\frac{i}{2}\vec{A}_j \cdot \vec{\sigma}$, with $\vec{\sigma} = (\sigma^M, \sigma^3)$, the Pauli matrices. The gauge field is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu + [A_\mu, A_\nu]$ and the gauge covariant derivative is given by $D_\mu = \partial_\mu + [A_\mu, \cdot]$. Note that only A_j, the

magnetic components of the $SU(2)$ connection $A_\mu = (A_0, A_j)$, appear in Eq. (1), since in the absence of a Chern-Simons term, the electric component of the connection A_0 vanishes, by virtue of the non-Abelian Julia-Zee theorem [18,19].

The energy of this model is not endowed with a topological lower bound and *a priori* we would not expect the resulting vortices to be topologically stable. But the question of stability is more subtle than this. The (genuinely) non-Abelian vortices we have constructed numerically, present bifurcations from the corresponding Abelian profiles, on plots of their energies *vs.* the Higgs self-interaction coupling constant λ. Remarkably, it turns out that each non-Abelian profile lies below the corresponding Abelian profile for all values of λ and hence cannot be expected to decay into the Abelian vortex with higher energy than it has. Indeed, we show that in the $\lambda \to \infty$ limit these non-Abelian vortices are described by the (stable) *self-dual* "instantons" [20] of the $O(3)$ sigma model on $\mathbb{R}^2$. However, since these Abelian vortices embedded in the non-Abelian theory at hand are known to be unstable, one cannot expect this feature of the non-Abelian vortices found here to imply stability. Furthermore, the Belavin-Polyakov "instantons" feature an arbitrary scale, which indicates instability. Thus, we would expect that our non-Abelian vortices are in effect, *sphalerons*. The quantitative stability anaysis will be carried out elsewhere.

The radial Ansatz we use is

$$
\begin{aligned}
\Phi &= vh\frac{\sigma_r^{(n)}}{2i} - vg\frac{\sigma^3}{2i}, \\
A_j &= -\frac{(\varepsilon\hat{x})_j}{r}\left(c\frac{\sigma_r^{(n)}}{2i} - (a + n)\frac{\sigma^3}{2i}\right), \qquad j = 1, 2,
\end{aligned}
\tag{2}
$$

where we denote $\sigma_r^{(n)} = \cos n\varphi\, \sigma^1 + \sin n\varphi\, \sigma^2$ and $(\varepsilon\hat{x})_j = (\sin\varphi, -\cos\varphi)$. Here, $\{a, c, g, h\}$ are functions of r only and the integer n is the vortex number. This Ansatz, previously used to construct non-Abelian Chern-Simons–Higgs vortices [21], is a consistent truncation of the most general Ansatz.

III. EQUATIONS OF MOTION

Subject to the Ansatz Eq. (2), the Euler-Lagrange equations reduce to the following set of nonlinear ordinary differential equations:

$$
\begin{aligned}
-r\left(\frac{a_r}{r}\right)_r &= -v^2(ah - cg)h, \\
-r\left(\frac{c_r}{r}\right)_r &= v^2(ah - cg)g, \\
(rh_r)_r &= \frac{1}{r}(ah - cg)a - 8v^2\lambda^2 r[1 - (h^2 + g^2)]h, \\
-(rg_r)_r &= \frac{1}{r}(ah - cg)c + 8v^2\lambda^2 r[1 - (h^2 + g^2)]g, \quad (3)
\end{aligned}
$$

together with the constraint equation

$$
v^2(hg_r - gh_r) - \frac{1}{r^2}(ac_r - ca_r) = 0. \tag{4}
$$

The subscript r denotes ordinary differentiation with respect to r. The energy density now reads

$$
\begin{aligned}
\mathcal{E} = \frac{1}{4r^2}(a_r^2 + c_r^2) + \frac{1}{4}v^2\Big[(h_r^2 + g_r^2) + \frac{1}{r^2}(ah - cg)^2 \\
+ 4v^2\lambda^2[1 - (h^2 + g^2)]^2 \Big],
\end{aligned}
\tag{5}
$$

the total energy E given by $E = 2\pi \int r\mathcal{E}dr$.

The embedded Abelian solutions, namely, the solutions to the embedded Abelian subsystem, correspond to the truncation $\{c = 0, g = 0\}$. These are the ANO vortices which play an important role in the classification of the non-Abelian vortices we have constructed. In particular we will study the dependence of these on the parameter λ, so it is pertinent at this point to note that the critical configuration of the Abelian vortices corresponds to the value $\lambda = \frac{1}{4}$. This is the Bogomol'nyi limit where the Abelian vortices do not interact.

IV. NUMERICAL RESULTS

In order to generate vortex solutions to Eqs. (3) and (4), we impose boundary conditions such that the energy of the solutions is finite and both the gauge and the Higgs field functions are regular at the origin. The system of equations is solved numerically by means of a collocation method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure.

The only free parameters are n and λ, since we fixed the unit of length by setting $v = 1$ in what follows. For fixed finite values of these parameters, only a finite number of regular solutions exist. There always exists one Abelian solution (ANO solution) for any n and $\lambda(\neq 0)$. For small values of λ, this is the only possible solution. However, as λ increases new non-Abelian solutions branch off the Abelian ones. With increasing λ, more and more non-Abelian branches appear, their number becoming infinite for $\lambda = \infty$. For given n, all of the non-Abelian solutions have energy lower than that of their Abelian counterparts for each value of λ. This branch structure of the solutions is exhibited in Fig. 1 where the energy per vortex number, E/n, is plotted versus the constant λ for $n = 1, 2$.

In this figure we observe the first two non-Abelian branches for $n = 1$ and the first three ones for $n = 2$. The lowest non-Abelian branch branches off the Abelian solutions at $\lambda \approx 3.705, 0.975$, for $n = 1, 2$, respectively. It is clearly seen that higher values of the vorticity n allow for new non-Abelian branches at lower values of λ. This is more explicitly shown in Fig. 2, where the locations of the first branching points for $n = 1, 2, 3$ are given.

The structure of non-Abelian branches may be labeled by the pair (n, m), n being the vorticity and the integer m indicating the specific non-Abelian branch for that vortic-
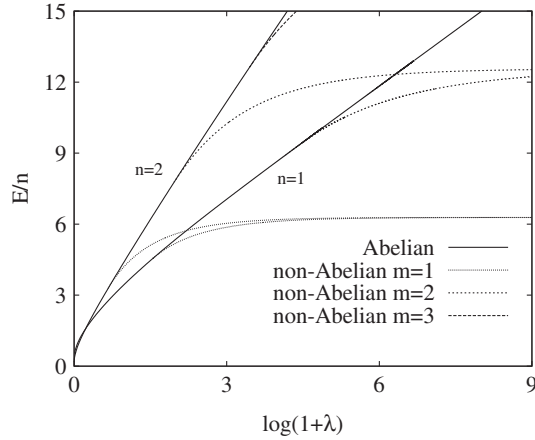
FIG. 1. Energy per vortex number E/n versus the Higgs potential coupling constant λ for YMH solutions with $n = 1, 2$.

ity n (lower m means lower energy). Notice that the Abelian solutions behave in a different way, which we will emphasize below. In fact, although Abelian solutions exist for any nonvanishing value of λ, for each non-Abelian branch there exists a minimal value of λ, $\lambda_{(n,m)}^{\min}$ (which depends on n and m), below which the non-Abelian branch ceases to exist. In fact, at that minimal value the non-Abelian branch matches the corresponding Abelian branch for that value of n, so $\lambda_{(n,m)}^{\min}$ corresponds to the branching points where non-Abelian branches start to exist.

A remarkable fact in Fig. 2 is that the gap between neighboring branching points is roughly constant for each n on logarithmic scale for λ. More precisely, the quantity $n[\log(1 + \lambda_{(n,m+1)}^{\min}) - \log(1 + \lambda_{(n,m)}^{\min})]$ is roughly constant and independent of (n, m). This feature becomes more accurate for large λ, revealing an underlying structure in the non-Abelian sector in the limit $\lambda \to \infty$. In fact, denoting the energy of the (n, m) non-Abelian solutions by $E_{(n,m)} = E_{(n,m)}(\lambda)$, one observes in Fig. 1 that for each m the energy per vortex number tends to a limit which does not depend on n but only on m.
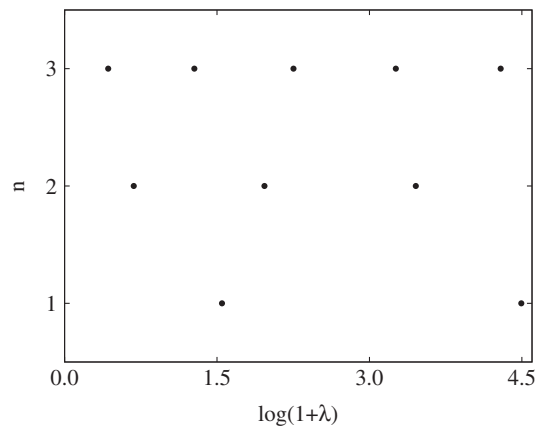
It turns out that

$$\lim_{\lambda \to \infty} \frac{E_{(n,m)}(\lambda)}{n} = 2\pi m, \qquad (6)$$

the energy *per unit* vorticity is equal to the energy of the *unit* vortex. Hence, non-Abelian vortices with given m are noninteracting in that limit. Figure 3 shows this limit for $n = 3$ solutions. In that figure it is clearly seen that in the large λ region the ratio $E/(2\pi n)$ approaches the integer value m that labels the non-Abelian branches. One observes an infinite number of non-Abelian branches for each n emerging from the logarithmically divergent Abelian profile, each converging to a finite limit.

One can understand this feature as follows. We have verified that in this limit the contribution of the potential term in Eq. (1) to the energy of the non-Abelian vortices vanishes. (This contrasts with the corresponding situation for the vortices of the Abelian Higgs model.) Thus, the YMH theory supporting the non-Abelian vortices becomes a $O(3)$ sigma model on $\mathbb{R}^2$ in this limit [22]. Likewise, in our case the vanishing of the Higgs potential leads to the $O(3)$ sigma model constraint, resulting in the $SO(3)$ gauged $O(3)$ model, which unlike in the Weinberg-Salam (WS) case [23], does not satisfy the Derrick scaling requirement for finite energy. To this end, we have verified that in this limit the contribution to the energy of the YM term $\mathrm{Tr}F^2$ in Eq. (1) also vanishes, consistently with the Derrick scaling requirement, and that indeed the YM potential becomes a *pure-gauge* in this limit. Thus, the only contribution comes from the $\mathrm{Tr}D\Phi^2$ term in Eq. (1), which in this case reduces to $\mathrm{Tr}\partial\Phi^2$ of the scale invariant $O(3)$ sigma model on $\mathbb{R}^2$. Our non-Abelian vortices in this limit are described by the radially symmetric vorticity-n subset of the noninteracting self-dual Belavin-Polyakov ferromagnetic vortices [20].

The effect of non-Abelianness on YMH solutions affects not only the energy values, which become lower for non-Abelian solutions, but also to the way the energy is distributed throughout space. Both for Abelian and non-
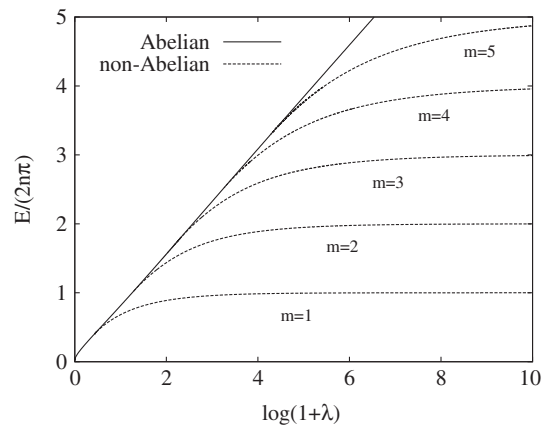

FIG. 2. Location of the first branching points for $n = 1, 2, 3$.

FIG. 3. Energy per vortex number E/n versus the Higgs potential coupling constant λ for YMH solutions with $n = 3$.
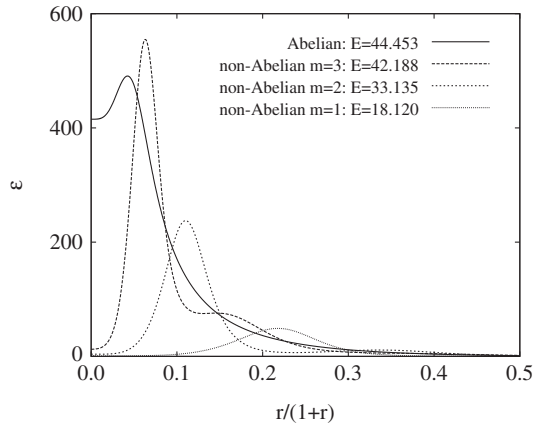
FIG. 4. Energy density for YMH solutions with $n = 3$ and $\lambda = 20.0$.

Abelian configurations, solutions are radial for $n = 1$ (their energy density having the global maximum at the origin) and circular for $n > 1$ (their energy density having the global maximum at a finite nonvanishing value of r). For ring-shaped configurations ($n > 1$), as the solutions are more non-Abelian (lower values of m) the energy density profile spreads: the maximum is moved to higher values of r and its height decreases. In addition, the value of the energy density at the origin tends to zero, the profile becoming more and more ringlike. This result is demonstrated in Fig. 4 where the energy density profiles of YMH solutions with $n = 3$ and $\lambda = 20.0$ are shown.

V. CONCLUSIONS

As a final comment on the possible physical status of our solutions, we emphasize that the model on $\mathbb{R}^2$ employed here is precisely that which supports monopoles on $\mathbb{R}^3$. Interestingly, this YMH model on $\mathbb{R}^3$ supports also monopole-antimonopole solutions, constructed in [26]. This describes a consistent picture where our vortices are candidates for flux tubes starting and ending on monopoles of opposite polarities. Our results are qualitatively consistent with the picture in [26]. In particular for vorticities $n \geq 3$, the energy density distribution in the monopole-antimonopole configuration presents a ring-shaped density situated on the symmetry plane (the $\mathbb{R}^2$ plane where our vortices exist) much like the circles in Fig. 4.

We conclude by noting that the vortices constructed are genuinely non-Abelian, but are not endowed with a topological lower bound. That leads us to expect that these non-Abelian solutions are unstable for finite values of λ, even though the limiting solutions, namely, the Belavin-Polyakov vortices for $\lambda \rightarrow \infty$ are stable.

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