

Casimir force in brane worlds: Coinciding results from Green's and zeta function approachesRomán Linares,^{1,*} Hugo A. Morales-Técotl,^{1,†} and Omar Pedraza^{2,‡}¹*Departamento de Física, Universidad Autónoma Metropolitana Iztapalapa, San Rafael Atlixco 186, Código Postal 09340, México Distrito Federal, México*²*Centro de Estudios en Física y Matemáticas Básicas y Aplicadas, Universidad Autónoma de Chiapas, 4a. Oriente Norte 1428, Tuxtla Gutiérrez, Chiapas, México*

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Casimir force encodes the structure of the field modes as vacuum fluctuations and so it is sensitive to the extra dimensions of brane worlds. Now, in flat spacetimes of arbitrary dimension the two standard approaches to the Casimir force, Green's function, and zeta function yield the same result, but for brane world models this was only assumed. In this work we show that both approaches yield the same Casimir force in the case of universal extra dimensions and Randall-Sundrum scenarios with one and two branes added by p compact dimensions. Essentially, the details of the mode eigenfunctions that enter the Casimir force in the Green's function approach get removed due to their orthogonality relations with a measure involving the right hypervolume of the plates, and this leaves just the contribution coming from the zeta function approach. The present analysis corrects previous results showing a difference between the two approaches for the single brane Randall-Sundrum; this was due to an erroneous hypervolume of the plates introduced by the authors when using the Green's function. For all the models we discuss here, the resulting Casimir force can be neatly expressed in terms of two four-dimensional Casimir force contributions: one for the massless mode and the other for a tower of massive modes associated with the extra dimensions.

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I. INTRODUCTION

Historically, the idea to consider our observable four-dimensional (4D) universe as a subspace of a higher dimensional spacetime has a long tradition that started with the works of G. Nördstrom [1], T. Kaluza [2], and O. Klein [3] (see e.g. [4], and references therein). Nowadays, there are two broad approaches one typically takes to address the possible consequences of extra dimensions in 4D physics. The top-down approach starts either from a fundamental theory or a low energy limit of it, for instance M /string theory or supergravity [5], and upon compactification of the extra dimensions, one hopes to find an effective theory in 4D containing as much of the physics we know (see e.g. [6], and references therein). In this approach, one favors the properties of the compactification manifold, and upon the requirement that the compactification be performed in a consistent way, one tracks the physical consequences that the geometry of the internal manifold has on the resulting lower dimensional theory, including, for instance, the gauge group and the matter content. However, in this approach we are unable to select the lower dimensional theory in a unique fashion. The standard model hopefully would correspond to a particular internal space or vacuum configuration chosen by nature by some still unknown mechanism (see e.g. [7], and references therein).

In contrast, the bottom-up approach relies on “model building,” where the requirements of having the low en-

ergy spectrum and interactions of the known 4D physics put restrictions on properties such as the types of singularities, curvature, symmetries, etc., supported by the internal space. The constraints are powerful because they hold for a large class of models without having to fully specify the compactification details. Of course they are only necessary conditions; nevertheless, they serve as a useful guide in the search for realistic models before a complete theory/model can be explicitly constructed. In this approach, one looks at the different well known physical phenomena and their corresponding experimental confirmations, and then, by requiring agreement between the contributions of the extra dimensions to the 4D physics and the experimental errors, one gets bounds to the higher dimensional free parameters. This information forms the core of the necessary knowledge for model building. Following this approach, most attention has been devoted to high energy physics (see e.g. [8,9], and references therein) and cosmology (see e.g. [10–13], and references therein). More recently, the possibility to obtain information from models with extra dimensions studying low energy physical phenomena such as the Casimir effect has also been addressed [14–34]. The interest in the Casimir force is twofold. First, the force between neutral perfect conducting plates predicted by Casimir [35], is experimentally well established [36–40], and nowadays, the increasing accuracy reached in its determination makes us think that constraining model parameters in this way is at the least complementary to those based on high energy experiments. Second, its theoretical analysis involves two aspects naturally appearing in the study of models with extra dimensions, namely, the mode structure

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of matter fields and the submillimeter length scale, of order $1 \mu\text{m}$, at which the force becomes noticeable and for which some extra-dimensional models have been conjectured to produce observable effects.

In this paper, we follow the model building approach to determine the Casimir force for a massless scalar field between two parallel plates. This situation mimics the actual experimental setup where the electromagnetic rather than a massless scalar field is considered. We model the plates as codimension one hypersurfaces in the extra-dimensional spacetime; therefore, what one really obtains is the force per unit of hypervolume of the plate, as it was established long ago for hyper-dimensional Minkowski spacetime [41]. In this way, the extra dimensions yield corrections to the usual 4D Casimir force. Remarkably, the resulting force can be expressed as the sum of two types of contributions: one that is given by the zero mode, thus producing the standard 4D Casimir force for a massless scalar field, and the other one that includes the addition of 4D Casimir forces corresponding to the massive modes.

The present work is aimed at showing that both Green's function and zeta function techniques (see e.g. [42], and references therein) yield the same Casimir force for some typical extra-dimensional scenarios. In particular it corrects a previous difference between the Casimir force for one-brane Randall-Sundrum models [43,44] using the *zeta function method* [16,19] and the one obtained using *Green's function approach* [17,18]. Such a difference was originated by an erroneous hypervolume factor for the plates considered in the setting in [17,18].

For the sake of clarity, we first study the case of universal extra dimensions in five dimensions. This corresponds to 4D Minkowski extended by an spatial compact extra dimension attached to each of its points. The topology of the extra dimension is an orbifold S^1/Z_2 . The Casimir effect in this geometry was studied using the zeta function regularization method in [14], but here we present the corresponding Green's function analysis. The second model we shall consider is the so called Randall-Sundrum II p model (RSII- p). These have a single $(3+p)$ -brane [45]. For this model, the Casimir effect was computed in [16,19] using the zeta function method and in [17,18] using the Green's function method. Finally, we shall consider the Randall-Sundrum I p model (RSI- p). These are defined by two $(3+p)$ -branes. In this case the Casimir force was studied in [19] using the zeta function technique. To the best of our knowledge, an analogous study is missing applying the Green's function approach, and in this paper we fill in this gap. We shall conclude that for all of the above extra-dimensional models, the Casimir force obtained by either of the approaches—zeta function regularization or Green's function—is the same.

The structure of the paper is as follows. Since it will be used frequently in Sec. II, we briefly recall the analysis of the Casimir force in 4D Minkowski spacetime for a mas-

sive scalar field; whereas, its extension to $d+1$ Minkowski spacetime is summarized in the Appendix. In Sec. III, we discuss the universal extra dimension model. Section IV is devoted to RSII- p , while Sec. V deals with RSI- p . Finally, Section VI contains the discussion of our results. Unless otherwise stated, we use units in which $\hbar = c = 1$.

II. SCALAR FIELD IN 4D MINKOWSKI SPACETIME

To make use of it in the sequel, we briefly review the analysis of the Casimir force for a massive scalar field in 4D Minkowski spacetime [46]. We start by computing the dispersion relation and then determine the Casimir force by the two approaches: zeta function and Green's function.

Let the scalar field to have mass μ and subject to Dirichlet boundary conditions at the planes $z = 0, l$. The starting point of the analysis is the Klein-Gordon's equation

$$(\square + \mu^2)\phi = 0, \quad \square = \partial_{tt} - \Delta, \quad (1)$$

with Δ the Laplacian in \mathbb{R}^3 and μ the mass of the scalar field. x represents a spacetime coordinate with components (t, x_1, x_2, z) , the last three being spatial and Cartesian. The 4D Minkowski metric is $\eta_{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$. By separating the dependence of the field in Cartesian coordinates $\vec{x} = (x_1, x_2, z)$ as $\phi(x_i, z, t) = \chi_i(x_i)Y(z)e^{i\omega t}$, $i = 1, 2$, we have the equivalent set of eigenvalue equations

$$-\partial_{ii}\chi_i(x_i) = k_i^2\chi_i(x_i), \quad i = 1, 2, \quad (2)$$

$$-\partial_{zz}Y(z) = k_3^2Y(z), \quad k_3^2 := \omega^2 - (\mu^2 + k_1^2 + k_2^2). \quad (3)$$

Here, k_i^2 , $i = 1, 2$, and k_3 are separation constants or eigenvalues. The physical plates are two-dimensional surfaces described by coordinates x_1 and x_2 so that χ_i in (2) can be subject to free boundary conditions. Coordinate z is transverse to the plates, so Y in (3) will be subject to Dirichlet boundary conditions at $z = 0, l$. The corresponding eigenfunctions are

$$\chi_i(x_i) = \frac{1}{\sqrt{2\pi}} e^{ik_i x_i}, \quad k_i \in \mathbb{R}, \quad i = 1, 2, \quad (4)$$

$$Y_N(z) = \sqrt{\frac{2}{l}} \sin \frac{N\pi z}{l}, \quad N = 1, 2, \dots, \quad (5)$$

and the resulting dispersion relation is

$$\omega^2 = \mu^2 + k_1^2 + k_2^2 + \frac{N^2\pi^2}{l^2}. \quad (6)$$

A. Zeta function approach

To compute the Casimir force, one can compute the Casimir energy between the plates E_{plates} , by summing up the zero-point energy per unit area $\hbar\omega/2$. There are two ingredients required to follow this strategy: the *dispersion relations* and the *mode structure* (which can be continuous, discrete, or an admixture). It turns out that E_{plates} contains a linear term in the separation l between planes which gives rise to a constant Casimir force. This term can be canceled by addition of a constant to the Hamiltonian density or by considering the energy E_0 in the absence of the plates which means $k_3 \in \mathbb{R}$ and $Y_N(z) = \frac{1}{\sqrt{2\pi}} e^{ik_3 z}$. Adopting the second option, the resulting finite expression for the Casimir energy per unit area of the plate is

$$\begin{aligned} \mathcal{E}_{4\text{D}}(\mu) &= \frac{E_{\text{plates}} - E_0}{L^2} \\ &= \frac{1}{2} \prod_{i=1,2} \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \left(\sum_{N=1}^{\infty} \omega_{k_1, k_2, N}(\mu) \right. \\ &\quad \left. - l \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \omega_{k_1, k_2, k_3}(\mu) \right), \end{aligned} \quad (7)$$

where

$$\omega_{k_1, k_2, N}(\mu) \equiv \sqrt{\mu^2 + k_1^2 + k_2^2 + \frac{N^2 \pi^2}{l^2}}, \quad (8)$$

$$\omega_{k_1, k_2, k_3}(\mu) \equiv \sqrt{\mu^2 + k_1^2 + k_2^2 + k_3^2}, \quad (9)$$

and L^2 is the area of a square shaped piece of the plates at $z = 0, l$. Notice the extra factor of l in the second term of (7); it comes from the fact that E_0 is the energy in the whole volume delimited by $z = 0, l$, in the transverse direction; whereas, \mathcal{E} is the energy per unit area L^2 . We are denoting explicitly the dependence of the density energy $\mathcal{E}_{4\text{D}}$ on the mass μ , to stress the fact that the 4D scalar field is massive.

We perform explicitly the integrals in the Appendix obtaining

$$\mathcal{E}_{4\text{D}}(\mu) = -\frac{\mu^2}{8\pi} \cdot \frac{1}{l} \sum_{N=1}^{\infty} \frac{1}{N^2} K_{-2}(2Nl\mu), \quad (10)$$

where K is the modified Bessel function of second type. The Casimir force is obtained from the Casimir energy simply deriving with respect to the separation between the plates: $F_{4\text{D}} = -\partial \mathcal{E}_{4\text{D}} / \partial l$, thus

$$\begin{aligned} f_{4\text{D}}(\mu) &= -\frac{\mu^2}{8\pi^2} \left[\frac{1}{l^2} \sum_{N=1}^{\infty} \frac{1}{N^2} K_2(2Nl\mu) \right. \\ &\quad \left. - \frac{2\mu}{l} \sum_{N=1}^{\infty} \frac{1}{N} K_3(2Nl\mu) \right], \end{aligned} \quad (11)$$

where a property of the derivative of the Bessel function

has been used. In general, this expression can not be simplified further and usually people computes it numerically for a given value of the mass μ . For the massless case, $\mu = 0$, however, the expression can be simplified to yield

$$f_{4\text{D}}(0) = -\frac{\pi^2}{480} \frac{1}{l^4}, \quad (12)$$

in which the appropriate approximation for small argument of the Bessel functions has been used and then the identification of a zeta function allows to evaluate the result.

B. Green's function approach

In the Green's function approach, once we are armed with the eigenfunctions (4) and (5), we can express the Green's function $G_{4\text{D}}$ for the problem $(\square + \mu^2)G_{4\text{D}}(x, x') = -\delta(x - x')$, subject to Dirichlet boundary conditions at $z = 0, l$, as

$$\begin{aligned} G_{4\text{D}}(x, x') &= \prod_{i=1,2} \int dk_i \chi_i^*(x_i) \chi_i(x'_i) \\ &\quad \times \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} g(z, z'), \end{aligned} \quad (13)$$

$$g(z, z') = \sum_{N=1}^{\infty} \frac{Y_N^*(z) Y_N(z')}{\frac{N^2 \pi^2}{l^2} - k_3^2}, \quad (14)$$

with $*$ denoting complex conjugation and g the so-called reduced Green's function. Notice that this expression of the Green function is valid only in the region between the plates. Now given the relation between the vacuum expectation value of the time ordered product of fields and the Green's function, $\langle T[\phi(x)\phi(x')] \rangle = \frac{1}{i} G_{4\text{D}}(x, x')$, the force per unit area on either plate can be obtained from the vacuum expectation value of the energy momentum tensor $T_{\nu\rho} = \partial_\nu \phi \partial_\rho \phi - \eta_{\nu\rho} \mathcal{L}$ with $\mathcal{L} = \frac{1}{2} \partial_\nu \phi \partial^\nu \phi - \frac{1}{2} \mu^2 \phi^2$. Since upon integration over all space the term $\eta_{\nu\rho} \mathcal{L}$ does not contribute by virtue of the Klein-Gordon's Eq. (1), one gets

$$f_{\text{in}}(\mu) = \langle T_{zz}^{\text{in}} \rangle = \lim_{x' \rightarrow x} \partial_z \partial_{z'} G_{4\text{D}}(x, x')|_{z=0, l} \quad (15)$$

$$= \prod_{i=1,2} \int dk_i \chi_i^*(x_i) \chi_i(x_i) \int \frac{d\omega}{2\pi} \lim_{z' \rightarrow z} \partial_z \partial_{z'} g(z, z')|_{z=0, l}. \quad (16)$$

In the coincident limit, $\chi_i^*(x_i) \chi_i(x_i) = \frac{1}{2\pi}$ and so the dependence on x_i , $i = 1, 2$, drops out, which should have been expected from the translational invariance of the parallel plates configuration along the x_i , $i = 1, 2$, directions.

Combining (5) together with (14) allows us to obtain the explicit form of $g(z, z')$, which upon substitution in (16) and after the change of variables, $\omega \rightarrow i\xi$ and $k_1^2 + k_2^2 + \xi^2 \rightarrow \rho^2$ produces

$$f_{\text{in}}(\mu) = \frac{1}{l} \int \frac{dk_1 dk_2}{(2\pi)^2} \int \frac{d\xi}{2\pi} \sum_{N=1}^{\infty} \frac{\frac{\pi^2 N^2}{l^2}}{\rho^2 + \mu^2 + \frac{\pi^2 N^2}{l^2}}. \quad (17)$$

This allows us to read (k_i, ξ) as three-dimensional Cartesian coordinates. This is not the final answer, because so far we have only considered the force to the left of $z = l$ or to the right of $z = 0$; actually the integral (17) diverges. We also have to include the flux of momentum for instance to the right of $z = l$. We shall not elaborate on this issue, for our purpose it is enough to mention that the normal-normal component of the stress tensor at $z = l$ is $\langle T_{zz}^{\text{out}} \rangle = i\sqrt{\rho^2 + \mu^2}/2$ [46]. The net forces at $z = l$ produce finally

$$f_{4\text{D}}(\mu) = \frac{1}{2} \int \frac{dk_1 dk_2}{(2\pi)^2} \int \frac{d\xi}{2\pi} \left(\frac{2}{l} \sum_{N=1}^{\infty} \frac{\frac{\pi^2 N^2}{l^2}}{\rho^2 + \mu^2 + \frac{\pi^2 N^2}{l^2}} + \sqrt{\rho^2 + \mu^2} \right). \quad (18)$$

We compute this integral explicitly in the Appendix, obtaining again the expression (11) for the 4D Casimir force. We could use the easier argument based on the discontinuity of the derivative of the reduced Green's function g above. Of course, the same $f_{4\text{D}}$ results from the discontinuity of the zz component of the energy momentum tensor on either plate at $z = 0, l$.

Notice that in order to obtain the force, the Green's function method includes a “bit” more of information than the previously used in the zeta function regularization method. We have used again the eigenvalues through the dispersion relations but we have also used explicitly the *eigenfunctions* and not only the mode structure. However, upon integration of the modes χ_i , $i = 1, 2$, the real input is again only the modes structure as in the zeta regularization method losing the bit of extra information. This property of the Green's method is the one that makes it equivalent to the zeta function regularization method. For instance, in the models with extra dimensions, as we will discuss, the eigenfunctions depending on the extra coordinates give us information about the localization of the field modes; however, since at the end these eigenfunctions are integrated out, the information on the localization is in some sense “lost.” This is in agreement with our concept of consistent compactifications, for which the extra-dimensional coordinates must disappear explicitly (see for instance [47], and references therein).

It is straightforward to generalize the result of the 4D Casimir force to a $(d + 2)$ D Minkowski spacetime (see the Appendix). In this case the scalar field is bound by hyperplanes of d dimensions and the force per unit d -dimensional volume between the hyperplanes is given by [41]

$$f_{(d+2)\text{D}}(\mu) = -2 \left(\frac{\mu}{4\pi} \right)^{(d+2)/2} \left[\frac{3}{l^{(d+2)/2}} \sum_{N=1}^{\infty} \frac{1}{N^{(d+2)/2}} \times K_{(d+2)/2}(2Nl\mu) + \frac{2\mu}{l^{d/2}} \sum_{N=1}^{\infty} \frac{1}{N^{d/2}} K_{d/2}(2Nl\mu) \right]. \quad (19)$$

For the forthcoming analysis, it is convenient to notice that this expression has the following limit values:

$$\lim_{\mu \rightarrow 0} f_{(d+2)\text{D}}(\mu) = -\frac{d}{l^{d+2}(4\pi)^{(d+2)/2}} \Gamma\left(\frac{d+2}{2}\right) \zeta(d+2), \quad (20)$$

and

$$\lim_{\mu \rightarrow \infty} f_{(d+2)\text{D}}(\mu) \rightarrow 0. \quad (21)$$

Once we have reviewed the way in which each method gives origin to the Casimir force, let us continue with the extra dimensions models.

III. UNIVERSAL EXTRA DIMENSIONS

A. The model

The Casimir force in a universal extra dimension (UXD) scenario [48] was considered in [14] for the case of a massless scalar field to probe the possible existence and size of an additional spatial dimension which is compactified on a S^1/Z_2 orbifold. This geometry restricts the possible vacuum fluctuations of the scalar field to have a wave vector along the extra dimension of the form $k_n = n/R$, with k_n being the wave vectors in the direction of the universal extra dimension and R the radius of S^1 .

Let us start with the five-dimensional (5D) action for a massive scalar field

$$S = \frac{1}{2} \int d^4x \int_0^\pi R d\theta \sqrt{g} (g^{\alpha\beta} \partial_\alpha \Phi(x, \theta) \partial_\beta \Phi(x, \theta) - m_5^2 \Phi^2(x, \theta)). \quad (22)$$

Here, $x^\alpha = (x^\mu, R\theta)$ are the coordinates with $\alpha = (\mu, 4)$ and $\mu = 0, \dots, 3$ are the indexes of our 4D spacetime. m_5 is the mass of the 5D field. πR is the size of the extra dimensions, and $g^{\alpha\beta}$ is the inverse of the metric defined by the interval

$$ds_5^2 = \eta_{\mu\nu} dx^\mu dx^\nu - R^2 d\theta^2. \quad (23)$$

In this metric, the 5D Klein-Gordon equation reads

$$\square_4 \Phi - \frac{1}{R^2} \partial_\theta^2 \Phi + m_5^2 \Phi = 0, \quad (24)$$

which separates through $\Phi(x, R\theta) = \phi(x) \psi(\theta)$ into

$$(\partial_\theta^2 + m_\theta^2 R^2) \psi(\theta) = 0, \quad (25)$$

$$\square_4 \phi + (m_5^2 + m_\theta^2) \phi(x) = 0. \quad (26)$$

When the extra dimension is Kaluza-Klein-type, the only condition on the fields is: $\Phi(x, \theta) = \Phi(x, \theta + 2\pi)$, which allows for a Fourier expansion taking the form

$$\Phi(x, \theta) = \frac{1}{\sqrt{\pi R}} \phi^{(0)}(x) + \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi R}} [\phi^{(n)}(x) \cos(n\theta) + \chi^{(n)}(x) \sin(n\theta)]. \quad (27)$$

When the extra dimension is instead an orbifold S^1/Z_2 , there is an additional parity condition on the 5D scalar field: $\Phi(x, \theta) = \pm \Phi(x, -\theta)$. In this case it is clear that the modes of the scalar field have definite parity, they are either even, denoted by (+), or odd, denoted by (-). Explicitly, they are

$$\phi(x, \theta)_+ = \frac{1}{\sqrt{\pi R}} \phi^{(0)}(x) + \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi R}} \phi^{(n)}(x) \cos(n\theta), \quad (28)$$

$$\phi(x, \theta)_- = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi R}} \chi^{(n)}(x) \sin(n\theta). \quad (29)$$

In models of extra dimensions, the zero mode is associated with the lower dimensional physics, and, as a consequence, we need to consider the even modes if we want to reproduce the lower dimensional physics. Therefore, we impose the additional parity condition $\Phi(x, -\theta) = +\Phi(x, \theta)$ on the 5D scalar field which leave us with the set of 4D scalars field $\{\phi^{(0)}(x), \phi^{(n)}(x)\}$. These fields satisfy the effective 4D equations

$$\square_4 \phi^{(0)}(x) + m_5^2 \phi^{(0)}(x) = 0, \quad (30)$$

$$\square_4 \phi^{(n)}(x) + \left(m_5^2 + \frac{n^2}{R^2}\right) \phi^{(n)}(x) = 0. \quad (31)$$

We interpret to the zero mode $\phi^{(0)}(x)$ as a 4D massive scalar field of mass m_5 and to each mode of the Kaluza-Klein tower $\phi^{(n)}(x)$ as 4D massive scalar fields of mass m_4 , given by

$$m_4 \equiv \sqrt{m_5^2 + \frac{n^2}{R^2}}, \quad n \in \mathbb{N} \cup \{0\}. \quad (32)$$

The modes for the coordinate θ are simply

$$\psi_0(\theta) = \frac{1}{\sqrt{\pi R}} \quad \text{and} \quad \psi_n(\theta) = \sqrt{\frac{2}{\pi R}} \cos(n\theta). \quad (33)$$

This means that the zero mode field is constant along the extra dimension whereas the massive modes are distributed harmonically in that direction.

B. Zeta function approach

In [14], the zeta function method was used to compute the Casimir force associated to a 5D massless scalar field. Here, we consider a 5D massive scalar field. As we have mentioned, in this formalism the relevant quantities are the frequency of the vacuum fluctuations and the modes structure, so if we decompose the 4D fields (i.e. the zero mode and the Kaluza-Klein tower) in the same way as in Eqs. (2) to (5), the energy per unit 3-volume ($L^2 \times \pi R$) of the hyperplanes is

$$\mathcal{E}_{\text{UXD}}(m_5) = \frac{1}{2} \prod_{i=1,2} \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \left(\sum_{N=1, n=0}^{\infty} \omega_{k_i, N, n}(m_4) - l \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \sum_{n=0}^{\infty} \omega_{k_i, k_3, n}(m_4) \right), \quad (34)$$

where

$$\omega_{k_i, N, n}(m_4) \equiv \sqrt{k_1^2 + k_2^2 + \left(\frac{\pi N}{l}\right)^2 + \frac{n^2}{R^2} + m_4^2}, \quad (35)$$

$$\omega_{k_i, k_3, n}(m_4) \equiv \sqrt{k_1^2 + k_2^2 + k_3^2 + \frac{n^2}{R^2} + m_4^2}, \quad (36)$$

and the mass m_4 is given by (32).

In order to get \mathcal{E}_{UXD} , we do not have to compute anything, we already have the answer. Notice that we can rewrite Eq. (34) as follows:

$$\mathcal{E}_{\text{UXD}}(m_5) = \mathcal{E}_{4\text{D}}(m_4|_{n=0}) + \sum_{n=1}^{\infty} \mathcal{E}_{4\text{D}}(m_4), \quad (37)$$

where $\mathcal{E}_{4\text{D}}(m_5)$ is given by (7) and whose analytical expression after integration is (10). Deriving this expression with respect to the separation between hyperplanes, we finally get

$$f_{\text{UXD}}(m_5) = f_{4\text{D}}(m_5) + \sum_{n=1}^{\infty} f_{4\text{D}}(m_4), \quad (38)$$

which reads

$$\begin{aligned} f_{\text{UXD}}(m_5) = & -\frac{m_5^2}{8\pi^2} \left[\frac{3}{l^2} \sum_{N=1}^{\infty} \frac{1}{N^2} K_2(2Nlm_5) + \frac{2m_5}{l} \right. \\ & \times \left. \sum_{N=1}^{\infty} \frac{1}{N} K_1(2Nlm_5) \right] - \frac{1}{8\pi^2} \sum_{n=1}^{\infty} \left(m_5^2 + \frac{n^2}{R^2} \right) \\ & \times \left[\frac{3}{l^2} \sum_{N=1}^{\infty} \frac{1}{N^2} K_2 \left(2Nl \sqrt{m_5^2 + \frac{n^2}{R^2}} \right) \right. \\ & \left. + \frac{2\sqrt{m_5^2 + \frac{n^2}{R^2}}}{l} \sum_{N=1}^{\infty} \frac{1}{N} K_1 \left(2Nl \sqrt{m_5^2 + \frac{n^2}{R^2}} \right) \right]. \end{aligned}$$

For the massless case ($m_5 = 0$) [14], one obtains

$$\begin{aligned}
 f_{\text{UXD}}(m_5 = 0) &= -\frac{\pi^2}{480} \frac{1}{l^4} + \sum_{n=1}^{\infty} f_{4\text{D}}\left(\frac{n}{R}\right) \\
 &= -\frac{\pi^2}{480} \frac{1}{l^4} - \frac{1}{8\pi^2} \sum_{n=1}^{\infty} \frac{n^2}{R^2} \left[\frac{3}{l^2} \sum_{N=1}^{\infty} \frac{1}{N^2} \right. \\
 &\quad \left. \times K_2\left(2Nl \frac{n}{R}\right) + \frac{2n}{lR} \sum_{N=1}^{\infty} \frac{1}{N} K_1\left(2Nl \frac{n}{R}\right) \right].
 \end{aligned} \tag{39}$$

In the setting with only one extra dimension, a good agreement with the data can only be obtained if the radius of such dimension is smaller than $R \leq 10$ nm [14]. This bound is weaker than others obtained from high energy physics which are around 10^{-9} nm.

C. Green's function approach

Consider now the 5D Green's function expressed in terms of the eigenfunctions (33)

$$G_{5\text{D}}(x, \theta, x', \theta') = \sum_{n=0}^{\infty} \psi_n(\theta) \psi_n(\theta') G_{4\text{D}}\left(x, x'; m_5^2 + \frac{n^2}{R^2}\right), \tag{40}$$

where $G_{4\text{D}}$ is the 4D Green's function given by (13). In terms of it, the Casimir force between the hyperplanes is

$$f_{\text{UXD}} = \frac{1}{L^2} \int d\tilde{x}_{\perp} \int_0^{\pi} R d\theta \sqrt{g} [\langle T_{zz}^{\text{in}} \rangle|_{z=l} - \langle T_{zz}^{\text{out}} \rangle|_{z=l}], \tag{41}$$

where

$$\langle T_{zz}^{\text{in/out}} \rangle|_{z=l} = \frac{1}{2i} \partial_z \partial_{z'} G_{5\text{D}}^{\text{in/out}}(x, \theta, x', \theta')|_{x_i \rightarrow x'_i, \theta \rightarrow \theta'}. \tag{42}$$

Just as with the zeta function method, we do not have to compute that much to get the answer. Notice that the 5D Green's function can be rewritten in terms of the 4D one in the way

$$\begin{aligned}
 G_{5\text{D}}(x, \theta, x', \theta') &= \psi_0(\theta) \psi_0(\theta') G_{4\text{D}}(x, x'; m_5^2) \\
 &\quad + \sum_{n=1}^{\infty} \psi_n(\theta) \psi_n(\theta') G_{4\text{D}}\left(x, x'; m_5^2 + \frac{n^2}{R^2}\right),
 \end{aligned} \tag{43}$$

and the expectation value (42) can be rewritten in terms of the expectation values in 4D

$$\begin{aligned}
 \langle T_{zz}^{\text{in/out}} \rangle|_{z=l} &= \psi_0^2(\theta) \frac{1}{2i} \partial_z \partial_{z'} G_{4\text{D}}^{\text{in/out}}(x, x')|_{x_i \rightarrow x'_i} \\
 &\quad + \sum_{n=1}^{\infty} \psi_n^2(\theta) \frac{1}{2i} \partial_z \partial_{z'} G_{4\text{D}}^{\text{in/out}}(x, x')|_{x_i \rightarrow x'_i}.
 \end{aligned} \tag{44}$$

Substituting this expression in (41), we can rewrite the force in terms of the Casimir force for scalars fields in 4D

$$\begin{aligned}
 f_{\text{UXD}}(m_5) &= f_{4\text{D}}(m_5) \int_0^{\pi} R d\theta \psi_0^2(\theta) \\
 &\quad + \sum_{n=1}^{\infty} f_{4\text{D}}\left(m_5^2 + \frac{n^2}{R^2}\right) \int_0^{\pi} R d\theta \psi_n^2(\theta).
 \end{aligned} \tag{45}$$

Because both integrals in θ are equal to 1, we get exactly the expression (38). We conclude that in the case of one universal extra dimension, the effective Casimir forces obtained by both methods coincide. Notice that the force we have computed is the force per unit volume, i.e. the force per unit area (L^2) of the plates and per unit length in the extra dimension. Physically, what we have is a couple of three-dimensional plates, with one dimension stretching along the extra dimension, but both embedded in four spatial dimensions. Such a setting is referred to as having plates of codimension one. An extension of this idea to Randall-Sundrum models actually holds and is what we show next.

IV. RANDALL-SUNDRUM II- p MODELS

The interest in the Randall-Sundrum II- p models comes from its property of localizing not only scalar and gravity fields but also gauge fields whenever there are p extra compact dimensions [45,49,50]. In the case of $p = 0$, the model only localizes scalar and gravity fields. The model corresponds to a $(3 + p)$ -brane with p compact dimensions and positive tension κ , embedded in a $(5 + p)$ spacetime whose metrics are two patches of anti-de Sitter (AdS_{5+p}) of curvature radius κ^{-1}

$$ds_{5+p}^2 = e^{-2\kappa|y|} \left[\eta_{\mu\nu} dx^\mu dx^\nu - \sum_{j=1}^p R_j^2 d\theta_j \right] - dy^2. \tag{46}$$

The Casimir force for a massless scalar field in the RSII setup ($p = 0$) was computed in [16] using the zeta function regularization method; whereas, in [17] the Casimir force was computed for both a massive and a massless scalar field in the RSII-1 model and then generalized to the RSII- p model by means of the Green's function approach in [18] and using the zeta function method in [19]. However, their results turned out different and seemingly depended on the method adopted—a situation clearly unacceptable. Here, we will show that such difference was originated by an erroneous hypervolume factor for the plates considered in the setting in [17,18]. We shall restrict ourselves to the case of a higher dimensional massless scalar field. The interested reader in the massive case can see [18] performing the corresponding modifications. A related calculation of the Casimir effect in de Sitter and anti-de Sitter braneworlds can be found in [51].

A. The mode structure

Let us consider the $(5 + p)$ D action for a massless scalar field Φ in the RSII- p metric (46)

$$S = \frac{1}{2} \int d^4x \prod_{j=1}^p R_j d\theta_j dy \sqrt{|g|} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi. \quad (47)$$

Here, $X^\alpha \equiv (x^\mu, R_i \theta_i, y)$ where x^μ are the coordinates of our 4D spacetime, the p coordinates θ_i are associated to the p compact S^1 's, and y is the noncompact coordinate transverse to the brane which is placed at $y = 0$. The field equation for the scalar field is given by

$$e^{2\kappa|y|} \square_4 \Phi - e^{2\kappa|y|} \sum_{j=1}^p \frac{1}{R_j^2} \partial_{\theta_j}^2 \Phi - \frac{1}{\sqrt{-g}} \partial_y [\sqrt{-g} \partial_y \Phi] = 0, \quad (48)$$

which separates through $\Phi(X) = \varphi(x) \prod_{j=1}^p \Theta_j(\theta_j) \psi(y)$ into

$$(\partial_{\theta_j}^2 + m_{\theta_j}^2 R_j^2) \Theta_j(\theta_j) = 0, \quad j = 1, \dots, p, \quad (49)$$

$$(\partial_y^2 - (4+p)\kappa \operatorname{sgn}(y) \partial_y + m^2 e^{2\kappa|y|}) \psi(y) = 0, \quad (50)$$

$$(\square_4 + m_4^2) \varphi(x) = 0. \quad (51)$$

The $(p+1)$ separation constants with units of mass, m_{θ_j} and m , correspond to the spectra of the modes for the compact and noncompact dimensions, respectively. They give rise in turn to the effective mass, m_4 , of the 4D modes in (51) through: $m_4^2 \equiv \sum_{j=1}^p m_{\theta_j}^2 + m^2$.

To find mode solutions to the above equations we shall incorporate three types of boundary conditions: (a) To implement the presence of the plates in $(4+p)$ -space, we simply set $\varphi(z=0, l) = 0$. The eigenfunctions and eigenvalues for this Dirichlet boundary conditions were already discussed in Sec. II. (b) To match the modes across the brane along the noncompact dimension, we impose $\psi(y=0^+) = \psi(y=0^-)$ and $\partial_y \psi(y=0^+) = \partial_y \psi(y=0^-)$. (c) To account for the compactness of the p dimensions, we set $\Theta_{n_j}(\theta_j) = \Theta_{n_j}(\theta_j + 2\pi)$. Here, we obtain explicitly the plates represented by two parallel planes in 3 space but stretching along the extra dimensions.

The allowed modes for the noncompact dimension are now a massless zero mode localized on the brane

$$\psi_0 = \sqrt{\frac{(2+p)\kappa}{2}}, \quad (52)$$

which satisfies the normalization condition,

$$\int_{-\infty}^{\infty} dy e^{-(p+2)\kappa|y|} \psi_0^2 = 1. \quad (53)$$

The localization comes from the fact that the 4D effective profile of the modes is given by $\tilde{\psi}_0 = e^{-(p+2)\kappa|y|/2} \psi_0$ which clearly is localized on the brane. The massive modes have the form

$$\psi_m(y) = e^{((4+p)/2)\kappa y} \sqrt{\frac{m}{2\kappa}} \left[a_m J_\gamma \left(\frac{me^{\kappa y}}{\kappa} \right) + b_m N_\gamma \left(\frac{me^{\kappa y}}{\kappa} \right) \right], \quad (54)$$

$$m > 0.$$

Here, J_γ and N_γ are the Bessel and Neumann functions, respectively. $\gamma = \frac{4+p}{2}$ and the coefficients a_m and b_m are given by

$$a_m = -\frac{A_m}{\sqrt{1+A_m^2}}, \quad b_m = \frac{1}{\sqrt{1+A_m^2}}, \quad (55)$$

where

$$A_m = \frac{N_{\gamma-1}(\frac{m}{\kappa})}{J_{\gamma-1}(\frac{m}{\kappa})}. \quad (56)$$

Notice that in this case, the localization of the massive modes on the brane is better for increasing p , since the modes are modulated exponentially in the form $e^{-p\kappa|y|/2}$. The normalization condition for the massive modes is

$$\int_{-\infty}^{\infty} dy e^{-(p+2)\kappa|y|} \psi_m(y) \psi_{m'}(y) = \delta(m-m'). \quad (57)$$

The modes in θ_j are:

$$\Theta_{n_j}(\theta_j) = \frac{1}{\sqrt{2\pi R_j}} e^{in_j \theta_j}, \quad \text{where } n_j = m_{\theta_j} R_j \in \mathbb{Z}. \quad (58)$$

Therefore, the contributions of the extra compact dimensions to m_4 , are given in terms of $m_{\theta_j}^2 = n_j^2 / R_j^2$

$$m_4^2 = \sum_{j=1}^p \frac{n_j^2}{R_j^2} + m^2. \quad (59)$$

B. Zeta function approach

In analogy with the cases studied above and due to the fact that the modes behave differently for the zero mode (52) and for the Kaluza-Klein modes (54), the energy density per unit of $(p+3)$ volume ($L^2 \times \prod_{j=1}^p (2\pi R_j) \times \frac{2}{(p+3)\kappa}$) for the scalar field is

$$\begin{aligned} \mathcal{E}_{\text{RSII}p} = & \frac{1}{2} \prod_{i=1,2} \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \sum_{\{n\}} \left(\sum_{N=1}^{\infty} \omega_{k_i, N, n_j}(m_4) \right. \\ & \left. - l \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \omega_{k_i, k_3, n_j}(m_4) \right) + \frac{1}{2} \prod_{i=1,2} \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \\ & \times \int_0^{\infty} \frac{dm}{\kappa} \sum_{\{n\}} \left(\sum_{N=1}^{\infty} \omega_{k_i, N, n_j, m}(m_4) \right. \\ & \left. - l \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \omega_{k_i, k_3, n_j, m}(m_4) \right), \quad (60) \end{aligned}$$

where m_4 is given by (59), $\{n\}$ denotes the set $\{n_1, n_2, \dots, n_p | n_1 \in \mathbb{Z}, \dots, n_p \in \mathbb{Z}\}$ and the dispersion re-

lations are

$$\omega_{k_i, N, n_j, m}(m_4) \equiv \sqrt{k_1^2 + k_2^2 + \left(\frac{\pi N}{l}\right)^2 + \sum_{j=1}^p \frac{n_j^2}{R_j^2} + m^2}, \quad (61)$$

and

$$\omega_{k_i, k_3, n_j, m}(m_4) \equiv \sqrt{k_1^2 + k_2^2 + k_3^2 + \sum_{j=1}^p \frac{n_j^2}{R_j^2} + m^2}. \quad (62)$$

It is important to stress that in this case each one of the different p sums in n_j goes from $-\infty$ to ∞ and not as in the UXD case from 0 to ∞ . The reason is that here, we are considering the Kaluza-Klein tower associated to S^1 ; whereas, in the UXD case the tower is due to the orbifold S^1/Z_2 . In fact, the Kaluza-Klein tower of S^1 is two copies the Kaluza-Klein tower of S^1/Z_2 .

Regarding the integration on the continuous massive modes m , it is possible to take advantage of the following trick:

$$\int_0^\infty dm f(m) = \frac{1}{2} \int_{-\infty}^\infty dm f(m) = \pi \int_{-\infty}^\infty \frac{dm}{2\pi} f(m), \quad (63)$$

which is valid whenever the function f be even: $f(-m) = f(m)$. Because of the fact that the frequency $\omega(m)$ satisfies this condition, one can consider the integration on m at the same footing that the integrals on k_i , $i = 1, 2$. As a consequence after integration, the result is simply

$$\mathcal{E}_{\text{RSII}_p}(0) = \sum_{\{n\}} \left(\mathcal{E}_{4\text{D}} \left(\sqrt{\sum_{j=1}^p \frac{n_j^2}{R_j^2}} \right) + \frac{\pi}{\kappa} \mathcal{E}_{5\text{D}} \left(\sqrt{\sum_{j=1}^p \frac{n_j^2}{R_j^2}} \right) \right), \quad (64)$$

where $\mathcal{E}_{5\text{D}}(\mu)$ can be obtained from (A8) and whose analytical expression after integration can be obtained from (A10). Deriving this expression with respect to the separation between hyperplanes, one gets

$$f_{\text{RSII}_p}(0) = \sum_{\{n\}} \left(f_{4\text{D}} \left(\sqrt{\sum_{j=1}^p \frac{n_j^2}{R_j^2}} \right) + \frac{\pi}{\kappa} f_{5\text{D}} \left(\sqrt{\sum_{j=1}^p \frac{n_j^2}{R_j^2}} \right) \right). \quad (65)$$

Interpretation of this result is straightforward, as the first term corresponds to an infinite sum of 4D Casimir forces, one of the terms corresponds to the Casimir force due to a massless scalar field (the one corresponding to the zero mode of all the p compact extra dimensions), plus an infinite sum of 4D Casimir forces corresponding to massive scalar fields (where the mass corresponds to all different combinations where there is at least a non zero mode), and the second term comes from the non zero modes of the noncompact extra dimensions and corresponds to a sum of

5D Casimir forces. Some examples previously discussed in the literature are

(i) Case $p = 0$.

In this case we simply have [16]

$$f_{\text{RSII}}(0) = f_{4\text{D}}(0) + \frac{\pi}{\kappa} f_{5\text{D}}(0). \quad (66)$$

At this point you can think units do not match, but they do because f_4 is a force per unit area and f_5 is a force per unit volume. Explicitly, we have

$$f_{\text{RSII}}(0) = -\frac{\pi^2}{480} \frac{1}{l^4} \left(1 + \frac{45}{4\pi^3} \zeta(5) \frac{1}{\kappa l} \right). \quad (67)$$

(ii) Case $p = 1$.

In this case we have [19]

$$\begin{aligned} f_{\text{RSII}_1}(0) &= \sum_{n=-\infty}^{\infty} \left(f_{4\text{D}} \left(\frac{n}{R} \right) + \frac{\pi}{\kappa} f_{5\text{D}} \left(\frac{n}{R} \right) \right) \\ &= f_{4\text{D}}(0) + \frac{\pi}{\kappa} f_{5\text{D}}(0) + 2 \sum_{n=0}^{\infty} \left(f_{4\text{D}} \left(\frac{n}{R} \right) \right. \\ &\quad \left. + \frac{\pi}{\kappa} f_{5\text{D}} \left(\frac{n}{R} \right) \right). \end{aligned} \quad (68)$$

C. Green's function approach

Let us now apply the Green's function method to the RSII- p models. As we have discussed, the force between the plates is obtained by integrating over coordinates ‘‘lateral’’ to the plates. In this case, \vec{x}_\perp , y , θ_j due to the fact that the normal-normal component of vacuum energy momentum tensor in $3 + 1 + p$ spatial dimensions has physical units of force per unit of ‘‘volume’’ of $2 + 1 + p$ space:

$$\begin{aligned} F &= \int_0^A d\vec{x}_\perp \int_{-\infty}^\infty dy \sqrt{|g_{\text{plate}}|} \left[\prod_{j=1}^p \int_0^{2\pi} R d\theta_j \right] \\ &\quad \times [\langle T_{zz}^{\text{in}} \rangle|_{z=l} - \langle T_{zz}^{\text{out}} \rangle|_{z=l}], \end{aligned} \quad (69)$$

where A is the area of the planes forming the plates in 3 space and g_{plates} is the induced metric on the physical plate located a $z = l$. Since the physical plate is a surface of $(p + 2)\text{D}$, the $\sqrt{|g_{\text{plates}}|}$ contributes an exponential of $-\kappa|y|(p + 2)$. Instead of this factor, the exponential in [17,18] contained an erroneous power given by $-\kappa|y|(p + 3)$.

The vacuum expectation values and the Green's function are related to the normal-normal components of the vacuum energy momentum tensor through

$$\begin{aligned} \langle T_{zz}^{\text{in/out}} \rangle|_{z=l} &= \frac{1}{2i} \partial_z \partial_{z'} G_{(5+p)\text{D}}^{\text{in/out}} \\ &\quad \times (x, y, \theta; x', y', \theta')|_{x_\perp \rightarrow x'_\perp, z \rightarrow z'=l, \theta_j \rightarrow \theta'_j}. \end{aligned} \quad (70)$$

As in the previous cases, we can rewrite the $(5 + p)\text{D}$

Green's function in terms of the 4D Green's function

$$G_{(5+p)\text{D}}(x, y, \theta; x', y', \theta') = \sum_{\{n\}} \prod_{j=1}^p \Theta_{n_j}^*(\theta_j) \Theta_{n_j}(\theta'_j) \psi_0(y) \psi_0(y') G_{4\text{D}}(x, x'; m_4 |_{m=0}) \\ + \sum_{\{n\}} \int \frac{dm}{\kappa} \prod_{j=1}^p \Theta_{n_j}^*(\theta_j) \Theta_{n_j}(\theta'_j) \psi_m(y) \psi_m(y') G_{4\text{D}}(x, x'; m_4), \quad (71)$$

where m_4 is given by (59). Introducing this expression in (70) and then in (69), we obtain

$$f = \sum_{\{n\}} f_{4\text{D}}(m_4 |_{m=0}) \int_{-\infty}^{\infty} dy e^{-2\kappa|y|(p+2)} \psi_0^2 \\ + \sum_{\{n\}} \int \frac{dm}{\kappa} f_{4\text{D}}(m_4) \int_{-\infty}^{\infty} dy e^{-2\kappa|y|(p+2)} \psi_m^2(y). \quad (72)$$

But by virtue of the relations (53) and (57), the dependence on the y coordinate drops out completely obtaining

$$f = \sum_{\{n\}} f_{4\text{D}}(m_4 |_{m=0}) + \sum_{\{n\}} \int_0^{\infty} \frac{dm}{\kappa} f_{4\text{D}}(m_4). \quad (73)$$

This expression coincides exactly with the one obtained by the zeta function method (65) once one performs the trick (63).

Before ending this section, some comments are in order. Notice that there is no factor depending on the number of compact dimensions p in front of the effective 4D Casimir force, as it was reported in [17,18]. Also, because the integration in the noncompact extra dimension has been carried out, it is not longer necessary to evaluate the eigenfunctions $\psi_m(y)$ on the brane. In fact, the contribution of the whole size of the plates in the y direction has been considered.

V. RANDALL-SUNDRUM $1p$ MODELS

A. Mode structure

As a final example, we discuss the case of a bulk scalar field in a RSI- p model. In this case the metric is given again by (46), but this time the coordinate y is compact ($0 \leq y \leq \pi r$). Thus, the setup allows two $(3+p)$ -branes to lie, respectively, at $y = 0, \pi r$. The Casimir effect for a massless scalar field in this model was computed using the zeta function method for $p = 0$ in [16] and for arbitrary p in [19]. We address this problem here for a massive scalar field of mass μ . As it is well known [52], when the higher dimensional scalar field is massive there does not exist a zero mode solution of the equations of motion with simple Neumann or Dirichlet boundary conditions. In order to overcome this problem, it is necessary to modify the boundary action and include boundary mass terms

$$S = S_{\Phi} + S_{\text{Brane}}, \quad (74)$$

$$S_{\Phi} = \frac{1}{2} \int d^4x dy \prod_{j=1}^p R_j d\theta_j \sqrt{-g} (g^{MN} \partial_M \Phi \partial_N \Phi - \mu^2 \Phi^2), \quad (75)$$

$$S_{\text{Brane}} = - \int d^4x dy \prod_{j=1}^p R_j d\theta_j \sqrt{-g} 2b\kappa [\delta(y) \\ - \delta(y - \pi r)] \Phi^2, \quad (76)$$

where b is a dimensionless constant parametrization the boundary mass in units of κ . The S_{Brane} allows to implement various boundary conditions corresponding to different mode's localizations in the y direction as we explain now. The resulting field equations are, by letting $\Phi(x, y, \theta_j) = \varphi(x) \psi(y) \prod_{j=1}^p \Theta_j(\theta_j)$,

$$(\partial_{\theta_j}^2 + m_{\theta_j}^2 R_j^2) \Theta_j(\theta_j) = 0, \quad (77)$$

$$(\partial_y^2 - (4+p)\kappa \text{sgn}(y) \partial_y + m^2 e^{2\kappa|y|} - \mu^2 - b\kappa(\delta(y) \\ - \delta(y - \pi r))) \psi(y) = 0, \quad (78)$$

$$(\square_4 + m_4^2) \phi(x) = 0, \quad (79)$$

with m_{θ_j}, m separations constants so that the effective mass of the scalar field can be read as $m_4^2 := m_{\theta_j}^2 + m^2$. Eigenfunctions and eigenvalues of the p coordinates θ_j are given by (58) and we do not elaborate further on them. In the y direction, the eigenfunctions are accounted for by subjecting them to the modified Neumann boundary conditions

$$\left[\frac{\partial \psi}{\partial y} - b\kappa \text{sgn}(y) \psi \right]_{y=0, \pi r} = 0. \quad (80)$$

If we write the higher dimensional mass in units of κ , i.e. $\mu^2 \equiv a\kappa^2$, the above equations depend on the two arbitrary mass parameters: a and b . For generic values of these parameters, there are not solutions to the boundary conditions; however, if $b = a \pm \gamma$ where $\alpha = \frac{4+p}{2}$ and $\gamma \equiv \sqrt{\alpha^2 + a}$, a zero mode solution exists. Assuming γ to be real, the only free parameter has a range $-\infty < b < \infty$ and using it, the scalar zero mode can be localized anywhere in the bulk

$$\psi_0(y) = \begin{cases} \sqrt{\frac{(b-[\alpha-1])\kappa}{(e^{2(b-[\alpha-1])\kappa r}-1)}} e^{(b-[\alpha-1])\kappa y}, & b-[\alpha-1] > 0 \text{ localized in IR brane} \\ \frac{1}{\sqrt{2\pi r}}, & b-[\alpha-1] = 0 \text{ no localization} \\ \sqrt{\frac{|b-[\alpha-1]|\kappa}{(1-e^{-2|b-[\alpha-1]|\kappa r})}} e^{(b-[\alpha-1])\kappa y}, & b-[\alpha-1] < 0 \text{ localized in UV brane} \end{cases}, \quad (81)$$

which satisfies the normalization condition

$$\int_0^{\pi r} dy e^{-(2+p)\kappa|y|} \psi_0^2(y) = 1. \quad (82)$$

These results generalize the case $p = 0$ ($\Rightarrow \alpha = 2$) [52].

As for the massive modes, we have

$$\psi_m(y) = e^{\alpha\kappa y} \left[c_1 J_\gamma \left(\frac{m}{\kappa} e^{\kappa y} \right) + c_2 Y_\gamma \left(\frac{m}{\kappa} e^{\kappa y} \right) \right], \quad (83)$$

where $c_{1,2}$ are arbitrary constants and J_γ, Y_γ are Bessel's functions of order γ . Imposing the boundary conditions in the low energy regime $m \ll \kappa$ with *large* $r, \kappa r \gg 1$ allows us to obtain the approximated Kaluza-Klein mass spectrum

$$\frac{m_n}{\kappa} = \left(n + \frac{\gamma}{2} - \frac{3}{4} \right) \pi e^{-\kappa\pi r}, \quad n = 1, 2, \dots \quad (84)$$

Thus, we have that the 4D mass is given by

$$m_4^2 = \begin{cases} \sum_{j=1}^p \frac{n_j^2}{R_j^2}, & n = 0, \\ \sum_{j=1}^p \frac{n_j^2}{R_j^2} + \left(n + \frac{\gamma}{2} - \frac{3}{4} \right)^2 \kappa^2 \pi^2 e^{-2\kappa\pi r}, & n = 1, 2, \dots \end{cases} \quad (85)$$

The constants in (83) are chosen in such a way that the orthogonality relations are

$$\int_0^{\pi r} dy e^{-(2+p)\kappa|y|} \psi_m(y) \psi_{m'}(y) = \delta_{mm'}. \quad (86)$$

B. Zeta function approach

Now the energy density per unit of $(p + 3)$ volume of the plate, which is given by $(L^2 \times \prod_{j=1}^p (2\pi R_j) \times \frac{2}{(p+3)\kappa} \times (1 - e^{-(p+3)\kappa\pi r}))$, takes the form

$$\begin{aligned} \mathcal{E}_{\text{RSI}p} &= \frac{1}{2} \prod_{i=1,2} \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \sum_{\{n_j\}} \left(\sum_{N=1}^{\infty} \omega_{k_i, N, n_j} \left(\sum_{j=1}^p \frac{n_j^2}{R_j^2} \right) \right. \\ &\quad - l \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \omega_{k_i, k_3, n_j} \left(\sum_{j=1}^p \frac{n_j^2}{R_j^2} \right) \Big) + \frac{1}{2} \prod_{i=1,2} \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \\ &\quad \times \sum_{n=1} \sum_{\{n_j\}} \left(\sum_{N=1}^{\infty} \omega_{k_i, N, n_j, m} \left(\sum_{j=1}^p \frac{n_j^2}{R_j^2} + m_n^2 \right) \right. \\ &\quad \left. - l \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \omega_{k_i, k_3, n_j, m} \left(\sum_{j=1}^p \frac{n_j^2}{R_j^2} + m_n^2 \right) \right), \quad (87) \end{aligned}$$

where $\{n_j\}$ denotes the set $\{n_1, n_2, \dots, n_p | n_1 \in \mathbb{Z}, \dots, n_p \in \mathbb{Z}\}$

$$\begin{aligned} \omega_{k_i, N, n_j, m_n} &\left(\sum_{j=1}^p \frac{n_j^2}{R_j^2} + m^2 \right) \\ &\equiv \sqrt{k_1^2 + k_2^2 + \left(\frac{\pi N}{l} \right)^2 + \sum_{j=1}^p \frac{n_j^2}{R_j^2} + m_n^2}, \quad (88) \end{aligned}$$

and

$$\begin{aligned} \omega_{k_i, k_3, n_j, m_n} &\left(\sum_{j=1}^p \frac{n_j^2}{R_j^2} + m_n^2 \right) \\ &\equiv \sqrt{k_1^2 + k_2^2 + k_3^2 + \sum_{j=1}^p \frac{n_j^2}{R_j^2} + m_n^2}. \quad (89) \end{aligned}$$

Next, upon integrating, we have

$$\begin{aligned} \mathcal{E}_{\text{RSI}p}(\mu) &= \sum_{\{n_j\}} \left(\mathcal{E}_{4\text{D}} \left(\sqrt{\sum_{j=1}^p \frac{n_j^2}{R_j^2}} \right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \mathcal{E}_{4\text{D}} \left(\sqrt{\sum_{j=1}^p \frac{n_j^2}{R_j^2} + m_n^2} \right) \right), \quad (90) \end{aligned}$$

where $\mathcal{E}_{4\text{D}}$ is given by (10). Deriving this expression with respect to the separation between plates, one gets

$$\begin{aligned} f_{\text{RSI}p}(\mu) &= \sum_{\{n_j\}} \left(f_{4\text{D}} \left(\sqrt{\sum_{j=1}^p \frac{n_j^2}{R_j^2}} \right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} f_{4\text{D}} \left(\sqrt{\sum_{j=1}^p \frac{n_j^2}{R_j^2} + m_n^2} \right) \right). \quad (91) \end{aligned}$$

As a particular case, we have $\mu = p = 0$ [16]

$$\begin{aligned} f_{\text{RSI}p}(0) &= f_{4\text{D}}(0) + \sum_{n=1}^{\infty} f_{4\text{D}} \left(\left(n + \frac{1}{4} \right) \pi e^{-\kappa\pi r} \right) \\ &= -\frac{\pi^2}{480l^4} - \frac{e^{-2\kappa\pi r}}{8} \sum_{n=1}^{\infty} \left(n + \frac{1}{4} \right)^2 \left[\frac{3}{l^2} \sum_{N=1}^{\infty} \frac{1}{N^2} K_2 \right. \\ &\quad \times \left(2Nl \left(n + \frac{1}{4} \right) \pi e^{-\kappa\pi r} \right) + \frac{2}{l} \left(n + \frac{1}{4} \right) \pi e^{-\kappa\pi r} \\ &\quad \left. \times \sum_{N=1}^{\infty} \frac{1}{N} K_1 \left(2Nl \left(n + \frac{1}{4} \right) \pi e^{-\kappa\pi r} \right) \right]. \quad (92) \end{aligned}$$

C. Green’s function approach

Once again the calculation using the Green’s function method relies on the fact that the $(5 + p)$ D Greens function can be written in terms of the 4D Green’s function via

$$G_{(5+p)D}(x, y, \theta; x', y', \theta') = \sum_{\{n_j\}} \prod_{j=1}^p \Theta_{n_j}^*(\theta_j) \Theta_{n_j}(\theta'_j) \psi_0(y) \psi_0(y') G_{4D}(x, x'; m_4|_{m=0}) + \sum_{\{n_j\}} \sum_{n=1}^{\infty} \prod_{j=1}^p \Theta_{n_j}^*(\theta_j) \Theta_{n_j}(\theta'_j) \psi_n(y) \psi_n(y') G_{4D}(x, x'; m_4), \tag{93}$$

where m_4 is given by (85). Introducing this expression in (70) and then in (69), we obtain

$$f = \sum_{\{n_j\}} f_{4D}(m_4|_{m=0}) \int_0^{\pi r} dy e^{-2\kappa|y|(p+2)} \psi_0^2 + \sum_{\{n_j\}} \sum_{n=1}^{\infty} f_{4D}(m_4) \int_0^{\pi r} dy e^{-2\kappa|y|(p+2)} \psi_n^2(y). \tag{94}$$

And now, by virtue of the orthogonality relations, Eqs. (82) and (86), the dependence on the y coordinate drops out to yield

$$f = \sum_{\{n\}} f_{4D}(m_4(\{n_j\}, n = 0)) + \sum_{\{n\}} \sum_{n=1} f_{4D}(m_4(\{n_j\}, n)). \tag{95}$$

This expression coincides exactly with the one obtained by the zeta function regularization method, Eq. (91).

VI. DISCUSSION

The old idea that our world is embedded in a spacetime with dimension higher than four has reemerged in brane world models which have revealed windows to look for deviations from standard physics mostly in high energy physics [8,9] and cosmology (e.g. [10–13]). Nevertheless, low energy tests may also provide some insight into possible imprints of extra dimensions including, in particular, the Casimir force [14–34]. Such force is sensitive to the mode structure of the field which in turn depends on the features of the background spacetime and bounds can be set for the values of the parameters of given brane world models which produce Casimir forces deviating from known data beyond the corresponding uncertainties.

To determine the Casimir force, one can made use of either of two well known approaches: Green’s function and zeta function. In the case of flat spacetimes both approaches yield the same result (see eg. [46]); however, for brane worlds one usually assumes this is the case. In this work we have actually shown Green’s function and zeta function yield the same Casimir force for the case of universal extra dimensions and Randall-Sundrum models with one or two branes added by p compact dimensions. These results correct, in particular, an erroneous difference

between the Casimir force obtained by Green’s function technique [17,18] and the one obtained from zeta function [16,19] for a massless scalar field in the case of a single brane Randall-Sundrum scenario added by p compact dimensions. The origin of the difference in this case was due to an incorrect hypervolume for the plates subject to the Casimir force in [17,18].

The coincidence of the above two approaches to the Casimir force can be understood as follows. Although the Green’s function technique involves further details of the mode decomposition of the corresponding fields, it is due to their orthogonality relations which involve the correct hypervolume factor of the plates that one can literally eliminate the mode eigenfunctions from the Casimir force. This is neatly seen in the case of UXD, Eq. (45). This holds similarly for the cases of RSII- p and RSI- p , as can be seen from Eqs. (72) and (94), respectively.

Hence, given the equivalence of the zeta function and Green’s function to determine the Casimir force in Randall-Sundrum and universal extra dimensions models, one can conclude that localization of the field modes does not play a role as far as the Casimir force is concerned. This is so due to the fact that zeta function is actually insensitive to the form of the mode eigenfunctions but only to dispersion relations. On the side of the Green’s function approach, while built explicitly on an eigenfunction expansion, it loses them by virtue of the orthogonality relations they fulfill which just absorbs the hypervolume of the plates. This is reassuring since it has been noticed that localizing all the fields of the standard model to the brane located at $y = \pi R$ leads to problems with the phenomenology of proton decay, flavor changing neutral currents effects and neutrino masses [52]. Indeed, although originally the p extra compact dimensions were added to the Randall-Sundrum models to produce localization of gauge fields, such a feature seems not to be required anymore.

It should be stressed an important pattern has put forward in this work for the resulting Casimir force in brane worlds, namely, it can be expressed as the sum of two terms each one having the specific form of the 4D Casimir force: the first one containing, in particular, the zero mode defined by the extra dimensions and the second one including the full Kaluza-Klein tower of massive modes. This feature

simplified importantly the analysis and, in particular, allowed to adopt the full machinery of previous results in $d + 2$ Minkowski spacetime [41].

It would be rather interesting to study other brane world models to test the equivalence we have here proved between the zeta function and Green's function approaches in the calculation of the Casimir force. Indeed, it should be possible to have a general proof of it at least for sufficiently symmetric spacetimes [53].

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APPENDIX: CASIMIR FORCE IN $D + 2$ MINKOWSKI SPACETIME

In this appendix, we compute the Casimir force for a massive scalar field of mass μ in $(d + 2)$ -dimensional Minkowski spacetime. We discuss the calculation using the two methods we are interested in: the zeta function method and the Green's function method. Although for a given mass $\mu \neq 0$ the force can be computed at the very end only numerically, it is possible to give an analytical expression of it for a generic mass. The aim to show this computation is twofold: (i) to avoid unnecessary repetitions of the calculation throughout the paper and (ii) to allow us comparison with recent results in the literature.

1. Zeta function approach

This computation was originally discussed in [41] and we adapt it here to our notation. Our starting point is Eq. (7)

$$\begin{aligned} \mathcal{E} &= \frac{E_{\text{plates}} - E_0}{L^2} \\ &= \frac{1}{2} \prod_{i=1,2} \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \left(\sum_{N=1}^{\infty} \omega_{k_1, k_2, N}(\mu) \right. \\ &\quad \left. - l \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \omega_{k_1, k_2, k_3}(\mu) \right). \end{aligned} \quad (\text{A1})$$

In terms of the integral

$$I_d(\alpha^2) \equiv \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \sqrt{k_d^2 + \alpha^2}, \quad (\text{A2})$$

we can rewrite Eq. (A1) as

$$\mathcal{E} = \sum_{N=1}^{\infty} I_{d=2} \left(\frac{N^2 \pi^2}{L^2} + \mu^2 \right) - I_{d=3}(\mu^2). \quad (\text{A3})$$

In order to be general, we shall evaluate the Casimir energy

for a scalar field in $D = d + 2$ dimensional Minkowski spacetime, between hyperplanes of dimension d . This problem is known as the Casimir effect of codimension one [41]

$$\mathcal{E}_{d+2} = \frac{E_{\text{plates}} - E_0}{L^d} = \sum_{N=1}^{\infty} I_d \left(\frac{N^2 \pi^2}{l^2} + \mu^2 \right) - I_{d+1}(\mu^2). \quad (\text{A4})$$

Using the Euler representation for the gamma function

$$\Gamma(z) = g^z \int_0^{\infty} e^{-gt} t^{z-1} dt, \quad (\text{A5})$$

the integral (A2) can be rewritten employing the Schwinger proper time representation for the square root as

$$I_d(\alpha^2) = \frac{1}{2\Gamma(-1/2)} \int \frac{d^d k}{(2\pi)^d} \int_0^{\infty} \frac{dt}{t} t^{-1/2} e^{-t(k_d^2 + \alpha^2)}. \quad (\text{A6})$$

Performing the Gaussian integral first and using (A5) again, we have

$$I_d(\alpha^2) = -\frac{1}{2} \frac{1}{(4\pi)^{(d+1)/2}} \Gamma\left(-\frac{d+1}{2}\right) \alpha^{d+1}, \quad (\text{A7})$$

where we have used the value $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$. Substituting this result in (A4), we obtain

$$\begin{aligned} \mathcal{E}_{d+2} &= -\frac{1}{2(4\pi)^{(d+1)/2}} \left(\Gamma\left(-\frac{d+1}{2}\right) \left(\frac{\pi}{l}\right)^{d+1} \right. \\ &\quad \left. \times \sum_{N=1}^{\infty} \left(N^2 + \frac{l^2 \mu^2}{\pi^2} \right)^{(d+1)/2} + \frac{\Gamma(-\frac{d+2}{2})}{\Gamma(-\frac{1}{2})} l \mu^{d+2} \right). \end{aligned} \quad (\text{A8})$$

In order to compute the sum, we use the Epstein-Hurwitz function which is defined as

$$\begin{aligned} \zeta_{\text{EH}}(s, a^2) &= \sum_{N=1}^{\infty} (N^2 + a^2)^{-s} \\ &= -\frac{(a^2)^{-s}}{2} + \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{2\Gamma(s)} (a^2)^{(1/2)-s} \\ &\quad + \frac{2\pi^s}{\Gamma(s)} (a^2)^{-(s/2)+(1/4)} \sum_{n=1}^{\infty} n^{s-(1/2)} \\ &\quad \times K_{s-(1/2)}(2\pi n \sqrt{a^2}), \end{aligned} \quad (\text{A9})$$

where K is the modified Bessel function of second type. In our case, $s = -\frac{d+1}{2}$ and $a = \frac{l\mu}{\pi}$. It turns out that the second term in (A9) cancels with the second term in (A8). Often this cancellation is not performed explicitly but by an equivalent argument the second term in the Epstein-Hurwitz function is discarded [41]. This is why people claim that in the zeta function regularization method it is not necessary to subtract any quantity and that a finite result comes directly considering only the first integral in

(A1). The final expression for the energy is

$$\begin{aligned} \mathcal{E}_{d+2} = & -\frac{1}{2(4\pi)^{(d+1)/2}} \left(-\frac{1}{2} \Gamma\left(-\frac{d+1}{2}\right) \mu^{d+1} + \frac{2}{\sqrt{\pi}} \right. \\ & \left. \times \frac{\mu^{(d+2)/2}}{l^{d/2}} \sum_{n=1}^{\infty} \frac{1}{n^{(d+2)/2}} K_{-(d+2)/2}(2nl\mu) \right). \end{aligned} \quad (\text{A10})$$

The first term is a constant energy and therefore we discard it, because it does not contribute to the Casimir force. Deriving the energy with respect to the separation l between the hyperplanes, we obtain the $(d+2)$ -dimensional Casimir force

$$\begin{aligned} f_{d+2} = & -\frac{d\mathcal{E}_{d+2}}{dl} \\ = & \frac{2}{(4\pi)^{(d+2)/2}} \mu^{(d+2)/2} \frac{d}{dl} \left[\frac{1}{l^{d/2}} \sum_{n=1}^{\infty} \frac{1}{n^{(d+2)/2}} \right. \\ & \left. \times K_{-(d+2)/2}(2nl\mu) \right], \end{aligned} \quad (\text{A11})$$

which can be evaluated explicitly. Using the properties $K_{-\nu}(z) = K_{\nu}(z)$ and $z\partial_z K_{\nu}(z) = -zK_{\nu-1}(z) - \nu K_{\nu}(z)$, we obtain finally

$$\begin{aligned} f_{(d+2)}(\mu) = & -2 \left(\frac{\mu}{4\pi} \right)^{(d+2)/2} \left[\frac{1}{l^{(d+2)/2}} \sum_{n=1}^{\infty} \frac{1}{n^{(d+2)/2}} \right. \\ & \times K_{(d+2)/2}(2nl\mu) - \frac{2\mu}{l^{d/2}} \sum_{n=1}^{\infty} \frac{1}{n^{d/2}} \\ & \left. \times K_{(d+4)/2}(2nl\mu) \right]. \end{aligned} \quad (\text{A12})$$

In particular, we are interested in the 4D Casimir force, which is obtained setting $d=2$ in the formula above

$$\begin{aligned} f_{4\text{D}}(\mu) = & -\frac{\mu^2}{8\pi^2} \left[\frac{1}{l^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2nl\mu) \right. \\ & \left. - \frac{2\mu}{l} \sum_{n=1}^{\infty} \frac{1}{n} K_3(2nl\mu) \right]. \end{aligned} \quad (\text{A13})$$

Sometimes this expression is presented in a slightly different way, which can be obtained by using the identity $K_{\nu}(z) = K_{\nu-2}(z) + \frac{2(\nu-1)}{z} K_{\nu-1}(z)$

$$\begin{aligned} f_{4\text{D}}(\mu) = & \frac{\mu^2}{8\pi^2} \left[\frac{3}{l^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2nl\mu) \right. \\ & \left. + \frac{2\mu}{l} \sum_{n=1}^{\infty} \frac{1}{n} K_1(2nl\mu) \right]. \end{aligned} \quad (\text{A14})$$

2. Green's function approach

Our starting point is the integral (18), but in the spirit of generality, in analogy with the zeta function method, we allow to have d transverse dimensions, namely,

$$\begin{aligned} f_{(d+2)}(\mu) = & \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \int \frac{d\xi}{2\pi} \left(\frac{2}{l} \sum_{k=1}^{\infty} \frac{\frac{\pi^2 k^2}{l^2}}{\rho^2 + \mu^2 + \frac{\pi^2 k^2}{l^2}} \right. \\ & \left. + \sqrt{\rho^2 + \mu^2} \right). \end{aligned} \quad (\text{A15})$$

This integral can be performed straightforward in polar coordinates, in terms of the volume of the unitary d -dimensional sphere that we denote by $\text{vol}(S^d)$

$$\begin{aligned} \int \frac{d^d k}{(2\pi)^d} \int \frac{d\xi}{2\pi} \rightarrow & \frac{\text{vol}(S^d)}{(2\pi)^{d+1}} \int_0^{\infty} \rho^d d\rho, \\ \text{with } \text{vol}(S^d) = & \frac{2\pi^{(d+1)/2}}{\Gamma(\frac{d+1}{2})}. \end{aligned} \quad (\text{A16})$$

In polar coordinates, Eq. (5) is rewritten as

$$\begin{aligned} f_{(d+2)}(\mu) = & \frac{\text{vol}(S^d)}{2(2\pi)^{d+1}} \left[\frac{2}{l} \sum_{k=1}^{\infty} \frac{\pi^2 k^2}{l^2} \int_0^{\infty} d\rho \frac{\rho^d}{\rho^2 + \mu^2 + \frac{\pi^2 k^2}{l^2}} \right. \\ & \left. + \int_0^{\infty} d\rho \rho^d \sqrt{\rho^2 + \mu^2} \right] \\ = & \frac{\text{vol}(S^d) \Gamma(\frac{d+1}{2})}{4(2\pi)^{d+1}} \left[\frac{2\Gamma(\frac{1-d}{2})}{l} \sum_{k=1}^{\infty} \frac{\pi^2 k^2}{l^2} \right. \\ & \left. \times \left(\mu^2 + \frac{\pi^2 k^2}{l^2} \right)^{(d-1)/2} + \frac{\Gamma(-\frac{d+2}{2})}{\Gamma(-\frac{1}{2})} \mu^{d+2} \right], \end{aligned} \quad (\text{A17})$$

where we have used the result

$$\begin{aligned} \int_0^{\infty} d\rho \frac{\rho^d}{(\rho^2 + c)^s} = & \frac{1}{2} \Gamma\left(\frac{d+1}{2}\right) \frac{\Gamma(s - \frac{1}{2} - \frac{d}{2})}{\Gamma(s)} \\ & \times c^{(d/2) + (1/2) - s}, \end{aligned} \quad (\text{A18})$$

with $s=1$ and $c = \mu^2 + \frac{\pi^2 k^2}{l^2}$ in the first integral, and $s = -1/2$ and $c = \mu^2$ in the second. The next step in the computation is to notice that the two terms together in (A17) can be rewritten in terms of the derivative of a Epstein-Hurwitz function (A9). In order to show this consider the infinite series in the first term which we shall denote as \mathcal{S}

$$\begin{aligned} \mathcal{S} = & \frac{1}{l} \sum_{k=1}^{\infty} \frac{\pi^2 k^2}{l^2} \left(\mu^2 + \frac{\pi^2 k^2}{l^2} \right)^{(d-1)/2} \\ = & -\frac{1}{d+1} \left(\frac{\pi}{l} \right)^{d+1} \frac{d}{dl} \sum_{k=1}^{\infty} \left(k^2 + \frac{l^2 \mu^2}{\pi^2} \right)^{(d-1)/2}, \end{aligned} \quad (\text{A19})$$

but the series written in this way is precisely the Epstein-Hurwitz zeta function (A9). In terms of it

$$\mathcal{S} = -\frac{1}{d+1} \left(\frac{\pi}{l} \right)^{d+1} \frac{d}{dl} \zeta_{\text{EH}} \left(-\frac{d+1}{2}, \left(\frac{l\mu}{\pi} \right)^2 \right), \quad (\text{A20})$$

and computing explicitly the derivative, we obtain

$$S = -\frac{1}{d+1} \frac{1}{\Gamma(\frac{d+1}{2})} \left[\frac{\Gamma(-\frac{d+2}{2})}{2\sqrt{\pi}} \mu^{d+2} + \frac{2}{\sqrt{\pi}} \mu^{(d+2)/2} \frac{d}{dl} \times \left(l^{-(d/2)} \sum_{n=1}^{\infty} n^{-(d+2)/2} K_{-(d+2)/2}(2nl\mu) \right) \right]. \quad (\text{A21})$$

Using the identity $-(d+1)\Gamma(-\frac{d+1}{2}) = 2\Gamma(-\frac{d-1}{2})$, and since $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, we obtain finally

$$2S\Gamma\left(\frac{1-d}{2}\right) + \frac{\Gamma(-\frac{d+2}{2})}{\Gamma(-\frac{1}{2})} \mu^{d+2} = \frac{2}{\sqrt{\pi}} \mu^{(d+2)/2} \frac{d}{dl} \times \left(l^{-(d/2)} \sum_{n=1}^{\infty} n^{-(d+2)/2} K_{-(d+2)/2}(2nl\mu) \right). \quad (\text{A22})$$

But these are precisely the terms inside the brackets in Eq. (A17), so we get the result

$$f_{d+2}(\mu) = \frac{\text{vol}(S^d)\Gamma(\frac{d+1}{2})\sqrt{\pi}}{(2\pi)^{d+2}} \mu^{(d+2)/2} \frac{d}{dl} \times \left(l^{-(d/2)} \sum_{n=1}^{\infty} n^{-(d+2)/2} K_{-(d+2)/2}(2nl\mu) \right). \quad (\text{A23})$$

We can evaluate explicitly the derivative. Using the properties $K_{-\nu}(z) = K_{\nu}(z)$ and $z\partial_z K_{\nu}(z) = -zK_{\nu-1}(z) - \nu K_{\nu}(z)$ and substituting the value of $\text{vol}(S^d)$, we obtain finally

$$f_{(d+2)}(\mu) = -2\left(\frac{\mu}{4\pi}\right)^{(d+2)/2} \left[\frac{1}{l^{(d+2)/2}} \sum_{n=1}^{\infty} \frac{1}{n^{(d+2)/2}} \times K_{(d+2)/2}(2nl\mu) - \frac{2\mu}{l^{d/2}} \sum_{n=1}^{\infty} \frac{1}{n^{d/2}} \times K_{(d+4)/2}(2nl\mu) \right]. \quad (\text{A24})$$

This expression of the force coincides exactly with the one obtained above using the zeta function approach.

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