PHYSICAL REVIEW D **81**, 127701 (2010)

Supersymmetry between Jackiw-Nair and Dirac-Majorana anyons

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(Received 21 April 2010; published 1 June 2010)

The Jackiw-Nair description of anyons combines spin-1 topologically massive fields with the discrete series representation of the Lorentz algebra, which has fractional spin. In the Dirac-Majorana formulation the spin-1 part is replaced by the spin $1/2$ planar Dirac equation. The two models are shown to belong to an $N = 1$ supermultiplet, which carries a super-Poincaré symmetry.

DOI: 10.1103/PhysRevD.81.127701 PACS numbers: 11.30.Pb, 03.65.Pm, 11.15.Wx, 11.30.−j

In the by now standard description of anyons due to Jackiw and Nair [1], the spin 1 representation carried by the topologically massive (TM) vector system [2,3] is combined with fractional spin. The latter is carried by an internal space (namely the Poincaré disc model of the Lobachevsky plane) [4], described by a complex coordinate z. A "Jackiw-Nair" (JN) wave function is,

$$F_\mu(z, x) = \sum_n f_n(z) F^n_\mu(x), \tag{1}$$

where the $f_n = c_n z^n$, $c_n = \sqrt{\Gamma(2\alpha + n)/\Gamma(2\alpha)\Gamma(n+1)}$, is restricted to the unit-disk [1,4]. The f_n, $n = 0, 1, 2, \ldots$, span an infinite dimensional orthonormal basis in internal space. $F^n_\mu(x)$ is, for each internal index n, a TM wave function. Then Jackiw and Nair propose to describe anyons by the equations

$$(P^\mu \mathfrak{J}^+_\mu - \beta_+ m)F = 0, \qquad \beta_+ = \alpha - 1, \tag{2}$$

where $\alpha > 0$.[1] Here $F = (F_\mu)$ and $\mathfrak{J}^+_\mu$ generates the direct sum of Lorentz algebras, $\mathfrak{J}^+_\mu = J^+_\mu + j_\mu$, where $(J^+_\mu)_\nu{}^\lambda = i\epsilon_{\mu\nu}{}^\lambda$ generates the spin 1 representation of the Lorentz algebra, and j_μ, carrying a fractional spin, belongs to the discrete series of the Lorentz algebra (D^+_α) [1]. J^+_μ acts on the vector index of the field F_μ, and j_μ acts on its "fractional" part, labeled by n. The "internal" representation can be realized as,

$$j_0 = z\partial_z + \alpha, \qquad j_1 = -\frac{1+z^2}{2}\partial_z - \alpha, \qquad j_2 = -i\frac{1-z^2}{2}\partial_z + i\alpha. \tag{3}$$

The Lorentz Casimir $j^\mu j_\mu = -\alpha(\alpha - 1)$ is constant, so the representation is irreducible.

Eqn. (2) fixes only one of the Casimirs of the planar Poincaré group, and must therefore be supplemented with subsidiary conditions. Those chosen by Jackiw and Nair [1] are equivalent to

$$P^\mu F_\mu = 0, \qquad \epsilon^{\mu\nu\lambda} P_\mu j_\nu F_\lambda = 0. \tag{4}$$

Equations (2) and (4) imply the Klein-Gordon equation with mass m, while (2) fixes the second Casimir operator of the Poincaré group,

$$P^\mu(-\epsilon_{\mu\nu\lambda} x^\nu P^\lambda + \mathfrak{J}^+_\mu) = \beta_+ m. \tag{5}$$

Hence, the spin is $\beta_+ = \alpha - 1$. Eqns. (2) and (4) imply the equation of the TM theory,

$$\mathfrak{D}_\mu{}^\nu F_\nu \equiv (-i\epsilon_{\mu\lambda}{}^\nu P^\lambda + m\delta_\mu{}^\nu)F_\nu = 0. \tag{6}$$

Conversely, it can be shown [5] that (2) and (4) together are equivalent to imposing the TM equations (6), augmented with the Majorana equation

$$(P^\mu j_\mu - \alpha m)F = 0. \tag{7}$$

The Jackiw-Nair theory is, hence, equivalent to the coupled TM-Majorana system (6) and (7).

In another, slightly different approach [6], the anyon field is described rather by a spinor,

$$\psi_a(x, z) = \sum_n g_n(z)\psi^n_a(x), \qquad (P^\mu \mathfrak{J}^-_\mu - \beta_- m)\psi = 0, \qquad \beta_- = \alpha - \tfrac{1}{2}, \tag{8}$$

where the $\mathfrak{J}^-_\mu = J^-_\mu + j_\mu$ with $(J^-_\mu)_a{}^b = -\frac{1}{2}(\gamma_\mu)_a{}^b$ generate the spin $1/2$ representation of the planar Lorentz group. Instead of the TM equation (6), the field is required to satisfy the planar Dirac equation,

$$\mathcal{D}_a{}^b \psi_b \equiv (P_\mu \gamma^\mu - m)_a{}^b \psi_b = 0. \tag{9}$$

Note that the Dirac (9) and Majorana (7) equations [with ψ replacing F] imply

$$(j_\mu \gamma^\mu + \beta_-)\psi = 0, \qquad i\epsilon^{\mu\nu\lambda} P_\mu j_\nu \gamma_\lambda \psi = 0 \tag{10}$$

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[1]The restriction to $\alpha > 0$ can in fact be removed, and leads to interpolating anyons which correspond to nonunitary representations [5].

as consistency conditions, which eliminate the redundant modes [6].

The aim of this paper is to show that the Jackiw-Nair and Dirac-Majorana approaches are two facets of the same supersymmetric system: they are in fact, superpartners. To this end, we note first that a fractional spin field can be described, in both approaches, by equations of the same form,

$$D^{\pm}\psi^{\pm}=0,$$
$$(P^{\mu}j_{\mu}-\alpha m)\psi^{\pm}=0,\quad \text{with spin}\begin{cases}\beta_{-}=\alpha-\frac{1}{2}\\ \beta_{+}=\alpha-1\end{cases},\qquad (11)$$

where $D^{+}=\mathfrak{D}$ and $D^{-}=\mathcal{D}$ are the operators in TM and the Dirac equations (6) and (9), respectively, and we put $\psi^{-}=\psi$ and $\psi^{+}=F$. We note for further reference that, in both frameworks, the posited first-order equations imply that the field satisfies the Klein-Gordon equation.

The (fractional) spins of the fields ψ^{-} and ψ^{+} are shifted by $\beta_{-}-\beta_{+}=1/2$, and have the same masses. They can therefore be unified into a supermultiplet along the same lines as done recently for the TM and Dirac fields [5,7]. We posit

$$(P_{\mu}\mathcal{J}^{\mu}-\hat{\alpha}m)\Psi=0,\qquad (P^{\mu}j_{\mu}-\alpha m)\Psi=0,\qquad (12)$$

where Ψ is formed by putting together the Dirac-Majorana and Jackiw-Nair (1) fields,

$$\Psi=\begin{pmatrix}\psi^{-}\\ \psi^{+}\end{pmatrix},$$

and $\hat{\alpha}$ is $\alpha_{-}=-\frac{1}{2}$ for

$$\psi=\begin{pmatrix}\psi^{-}\\ 0\end{pmatrix},$$

and $\alpha_{+}=-1$, for

$$\psi=\begin{pmatrix}0\\ \psi^{+}\end{pmatrix}.$$

The first equation in (12) here is the supersymmetric equation which unifies the Dirac and TM equations [7], and is supplemented by the Majorana equation.

The total spin operator, $\hat{\beta}=\hat{\alpha}+\alpha=\mathrm{diag}(\beta_{-}\mathbb{1}_2,\beta_{+}\mathbb{1}_3)$ takes the value $\beta_{\pm}$ on the subspaces spanned by the Dirac-Majorana and the Jackiw-Nair field, respectively.

Lorentz transformations are generated by $\mathfrak{M}_{\mu}=-\epsilon_{\mu\nu\lambda}x^{\nu}P^{\lambda}+\mathfrak{J}_{\mu}$ where $\mathfrak{J}_{\mu}=\mathrm{diag}(\mathfrak{J}_{\mu}^{-},\mathfrak{J}_{\mu}^{+})$ is block-diagonal with irreducible components acting on the Dirac-Majorana and Jackiw-Nair components of the supermultiplet. Augmented with translations P_{μ} yields the Poincaré algebra,

$$[P_{\mu},P_{\nu}]=0,\qquad [\mathfrak{M}_{\mu},P_{\nu}]=-i\epsilon_{\mu\nu\lambda}P^{\lambda},$$
$$[\mathfrak{M}_{\mu},\mathfrak{M}_{\nu}]=-i\epsilon_{\mu\nu\lambda}\mathfrak{M}^{\lambda}.\qquad (13)$$

The Lorentz algebra generated by $\mathcal{J}_{\mu}$ can be extended into the *superalgebra* $\mathfrak{osp}(1|2)$ by adding the off-diagonal matrices[2]

$$L_{\underline{A}}=\sqrt{2}\begin{pmatrix}0 & Q_{\underline{A}a}{}^{\mu}\\ Q_{\underline{A}\mu}{}^{a} & 0\end{pmatrix},\qquad Q_{\underline{1}a}{}^{\mu}=\begin{pmatrix}0&1&i\\1&0&0\end{pmatrix},$$
$$Q_{\underline{2}a}{}^{\mu}=\begin{pmatrix}1&0&0\\0&1&-i\end{pmatrix}.\qquad (14)$$

The operators $L_{\underline{A}}$ form a Lorentz spinor, $[\mathfrak{J}_{\mu},L_{\underline{A}}]=\frac{1}{2}(\gamma_{\mu})_{\underline{A}}{}^{\underline{B}}L_{\underline{B}}$ and interchange ψ and F (where we returned to our original notations).

They do not preserve the physical states defined as solutions of the Dirac and TM equations, respectively. Consider instead the supercharges

$$\mathcal{Q}_{\underline{A}}=\frac{1}{2\sqrt{m}}(P_{\mu}\gamma^{\mu}-Rm)_{\underline{A}}{}^{\underline{B}}L_{\underline{B}},\qquad (15)$$

where $R=\mathrm{diag}(-\mathbb{1}_2,\mathbb{1}_3)$ is the reflection operator, $\{R,L_{\underline{A}}\}=0$, which transform a two-component Dirac field into a three-component TM field F' and conversely. Explicitly,

$$\mathcal{Q}_{\underline{A}}\Psi=\begin{pmatrix}\psi'_a\\ F'_{\mu}\end{pmatrix}=\begin{pmatrix}\sum_n f_n(z)\mathcal{Q}_{\underline{A}a}{}^{\mu}F_{\mu}(x)\\ \sum_n g_n(z)\mathcal{Q}_{\underline{A}\mu}{}^{a}\psi_a^n(x)\end{pmatrix}.\qquad (16)$$

A general SUSY transformation is a linear combination of the $\mathcal{Q}_{\underline{A}}$s, $\mathcal{Q}=\zeta^{\underline{A}}\mathcal{Q}_{\underline{A}}$. Moreover,

$$\mathcal{D}_a{}^{b}\psi'_b=\zeta^{\underline{A}}\Big(\mathcal{Q}_{\underline{A}a}{}^{\mu}\mathfrak{D}_{\mu}{}^{\nu}F_{\nu}+\frac{1}{2\sqrt{m}}\mathcal{Q}_{\underline{A}a}{}^{\mu}(P^2+m^2)F_{\mu}\Big),\qquad (17)$$

$$\mathfrak{D}_{\mu}{}^{\nu}F'_{\nu}=\zeta^{\underline{A}}\Big(-\frac{1}{2}\mathcal{Q}_{\underline{A}\mu}{}^{a}\mathcal{D}_a{}^{b}\psi_b-\frac{1}{2\sqrt{m}}\mathcal{Q}_{\underline{A}\mu}{}^{a}(P^2+m^2)\psi_a\Big),\qquad (18)$$

showing that ψ'_b satisfies the Dirac equation if and only if F'_{ν} satisfies the TM equation.

Now the Majorana equations are intertwined by the SUSY transformation,

$$(P^{\mu}j_{\mu}-\alpha m)\psi'_a=\zeta^{\underline{A}}(\mathcal{Q}_{\underline{A}a}{}^{\mu}(P^{\nu}j_{\nu}-\alpha m)F_{\mu})=0,\qquad (19)$$

$$(P^{\mu}j_{\mu}-\alpha m)F'_{\mu}=\zeta^{\underline{A}}(\mathcal{Q}_{\underline{A}\mu}{}^{a}(P^{\nu}j_{\nu}-\alpha m)\psi_a)=0,\qquad (20)$$

allowing us to conclude that the $\mathcal{Q}_{\underline{A}}$ in (15) generate indeed a supersymmetry transformation between the two, DM and JN, sectors.

[2]Underlined capitals denote $\mathfrak{osp}(1|2)$ spinors.

Completing the Poincaré algebra (13) by the supercharges (15),

$$[P_\mu, \mathcal{Q}_{\underline{A}}] = 0, \qquad [\mathfrak{M}_\mu, \mathcal{Q}_{\underline{A}}] = -\tfrac{1}{2}(\gamma_\mu)_{\underline{A}}{}^{\underline{B}}\mathcal{Q}_{\underline{B}}, \tag{21}$$

$$\{\mathcal{Q}_{\underline{A}}, \mathcal{Q}_{\underline{B}}\} = 2(P\gamma)_{\underline{A}\,\underline{B}} + \frac{1}{2m}[(\mathcal{J}\gamma)_{\underline{A}\,\underline{B}}(P^2 + m^2) - 2(P\gamma)_{\underline{A}\,\underline{B}}(P\mathcal{J} - \hat{\alpha}m)], \tag{22}$$

where $(P\gamma)_{\underline{A}\,\underline{B}}$ means $P^\mu \gamma_{\mu\underline{A}}{}^{\underline{C}}\epsilon_{\underline{C}\,\underline{B}}$.

On shell, the unified system carries therefore an $N = 1$ super-Poincaré symmetry. The super-Casimir is $C = P^\mu \mathcal{J}_\mu - \frac{1}{16}[\mathcal{Q}_{\underline{1}}, \mathcal{Q}_{\underline{2}}] = m(\alpha - 3/4)$ a constant, showing that the representation is indeed irreducible.

We note, in conclusion, that the supersymmetry of the two, Dirac-Majorana (DM) and Jackiw-Nair (JN) types of anyons proved here explicitly had to be expected from that of the "carrying" spin 1/2 and spin 1 spaces [2,5].

Partial support by the FONDECYT (Chile) under the Grant No. 1095027 and by DICYT (USACH), and by Spanish Ministerio de Educación under Project No. SAB2009-0181 (sabbatical grant of MSP), is acknowledged. MV has been supported by CNRS postdoctoral grant (Contract No. 87366).

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