

Dynamical solutions in the 3-form field background in the Nishino-Salam-Sezgin modelMasato Minamitsuji,^{1,2,*} Nobuyoshi Ohta,^{3,†} and Kunihito Uzawa³¹*Center for Quantum Spacetime, Sogang University, Shinsu-dong 1, Mapo-gu, Seoul, 121-742 South Korea*²*Department of Physics, Graduate School of Science and Technology, Kwansai Gakuin University, Sanda 669-1337, Japan*³*Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*

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We investigate the dynamical 3-form flux compactifications and their implications for cosmology in our brane world in the six-dimensional Nishino-Salam-Sezgin model, which is usually referred to as the Salam-Sezgin model. We take the background of the 3-form field acting on the internal space and timelike dimensions without the $U(1)$ gauge field strength. The first class of solutions we discuss is the dynamical generalization of the static solutions obtained recently. In this class, we find that the time evolution is restricted only in the azimuthal dimension of the internal space and not in the ordinary three-dimensional one, which does not give a cosmological evolution. The second class of solutions is obtained by exchanging the roles of the radial coordinate and the time coordinate from the assumptions in the first class. At the center of the internal space, there is a conical singularity which may be interpreted as our 3-brane world, but it is difficult to realize a warped structure in the direction of the ordinary 3-space. Except for the oscillating solutions, a dynamical evolution leads to expanding or contracting ordinary 3-space, depending on the choice of time direction. Furthermore, in the expanding solutions, there are decelerating and accelerating ones: In the latter solution, the evolution is sustained for a finite proper time and the scale factor of the 3-space diverges.

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I. INTRODUCTION

Brane world models have attracted particular interest in recent years. In the cosmological aspects, six-dimensional brane world models may be useful for a resolution of the cosmological constant problem [1] (see, however, [2]). A six-dimensional model is also recognized as an important playground to study cosmology and gravity with stabilized extra dimensions by fluxes of antisymmetric tensor fields. In the simplest realization of the flux compactifications in six dimensions, the internal space has the shape of a rugby ball [1,2], where codimension-two branes are placed at the poles. The warped generalizations of the rugby ball solutions were reported in the Nishino-Salam-Sezgin (NSS) six-dimensional supergravity [3] and in the Einstein-Maxwell theory [4]. See [5] for the NSS model. Note that this model is usually referred to as the Salam-Sezgin model especially when compactification is discussed. However, the compactification on S^2 , which was found by Salam and Sezgin, was constructed in the theory first proposed by Nishino and Sezgin. Therefore, it is more suitable for us to refer to the model as the NSS model. In Ref. [6], a study of the cosmology was presented on the analogy of the classical mechanics. It is also recognized that a 3-brane in six or higher dimensions faces a problem about the localization of the matter, because such a brane generically produces a curvature singularity. A way to

circumvent this problem is to introduce a thickness of the brane, as proposed in [7,8].

In order to explore the cosmology, one useful approach is to use the dynamical solutions. In this direction, exact dynamical solutions were obtained in [9–12]. The field components of the bosonic part of the NSS model are the gravity, the $U(1)$ -gauge field, Kalb-Ramond 2-form (i.e., 3-form field strength), and the dilaton. In Refs. [9,10], warped time-dependent solutions with the $U(1)$ -field strength, without the Kalb-Ramond 3-form field strength in the NSS model, were explored. The solutions have all six dimensions evolving in time. There are three types of the exact time-dependent solutions: One of them is the scaling solution, in which both the scale factors of the external and internal spaces have the linear dependence on the proper time. The other two solutions have more complicated time dependence. In the earlier times, these solutions can be seen as the generalizations of the solutions obtained in Ref. [13], but in the later times both approach the above scaling solution. The scaling solution is the generalization of the static solution obtained in [3] and becomes an attractor. In this sense, the static brane solution in the 2-form background is always unstable. An extension of the time-dependent 2-form solution in the NSS model to more general Einstein-Maxwell-dilaton models was considered in Ref. [11]. The exact wave solution was also obtained in [12].

The purpose of this work is to present the other classes of time-dependent solutions in the NSS model in the presence of the nonvanishing 3-form field strength [but in

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the absence of the $U(1)$ field strength], and discuss their implications for the brane world cosmology. The first class of solutions is the dynamical generalization of the static solutions obtained recently in Ref. [14], where the internal space has a torus topology with two cusps, along the line of the similar generalization of brane solutions [15]. We will find that our solutions are stable, in contrast to the 2-form solutions. The second class is obtained by exchanging the roles of the radial coordinate and the time coordinate in the first class. We also discuss the dynamical generalization of the black 1-brane solution in the NSS model.

The paper is constructed as follows. In Sec. II, we briefly review the NSS model. In Sec. III, the time-dependent generalization of the solutions and their application in constructing the brane world model are discussed. In Sec. IV, the S -brane-like solution, obtained by exchanging the roles of space and time, and their cosmological properties are discussed. The final section is devoted to giving a brief summary and conclusion. In the Appendix, we discuss the black 1-brane solution.

II. NISHINO-SALAM-SEZGIN MODEL

The action of the bosonic part of the NSS supergravity [5] is given by

$$I = \int d^6x \sqrt{-g} \left[\frac{1}{2\kappa^2} (R - (\partial\phi)^2) - \frac{1}{4} e^{-\phi} F_{\mu\nu}^2 - \frac{1}{12} e^{-2\phi} H_{\mu\nu\rho}^2 - \frac{2g^2}{\kappa^4} e^\phi \right], \quad (2.1)$$

where ϕ is the dilaton, $H = dB$ is the field strength for the Kalb-Ramond field B , and $F = dA$ is the field strength for the $U(1)$ gauge field A . The parameters g and κ are gauge and gravitational coupling constants, respectively. Variation of the action (2.1) gives the field equations

$$R_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \kappa^2 e^{-\phi} \left(F_{\mu\nu}^2 - \frac{1}{8} F^2 g_{\mu\nu} \right) + \frac{\kappa^2}{2} e^{-2\phi} \left((H^2)_{\mu\nu} - \frac{1}{6} H^2 g_{\mu\nu} \right) + \frac{g^2}{\kappa^2} e^\phi g_{\mu\nu}, \quad (2.2)$$

$$\square\phi + \frac{\kappa^2}{4} e^{-\phi} F^2 + \frac{\kappa^2}{6} e^{-2\phi} H^2 - \frac{2g^2}{\kappa^2} e^\phi = 0, \quad (2.3)$$

$$\partial_\mu (\sqrt{-g} e^{-2\phi} H^{\mu\nu\rho}) = 0, \quad (2.4)$$

$$\partial_\mu (\sqrt{-g} e^{-\phi} F^{\mu\nu}) = 0. \quad (2.5)$$

Most of the preceding works have discussed the $U(1)$ field acting on the internal space dimensions, $F \neq 0$, with the vanishing 3-form $H = 0$. Instead, in this work, we consider the nonvanishing 3-form field acting on the internal space and timelike dimensions $H \neq 0$, with the vanishing $U(1)$ field $F = 0$.

III. GENERALIZED WARPED COMPACTIFICATIONS BY THE 3-FORM

A. Ansatz and equations

In this section, we take the following metric ansatz [15]:

$$ds_6^2 = -e^{2u_0(t,y)} dt^2 + e^{2u_1(t,y)} \sum_{i=1}^3 (dx^i)^2 + e^{2v(t,y)} dy^2 + e^{2w(t,y)} d\theta^2, \quad (3.1)$$

where the coordinates t and x^i describe our four-dimensional worldvolume, and the remaining y and θ are transverse to it. The metric components u_0 , u_1 , v , w , and the dilaton ϕ are assumed to be functions of both t and y . For convenience of our derivation, we also define

$$U \equiv u_0 + 3u_1 - v + w. \quad (3.2)$$

We assume that for the 3-form field strength there is only the y dependence and also the $U(1)$ field strength vanishes:

$$H = E'(t, y) dt \wedge dy \wedge d\theta, \quad F = 0. \quad (3.3)$$

Throughout the main text in this paper, the dot and prime denote derivatives with respect to t and y . The Einstein equations. (2.2) become

$$e^{2(u_0-v)} (u_0'' + U' u_0') - \ddot{U} + \ddot{u}_0 - 2\ddot{v} + \dot{u}_0 (\dot{U} - \dot{u}_0 + 2\dot{v}) - 3\dot{u}_1^2 - \dot{v}^2 - \dot{w}^2 = \dot{\phi}^2 + \frac{\kappa^2}{2} e^{-2\phi-2v-2w} E'^2 - \frac{g^2}{\kappa^2} e^{\phi+2u_0}, \quad (3.4)$$

$$-3\dot{u}_1' - \dot{w}' + u_0' (3\dot{u}_1 + \dot{w}) - 3u_1' (\dot{u}_1 - \dot{v}) + w' (\dot{v} - \dot{w}) = \dot{\phi} \phi', \quad (3.5)$$

$$e^{2(u_1-v)} (u_1'' + U' u_1') - e^{-2(u_0-u_1)} [\ddot{u}_1 + \dot{u}_1 (\dot{U} - 2\dot{u}_0 + 2\dot{v})] = -\frac{\kappa^2}{2} e^{-2\phi-2v-2w} E'^2 - \frac{g^2}{\kappa^2} e^{\phi+2u_0}, \quad (3.6)$$

$$U'' + v'' + u_0'^2 + 3u_1'^2 + w'^2 - v' (U' + v') - e^{-2(u_0-v)} [\ddot{v} + \dot{v} (\dot{U} - 2\dot{u}_0 + 2\dot{v})] = -\phi'^2 + \frac{\kappa^2}{2} e^{-2\phi-2u_0-2w} E'^2 - \frac{g^2}{\kappa^2} e^{\phi+2v}, \quad (3.7)$$

$$e^{2(w-v)} (w'' + U' w') - e^{2(w-u_0)} [\ddot{w} + \dot{w} (\dot{U} - 2\dot{u}_0 + 2\dot{v})] = \frac{\kappa^2}{2} e^{-2\phi-2u_0-2v} E'^2 - \frac{g^2}{\kappa^2} e^{\phi+2w}. \quad (3.8)$$

The dilaton Eq. (2.3) and the equation for the 3-form field (2.4) give

$$(e^U \phi')' - (e^{U-2u_0+2v} \dot{\phi})' - \kappa^2 e^{-u_0+3u_1-v-w-2\phi} E'^2 - \frac{2g^2}{\kappa^2} e^\phi = 0, \quad (3.9)$$

$$(e^{U-2u_0-2w-2\phi}E')' = 0, \quad (3.10)$$

$$(e^{U-2u_0-2w-2\phi}E')^\cdot = 0. \quad (3.11)$$

The latter Eqs. (3.10) and (3.11) give

$$e^{U-2u_0-2w-2\phi}E' = c, \quad (3.12)$$

where c is an integration constant. Substituting (3.12) into Eqs. (3.4), (3.6), (3.8), and (3.9), we obtain

$$\begin{aligned} \left(e^U u'_0 - \frac{\kappa^2}{2} cE\right)' &= e^{U+2(v-u_0)}[\dot{U} - \ddot{u}_0 + 2\dot{v} \\ &\quad - \dot{u}_0(\dot{U} - \dot{u}_0 + 2\dot{v}) + 3\dot{u}_1^2 + \dot{v}^2 + \dot{w}^2 \\ &\quad + \dot{\phi}^2] - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \end{aligned} \quad (3.13)$$

$$\left(e^U u'_1 + \frac{\kappa^2}{2} cE\right)' = (\dot{u}_1 e^{U+2(v-u_0)})^\cdot - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (3.14)$$

$$\left(e^U w' - \frac{\kappa^2}{2} cE\right)' = (\dot{w} e^{U+2(v-u_0)})^\cdot - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (3.15)$$

$$(e^U \phi' - \kappa^2 cE)' = (\dot{\phi} e^{U+2(v-u_0)})^\cdot + \frac{2g^2}{\kappa^2} e^{\phi+2v+U}. \quad (3.16)$$

Let us assume that the solutions satisfy the following equations:

$$\left(e^U u'_0 - \frac{\kappa^2}{2} cE\right)' = -\frac{g^2}{\kappa^2} e^{\phi+2v+U} + \ell_0, \quad (3.17)$$

$$\left(e^U u'_1 + \frac{\kappa^2}{2} cE\right)' = -\frac{g^2}{\kappa^2} e^{\phi+2v+U} + \ell_1, \quad (3.18)$$

$$\left(e^U w' - \frac{\kappa^2}{2} cE\right)' = -\frac{g^2}{\kappa^2} e^{\phi+2v+U} + \ell_w, \quad (3.19)$$

$$(e^U \phi' - \kappa^2 cE)' = \frac{2g^2}{\kappa^2} e^{\phi+2v+U} - 2\ell_\phi, \quad (3.20)$$

where ℓ_i ($i = 0, 1, w, \phi$) are constants corresponding to the time-derivative parts in Eqs. (3.13), (3.14), (3.15), and (3.16). Relaxing these separability conditions is a very interesting issue, but we find it difficult to do so. As in Ref. [15], we also assume that U is independent of y . In fact, it is known that some nontrivial time-dependent solutions can be obtained even with this restriction. Then using $v' = u'_0 + 3u'_1 + w'$ obtained from $U' = 0$ and Eqs. (3.17), (3.18), and (3.19), we find

$$(2v + \phi)'' = -\frac{8g^2}{\kappa^2} e^{2v+\phi} + 2(\ell_0 + 3\ell_1 + \ell_w - \ell_\phi) e^{-U}. \quad (3.21)$$

It is hard to find a solution unless we assume

$$\ell_0 + 3\ell_1 + \ell_w - \ell_\phi = 0. \quad (3.22)$$

When this is obeyed, the solution to Eq. (3.21) is

$$e^{-(2v+\phi)} = \frac{4g^2}{\kappa^2 f_1^2} \cosh^2[f_1(y - y_1)], \quad (3.23)$$

where f_1 and y_1 are constants. Although relaxing the condition Eq. (3.22) would be also an interesting issue, we leave it for future study.

It follows from Eqs. (3.17), (3.18), (3.19), and (3.20) that

$$u'_0 = \frac{\kappa^2}{2} c e^{-U} E - \frac{f_1}{4} \tanh[f_1(y - y_1)] - G_0 + \ell_0 e^{-U} y, \quad (3.24)$$

$$\begin{aligned} u'_1 &= -\frac{\kappa^2}{2} c e^{-U} E - \frac{f_1}{4} \tanh[f_1(y - y_1)] - G_1 \\ &\quad + \ell_1 e^{-U} y, \end{aligned} \quad (3.25)$$

$$w' = \frac{\kappa^2}{2} c e^{-U} E - \frac{f_1}{4} \tanh[f_1(y - y_1)] - G_w + \ell_w e^{-U} y, \quad (3.26)$$

$$\begin{aligned} \phi' &= \kappa^2 c e^{-U} E + \frac{f_1}{2} \tanh[f_1(y - y_1)] + 2G_\phi \\ &\quad - 2\ell_\phi e^{-U} y, \end{aligned} \quad (3.27)$$

where we have chosen G_i to be constant for simplicity. We then find from (3.23) and (3.27)

$$\begin{aligned} v' &= -\frac{\kappa^2}{2} c e^{-U} E - \frac{5f_1}{4} \tanh[f_1(y - y_1)] - G_\phi \\ &\quad + \ell_\phi e^{-U} y. \end{aligned} \quad (3.28)$$

The condition $U' = 0$ imposes

$$G_0 + 3G_1 + G_w - G_\phi = 0. \quad (3.29)$$

For convenience, we define

$$Y \equiv 2\kappa^2 c e^{-U} E. \quad (3.30)$$

By substituting the above relations into Eq. (3.7), we find

$$\begin{aligned} 0 &= \frac{1}{2}(Y^2 - Y') - f_1^2 + e^{-U}(\ell_\phi - \ell_v) + 3A_1^2 + A_w^2 \\ &\quad + 3A_\phi^2 + (3A_1 + A_w - A_\phi)^2 + Y(3A_1 + A_\phi), \end{aligned} \quad (3.31)$$

where we have assumed $(\dot{v} e^{U-2u_0+2v})^\cdot = \ell_v$ (ℓ_v is a constant), and have defined

$$A_i = G_i - \ell_i e^{-U} y, \quad (i = 1, w, \phi). \quad (3.32)$$

At this stage, except for the cases where $\ell_i = 0$, it is still not easy to find analytically tractable solutions. Thus, in this paper, we focus on the case $\ell_i = 0$, leaving study of more general cases for the future. Then, without loss of generality, we may set $3G_1 + G_\phi = 0$. Now Eq. (3.31) reduces to

$$Y^2 - Y' - f_2^2 = 0, \quad (3.33)$$

where we have defined

$$f_2^2 = 2f_1^2 - 4(G_\phi - G_w)^2 - \frac{32}{3}G_\phi^2. \quad (3.34)$$

f_2^2 can take either a positive, zero, and negative value, and the solutions for $Y(y)$ can be classified into the three types, depending on the sign of f_2^2 . We will give the solutions shortly. Once $Y(y)$ is obtained, by integrating Eqs. (3.24), (3.25), (3.26), (3.27), and (3.28), we can find the metric components and dilaton as

$$u_0 = -\frac{1}{4}P(y) - \frac{1}{4}\text{Incosh}[f_1(y - y_1)] - (2G_\phi - G_w)y + h_0(t), \quad (3.35)$$

$$u_1 = \frac{1}{4}P(y) - \frac{1}{4}\text{Incosh}[f_1(y - y_1)] + \frac{1}{3}G_\phi y + h_1(t), \quad (3.36)$$

$$v = \frac{1}{4}P(y) - \frac{5}{4}\text{Incosh}[f_1(y - y_1)] - G_\phi y + h_v(t), \quad (3.37)$$

$$w = -\frac{1}{4}P(y) - \frac{1}{4}\text{Incosh}[f_1(y - y_1)] - G_w y + h_w(t), \quad (3.38)$$

$$\phi = -\frac{1}{2}P(y) + \frac{1}{2}\text{Incosh}[f_1(y - y_1)] + 2G_\phi y + h_\phi(t), \quad (3.39)$$

where the function $P(y)$ depends on the sign of f_2^2 ; hence, the type of solutions, and h_i ($i = 0, 1, v, w, \phi$) could be functions of t . When all h_i vanish, the solution becomes static.

Now the three types of solutions are given as follows:

The sinh solutions: For a positive f_2^2 , the solution to Eq. (3.33) is given by

$$Y(y) = -f_2 \coth[f_2(y - y_2)]. \quad (3.40)$$

After integration, we obtain

$$P(y) = \ln|\sinh[f_2(y - y_2)]|. \quad (3.41)$$

The sin solutions: For a negative f_2^2 , the solution to Eq. (3.33) is given by

$$Y(y) = -|f_2| \cot[|f_2|(y - y_2)]. \quad (3.42)$$

After integration, we obtain

$$P(y) = \ln|\sin[|f_2|(y - y_2)]|. \quad (3.43)$$

The linear solutions: For $f_2^2 = 0$, the solution to Eq. (3.33) is given by

$$Y(y) = -\frac{1}{y - y_2}. \quad (3.44)$$

After integration, we obtain

$$P(y) = \ln|y - y_2|. \quad (3.45)$$

These give the dynamical generalization of the static solutions found in [14]. The correspondence is given by the following reparameterizations of $f_1 \leftrightarrow \lambda_2$, $f_2 \leftrightarrow \lambda_1$, and $q \leftrightarrow c$. Note, however, that Eq. (3.34) does not completely agree with Eq. (2.10) of [14], in some factors.

B. General time-dependent solutions

Now our task is to solve the remaining time-derivative parts of (3.14), (3.15), (3.16), and (3.7). The remaining time-dependent parts of these equations give

$$\ddot{h}_i = (\dot{h}_0 - 3\dot{h}_1 - \dot{h}_v - \dot{h}_w)\dot{h}_i, \quad (3.46)$$

where $i = 1, v, w, \phi$. They lead to

$$\dot{h}_i e^{-h_0 + 3h_1 + h_v + h_w} = c_i, \quad (3.47)$$

where all c_i are constants. The off-diagonal components of the Einstein equation (3.5) imposes the constraints among the time-dependent functions

$$\begin{aligned} \dot{h}_v &= -3\dot{h}_1, & \dot{h}_\phi &= 6\dot{h}_1, \\ (11G_\phi - 3G_w)\dot{h}_1 &= (G_w - G_\phi)\dot{h}_w. \end{aligned} \quad (3.48)$$

It is straightforward to confirm that except for the case of $G_w = G_\phi$ all h_i cannot have any nontrivial time dependence. To show this, for the moment let us set $h_0(t) = 0$, which is always possible by an appropriate rescaling for the time coordinate. From Eq. (3.48), we have $c_v = -3c_1$ and $c_\phi = 6c_1$.

In the case of $11G_\phi \neq 3G_w$, Eq. (3.47) tells us that

$$\begin{aligned} \dot{h}_1 &= \frac{G_w - G_\phi}{(11G_\phi - 3G_w)t}, & \dot{h}_v &= -3\frac{G_w - G_\phi}{(11G_\phi - 3G_w)t}, \\ \dot{h}_w &= \frac{1}{t}, & \dot{h}_\phi &= 6\frac{G_w - G_\phi}{(11G_\phi - 3G_w)t}. \end{aligned} \quad (3.49)$$

It follows from Eq. (3.13) that

$$0 = \frac{48(G_\phi - G_w)^2}{(11G_\phi - 3G_w)^2 t^2}. \quad (3.50)$$

Thus, the consistency requires $G_w = G_\phi$. In the case of $11G_\phi = 3G_w$, $\dot{h}_w = 0$. We may set $h_w = 0$ and then $3h_1 + h_v = \text{const}$. From Eq. (3.47), we have $h_1 = Ct$, $h_v = -3Ct$, and $h_\phi = 6Ct$, where C is a constant. We then find from Eq. (3.13)

$$C^2 = 0, \quad (3.51)$$

which leads to a trivial solution. Thus, $G_w = G_\phi$ must be satisfied to have nontrivial solutions.

For $G_w = G_\phi$,

$$f_2^2 = 2f_1^2 - \frac{32}{3}G_w^2. \quad (3.52)$$

Then, $\dot{h}_v = \dot{h}_\phi = \dot{h}_1 = 0$ but $\dot{h}_w \neq 0$. Thus, we obtain

$$h_1 = k_1, \quad h_v = k_v, \quad h_\phi = k_\phi, \quad (3.53)$$

where k_1 , k_v , and k_ϕ are integration constants. Therefore, the ordinary 3-space metric and the dilaton do not depend on time. From Eq. (3.23), we obtain

$$k_\phi = -2 \left(k_v + \ln \left(\frac{2g}{\kappa f_1} \right) \right). \quad (3.54)$$

h_0 and h_w are still undetermined except for the relation (3.47), which gives $(e^{h_w})' = c_w e^{k_v+3k_1} e^{h_0}$. The two-dimensional metric in the t and θ directions now becomes

$$\begin{aligned} ds^2 &= e^{2F(y)} (-e^{2h_0} dt^2 + e^{2h_w} d\theta^2) \\ &= \frac{e^{2F(y)}}{c_w^2 e^{2(k_v+3k_1)}} (-(e^{h_w})' dt)^2 + c_w^2 e^{2(k_v+3k_1)} e^{2h_w} d\theta^2 \\ &= \frac{e^{2F(y)}}{c_w^2 e^{2(k_v+3k_1)}} (-dT^2 + c_w^2 e^{2(k_v+3k_1)} T^2 d\theta^2), \end{aligned} \quad (3.55)$$

where $F(y)$ is the common y -dependent part, and $T = e^{h_w(t)}$ is the proper time. Thus, the time dependence only appears in the θ direction and not in the ordinary 3 directions, which cannot present any cosmological evolution. These results imply that the solution is dynamically unstable for the evolution along the θ direction, but stable for the others. In terms of the four-dimensional effective theory, the effective potential would contain one flat direction associated with the motion in the θ direction.

One might guess that more generalized time-dependent solutions could be obtained by allowing for the time variations of f_i and G_j . However, in such cases, the time dependence of the Einstein and dilaton equations cannot be separated out from the dependence on y . Thus, it seems to be impossible to find consistent time-dependent solutions.

C. Comparison with 2-form flux compactification

It would be interesting to compare our result with the dynamical solutions with the 2-form background in the NSS model. In the case of the $U(1)$ background, there is the so-called scaling solution [9,10], which can be the attractor. In this solution, both the internal and external dimensions have the time dependence linear in time and evolve uniformly. It means that the static solutions in the $U(1)$ background are not stable. Thus, the scaling symmetry always requires some additional mechanism to stabilize the extra dimensions, for example, as discussed in [16,17].

In our 3-form background, the analyses in the previous subsection indicate that there may be no time-dependent generalizations of the static solutions in the case of $G_w \neq$

G_ϕ . For the case $G_w = G_\phi$, the time evolution is allowed only in the θ direction and is forbidden into the other directions. Thus, in either case, this seems to imply that the 3-form compactification is stable in comparison with the case of the $U(1)$ -background. Here, one remark is in order. In deriving the solutions, we made some assumptions on the time dependence, and relaxation of them might lead to additional possible evolutions of space. At the moment, we do not have the complete proof on the stability of such general solutions.

D. Application to brane world model

1. Singularity

Reference [14] investigated the construction of the brane world by using the static solution in the background of the 3-form field strength. A curvature singularity exists at $y = y_2$, irrespective of the type of solutions. Note that this is a real curvature singularity, which is more severe than a conical one and may not be identified as our 3-brane world. To see this, we define $\bar{y} := y - y_2$, and the approximate spacetime metric near the singularity $\bar{y} = 0$ is given by

$$ds^2 \sim c_y^2 \bar{y}^{1/2} d\bar{y}^2 + c_\theta^2 \frac{d\theta^2}{\bar{y}^{1/2}} - c_t^2 \frac{dt^2}{\bar{y}^{1/2}} + c_x^2 \bar{y}^{1/2} \sum_{i=1}^3 (dx^i)^2, \quad (3.56)$$

where c_i ($i = y, \theta, t, x$) are unimportant coefficients. This metric leads to a divergent scalar curvature $R \sim \bar{y}^{-5/2}$ near $\bar{y} = 0$. Thus, as we will discuss in the next subsection, to construct a brane world, the singularity is firstly removed from the original spacetime in [14]. Then, an identical copy of the remaining piece of the spacetime is attached to the opposite side across the codimension-one boundary at $y = y_2 + \epsilon$, where $\epsilon > 0$, which wraps the axis of the rotational symmetry of the internal space. This boundary may be identified as our brane world. There are jumps of the physical quantities across the boundary.

2. Construction of the brane world

We now construct the brane world by employing the dynamical solution in the 3-form background. First, the original spacetime is cut at $y = y_0 > y_1 (> y_2)$, and the singular part of $y_2 < y < y_0$ is removed. Then, an identical copy of the remaining piece is glued with the original one, at the codimension-one boundary $y = y_0$.

The induced metric on the boundary is given by

$$ds_5^2 = e^{2w(t,y_0)} d\theta^2 - e^{2u_0(t,y_0)} dt^2 + e^{2u_1(t,y_0)} \sum_{i=1}^3 (dx^i)^2. \quad (3.57)$$

The extrinsic curvature to the boundary is given by

$$K_{ab} = \frac{\epsilon}{2e^v} (g_{ab})'|_{y_0}, \quad (3.58)$$

where $\epsilon = +1$ or -1 denotes the direction of the normal vector toward increasing or decreasing y , respectively. In our case, we choose $\epsilon = +1$. Now the extrinsic curvature is discontinuous across the boundary and its jump is determined by the Israel junction condition:

$$[g_{ab}] = 0, \quad [\bar{K}_{ab}] = [K_{ab} - g_{ab}K] = -\kappa^2 S_{ab}, \quad (3.59)$$

where $[A]$ denotes the jump of a physical quantity A across $y = y_0$, and S_{ab} is the stress-energy tensor of the matter localized on the boundary. This codimension-one boundary can be identified as our brane world.

For our background metric, we obtain

$$\bar{K}^\theta_\theta = -e^{-\nu}(u'_0 + 3u'_1), \quad \bar{K}^t_t = -e^{-\nu}(w' + 3u'_1), \\ \frac{1}{3}\bar{K}^i_i = -e^{-\nu}(u'_0 + w' + 2u'_1). \quad (3.60)$$

S_{ab} also can be decomposed into the components $S^t_t = -\rho$, $S^\theta_\theta = p_\theta$ and $(1/3)S^i_i = p$. The junction condition at $y = y_0$ is given by

$$(e^{-\nu}(u'_0 + 3u'_1))|_{y=y_0} = \frac{\kappa^2}{2} p_\theta, \quad (3.61)$$

$$(e^{-\nu}(w' + 3u'_1))|_{y=y_0} = -\frac{\kappa^2}{2} \rho, \quad (3.62)$$

$$(e^{-\nu}(w' + u'_0 + 2u'_1))|_{y=y_0} = \frac{\kappa^2}{2} p. \quad (3.63)$$

They can be rewritten as

$$(e^{-\nu}(u'_1 - w'))|_{y=y_0} = \frac{\kappa^2}{2} (p_\theta - p), \quad (3.64)$$

$$(e^{-\nu}(u'_1 - u'_0))|_{y=y_0} = -\frac{\kappa^2}{2} (\rho + p), \quad (3.65)$$

$$(e^{-\nu}(u'_0 + 3u'_1))|_{y=y_0} = \frac{\kappa^2}{2} p_\theta. \quad (3.66)$$

Our discussion in this subsection can be applied to the sinh and *linear* solutions discussed in the previous section, and the form of $P(y)$ in Eqs. (3.35), (3.36), (3.37), (3.38), and (3.39) is not specified. It is now clear that the zero-thickness limit of $y_0 \rightarrow y_2$ is not well behaved, since the left-hand side of the junction Eqs. (3.64), (3.65), and (3.66) becomes singular. Note that the sin solution contains another curvature singularity at $y = \pi/|f_2| + y_2$ and is excluded from our consideration. In addition, for the sinh or *linear* solutions, the finiteness of bulk volume is not ensured. Thus, following Ref. [14], here we put the second brane world at some $y = L > y_0$. Note that the second brane takes $\epsilon = -1$. The junction condition at $y = L$ is given in a similar way with the opposite sign of the right-hand side of Eqs. (3.61), (3.62), and (3.63).

In order to obtain a dynamical solution, we must choose $G_w = G_\phi$. Also, we may choose the integration constants as

$$k_1 = 0, \quad k_\nu = 0, \quad k_\phi = -2 \ln\left(\frac{2g}{\kappa f_1}\right). \quad (3.67)$$

and thus $(e^{h_w})^\cdot = c_w^{-1} e^{h_0}$. By introducing the proper coordinate $T = c_w e^{h_w}$, after some computations, we obtain

$$u_1 - w = \frac{1}{2}P(y) + \frac{4}{3}G_\phi y - \ln T, \quad (3.68)$$

$$u_1 - u_0 = \frac{1}{2}P(y) + \frac{4}{3}G_\phi y, \quad (3.69)$$

$$3u_1 + u_0 = \frac{1}{2}P(y) - \ln \cosh[f_1(y - y_1)]. \quad (3.70)$$

The junction conditions are reduced to

$$\left\{ e^{-\nu} \left(\frac{1}{2}P'(y_0) + \frac{4}{3}G_\phi \right) \right\} = \frac{\kappa^2}{2} (p_\theta - p), \quad (3.71)$$

$$\left\{ e^{-\nu} \left(\frac{1}{2}P'(y_0) + \frac{4}{3}G_\phi \right) \right\} = -\frac{\kappa^2}{2} (p + \rho), \quad (3.72)$$

$$\left\{ e^{-\nu} (P'(y_0) - f_1 \tanh[f_1(y_0 - y_1)]) \right\} = \frac{\kappa^2}{2} p_\theta. \quad (3.73)$$

Noting that ν is not an explicit function of T , the left-hand side of the junction equations becomes static, and ρ , p , and p_θ remain constant. In the static case without the condition $G_w = G_\phi$, generically the three quantities ρ , p , and p_θ satisfy the different equations of state from the tension. Now, however, from Eqs. (3.71) and (3.72), the inclusion of the time dependence, i.e., the condition $G_w = G_\phi$, induces the relation $\rho = -p_\theta$.

Furthermore, the requirement of the recovery of the Lorentz symmetry at the brane $y = y_0$, i.e., $u'_1 = u'_0$, together with Eq. (3.69) leads to

$$3P'(y_0) = -8G_\phi. \quad (3.74)$$

By combining this with the previous relations, $u'_0 = u'_1 = w'$ is obtained. This relation suggests that $\rho = -p = -p_\theta$. Thus, we are lead to the conclusion that the brane with the ordinary four-dimensional spacetime with the Lorentz symmetry is supported only by the tension. Note that this conclusion is derived with some assumptions concerning time dependence, and without them we might be able to construct the brane world with the Lorentz symmetry, supported by some matter other than the tension.

Before closing this section, following Ref. [14], the four-dimensional effective theory on the brane world should be discussed. In the standard warped solutions, the four-dimensional Planck mass is obtained by integrating over the two-internal space. However, in our case, in general, the warp function in the timelike direction is different from that of the ordinary 3-space. Then, the six-dimensional

Einstein-Hilbert term now reduces to

$$M_6^4 \int d^6x \sqrt{-g} R \sim 2(2\pi)M_6^4 \int d^4x \sqrt{-g^{(4)}} \times \left(\int_{y_0}^L dy e^{-u_0+3u_1+v+w} g^{(4)tt} R_{tt}^{(4)} + \int_{y_0}^L dy e^{u_0+u_1+v+w} g^{(4)ij} R_{ij}^{(4)} \right), \quad (3.75)$$

where $M_6 := (\frac{1}{2\kappa^2})^{1/4}$ is the six-dimensional Planck mass. Note that the factor 2 in front of the integration come from the Z_2 symmetry, and (2π) is from the integration over the azimuthal direction. To define the unique four-dimensional effective Planck mass M_p , we need to require that

$$2(2\pi)M_6^4 \int_{y_0}^L dy e^{-u_0+3u_1+v+w} = 2(2\pi)M_6^4 \int_{y_0}^L dy e^{u_0+u_1+v+w} =: M_p^2, \quad (3.76)$$

or more explicitly

$$\int_{y_0}^L dy \frac{P(y)e^{2(G_\phi - G_w)y}}{\cosh^2[f_1(y - y_1)]} = \int_{y_0}^L dy \frac{e^{-(8/3)G_\phi y}}{\cosh^2[f_1(y - y_1)]}. \quad (3.77)$$

In the case of $G_\phi \neq G_w$ there is no time-dependent solution. The cutoff parameter y_0 appears in the definition of M_p^2 . In other words, the effect of the brane thickness is renormalized into the four-dimensional Planck mass. In the case of $G_\phi = G_w$, $e^w \propto T$ and hence $M_p^2 \propto T$. In this case, we should move to the Einstein frame. Then, the conformal transformation to the Einstein frame, $g_{\mu\nu}^{(E)} = T g_{\mu\nu}^{(4)}$, leads to the cosmic expansion of $\tau^{1/3}$, where τ is the cosmic proper time defined in the Einstein frame. However, this power in the expansion law is not sufficient for obtaining realistic cosmology. Realistic cosmological evolutions may be obtained by considering the time-dependent matter on the brane, through the induced motion of the brane in the bulk.

IV. S-BRANE-LIKE SOLUTIONS

A. Field equation

Assuming the same metric ansatz (3.1) as in the previous section, we investigate another class of the time-dependent solutions by exchanging the roles of the t coordinate with the y coordinate as well as those of u_0 with v . The way of constructions of solutions are very similar to the case of S branes, see e.g., [18–21]. We also assume that the 3-form field strength is the function of time

$$H = \dot{E}(t, y) dt \wedge dy \wedge d\theta. \quad (4.1)$$

The equations of motion for the flux are given by

$$(e^{U-2u_0-2w-2\phi}\dot{E})' = 0, \quad (4.2)$$

$$(e^{U-2u_0-2w-2\phi}\dot{E})' = 0. \quad (4.3)$$

Equations (4.2) and (4.3) give

$$e^{\Phi-2v-2w-2\phi}\dot{E} = c, \quad (4.4)$$

where c is a constant and

$$\Phi := U + 2v - 2u_0.$$

In this section, we assume $\Phi = 0$ is independent of t , $\dot{\Phi} = 0$. Employing (4.4), the Einstein and dilaton equations are now given by

$$(e^U u_0')' = e^\Phi (\ddot{u}_0 - \dot{u}_0^2 + 3\dot{u}_1^2 + \dot{v}^2 + \dot{w}^2 + \dot{\phi}^2) + \frac{1}{2}\kappa^2 c \dot{E} - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (4.5)$$

$$(e^U u_1')' = \left(\dot{u}_1 e^\Phi - \frac{1}{2}\kappa^2 c E \right)' - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (4.6)$$

$$e^U (v'' + U'' - v'^2 - v'U' + 3u_1'^2 + w'^2 + u_0'^2 + \phi'^2) = \left(\dot{v} e^\Phi + \frac{1}{2}\kappa^2 c E \right)' - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (4.7)$$

$$(e^U w')' = \left(\dot{w} e^\Phi + \frac{1}{2}\kappa^2 c E \right)' - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (4.8)$$

$$(e^U \phi')' = (\dot{\phi} e^\Phi + \kappa^2 c E)' + \frac{2g^2}{\kappa^2} e^{\phi+2v+U}. \quad (4.9)$$

Let us look for the solutions with the following assumptions that

$$0 = e^\Phi (\ddot{u}_0 - \dot{u}_0^2 + 3\dot{u}_1^2 + \dot{v}^2 + \dot{w}^2 + \dot{\phi}^2) + \frac{1}{2}\kappa^2 c \dot{E} - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (4.10)$$

$$0 = \left(\dot{u}_1 e^\Phi - \frac{1}{2}\kappa^2 c E \right)' - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (4.11)$$

$$0 = \left(\dot{v} e^\Phi + \frac{1}{2}\kappa^2 c E \right)' - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (4.12)$$

$$0 = \left(\dot{w} e^\Phi + \frac{1}{2}\kappa^2 c E \right)' - \frac{g^2}{\kappa^2} e^{\phi+2v+U}, \quad (4.13)$$

$$0 = (\dot{\phi} e^\Phi + \kappa^2 c E)' + \frac{2g^2}{\kappa^2} e^{\phi+2v+U}. \quad (4.14)$$

By combining Eqs. (4.11), (4.12), (4.13), and (4.14), with $\dot{\Phi} = 0$ and hence $\dot{u}_0 = \dot{v} + 3\dot{u}_1 + \dot{w}$, we find

$$(2u_0 + \phi)'' = \frac{8g^2}{\kappa^2} e^{2u_0 + \phi}. \quad (4.15)$$

For Eq. (4.15), there are three classes of solutions, namely, the sinh, sin and *linear* solutions. We discuss these solutions in order.

The sinh solutions: A solution to Eq. (4.15) is given by

$$e^{-(2u_0+\phi)} = \frac{4g^2}{\kappa^2 f_1^2} \sinh^2[f_1(t-t_1)], \quad (4.16)$$

where f_1 and t_1 are constants. Assuming that the right-hand sides of Eqs. (4.5), (4.6), (4.7), (4.8), and (4.9) vanish, we obtain

$$\dot{v} = -\frac{\kappa^2}{2} c e^{-\Phi} E - \frac{f_1}{4} \coth[f_1(t-t_1)] - G_v, \quad (4.17)$$

$$\dot{u}_1 = \frac{\kappa^2}{2} c e^{-\Phi} E - \frac{f_1}{4} \coth[f_1(t-t_1)] - G_1, \quad (4.18)$$

$$\dot{w} = -\frac{\kappa^2}{2} c e^{-\Phi} E - \frac{f_1}{4} \coth[f_1(t-t_1)] - G_w, \quad (4.19)$$

$$\dot{\phi} = -\kappa^2 c e^{-\Phi} E + \frac{f_1}{2} \coth[f_1(t-t_1)] + 2G_\phi, \quad (4.20)$$

where G_i ($i = v, 1, w, \phi$) are also constants. Because of the assumption that $\dot{\Phi} = 0$, we obtain

$$\dot{u}_0 = \frac{\kappa^2}{2} c e^{-\Phi} E - \frac{5f_1}{4} \coth[f_1(t-t_1)] - G_\phi, \quad (4.21)$$

and

$$3G_1 - G_\phi + G_v + G_w = 0. \quad (4.22)$$

Substituting these into Eq. (4.5), we find

$$\frac{1}{2}(Y^2 + \dot{Y}) - (3G_1 + G_\phi)Y - f_1^2 + (-3G_1 + G_\phi - G_w)^2 + 3G_1^2 + G_w^2 + 3G_\phi^2 = 0, \quad (4.23)$$

where Y is defined as in the previous section:

$$Y \equiv 2\kappa^2 c E e^{-\Phi}. \quad (4.24)$$

Without loss of generality, we may impose the further condition that

$$3G_1 + G_\phi = 0. \quad (4.25)$$

Then Eq. (4.23) reduces to

$$Y^2 + \dot{Y} + f_2^2 = 0, \quad (4.26)$$

where we defined

$$f_2^2 = -2f_1^2 + 4(G_\phi - G_w)^2 + \frac{32}{3}G_\phi^2. \quad (4.27)$$

Note that f_2^2 can be either positive, negative or zero.

For a positive, negative and vanishing f_2^2 , the solution to Eq. (4.26) is given by

$$Y = f_2 \cot[f_2(t-t_2)], \quad |f_2| \coth[|f_2|(t-t_2)], \quad \frac{1}{t-t_2}, \quad (4.28)$$

respectively. The corresponding solutions with the possible y dependence are given by

$$u_0 = \frac{1}{4}Q(t) - \frac{5}{4}\ln|\sinh[f_1(t-t_1)]| - G_\phi t + h_0(y), \quad (4.29)$$

$$u_1 = \frac{1}{4}Q(t) - \frac{1}{4}\ln|\sinh[f_1(t-t_1)]| + \frac{1}{3}G_\phi t + h_1(y), \quad (4.30)$$

$$v = -\frac{1}{4}Q(t) - \frac{1}{4}\ln|\sinh[f_1(t-t_1)]| - (2G_\phi - G_w)t + h_v(y), \quad (4.31)$$

$$w = -\frac{1}{4}Q(t) - \frac{1}{4}\ln|\sinh[f_1(t-t_1)]| - G_w t + h_w(y), \quad (4.32)$$

$$\phi = -\frac{1}{2}Q(t) + \frac{1}{2}\ln|\sinh[f_1(t-t_1)]| + 2G_\phi t + h_\phi(y), \quad (4.33)$$

where for a positive, negative or vanishing f_2^2 , $Q(t)$ is defined by

$$Q(t) := \ln|\sin[f_2(t-t_2)]|, \quad \ln|\sinh[|f_2|(t-t_2)]|, \quad \ln|t-t_2|, \quad (4.34)$$

respectively.

The sin solutions: Another solution to Eq. (4.15) is the oscillatory type:

$$e^{-(2u_0+\phi)} = \frac{4g^2}{\kappa^2 f_1^2} \sin^2[f_1(t-t_1)]. \quad (4.35)$$

Then, defining the new parameter

$$f_2^2 = 2f_1^2 + 4(G_\phi - G_w)^2 + \frac{32}{3}G_\phi^2, \quad (4.36)$$

which is always positive for a positive f_1^2 , we get the corresponding solutions with the possible y dependence:

$$u_0 = \frac{1}{4}\ln|\sin[f_2(t-t_2)]| - \frac{5}{4}\ln|\sin[f_1(t-t_1)]| - G_\phi t + h_0(y), \quad (4.37)$$

$$u_1 = \frac{1}{4}\ln|\sin[f_2(t-t_2)]| - \frac{1}{4}\ln|\sin[f_1(t-t_1)]| + \frac{1}{3}G_\phi t + h_1(y), \quad (4.38)$$

$$v = -\frac{1}{4}\ln|\sin[f_2(t-t_2)]| - \frac{1}{4}\ln|\sin[f_1(t-t_1)]| - (2G_\phi - G_w)t + h_v(y), \quad (4.39)$$

$$w = -\frac{1}{4}\ln|\sin[f_2(t-t_2)]| - \frac{1}{4}\ln|\sin[f_1(t-t_1)]| - G_w t + h_w(y), \quad (4.40)$$

$$\begin{aligned} \phi = & -\frac{1}{2} \ln |\sin[f_2(t - t_2)]| + \frac{1}{2} \ln |\sin[f_1(t - t_1)]| \\ & + 2G_\phi t + h_\phi(y). \end{aligned} \quad (4.41)$$

The linear solutions: The last solution to Eq. (4.15) takes the form

$$e^{-(2u_0+\phi)} = \frac{4g^2}{\kappa^2} (t - t_1)^2. \quad (4.42)$$

Then, by defining

$$f_2^2 = 4(G_\phi - G_w)^2 + \frac{32}{3}G_\phi^2, \quad (4.43)$$

which is always positive, we get the corresponding solutions with the possible y dependence:

$$u_0 = \frac{1}{4} \ln |\sin[f_2(t - t_2)]| - \frac{5}{4} \ln |t - t_1| - G_\phi t + h_0(y), \quad (4.44)$$

$$u_1 = \frac{1}{4} \ln |\sin[f_2(t - t_2)]| - \frac{1}{4} \ln |t - t_1| + \frac{1}{3}G_\phi t + h_1(y), \quad (4.45)$$

$$\begin{aligned} v = & -\frac{1}{4} \ln |\sin[f_2(t - t_2)]| - \frac{1}{4} \ln |t - t_1| - (2G_\phi - G_w)t \\ & + h_v(y), \end{aligned} \quad (4.46)$$

$$w = -\frac{1}{4} \ln |\sin[f_2(t - t_2)]| - \frac{1}{4} \ln |t - t_1| - G_w t + h_w(y), \quad (4.47)$$

$$\begin{aligned} \phi = & -\frac{1}{2} \ln |\sin[f_2(t - t_2)]| + \frac{1}{2} \ln |t - t_1| + 2G_\phi t \\ & + h_\phi(y), \end{aligned} \quad (4.48)$$

where

$$f_2^2 = 2f_1^2 + 4(G_\phi - G_w)^2 + \frac{32}{3}G_\phi^2. \quad (4.49)$$

In all the three types of solutions, setting all $h_i = 0$ leads to purely time-dependent solutions.

B. Generalization to the y -dependent cases

Following the similar arguments as done in the previous section, we find that the y dependence can be included only for the choice of $G_\phi = G_w$. In any case of the sinh, sin, and linear solutions shown in the previous subsection, the y -dependent functions satisfy

$$\begin{aligned} h_0 = k_0, \quad h_\phi = k_\phi, \quad h_1 = k_1, \\ (e^{h_w})' = c_w e^{-k_0 - 3k_1} e^{h_w}, \end{aligned} \quad (4.50)$$

where k_0 , k_v and k_1 are integration constants satisfying

$$k_\phi + 2k_0 = -2 \ln \left(\frac{2g}{\kappa f_1} \right). \quad (4.51)$$

Thus, even taking into account the y dependence, the spacetime structure in the ordinary four dimensions does not contain a warped structure. The internal space metric can be written as

$$\begin{aligned} ds_2^2 = & F(t)^2 (e^{2h_v} dy^2 + e^{2h_w} d\theta^2) \\ = & \frac{e^{2(k_0+3k_1)} F(t)^2}{c_w^2} \left(((e^{h_w})' dy)^2 + \frac{c_w^2}{e^{2(k_0+3k_1)}} (e^{h_w})^2 d\theta^2 \right) \\ = & \frac{e^{2(k_0+3k_1)} F(t)^2}{c_w^2} \left(dR^2 + \frac{c_w^2}{e^{2(k_0+3k_1)}} R^2 d\theta^2 \right), \end{aligned} \quad (4.52)$$

where $R = e^{h_w}$ is the proper radial coordinate. Assuming the standard periodicity 2π for the angular coordinate, there is generically a conical singularity at the center

$$\Delta = 2\pi \left(1 - \frac{c_w}{e^{k_0+3k_1}} \right). \quad (4.53)$$

This conical singularity at $R = 0$ can be seen as our brane world. The ordinary 3-space metric cannot depend on the internal space coordinate and therefore it is impossible to realize a warped structure.

C. Cosmological behaviors on the brane

Let us now discuss the cosmological behaviors of our solutions. In our real Universe, the scale factor is not oscillating. Thus, in this subsection we do not consider the solutions with oscillating scale factor and focus only on the sinh solutions of $G_w = G_\phi$ with $f_2^2 \leq 0$. Without loss of generality, we may assume $f_1 > 0$. For simplicity, we discuss the behaviors in the large t limit:

$$\begin{aligned} u_0 & \approx \frac{1}{4} \left(\sqrt{2 \left(1 - \frac{16}{3} g_\phi^2 \right)} - 5 - 4g_\phi \right) f_1 t =: p_0 f_1 t, \\ u_1 & \approx \frac{1}{4} \left(\sqrt{2 \left(1 - \frac{16}{3} g_\phi^2 \right)} - 1 + \frac{4}{3} g_\phi \right) f_1 t =: p_1 f_1 t, \\ v = w & \approx -\frac{1}{4} \left(\sqrt{2 \left(1 - \frac{16}{3} g_\phi^2 \right)} + 1 + 4g_\phi \right) f_1 t =: p_v f_1 t, \end{aligned} \quad (4.54)$$

where we also define $g_\phi := G_\phi/f_1$. Note that $|g_\phi| \leq \sqrt{3}/4 \approx 0.433$. For $-0.196 < g_\phi < 0.410$, $p_1 > 0$ and for $g_\phi > -0.395$, $p_v < 0$ (see Figs. 1 and 2).

In one choice of the proper time coordinate, $dT = -e^{u_0} dt$, the approximate spacetime metric is given by

$$\begin{aligned} ds^2 = & (-T)^{2q_v} \left(dR^2 + R^2 \left(1 - \frac{\Delta}{2\pi} \right)^2 d\theta^2 \right) - dT^2 \\ & + (-T)^{2q_1} \sum_{i=1}^3 (dx^i)^2, \end{aligned} \quad (4.55)$$

where unimportant constants are eliminated by the proper rescalings of R and x^i and Δ is defined in Eq. (4.53). We also defined the powers

$$q_1 := -\frac{p_1}{|p_0|}, \quad q_v := -\frac{p_v}{|p_0|}. \quad (4.56)$$

In the other choice of the proper time coordinate, $dT = e^{u_0} dt$, the approximate spacetime metric becomes

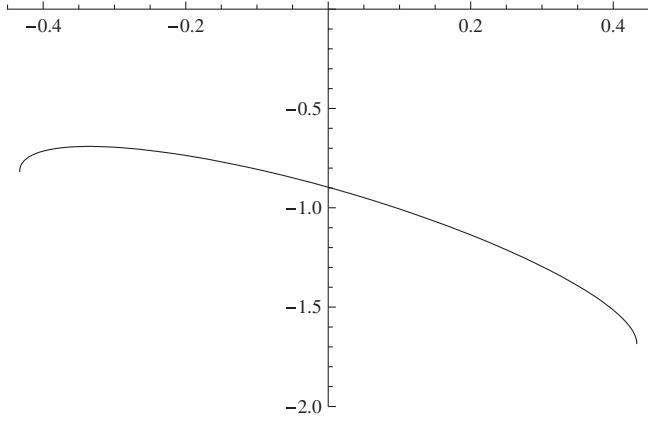


FIG. 1. The plots for p_0 are shown as a function of g_ϕ for $f_1 > 0$.

$$ds^2 = T^{2q_v} \left(dR^2 + R^2 \left(1 - \frac{\Delta}{2\pi} \right)^2 d\theta^2 \right) - dT^2 + T^{2q_1} \sum_{i=1}^3 (dx^i)^2. \quad (4.57)$$

In Fig. 3, we show the behavior of q_1 and q_v . For $-0.196 < g_\phi < 0.410$, $q_1 < 0$. For $g_\phi > -0.395$, $q_v > 0$. For $-\sqrt{3}/4 < g_\phi < -0.335$, $q_1 > q_v$, while for the rest $q_1 < q_v$.

For $-0.196 < g_\phi < 0.410$, $q_1 < 0$ and $q_v > 0$. Thus, in the metric (4.55), in the $T \rightarrow 0^-$ limit, the size of the ordinary 3-space increases and diverges within a finite time, while that of the internal space is shrinking to zero.

For $-\sqrt{3}/4 < g_\phi < -0.335$, $q_1 > q_v$ and $q_1 > 0$. Thus, for this range of g_ϕ , the metric (4.57) describes the ordinary 3-space which is expanding faster than the internal space dimensions. In particular, for $g_\phi < -0.395$, the internal space is contracting. But since $q_1 \leq 0.406$, an accelerating expansion, as in the inflationary or dark energy Universe, cannot happen.

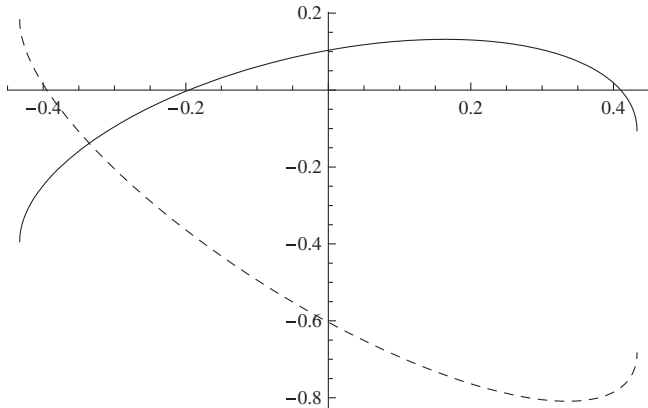


FIG. 2. The plots for p_1 (the solid curve) and p_v (the dashed curve) are shown as a function of g_ϕ for $f_1 > 0$.

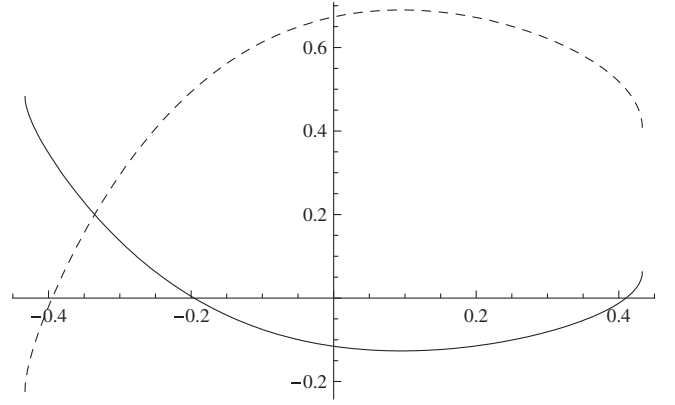


FIG. 3. The plots for q_1 (the solid curve) and q_v (the dashed curve) are shown as a function of g_ϕ for $f_1 > 0$.

The Einstein frame: Let us briefly discuss how our solution behaves in the Einstein frame. Rewriting the original metric in this form,

$$ds^2 = g_{\mu\nu}^{(4)} dx^\mu dx^\nu + e^{2v} dy^2 + e^{2w} d\theta^2, \quad (4.58)$$

we obtain

$$R = R^{(4)} + 2v^2 + 2w^2 + \dots, \quad (4.59)$$

where $R^{(4)}$ is the Ricci scalar associated with the metric $g_{\mu\nu}^{(4)}$. The gravity action can reduce to

$$\begin{aligned} \int d^6x \sqrt{-GR} &\sim V_2 \int d^4x \sqrt{-g^{(4)}} e^{v+w} R^{(4)} \\ &\sim V_2 \int d^4x \sqrt{-g^{(E)}} R^{(E)}, \end{aligned} \quad (4.60)$$

where $V_2 = \int dv dw$ is the comoving volume of the internal space, which is assumed to be finite for instance by introducing a cutoff. The Einstein frame metric is given by

$$\begin{aligned} ds_E^2 &:= g_{\mu\nu}^{(E)} dx^\mu dx^\nu = e^{v+w} g_{\mu\nu}^{(4)} dx^\mu dx^\nu \\ &= -e^{v+w+2u_0} dt^2 + e^{v+w+2u_1} \sum_{i=1}^3 (dx^i)^2. \end{aligned} \quad (4.61)$$

Using the solutions, we find

$$\begin{aligned} v + w + 2u_1 &= \frac{1}{3}(v + w + 2u_0) \\ &= -\ln |\sinh[f_1(t - t_1)]| - \frac{4}{3} G_\phi t. \end{aligned} \quad (4.62)$$

Thus, in the proper time coordinate system $d\tau = \pm e^{v+w+2u_0} dt$, the Einstein frame metric can be written as

$$ds_E^2 \approx -d\tau^2 + (\mp \tau)^{2/3} \sum_{i=1}^3 (dx^i)^2, \quad (4.63)$$

which corresponds to a contracting or expanding Universe filled by the stiff matter.

V. CONCLUSIONS

We have investigated the dynamical 3-form flux compactifications and their implications for brane world cosmology in the six-dimensional Nishino-Salam-Sezgin model. We take the background of the 3-form field acting on the internal space and timelike dimensions without the $U(1)$ gauge field strength.

The first class of solutions we discussed was the dynamical generalization [15] of the static solutions, recently obtained in [14]. It turned out that the dynamical generalization is possible only for the special case. In this class, we found that the time evolution is restricted in the azimuthal dimension of the internal space. The ordinary three-dimensional space is not dynamical and hence does not give rise to a cosmological evolution. There is a curvature singularity at the boundary of the internal space, which is more severe than the conical one. To construct the brane world model, following [14], we first removed the singular part from the original solution and then glued the remaining piece of the spacetime with an identical copy at the codimension-one boundary wrapping the axis of the rotational symmetry. This boundary may be regarded as our 3-brane world. Generically, because of the presence of the 3-form field the Lorentz symmetry in the ordinary four-dimensional spacetime is broken. In general, such a brane can be supported by the matter with the energy density ρ , the pressure in the ordinary 3-space p and that in the azimuthal dimension p_θ , which would be different from the pure tension. Since the inclusion of the time dependence reduces the number of the parameters, the brane can be embedded by the matter with $p_\theta = -\rho$ (but still $p \neq -\rho$). However, at the place where the Lorentz symmetry is recovered, the boundary brane world can be supported only by the pure tension, i.e., $p = p_\theta = -\rho$. Note the solutions reported here and related implications for the brane world model were obtained under several assumptions. It is very interesting to explore the time-dependent solutions and associated brane world models without these assumptions.

The second class of models was obtained by exchanging the roles of the radial coordinate and the time coordinate from the first class one. At the center of the internal space, there is a conical singularity which may be interpreted as codimension two our 3-brane world. But it does not seem to be possible to realize a warped structure in the external dimensions, as expected in analogy with the previous case. Except for the oscillating dynamical solutions, the solutions led to the expanding or contracting ordinary 3-space, depending on the choice of time direction. Among the expanding solutions, there are decelerating and accelerating ones: In the latter solution, the scale factor in the ordinary 3-space diverges within a finite time. In the Einstein frame, however, the Universe always followed the expansion law of the one filled by the stiff matter, irrespective of the choice of parameters.

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APPENDIX: DYNAMICAL BLACK 1-BRANES

In this Appendix, we discuss another class of the dynamical solutions with the nonvanishing 3-form field strength. The solution represents the black 1-brane extended along the y direction in the presence of a cosmological constant. We now take the following ansatz for the metric

$$ds_6^2 = e^{2u(t,y,r)}(-dt^2 + dy^2) + e^{2v(t,y,r)}(dr^2 + r^2 d\Omega_3^2), \quad (\text{A1})$$

the dilaton $\phi = \phi(t, y, r)$ and the 3-form field strength

$$H = E(t, y, r),, dt \wedge dr \wedge dy. \quad (\text{A2})$$

The (t, y) coordinates cover the worldvolume of the 1-brane, while the rest does the transverse spatial dimensions. The equations for the 3-form field are given by

$$\begin{aligned} (e^{-2u+2v-2\phi} r^3 E_r), &= (e^{-2u+2v-2\phi} r^3 E_r),_y \\ &= (e^{-2u+2v-2\phi} r^3 E_r),_r = 0. \end{aligned} \quad (\text{A3})$$

Defining $V \equiv 2u + 2v$, the integration of Eq. (A3) gives

$$r^3 e^{V-4u-2\phi} E_r = c, \quad (\text{A4})$$

where c is an integration constant. Then, the diagonal components of Einstein equations and the dilaton equation of motion are given by

$$\begin{aligned} e^{-2u}(-u_{,yy} - 4u_{,y}v_{,y} - \ddot{u} + 2\ddot{v} + \dot{V} - 2\dot{u}\dot{v} - \dot{u}\dot{V} \\ + 2\dot{u}^2 + 4\dot{v}^2) + e^{-2v}\left(-u_{,r}V_{,r} - u_{,rr} - \frac{3}{r}u_{,r}\right) \\ = -e^{-2u}\dot{\phi}^2 - \frac{\kappa^2 c}{2r^3}e^{-V-2v}E_r + \frac{g^2}{\kappa^2}e^\phi, \end{aligned} \quad (\text{A5})$$

$$\begin{aligned}
& e^{-2u}(u_{,yy} - 2v_{,yy} - V_{,yy} + 2u_{,y}v_{,y} + u_{,y}V_{,y} - 2u_{,y}^2 - 4v_{,y}^2 \\
& \quad + \ddot{u} + \dot{u}(\dot{V} - 2\dot{v} + 2\dot{u})) \\
& \quad + e^{-2v}\left(-u_{,r}V_{,r} - u_{,rr} - \frac{3}{r}u_{,r}\right) \\
& = e^{-2u}\phi_{,y}^2 - \frac{\kappa^2 c}{2r^3}e^{-V-2v}E_{,r} + \frac{g^2}{\kappa^2}e^\phi, \tag{A6}
\end{aligned}$$

$$\begin{aligned}
& e^{-2u}(\ddot{v} + \dot{v}(\dot{V} - 2\dot{u} + 2\dot{v}) - v_{,yy} \\
& \quad - v_{,y}(V_{,y} - 2u_{,y} + 2v_{,y})) \\
& \quad + e^{-2v}\left(-V_{,rr} - v_{,rr} - \frac{3}{r}v_{,r} + v_{,r}V_{,r} - 2(u_{,r}^2 + v_{,r}^2)\right) \\
& = e^{-2v}\phi_{,r}^2 - \frac{\kappa^2 c}{2r^3}e^{-V-2v}E_{,r} + \frac{g^2}{\kappa^2}e^\phi, \tag{A7}
\end{aligned}$$

$$\begin{aligned}
& e^{-2u}(\ddot{v} + \dot{v}(\dot{V} - 2\dot{u} + 2\dot{v}) - v_{,yy} \\
& \quad - v_{,y}(V_{,y} - 2u_{,y} + 2v_{,y})) \\
& \quad + e^{-2v}\left(-v_{,rr} - \frac{3}{r}v_{,r} - \left(v_{,r} + \frac{1}{r}\right)V_{,r}\right) \\
& = \frac{\kappa^2 c}{2r^3}e^{-V-2v}E_{,r} + \frac{g^2}{\kappa^2}e^\phi, \tag{A8}
\end{aligned}$$

$$\begin{aligned}
& e^{-2u}(-\ddot{\phi} - (\dot{V} - 2\dot{u} + 2\dot{v})\dot{\phi} + \phi_{,yy} \\
& \quad + (V_{,y} - 2u_{,y} + 2v_{,y})\phi_{,y}) \\
& \quad + e^{-2v}\left(\phi_{,rr} + \left(V_{,r} + \frac{3}{r}\right)\phi_{,r}\right) - \frac{\kappa^2 c}{r^3}e^{-V-2v}E_{,r} \\
& \quad - \frac{2g^2}{\kappa^2}e^\phi = 0. \tag{A9}
\end{aligned}$$

We require in Eqs. (A5) and (A6)

$$u_{,r} = \frac{\kappa^2}{2r^3}ce^{-V}E. \tag{A10}$$

Similarly in Eqs. (A8) and (A9),

$$v_{,r} = -\frac{\kappa^2}{2r^3}ce^{-V}E, \quad \phi_{,r} = \frac{\kappa^2}{r^3}ce^{-V}E, \tag{A11}$$

where we assume that V does not depend on r . Eliminating r -derivative terms in Eq. (A7), by Eqs. (A10) and (A11), we obtain

$$\begin{aligned}
& e^{2v-2u}[\ddot{v} + \dot{v}(\dot{V} - 2\dot{u} + 2\dot{v}) - v_{,yy} \\
& \quad - v_{,y}(V_{,y} - 2u_{,y} + 2v_{,y})] - \frac{g^2}{\kappa^2}e^{\phi+2v} \\
& = -\frac{1}{r^3}\left(\tilde{E}_{,r} - \frac{2\tilde{E}^2}{r^3}\right), \tag{A12}
\end{aligned}$$

where we defined $\tilde{E} := \kappa^2 ce^{-V}E$. Setting the right-hand side of Eq. (A12) to be zero, we obtain

$$\tilde{E} = \frac{Q}{H(t, y, r)}, \quad H(t, y, r) \equiv h(t, y) + \frac{Q}{r^2}, \tag{A13}$$

where Q is a constant. Then Eqs. (A10) and (A11) give that the metric functions and dilaton can be written as

$$\begin{aligned}
u & = -\frac{1}{4}\ln H(t, y, r), & v & = \frac{1}{4}\ln H(t, y, r), \\
\phi & = -\frac{1}{2}\ln H(t, y, r), \tag{A14}
\end{aligned}$$

where the integration constants are set to be zero. It is straightforward to check that the off-diagonal components of the Einstein equation

$$4u_{,r}\dot{v} - \dot{u}_{,r} - 3\dot{v}_{,r} - \phi_{,r}\dot{\phi} = 0, \tag{A15}$$

$$4u_{,r}v_{,y} - u_{,ry} - 3v_{,ry} - \phi_{,r}\phi_{,y} = 0,$$

are satisfied. We find that the time-dependent parts of Eqs. (A5)–(A7) reduce to

$$h_{,yy} + 3\ddot{h} = \frac{4g^2}{\kappa^2}, \tag{A16}$$

$$-\ddot{h} - 3h_{,yy} = \frac{4g^2}{\kappa^2}, \tag{A17}$$

$$\ddot{h} - h_{,yy} = \frac{4g^2}{\kappa^2}, \tag{A18}$$

respectively. Both of Eqs. (A8) and (A9) reduce to the same equation as Eq. (A18). The solution of Eqs. (A16)–(A18) is given by

$$h(t, y) = \frac{g^2}{\kappa^2}(t^2 - y^2) + a_t t + a_{y,y} + a_0, \tag{A19}$$

where a_i ($i = t, y, 0$) are integration constants. Then, the left-hand side of the remaining off-diagonal component of the Einstein equation

$$-4v_{,y}\dot{v} - 4\dot{v}_{,y} + 4u_{,y}\dot{v} + 4\dot{u}v_{,y} - \dot{\phi}\phi_{,y} = 0, \tag{A20}$$

is proportional to $\dot{h}_{,y}$, which vanishes because of Eq. (A19). As seen from Eq. (A19), h is no longer linear in the worldvolume coordinates but quadratic in them, which also happens in p -brane solutions with trivial or vanishing dilaton [22].

Let us summarize the solutions obtained here:

$$\begin{aligned}
u(t, y, r) & = -v(t, y, r) \\
& = -\frac{1}{4}\ln\left(\frac{Q}{r^2} + \frac{g^2}{\kappa^2}(t^2 - y^2) + a_t t + a_{y,y} + a_0\right), \tag{A21}
\end{aligned}$$

$$\begin{aligned}
\phi(t, y, r) & = -\frac{1}{2}\ln\left(\frac{Q}{r^2} + \frac{g^2}{\kappa^2}(t^2 - y^2) + a_t t + a_{y,y} + a_0\right). \tag{A22}
\end{aligned}$$

Because of the dependence on the spatial worldvolume

coordinate y as well as on the time coordinate t , the brane direction is not compact, and this solution may not be suitable for constructing the brane world. One might imagine that there may be a solution with a quadratic order dependence on time in the case of the 2-form field (i.e., 0-brane). For the dilaton coupling parameters in the NSS model, such a dynamical solution does not exist. But it does for the other special coupling parameters. Our solution can be interpreted as one of the dynamical p -brane

solutions, in a class of the theory with a cosmological constant. These solutions will be discussed in detail in [23]. In particular, among these solutions, the 0-brane solution in the four-dimensional spacetime behaves as a black hole embedded into a Friedmann-Robertson-Walker universe filled by the matter with the equation of state $w = -\frac{1}{3}$, where w is the ratio of the pressure with the energy density.

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