

Nonperturbative gluon and ghost propagators for $d = 3$ Yang-Mills theoryA. C. Aguilar,¹ D. Binosi,² and J. Papavassiliou³¹*Federal University of ABC, CCNH, Rua Santa Adélia 166, CEP 09210-170, Santo André, Brazil*²*European Centre for Theoretical Studies in Nuclear Physics and Related Areas (ECT*), Villa Tambosi, Strada delle Tabarelle 286, I-38050 Villazzano (TN), Italy*³*Department of Theoretical Physics and IFIC, University of Valencia-CSIC, E-46100, Valencia, Spain*
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We study a manifestly gauge-invariant set of Schwinger-Dyson equations to determine the nonperturbative dynamics of the gluon and ghost propagators in $d = 3$ Yang-Mills theory. The use of the well-known Schwinger mechanism, in the Landau gauge leads to the dynamical generation of a mass for the gauge boson (gluon in $d = 3$), which, in turn, gives rise to an infrared finite gluon propagator and ghost dressing function. The propagators obtained from the numerical solution of these nonperturbative equations are in very good agreement with the results of $SU(2)$ lattice simulations.

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I. INTRODUCTION

QCD in three space-time dimensions (QCD₃ for short) has received increasing attention in recent years, not only because it is the infinite-temperature limit of its four-dimensional counterpart (QCD₄), but also because, at zero temperature, these two theories, despite a number of important differences, seem to share a variety of important nonperturbative features [1–28].

QCD₃ differs from QCD₄ in several aspects. For example, the fact that QCD₃ lives in an odd-dimensional space allows the appearance of phenomena that are not possible in even-dimensional spaces, such as the parity violating gauge-boson masses from a Chern-Simons term [2,4,5]. Moreover, given that in $d = 3$ the square of the coupling constant has dimensions of mass, QCD₃ is superrenormalizable, having a trivial renormalization group. Finally, there are no finite-action classical solitons in QCD₃ (i.e., no instantons) (see [9] for a brief review).

On the other hand, both theories confine display area laws for Wilson loops in the fundamental representation, and develop nonperturbative vacuum condensates, such as $\text{Tr}\langle\mathcal{G}_{ij}^2\rangle$; in fact, in $d = 3$ one can actually prove [11] the existence of a $\text{Tr}\langle\mathcal{G}_{ij}^2\rangle$ condensate, associated with the minimum of the zero-momentum effective action, simply on the hypothesis that the full theory possesses a unique mass scale (that of the gauge coupling). In addition, and more importantly for the purposes of the present work, both theories appear to cure their infrared (IR) instabilities through the dynamical generation of a gauge-boson (gluon) mass, usually referred to also as “magnetic” mass, without affecting the local gauge invariance, which remains intact [29]. The nonperturbative dynamics that gives rise to the generation of such a mass is rather complex, and can be ultimately traced back to a subtle realization of the Schwinger mechanism [30–37].

The gluon mass generation manifests itself at the level of the fundamental Green’s functions of the theory in a very

distinct way, giving rise to an IR behavior that would be difficult to explain otherwise. Specifically, in the Landau gauge, both in $d = 3, 4$, the gluon propagator and the ghost-dressing function reach a finite value in the deep IR [38,39]. However, the gluon propagator of QCD₃ displays a local maximum at relatively low momenta, before reaching a finite value at $q = 0$. This characteristic behavior is qualitatively different from what happens in $d = 4$, where the gluon propagator is a monotonic function of the momentum in the entire range between the IR and UV fixed points [38].

It should also be mentioned that a qualitatively similar situation emerges within the “refined” Gribov-Zwanziger formalism [40,41], presented in [42]. In this latter framework the gluon mass is obtained through the addition of appropriate condensates to the original Gribov-Zwanziger action.

Even though several aspects of QCD₃ have been studied in a variety of works, the recent theoretical developments associated with the pinch technique (PT), together with the high-quality lattice results produced, motivate the detailed study of the entire shape of the gluon and ghost propagators in $d = 3$. Specifically, given that the gluon mass generation is a purely nonperturbative effect, in the continuum it has to be addressed within the framework of the Schwinger-Dyson equations (SDE). These complicated dynamical equations are best studied in a gauge-invariant framework based on the PT [29,43–46], and its profound correspondence with the background field method (BFM) [47]. As has been explained in detail in the recent literature [48,49], this latter formalism allows for a gauge-invariant truncation of the SD series in the sense that it preserves manifestly and at every step the transversality of the gluon self-energy.

In the present work we study the dynamics of the gluon and ghost propagators of pure Yang-Mills theory in $d = 3$, using the SDEs of the PT-BFM formalism in the Landau gauge. Even though our results are valid for every gauge

group, we will eventually focus on the group $SU(2)$, in order to make contact with available lattice simulations [50]. The crucial ingredient in this analysis, which accounts for the type of solutions obtained, is the gauge-invariant introduction of a gluon mass. The way gauge invariance is maintained is through the inclusion of Nambu-Goldstone-like (composite) massless excitations into the nonperturbative three-gluon vertex [29]. As a result, the fundamental Ward identities of the theory, which encode the underlying gauge symmetry, remain intact. The results obtained from our SDE analysis, presented in Sec. IV, compare rather well with the available lattice data (see, in particular, Figs. 6 and 8).

In addition, as a necessary intermediate step, we calculate an auxiliary function, denoted by $G(q)$, which plays an instrumental role in the PT-BFM framework (see next section). Interestingly enough, and in the Landau gauge *only*, $G(q)$ coincides with the so-called Kugo-Ojima (KO) function; this latter function, and, in particular, its value in the deep IR, is intimately connected to the corresponding and well-known confinement criterion [51].

The article is organized as follows. In Sec. II we briefly review the salient features of the SDEs within the PT-BFM framework. Section III contains a general discussion of the main conceptual issues related with the dynamical mass generation through the Schwinger mechanism. Particular attention is paid to the specific form of the three-gluon vertex that must be employed in order to maintain gauge invariance in the form of the Ward identities. In addition, we give a qualitative discussion of some of the main features expected for the gluon propagator in the presence of a gluon mass. Section IV contains the main results of this work. After setting up the corresponding SDE for the gluon propagator and the auxiliary function $G(q)$, we give explicit closed expressions for the latter quantities. The two available free parameters appearing in the expression for the gluon propagator, namely, the gauge coupling g and the mass m are then varied, in order to obtain the best possible agreement with the lattice data. The ghost-dressing function is also obtained from the self-consistent solution of the corresponding SDE; it too shows a good agreement with the lattice. Finally, in Sec. V we present our conclusions.

II. THE PT-BFM FRAMEWORK

In this section we remind the reader of the basic characteristics of the SD framework that is based on the PT-BFM formalism; for an extended review of the subject see [45].

We start by introducing the necessary notation. The gluon propagator $\Delta_{\mu\nu}(q)$ in the covariant gauges assumes the form

$$\Delta_{\mu\nu}(q) = -i \left[P_{\mu\nu}(q) \Delta(q) + \xi \frac{q_\mu q_\nu}{q^4} \right], \quad (2.1)$$

where ξ denotes the gauge-fixing parameter, $P_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu / q^2$ is the usual transverse projector, and $\Delta^{-1}(q) = q^2 + i\Pi(q)$, with $\Pi_{\mu\nu}(q) = P_{\mu\nu}(q)\Pi(q)$ the gluon self-energy. We also define the dimensionless vacuum polarization $\mathbf{\Pi}(q)$, as $\Pi(q) = q^2\mathbf{\Pi}(q)$. In addition, the full ghost propagator, $D(p)$, and its dressing function, $F(p)$, are related by $D(p) = iF(p)/p^2$.

The truncation scheme for the SDEs of Yang-Mills theories based on the PT respects gauge invariance (i.e., the transversality of the gluon self-energy) at every level of the ‘‘dressed-loop’’ expansion. This becomes possible due to the drastic modifications implemented in the building blocks of the SD series, i.e., the off-shell Green’s functions themselves, following the general methodology of the PT [29,43,46]. The PT is a well-defined algorithm that exploits systematically the BRST symmetry in order to construct new Green’s functions endowed with very special properties, in particular, the crucial property of gauge invariance, for they satisfy Abelian Ward identities instead of the usual Slavnov-Taylor identities. The PT may be used to rearrange systematically the entire SD series [48]. In the case of the gluon self-energy it gives rise to a new SDE, shown schematically in Fig. 1.

Note that the quantity that appears on the left-hand side of Fig. 1 is *not* the conventional self-energy $\Pi_{\mu\nu}$, but rather the PT-BFM self-energy, denoted by $\tilde{\Pi}_{\mu\nu}$. The graphs appearing on the right-hand side contain the conventional self-energy $\Pi_{\mu\nu}$ as before, but are composed out of two types of vertices:

- (i) The conventional vertices, where all incoming fields are quantum fields, i.e., they carry the virtual loop momenta; these vertices are all ‘‘internal’’, i.e., the external gluons cannot be one of their legs, and will be generally denoted by Γ .
- (ii) A new set of vertices, with one of their legs being the external gluon, carrying physical momentum q ; these new vertices, to be generally denoted by $\tilde{\Gamma}$, correspond precisely to the Feynman rules of the BFM [47], i.e., it is as if the external gluon had been converted dynamically into a background gluon.

As a result, the full vertices $\tilde{\Gamma}_{\alpha\mu\nu}^{amn}(q, k_1, k_2)$, $\tilde{\Gamma}_{\alpha}^{amn}(q, k_1, k_2)$, $\tilde{\Gamma}_{\alpha\mu\nu\rho}^{amnr}(q, k_1, k_2, k_3)$, and $\tilde{\Gamma}_{\alpha\mu}^{amnr}(q, k_1, k_2, k_3)$ appearing on the right-hand side of the SDE shown in Fig. 1 satisfy the simple Ward identities

$$\begin{aligned} q^\alpha \tilde{\Gamma}_{\alpha\mu\nu}^{amn} &= g f^{amn} [\Delta_{\mu\nu}^{-1}(k_1) - \Delta_{\mu\nu}^{-1}(k_2)], \\ q^\alpha \tilde{\Gamma}_{\alpha}^{amn} &= i g f^{amn} [D^{-1}(k_1) - D^{-1}(k_2)], \\ q^\alpha \tilde{\Gamma}_{\alpha\mu\nu\rho}^{amnr} &= g f^{adr} \Gamma_{\nu\rho\mu}^{drm}(q + k_2, k_3, k_1) + \text{c.p.}, \\ q^\alpha \tilde{\Gamma}_{\alpha\mu}^{amnr} &= g f^{aem} \Gamma_{\mu}^{enr}(q + k_1, k_2, k_3) + \text{c.p.}, \end{aligned} \quad (2.2)$$

where c.p. stands for cyclic permutations. Using these identities, it is straightforward to show that the crucial

transversality condition $q^\mu \hat{\Pi}_{\mu\nu}(q) = 0$ is enforced “blockwise” [48], i.e.,

$$\begin{aligned} q^\mu [(a_1) + (a_2)]_{\mu\nu} &= 0, \\ q^\mu [(a_3) + (a_4)]_{\mu\nu} &= 0, \\ q^\mu [(a_5) + (a_6)]_{\mu\nu} &= 0, \\ q^\mu [(a_7) + (a_8) + (a_9) + (a_{10})]_{\mu\nu} &= 0, \end{aligned} \quad (2.3)$$

which allow for a self-consistent truncation of the full gluon SDE given in Fig. 1.

Quite interestingly, the conventional $\Delta(q)$ and its PT-BFM counterpart $\hat{\Delta}(q)$ are connected by the following *background-quantum* identity [52]

$$\Delta(q) = [1 + G(q)]^2 \hat{\Delta}(q), \quad (2.4)$$

where the function $G(q)$ is the $g_{\mu\nu}$ component of the auxiliary two-point function $\Lambda_{\mu\nu}(q)$, defined as

$$\begin{aligned} \Lambda_{\mu\nu}(q) &= -ig^2 C_A \int_k H_{\mu\rho}^{(0)} D(k+q) \Delta^{\rho\sigma}(k) H_{\sigma\nu}(k, q) \\ &= g_{\mu\nu} G(q) + \frac{q_\mu q_\nu}{q^2} L(q), \end{aligned} \quad (2.5)$$

where C_A is the Casimir eigenvalue of the adjoint representation [$C_A = N$ for $SU(N)$] and $\int_k \equiv \mu^{2\epsilon} (2\pi)^{-d} \times \int d^d k$, with d the dimension of space-time. The function $H_{\sigma\nu}$ is given diagrammatically in Fig. 2. Note that it is related to the full gluon-ghost vertex by $q^\sigma H_{\sigma\nu}(p, r, q) = -i\Gamma_\nu(p, r, q)$; at tree-level, $H_{\sigma\nu}^{(0)} = ig_{\sigma\nu}$.

The identity (2.4) allows us to express the SDE of Fig. 1 as an integral equation involving only $\Delta(q)$, namely,

$$\Delta^{-1}(q) P_{\mu\nu}(q) = \frac{q^2 P_{\mu\nu}(q) + i \sum_{i=1}^{10} (a_i)_{\mu\nu}}{[1 + G(q)]^2}. \quad (2.6)$$

Finally, as shown in Fig. 3, the ghost SDE is the same as in the conventional formulation, namely,

$$iD^{-1}(q) = q^2 + ig^2 C_A \int_k \Gamma^\mu \Delta_{\mu\nu}(k) \Gamma^\nu(q, k) D(q+k), \quad (2.7)$$

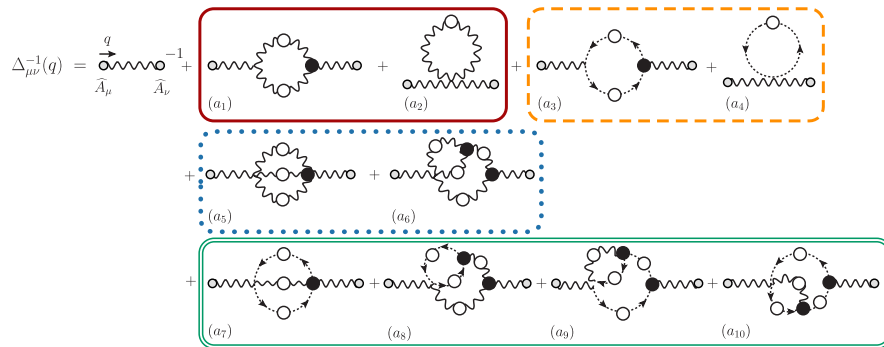


FIG. 1 (color online). The full SDE for the gluon self-energy in the PT-BFM framework. By virtue of the special Abelian-like Ward identities satisfied by the various fully-dressed vertices, the contributions of each block are individually transverse.

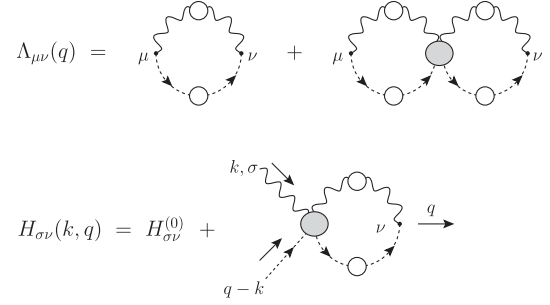


FIG. 2. Diagrammatic representation of the functions Λ and H .

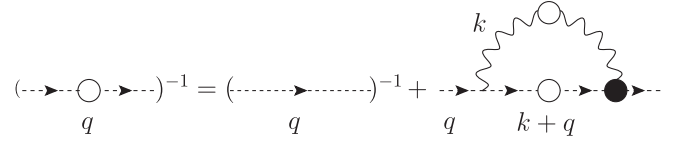


FIG. 3. The SDE satisfied by the ghost propagator.

where Γ_μ is the standard (asymmetric) gluon-ghost vertex at tree level, and Γ^μ its fully dressed counterpart.

III. MASS GENERATION IN $d = 3$ YANG-MILLS THEORY

It is well known that, just as happens at $d = 4$, the Yang-Mills dynamics in $d = 3$ generates an effective gauge-boson mass that cures all IR instabilities. The underlying mechanism that leads to the generation of such a dynamical mass, both in $d = 3, 4$, is the Schwinger mechanism, the only known procedure for obtaining massive gauge bosons while maintaining the gauge-symmetry intact.

As Schwinger pointed out a long time ago [30], the gauge invariance of a vector field does not necessarily imply zero mass for the associated particle if the current vector coupling is sufficiently strong. According to Schwinger’s fundamental observation, if $\Pi(q)$ acquires a pole at the zero-momentum transfer, then the vector meson becomes massive, even if the gauge symmetry forbids a mass at the level of the fundamental Lagrangian. Indeed, it is clear that if the vacuum polarization $\Pi(q)$ has a pole at

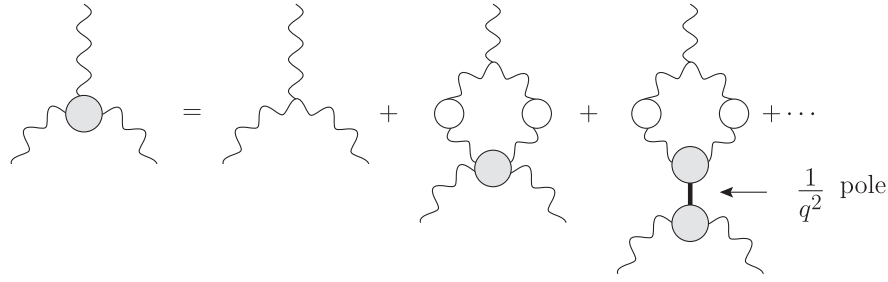


FIG. 4. Vertex with nonperturbative massless excitations triggering the Schwinger mechanism.

$q^2 = 0$ with positive residue μ^2 , i.e., $\mathbf{\Pi}(q) = \mu^2/q^2$, then (in Euclidean space) $\Delta^{-1}(q) = q^2 + \mu^2$. Thus, the vector meson becomes massive, $\Delta^{-1}(0) = \mu^2$, even though it is massless in the absence of interactions ($g = 0$). There is no physical principle which would preclude $\mathbf{\Pi}(q)$ from acquiring such a pole, even in the absence of elementary scalar fields. In a *strongly-coupled* theory, like nonperturbative Yang-Mills theory in $d = 3, 4$, this may happen for purely dynamical reasons, since strong binding may generate zero-mass bound-state excitations [32]. The latter act like dynamical Nambu-Goldstone bosons, in the sense that they are massless, composite, and *longitudinally coupled*; but, at the same time, they differ from Nambu-Goldstone bosons as far as their origin is concerned: they do not originate from the spontaneous breaking of any global symmetry [29]. In what follows we will assume that theory can indeed generate the required bound-state poles; the demonstration of the existence of a bound state, and, in particular, of a zero-mass bound state, is a difficult dynamical problem, usually studied by means of integral equations known as Bethe-Salpeter equations (see, e.g., [53]). Note also that the generation of a dynamical mass (both in $d = 3, 4$) requires (and, correspondingly, gives rise to) the formation of a gluon condensate.

The Schwinger mechanism is incorporated into the SDE of the gluon propagator essentially through the form of the fully dressed, nonperturbative three-gluon vertex (see Fig. 4). In fact, since the generation of the mass does not interfere with the gauge symmetry, which remains intact, the three-gluon vertex must satisfy the same Ward identity as in the massless case [viz. Eq. (2.2)], but now with massive, as opposed to massless, gluon propagators on its right-hand side. The way this crucial requirement is enforced is precisely through the incorporation into the three-gluon vertex of the Nambu-Goldstone (composite) massless excitations mentioned above. To see how this works with a simple example, let us consider the standard tree-level vertex

$$\Gamma_{\mu\alpha\beta}(q, p, r) = (q-p)_\beta g_{\mu\alpha} + (p-r)_\mu g_{\alpha\beta} + (r-q)_\alpha g_{\mu\beta}, \quad (3.1)$$

which satisfies the simple Ward identity

$$q^\mu \Gamma_{\mu\alpha\beta}(q, p, r) = P_{\alpha\beta}(r) \Delta_0^{-1}(r) - P_{\alpha\beta}(p) \Delta_0^{-1}(p), \quad (3.2)$$

where $\Delta_0^{-1}(q) = q^2$ is the inverse of the tree-level propagator. After the dynamical mass generation, the inverse gluon propagator becomes, roughly speaking,

$$\Delta_m^{-1}(q) = q^2 - m^2(q^2), \quad (3.3)$$

and the new vertex, $\Gamma_{\mu\alpha\beta}^m(q, p, r)$ that replaces $\Gamma_{\mu\alpha\beta}(q, p, r)$ must still satisfy the Ward identity of (3.2), but with $\Delta_0^{-1} \rightarrow \Delta_m^{-1}$ on the right-hand side. This is accomplished if

$$\Gamma_{\mu\alpha\beta}^m(q, p, r) = \Gamma_{\mu\alpha\beta}(q, p, r) + V_{\mu\alpha\beta}(q, p, r), \quad (3.4)$$

where $V_{\mu\alpha\beta}(q, p, r)$ contains the massless poles. A standard Ansatz for $V_{\mu\alpha\beta}(q, p, r)$ is [7]

$$V_{\mu\alpha\beta}(q, p, r) = m^2(r) \frac{q_\mu P_\alpha(q-p)_\rho}{2q^2 p^2} P_\beta^\rho(r) - [m^2(p) - m^2(q)] \frac{r_\beta}{r^2} P_\rho^\mu(q) P_\alpha^\rho(p) + \text{c.p.} \quad (3.5)$$

It is easy to check that

$$q^\mu V_{\mu\alpha\beta}(q, p, r) = P_{\alpha\beta}(p) m^2(p) - P_{\alpha\beta}(r) m^2(r), \quad (3.6)$$

and cyclic permutations. Therefore, one has

$$q^\mu \Gamma_{\mu\alpha\beta}^m(q, p, r) = P_{\alpha\beta}(r) \Delta_m^{-1}(r) - P_{\alpha\beta}(p) \Delta_m^{-1}(p), \quad (3.7)$$

as announced. Note that for constant masses [$m(q) = m(p) = m(r) = m$] the vertex of (3.5) reduces to

$$V_{\mu\alpha\beta}(q, p, r) = \frac{m^2}{2} \left[\frac{q_\mu P_\alpha(q-p)_\rho}{q^2 p^2} P_\beta^\rho(r) + \frac{P_\alpha r_\beta (p-r)_\rho}{p^2 r^2} P_\mu^\rho(q) + \frac{r_\beta q_\mu (r-q)_\rho}{r^2 q^2} P_\alpha^\rho(p) \right]. \quad (3.8)$$

Even though the precise implementation at the level of the complicated integral equations is rather subtle, the final

upshot of introducing a vertex such as Γ^m (or more sophisticated versions of it) into the SDE for the gluon self-energy is that one finally obtains, gauge invariantly, a nonvanishing $\hat{\Delta}^{-1}(0)$ and $\Delta^{-1}(0)$. Qualitatively speaking, in Euclidean space and d space-time dimensions, the (background) gluon propagator is given by

$$\hat{\Delta}^{-1}(q) = q^2 + \hat{\Pi}(q) + \hat{\Delta}^{-1}(0), \quad (3.9)$$

where $\hat{\Pi}(q)$ has the general form

$$\begin{aligned} \hat{\Pi}(q) = & c_1 g^2 \int_k \Delta(k) \Delta(k+q) K_1(q, k) \\ & + c_2 g^2 \int_k D(k) D(k+q) K_2(q, k). \end{aligned} \quad (3.10)$$

The functions $K_1(q, k)$ and $K_2(q, k)$ are SD kernels, whose closed form depends, among other things, on the dimensionality of space-time, the details of the vertices employed, and the gauge chosen, as do, in general, the constants c_1 and c_2 . Setting $\hat{\Delta}^{-1}(0) = m^2$, one then obtains

$$\hat{\Delta}^{-1}(q) = q^2 + m^2 + \hat{\Pi}(q). \quad (3.11)$$

To obtain the perturbative (one-loop) expression for $\hat{\Delta}(q)$ one must substitute in the integral on the right-hand side of (3.10) the tree-level values for Δ , D , K_1 , and K_2 , which is a good approximation for large values of the physical momentum q . However, for low values of q , one must solve the integral equation, which, under suitable assumptions, will furnish massive (IR finite) solutions for $\hat{\Delta}(q)$.

An easy way to qualitatively appreciate the effect of the mass on the solutions for $\hat{\Delta}(q)$ is to substitute $\Delta \rightarrow \Delta_m$ in the first integral on the right-hand side of (3.10), assuming for simplicity a constant mass m , and use tree-level expressions for all other terms. This will furnish an approximate expression for $\hat{\Pi}(q)$, to be denoted by $\hat{\Pi}_m(q)$, and the resulting $\hat{\Delta}^{-1}(q)$ will read

$$\hat{\Delta}^{-1}(q) = q^2 + m^2 + \hat{\Pi}_m(q). \quad (3.12)$$

In $d = 4$ the corresponding $\hat{\Pi}_m(q)$ will have the form

$$\hat{\Pi}_m^{(4)}(q) = b g^2 q^2 \int_0^1 dx \ln[q^2 x(1-x) + m^2]. \quad (3.13)$$

For $m \rightarrow 0$, or $q^2 \ll m^2$, one recovers the usual one-loop logarithm $b g^2 \ln(q^2)$, with b being the first coefficient of the QCD one-loop β function, $b = 11C_A/48\pi^2$. As explained in the literature, the presence of the mass inside the logarithm tames the Landau pole, and gives eventually rise to an IR finite value for the QCD effective charge

Similarly, in $d = 3$ we have [see the integral R_1 in Eq. (4.8)]

$$\hat{\Pi}_m^{(3)}(q) = -2b_3 g^2 q \arctan\left(\frac{q}{m}\right), \quad (3.14)$$

which in the limit $m \rightarrow 0$ assumes the one-loop perturba-

tive form [see also the integral I_1 in Eq. (4.8)]

$$\hat{\Pi}_{\text{pert}}^{(3)}(q) = -\pi b_3 g^2 q. \quad (3.15)$$

In this case however, and unlike in $d = 4$, b_3 is a numerical coefficient that depends explicitly on the value of the gauge parameter chosen; in the Feynman gauge, $b_3 = 15C_A/32\pi$ (we will return to this point in the next section).

Let us now briefly compare the versions of the gluon propagator obtained by substituting $\Pi_{\text{pert}}^{(3)}(q)$ or $\Pi_m^{(3)}(q)$ into (3.12). For the perturbative case we have

$$\hat{\Delta}_{\text{pert}}(q) = \frac{1}{q^2 - \pi b_3 g^2 q}. \quad (3.16)$$

There two points to notice: (i) $\hat{\Delta}_{\text{pert}}(q)$ has a Landau pole at $\bar{q} = \pi b_3 g^2$, and (ii) it displays a maximum value at $q^* = \bar{q}/2$. On the other hand, the gluon propagator corresponding to $\hat{\Pi}_m^{(3)}(q)$ becomes

$$\hat{\Delta}(q) = \frac{1}{q^2 + m^2 - 2b_3 g^2 q \arctan\left(\frac{q}{m}\right)}. \quad (3.17)$$

It is clear that the presence of the mass regulates the denominator for all values of q , provided that it exceeds a certain critical value (in units of g^2). In addition, $\hat{\Delta}(q)$ may or may not display a maximum, depending on the ratio g^2/m ; in general, its position is displaced with respect to q^* .

IV. RESULTS AND COMPARISON WITH THE LATTICE

In order to make contact with the $d = 3$ lattice results of [50], we must next determine the form of the relevant SDEs in the Landau gauge ($\xi = 0$). The three quantities of interest are

- (i) The gluon propagator, $\Delta(q)$ given in (2.6);
- (ii) The Kugo-Ojima function $G(q)$, given in (2.5), which connects the conventional and background gluon propagators;
- (iii) The ghost propagator, given in (2.7), and, in particular, its dressing function, $F(q)$.

A. Calculating the gluon propagator(s) and the KO function

In the ‘‘one-loop dressed’’ approximation, the PT-BFM gluon self-energy is given by the following (gauge-invariant) subset of diagrams:

$$\hat{\Pi}_{\mu\nu}(q) = [(a_1) + (a_2) + (a_3) + (a_4)]_{\mu\nu}. \quad (4.1)$$

When evaluating the diagrams (a_i) one should use the BFM Feynman rules [47], noticing, in particular, that the bare three- and four-gluon vertices depend explicitly on $1/\xi$, the coupling of the ghost to a background gluon is *symmetric* in the ghost momenta, and, finally, that there is a

four-field coupling between two background gluons and two ghosts.

As explained in [38], the limit $\xi \rightarrow 0$ of the diagrams (a_1) and (a_2) must be taken with care, due to the terms proportional to $1/\xi$ coming from the tree-level vertices. Introducing $\Delta_{\mu\nu}^t(q) = P_{\mu\nu}(q)\Delta(q)$, one obtains

$$\begin{aligned} [(a_1) + (a_2)]_{\mu\nu} &= g^2 C_A \left[\frac{1}{2} \int_k \Gamma_{\mu}^{\alpha\beta} \Delta_{\alpha\rho}^t(k) \Delta_{\beta\sigma}^t(k+q) \mathbf{L}_{\nu}^{\rho\sigma} \right. \\ &\quad \left. - \frac{4}{3} g_{\mu\nu} \int_k \Delta(k) + \int_k \Delta_{\alpha\mu}^t(k) \frac{(k+q)_{\beta}}{(k+q)^2} \right. \\ &\quad \left. \times [\Gamma + \mathbf{L}]_{\nu}^{\alpha\beta} + \int_k \frac{k_{\mu}(k+q)_{\nu}}{k^2(k+q)^2} \right]. \end{aligned} \quad (4.2)$$

The vertex $\mathbf{L}_{\mu\alpha\beta}$ is the fully dressed counterpart of $\Gamma_{\mu\alpha\beta}$ (in the Landau gauge); it satisfies the Ward identity

$$q^{\mu} \mathbf{L}_{\mu\alpha\beta} = P_{\alpha\beta}(k+q) \Delta^{-1}(k+q) - P_{\alpha\beta}(k) \Delta^{-1}(k). \quad (4.3)$$

It is then easy to verify that the right-hand side of (4.2) vanishes when contracted by q^{μ} , thus explicitly confirming the validity of the first equation in (2.3), for the special case of $\xi = 0$.

Similarly,

$$\begin{aligned} [(a_3) + (a_4)]_{\mu\nu} &= -g^2 C_A \left[\int_k \tilde{\Gamma}_{\mu} D(k) D(k+q) \tilde{\Gamma}_{\nu} \right. \\ &\quad \left. - 2i g_{\mu\nu} \int_k D(k) \right], \end{aligned} \quad (4.4)$$

with $\tilde{\Gamma}_{\mu}(q, p, r) \sim (r-p)_{\mu}$. The vertex $\tilde{\Gamma}_{\mu}$ satisfies the second Ward identity in (2.2), which leads immediately to the transversality of this block, i.e., the second equation in (2.3).

Finally, using tree-level values for the auxiliary function $H_{\sigma\nu}$ in (2.5) and for the vertex Γ^{ν} in (2.7), we obtain for the

Kugo-Ojima function

$$G(q) = \frac{g^2 C_A}{2} \int_k \left[1 + \frac{(k \cdot q)^2}{k^2 q^2} \right] \Delta(k) D(k+q), \quad (4.5)$$

while for $L(q)$ one has

$$L(q) = \frac{g^2 C_A}{2} \int_k \left[1 - 3 \frac{(k \cdot q)^2}{k^2 q^2} \right] \Delta(k) D(k+q). \quad (4.6)$$

The way we proceed is the following. Instead of actually solving the system of coupled integral equation, we will adopt an approximate procedure, which is operationally less complicated, and seems to capture rather well the underlying dynamics.

Specifically, we will assume that the gluon propagator has the form given in (3.12), and will determine the function $\Pi_m^{(3)}(q)$ by calculating the expressions given in (4.2) and (4.4) using inside the corresponding integrals $\Delta \rightarrow \Delta_m$ and $D \rightarrow D_0$. In order to maintain gauge invariance intact, we will set

$$\mathbf{L}_{\mu\alpha\beta}(q, p, r) = \Gamma_{\mu\alpha\beta}^m(q, p, r), \quad (4.7)$$

with $\Gamma_{\mu\alpha\beta}^m(q, p, r)$ given in (3.4). The vertex $V_{\mu\alpha\beta}^m(q, p, r)$ entering into $\Gamma_{\mu\alpha\beta}^m(q, p, r)$ will be that of Eq. (3.8), i.e., we will assume a constant mass m throughout.

From the final expressions appearing in the rest of the paper we will use Euclidean momenta. To that end we set $q^2 = -q_E^2$, with $q_E^2 > 0$ the positive square of a Euclidean four-vector, and $q_E = \sqrt{q_E^2}$. The Euclidean propagator is defined as $\hat{\Delta}_E(q_E^2) = -\hat{\Delta}(-q_E^2)$. To avoid notational clutter, we will suppress the subscript E in what follows.

The results of all our calculations will be expressed in terms of the following six basic integrals:

$$\begin{aligned} R_0 &= \int_k \frac{1}{k^2 - m^2} = \left(\frac{i}{4\pi} \right) m, & R_1 &= \int_k \frac{1}{(k^2 - m^2)[(k+q)^2 - m^2]} = \left(\frac{i}{4\pi} \right) \frac{1}{q} \arctan\left(\frac{q}{2m} \right), \\ I_1 &= \int_k \frac{1}{k^2(k+q)^2} = \left(\frac{i}{8} \right) \frac{1}{q}, & I_2 &= \int_k \frac{1}{(k^2 - m^2)(k+q)^2} = \left(\frac{i}{4\pi} \right) \frac{1}{q} \arctan\left(\frac{q}{m} \right), \\ I_3 &= \int_k \frac{1}{k^2(k^2 - m^2)} = \left(\frac{i}{4\pi} \right) \frac{1}{m}, & I_4 &= \int_k \frac{q \cdot k}{(k^2 - m^2)(k+q)^2} = \left(\frac{i}{8\pi} \right) \left[m + \frac{q^2 - m^2}{q} \arctan\left(\frac{q}{m} \right) \right], \end{aligned} \quad (4.8)$$

where the momentum q appearing in the integrals on the left-hand side is Minkowskian, while the momentum q appearing in the results on the right-hand side is Euclidean.

To facilitate the calculation, and since the transversality of $\hat{\Pi}_{\mu\nu}(q)$ is guaranteed, one may set in (4.1) $\hat{\Pi}_{\mu\nu}(q) = P_{\mu\nu}(q) \hat{\Pi}_m(q)$, and isolate $\hat{\Pi}_m(q)$ by taking the trace of both sides, i.e.,

$$(d-1) \hat{\Pi}_m(q) = [(a_1) + (a_2) + (a_3) + (a_4)]_{\mu}^{\mu}. \quad (4.9)$$

For the different four contributions shown in Eq. (4.2) we obtain the following results:

$$\begin{aligned}
\frac{1}{2} \int_k \Gamma_{\mu}^{\alpha\beta} \Delta_{\alpha\rho}^t(k) \Delta_{\beta\sigma}^t(k+q) \mathbf{L}^{\mu\rho\sigma} &= 9R_0 + \left(\frac{1}{4} \frac{q^6}{m^4} - 2 \frac{q^4}{m^2} - 10q^2 + 8m^2 \right) R_1 + \frac{1}{4} \frac{q^6}{m^4} I_1 \\
&\quad - \left(\frac{1}{2} \frac{q^6}{m^4} - 2 \frac{q^4}{m^2} - \frac{11}{2} q^2 - 3m^2 \right) I_2 - \left(\frac{5}{2} q^2 + 4m^2 \right) I_3, \\
\frac{4}{3} g_{\mu}^{\mu} \int_k \Delta(k) &= 4R_0, \\
\int_k \Delta_{\alpha\mu}^t(k) \frac{(k+q)_{\beta}}{(k+q)^2} [\Gamma + \mathbf{L}]^{\mu\alpha\beta} &= - \left(\frac{1}{2} + \frac{1}{4} \frac{m^2}{q^2} \right) R_0 - \left(\frac{1}{2} \frac{q^4}{m^2} + \frac{1}{4} q^2 \right) I_1 + \left(\frac{1}{2} \frac{q^2}{m^2} - \frac{11}{4} q^2 - 3m^2 + \frac{1}{4} \frac{m^4}{q^2} \right) I_2 \\
&\quad + \left(\frac{1}{2} q^2 + \frac{m^2}{4} \right) I_3, \\
\int_k \frac{k^{\mu}(k+q)_{\mu}}{k^2(k+q)^2} &= \frac{1}{2} q^2 I_1. \quad (4.10)
\end{aligned}$$

Next, let us turn to the diagrams (a_3) and (a_4) of Fig. 1, which contain a ghost loop. Since we will treat the ghost as a massless particle, the ‘‘tadpole’’ diagram (a_4) vanishes identically in dimensional regularization; from diagram (a_3) we get instead (after taking the trace)

$$(a_3)_{\mu}^{\mu} = -g^2 C_A \int_k \frac{(2k+q)^{\mu}(2k+q)_{\mu}}{k^2(k+q)^2} = -g^2 C_A q^2 I_1. \quad (4.11)$$

From the results above it is relatively straightforward to check, taking appropriate limits, that in the deep IR

$$\hat{\Pi}(0) = -i \frac{g^2 C_A}{6\pi} m. \quad (4.12)$$

Therefore, in order for the (Euclidean) $\hat{\Delta}(0)^{-1} = m^2 - i\hat{\Pi}(0)$ to be positive definite, m and g must satisfy the condition

$$\frac{m}{C_A g^2} > \frac{1}{6\pi}. \quad (4.13)$$

In the opposite limit, namely, for asymptotically large momenta, the addition of all terms given in (4.10) exposes a vast cancellation of all powers q^n , with $n > 1$. After all such cancellations taking place, one is left with a linear contribution, given by

$$\hat{\Pi}(q) \xrightarrow{q \rightarrow \infty} -i \frac{g^2 C_A}{32} q \left(15 - \frac{7}{2} \right). \quad (4.14)$$

The reason for writing the numerical coefficient in front of the leading contribution as a deviation from 15 is the following. The expression (4.14) should coincide with the $d = 3$ one-loop BFM self-energy calculated in the Landau gauge. For any dimension d and any value of the gauge-fixing parameter ξ_Q , the latter reads [45]

$$\begin{aligned}
\hat{\Pi}(q) &= \frac{g^2 C_A}{2} \left(\frac{7d-6}{d-1} \right) q^2 \int_k \frac{1}{k^2(k+q)^2} \\
&\quad - g^2 C_A q^2 (1 - \xi_Q) \left[\frac{1 - \xi_Q}{2} q^2 P^{\mu\nu}(q) \right. \\
&\quad \left. \times \int_k \frac{k_{\mu} k_{\nu}}{k^4(k+q)^4} + \int_k \frac{2q \cdot k}{k^4(k+q)^2} \right]. \quad (4.15)
\end{aligned}$$

In the Feynman gauge of the BFM, $\xi_Q = 1$, $\hat{\Pi}(q)$ collapses to the PT answer for the gauge-independent gluon self-energy; specifically, for $d = 3$,

$$\hat{\Pi}(q)|_{\xi_Q=1} = -i \frac{g^2 C_A}{32} q(15). \quad (4.16)$$

Away from $\xi_Q = 1$ the terms in the second line of (4.15) give additional contributions, which may be easily calculated using the basic results

$$\begin{aligned}
\int_k \frac{k_{\mu} k_{\nu}}{k^4(k+q)^4} &= -\frac{i}{32} \frac{1}{q^3} g_{\mu\nu} + \dots, \\
\int_k \frac{q \cdot k}{k^4(k+q)^2} &= -\frac{1}{16q}, \quad (4.17)
\end{aligned}$$

where the dots in the first integral indicate longitudinal parts. In particular, it is easy to verify that at $\xi_Q = 0$ these additional terms account precisely for the term $-\frac{7}{2}$ appearing in Eq. (4.14).

The above discussion reveals an important difference between the $d = 3$ and $d = 4$ cases. Specifically, in $d = 4$ the coefficient in front of the leading one-loop contribution to $\hat{\Pi}(q)$ is independent of the gauge-fixing parameter ξ_Q . This well-known BFM result can be easily deduced from (4.15), since both integrals proportional to $(1 - \xi_Q)$ are UV finite, i.e., they do not furnish logarithms. The coefficient in front of the logarithm is completely determined by the first integral, multiplied by the factor $\frac{g^2 C_A}{2} \left(\frac{7d-6}{d-1} \right)$, which, at $d = 4$, reduces to $16\pi^2 b = (11/3)g^2 C_A$, namely, the first coefficient of the Yang-Mills theory β function. As we have just demonstrated, things are different in $d = 3$,

where no renormalization is needed; the leading (linear) contribution depends explicitly on the value of ξ_Q .

Next, we determine an approximate expression for the function $G(q)$. To that end, we turn to (4.5) and substitute in the integral on the right-hand side, $\Delta \rightarrow \Delta_m$ and $D \rightarrow D_0$. One has then

$$\begin{aligned} G(q) &= \frac{ig^2 C_A}{2} \int_k \frac{1}{(k+q)^2(k^2-m^2)} \left[1 + \frac{(k \cdot q)^2}{k^2 q^2} \right] \\ &= \frac{ig^2 C_A}{8} \left[-\frac{2}{q^2} I_4 + 5I_2 - I_3 + \frac{q^2}{m^2} (I_2 - I_1) \right], \end{aligned} \quad (4.18)$$

which gives

$$\begin{aligned} G(q) &= -\frac{g^2 C_A}{32\pi m} \left[\frac{\pi}{2} \frac{q}{m} + \frac{m^2}{q^2} - 1 \right. \\ &\quad \left. + \frac{m}{q} \left(6 - \frac{m^2}{q^2} - \frac{q^2}{m^2} \right) \arctan\left(\frac{q}{m}\right) \right]. \end{aligned} \quad (4.19)$$

In the deep IR ($q \rightarrow 0$), and for asymptotically large momenta ($q \rightarrow \infty$), one finds

$$G(q) \xrightarrow{q \rightarrow 0} -\frac{g^2 C_A}{6\pi m}, \quad G(q) \xrightarrow{q \rightarrow \infty} 0. \quad (4.20)$$

From the expressions for $\hat{\Pi}_m(q)$ and $G(q)$ obtained above, we can determine the conventional gluon propagator, $\Delta(q)$, in the Landau gauge; the latter can then be compared to the lattice data. To that end, let us first employ the crucial identity of (2.4) to write

$$\Delta(q) = \frac{[1 + G(q)]^2}{q^2 + m^2 + \hat{\Pi}_m(q)}. \quad (4.21)$$

Then, by virtue of (4.20), $\Delta(q)$ has the same asymptotic behavior as $\hat{\Delta}(q)$.

Notice that in $d = 3$ the gluon and ghost propagators have the basic scaling property

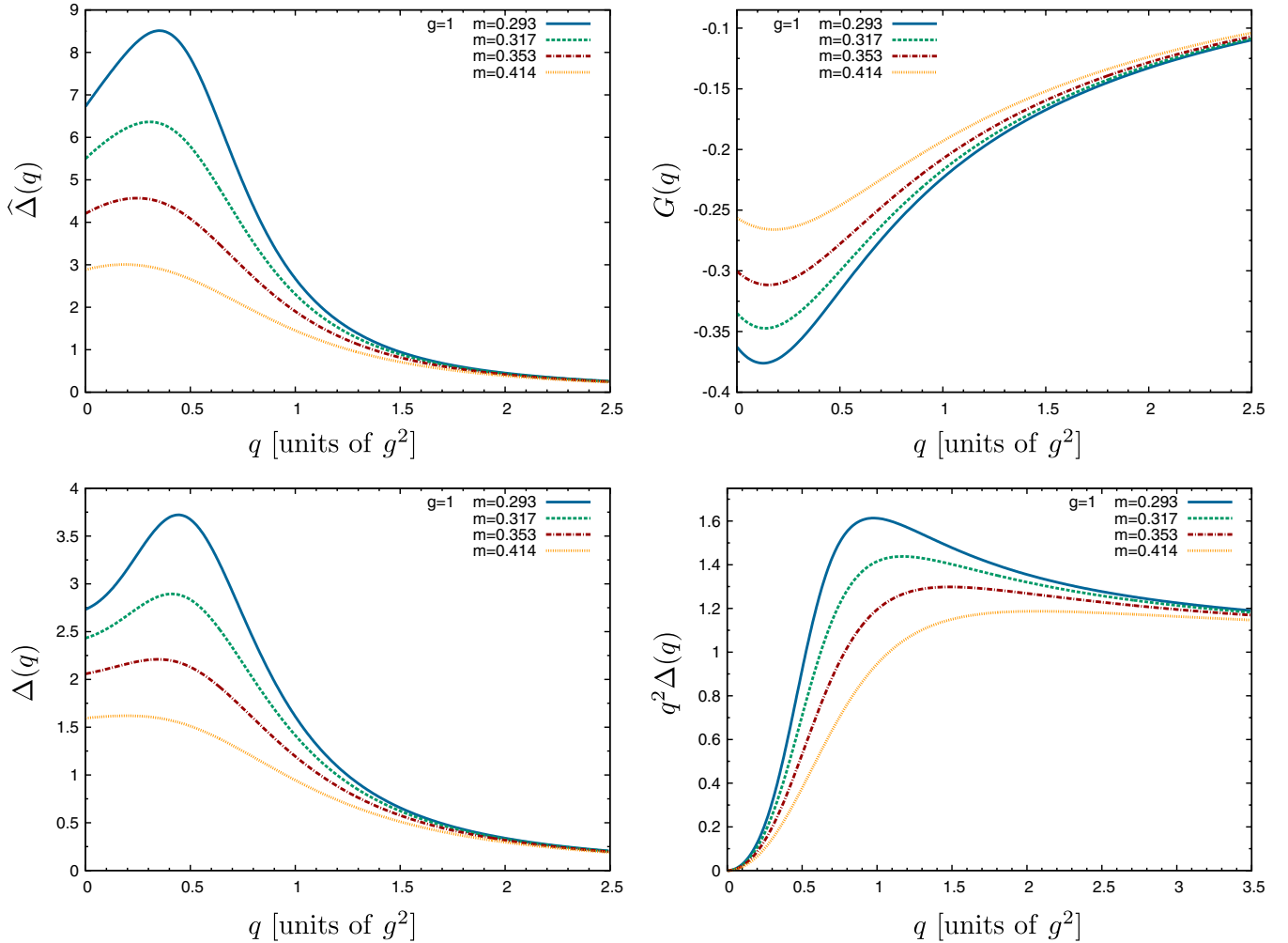


FIG. 5 (color online). Results for the massive one-loop approximation for the $d = 3$ gluon propagator. In the upper panels we show the plots for different values of the hard-mass parameter m for the background-quantity identity ingredients $\hat{\Delta}(q)$ (left) and the Kugo-Ojima function $G(q)$ (right). In the lower panels we show the conventional propagator $\Delta(q)$ (left) and its corresponding dressing function $q^2\Delta(q)$ (right).

$$\begin{aligned}\Delta(q, g, m) &= a^2 \Delta(aq, \sqrt{ag}, am), \\ D(q, g, m) &= a^2 D(aq, \sqrt{ag}, am),\end{aligned}\quad (4.22)$$

where a is a positive real number. Of course, the corresponding dressing functions (being dimensionless quantities) are invariant under such a combined rescaling; for example, the ghost-dressing function satisfies $F(q, g, m) = F(aq, \sqrt{ag}, am)$, and so does the gluon dressing function $q^2 \Delta(q)$ and the Kugo-Ojima function $G(q)$. One can then make use of these scaling properties to set g (respectively, m) equal to unity, and vary m (respectively, g) in order to study the shape of the solutions found so far. The results (when setting $g = 1$ and varying m) are shown in Fig. 5.

B. Comparing the gluon propagator with $SU(2)$ lattice data

We next compare the result of our calculation for the conventional gluon propagator $\Delta(q)$ with the lattice results of [50]. In order to do that, the lattice data must be first properly normalized (or, equivalently, the theoretical prediction must be suitably rescaled) Specifically, in the absence of any physical input that would fix the physical scale, one uses the scaling property (4.22) and determine the scaling factor a in such a way that the asymptotic (large momentum) segment of the lattice data coincides with that obtained from our calculation; indeed the two “tails” should coincide, given that perturbation theory is reliable in that region of momenta. The result of this procedure is shown in Fig. 6; evidently, the matching between the theoretical curve and the lattice data is very good. The best-fit curve furnishes the ratio

$$\frac{m}{2g^2} \approx 0.146, \quad (4.23)$$

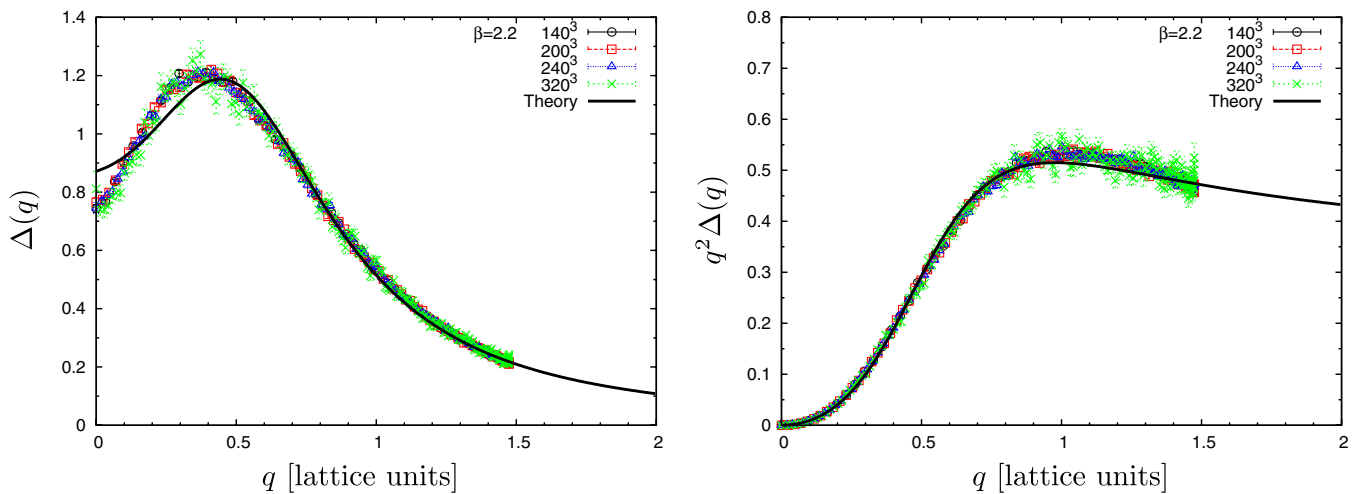


FIG. 6 (color online). Comparison of the lattice results of [50] with the gluon propagator (left) and the gluon dressing function (right) obtained within the massive one-loop approximation adopted in this paper. In passing, notice that the dressing function does not tend to 1 for asymptotically large q which also motivates the momentum rescaling procedure employed.

which appears to be in rather good agreement with previous theoretical and lattice studies [14].

C. Ghost-dressing function and lattice data

We next proceed to calculate the theoretical prediction for the ghost-dressing function $F(q)$. In a spirit similar to that adopted for the gluon propagator, as first approach in this direction, we simply compute the diagram for the ghost propagator (see Fig. 3) using as inputs on the right-hand side $\Delta \rightarrow \Delta_m$ and $D \rightarrow D_0$. The result of this calculation is

$$\begin{aligned}F^{-1}(q) &= 1 + ig^2 C_A \int_k \frac{1}{(k+q)^2(k^2-m^2)} \left[1 - \frac{(k \cdot q)^2}{k^2 q^2} \right] \\ &= 1 + \frac{ig^2 C_A}{4} \left[\frac{2}{q^2} I_4 + 3I_2 + I_3 - \frac{q^2}{m^2} (I_2 - I_1) \right].\end{aligned}\quad (4.24)$$

At this point, and before attempting a comparison with the corresponding lattice data, we note that, in the Landau gauge only, the ghost-dressing function $F(q)$, and the two form factors $G(q)$ and $L(q)$ defined in Eq. (2.5), are related (for any d) by the following important identity:

$$1 + G(q) + L(q) = F^{-1}(q). \quad (4.25)$$

The relation of Eq. (4.25) has been first obtained in [54], and some years later in [55], in the framework of the Batalin-Vilkovisky quantization formalism; as was shown there, this relation is a direct consequence of the fundamental BRST symmetry. Recently, the same identity has been derived exactly from the SDEs of the theory [56], and the important property $L(0) = 0$, usually assumed in the literature, was shown to be valid for any value of the space-time dimension d ; indeed setting $\Delta \rightarrow \Delta_m$ and $D \rightarrow D_0$ on the right-hand side of Eq. (4.6), one has

$$\begin{aligned}
L(q) &= \frac{ig^2 C_A}{2} \int_k \frac{1}{(k+q)^2(k^2-m^2)} \left[1 - 3 \frac{(k \cdot q)^2}{k^2 q^2} \right] \\
&= \frac{ig^2 C_A}{8} \left[\frac{6}{q^2} I_4 + 3I_3 + I_2 - 3 \frac{q^2}{m^2} (I_2 - I_1) \right] \\
&= \frac{3g^2 C_A}{32\pi m} \left[\frac{\pi}{2} \frac{q}{m} + \frac{m^2}{q^2} - 1 \right. \\
&\quad \left. + \frac{m}{q} \left(\frac{2}{3} - \frac{m^2}{q^2} - \frac{q^2}{m^2} \right) \arctan\left(\frac{q}{m}\right) \right], \quad (4.26)
\end{aligned}$$

and therefore

$$L(q) \xrightarrow{q \rightarrow 0} 0, \quad L(q) \xrightarrow{q \rightarrow \infty} 0. \quad (4.27)$$

It is then straightforward to verify from the result above and the closed expressions given in Eqs. (4.19) and (4.24), that Eq. (4.25) holds exactly within the approximation scheme we are using (see also the left panel of Fig. 7).

We next vary the parameters g and m in the expression given in Eq. (4.24) in order to reproduce the lattice data for $F(q)$. As a natural starting point we use the values that have resulted in the best fit for the gluon propagator, namely, $g = 1.285$ and $m = 0.480$. However, as is clear from the red dashed curve shown in Fig. 7 (right panel), the result obtained is in poor agreement with the lattice. If instead we allow the parameters to vary freely, i.e., we disregard the gluon data and attempt to only reproduce the ghost data, the best possible curve is shown by the black continuous line of Fig. 7, being obtained for the values $g = 2.049$, $m = 0.543$ (giving $m/2g^2 \approx 0.065$).

It is clear from this analysis that, within the approximation scheme employed, the lattice data may be well reproduced if treated independently, but it is not possible to arrive at a reasonable simultaneous fit, i.e., to fit both curves using a unique set of parameters.

D. Combined treatment: gluon propagator and ghost-dressing function vs lattice

To remedy this situation, we will improve the approximation used for obtaining the theoretical prediction for the ghost-dressing function. Specifically, we will study an approximate version of the ghost SDE given in Eq. (2.7), and we will solve self-consistently for the unknown function $F(q)$, instead of simply calculating its right-hand side, as was done above for obtaining the expression in Eq. (4.24).

Given that Eq. (2.7) contains $\Delta(k)$ as one of its basic ingredients, the general matching procedure becomes more subtle. In particular, instead of freely fitting just one set of data (that for the gluon propagator) one must now attempt to fit simultaneously both the gluon and ghost data as well as possible. As we will see, this more complicated procedure furnishes finally a very good agreement with the combined set of lattice data, but one has to settle for a slightly less accurate description of the gluon data compared to the one obtained in Fig. 6

After approximating the gluon-ghost vertex Γ_μ by its tree-level value, we arrive at the following integral equation for the ghost-dressing function F :

$$F^{-1}(q) = 1 + g^2 C_A \int_k \left[1 - \frac{(k \cdot q)^2}{k^2 q^2} \right] \frac{\Delta(k) F(k+q)}{(k+q)^2}. \quad (4.28)$$

The general idea now is to solve Eq. (4.28) numerically for $F(q)$, using as input for the $\Delta(k)$ under the integral sign the theoretical curve that, after the rescaling mentioned earlier, provides the best possible fit to the gluon data and, at the same time, allows for the numerical convergence of Eq. (4.28). We note in passing that this procedure permits us, after a shift of the integration variable, to pass all

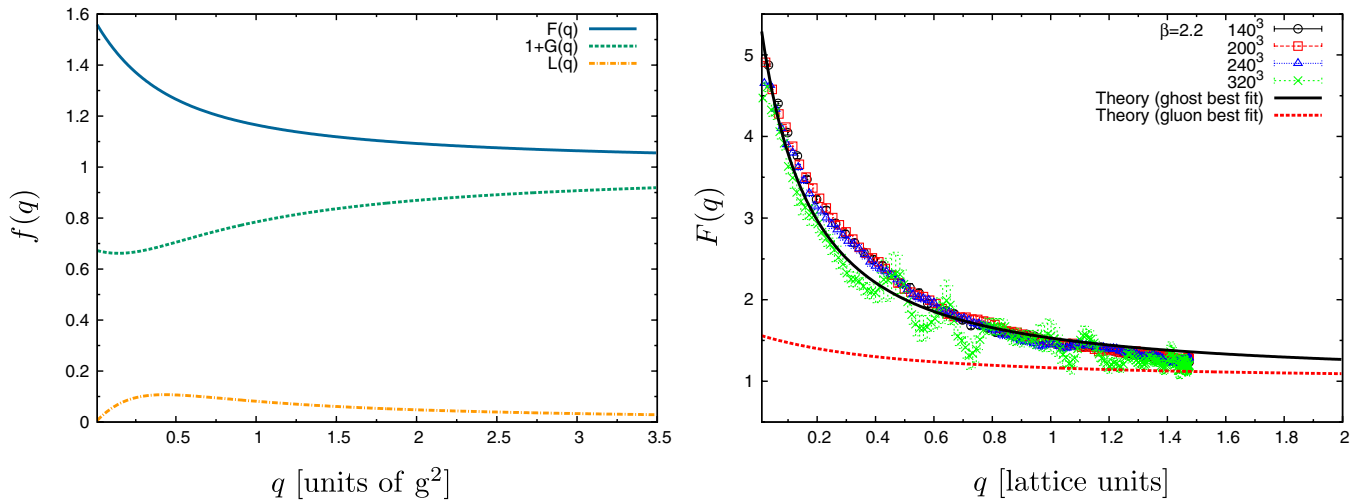


FIG. 7 (color online). (Left panel) Values of $L(q)$, $1 + G(q)$, and $F(q) = [1 + G(q) + L(q)]^{-1}$ within our approximation for $g = 1.287$ and $m = 0.539$. (Right panel) Comparison of the ghost-dressing function with the one calculated within our approximation for two sets of values corresponding to the gluon propagator best fit ($g = 1.287$ and $m = 0.539$, red dashed line) and to the best fit to the lattice data ($g = 2.049$, $m = 0.543$).

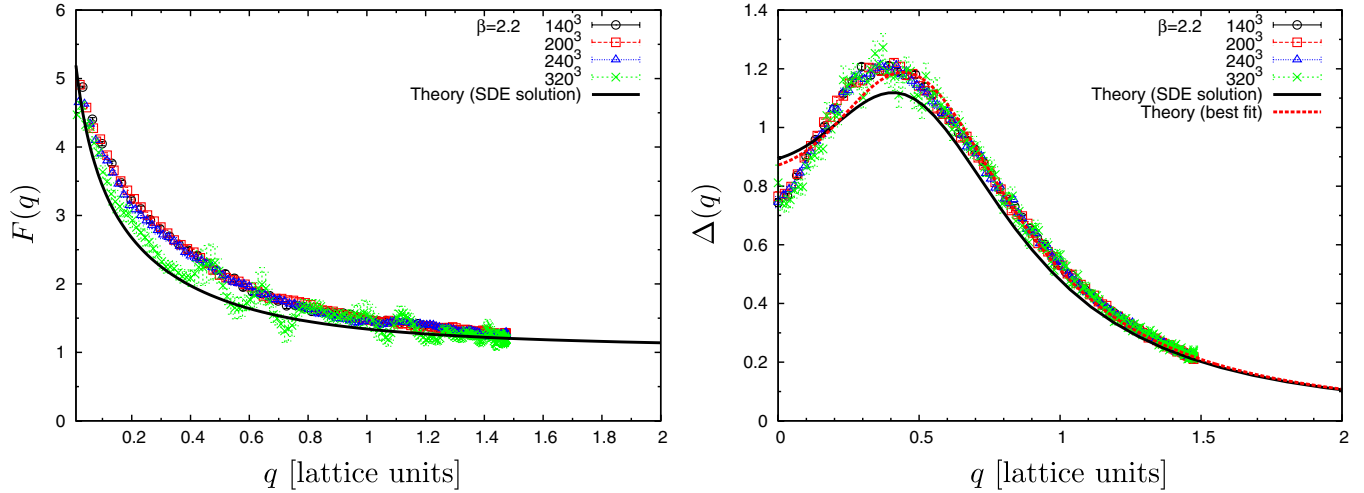


FIG. 8 (color online). Comparison of the lattice results of [50] with the ghost-dressing function obtained through the solution of the ghost SDE (left); on the right we show the gluon propagator for the same value of $m/2g^2 \approx 0.153$.

angular dependence from $F(k+q)$ to $\Delta(k)$, whose functional form is considered known; as a result, one does not need to resort to further approximations for the angular part of the integral equation.

The general observation regarding the numerical treatment of Eq. (4.28) is that it appears to be extremely sensitive to the precise shape of $\Delta(k)$ and the value of g ; minute variations of these quantities give rise to large disparities in the resulting $F(q)$.

The best possible solution that we have obtained is shown in the left panel of Fig. 8. As announced, the accuracy achieved in matching the lattice data for the gluon propagator is slightly inferior to that of our best fit (Fig. 8, right panel), but is still very good.

V. CONCLUSIONS

In this work we have presented a nonperturbative study of the (Landau gauge) gluon and ghost propagator for $d = 3$ Yang-Mills theory, using the one-loop dressed SDEs of the PT-BFM formalism. One of the most powerful features of this framework is that the transversality of the truncated gluon self-energy is guaranteed, by virtue of the QED-like Ward identities satisfied by the fully-dressed vertices entering into the dynamical equations.

The central dynamical ingredient of our analysis is the assumption that the famous Schwinger mechanism, namely, the dynamical formation of zero-mass Nambu-Goldstone-boson-like composite excitations, which allow the gauge-invariant generation of a gauge-boson mass, is indeed realized in $d = 3$ Yang-Mills theory. The way this dynamical scenario is incorporated into the SDEs is through the form of the three-gluon vertex. Specifically, in order to satisfy the correct Ward identity, as required by gauge-invariance, this vertex must contain massless, longitudinally coupled poles, representative precisely of the aforementioned composite excitations.

It should be emphasized that the approach followed here is approximate, not only in the sense that we consider the one-loop dressed version of the SDE, omitting (gauge-invariantly) higher orders [i.e., the third and fourth block of Fig. 1], but also because we do not actually solve simultaneously the full system of resulting equations. Specifically, as explained in Sec. IV, when evaluating the gluon self-energy we have used tree-level expressions for the ghost propagators appearing in diagram (a_3) of Fig. 1, and the same approximation is used also in the determination of the KO function $G(q)$. We have then used the resulting gluon propagator as an input into Eq. (4.28) to obtain the improved $F(q)$. Of course, this two-step procedure is bound to result in a considerable discrepancy between the “one-loop” $G(q)$ and the $F(q)$ obtained from solving its corresponding SDE; evidently, the identity of Eq. (4.25) cannot be fulfilled any longer. In addition, the dynamical gluon mass m has been treated for simplicity as a constant. However, a more thorough study should eventually include the important feature that the mass depends nontrivially on the momentum, in accordance with general considerations [57]; in fact, a complete SDE treatment ought to actually determine the precise way the mass is running [58]. The fact that, despite these simplifications, the lattice results for the gluon and ghost propagator are so well reproduced, suggests that the full treatment may reveal a number of subtle cancellations, caused by the highly nonlinear nature of the SDE equations, yielding finally results very similar to those reported here.

Let us outline briefly some of the modifications and additional field-theoretic inputs that such a full SDE treatment would entail. To begin with, a more complete Ansatz for the three-gluon vertex $\tilde{\Gamma}_{\alpha\mu\nu}$, appearing in the SDE of the gluon propagator [graph (a_1) in Fig. 1], must be devised. Such an Ansatz must not contain only the part of the massless poles, as Eq. (3.5) does, which only accounts for

the massive part of the propagator, but should also make explicit reference to the entire Δ , in the spirit of the analysis already presented in [58] (for $d = 4$). In addition, a similar Ansatz must be also introduced for the full gluon-ghost vertex $\tilde{\Gamma}_\alpha$ appearing in the graph (a_3) of Fig. 1. In order to maintain explicit gauge invariance, $\tilde{\Gamma}_\alpha$ must be such that the second Ward identity of Eq. (2.2) is automatically satisfied. Note that $\tilde{\Gamma}_\alpha$ is not the same as the conventional gluon-ghost vertex Γ_α that appears in Eq. (2.7) and in Fig. 3. Given that $\tilde{\Gamma}_\alpha$ and Γ_α are different, and that only the former is crucial for the transversality of the gluon SDE, one may approximate Γ_α by its tree-level value, without clashing with gauge-invariance, a freedom that exists only within the PT-BFM scheme. Should one opt for a more sophisticated treatment of Γ_α , then an appropriate Ansatz may have to be devised. Given that Γ_α satisfies a complicated Slavnov-Taylor identity, instead of the simple Ward identity of $\tilde{\Gamma}_\alpha$, further approximations may be necessary. We hope to be able to implement some of the aforementioned improvements in the near future.

Finally, it is important to clarify some pertinent conceptual points regarding the nature of the gluon mass that we have encountered in our analysis, and, in particular, its gauge (in)variance. Before proceeding any further, it is important to establish from the outset a clear distinction between the notions of gauge invariance and “gauge independence.” Gauge invariance is used for indicating that a Green’s function satisfies the Ward identity (or a Slavnov-Taylor identity) imposed by the gauge (or BRST) symmetry of the theory. On the other hand, the gauge (in)dependence of a Green’s function is related with the (in)dependence on the gauge-fixing parameter (e.g. ξ) used to quantize the theory. Evidently, an off-shell Green’s function may be gauge invariant but gauge dependent: for example, the QED photon-electron vertex, $\Gamma_\mu(p, p + q)$, depends explicitly on ξ , but satisfies (for every value of ξ) the classic Ward identity $q^\mu \Gamma_\mu(p, p + q) = S^{-1}(p + q) - S^{-1}(p)$. A celebrated example of a Green’s function that is both gauge invariant and gauge independent is the photon self-energy (vacuum polarization), which is both transverse and ξ independent. Returning to the issue of the gluon mass found in this article, it is clear that it has emerged from a gauge-invariant analysis, because, as explained in the text, the SDEs we have solved are manifestly gauge invariant, in the sense that the transversality of the $\Pi_{\mu\nu}(q)$ is guaranteed. However, it is also clear that a particular gauge choice has been implemented from the beginning, namely, that of the Landau gauge. Thus, the value for the gluon mass so obtained is particular to that gauge, and we are not aware of any quantitative studies, or even robust qualitative arguments, supporting the notion that the same value would emerge if the SDE analysis were to be repeated, for example, in the Feynman gauge. In particular, there is no

known analogy to what happens in the case of the gauge-boson propagators of the electroweak sector, where the pole position (furnishing the on-shell mass and width of the particle) is gauge independent and can be obtained from the gauge-boson propagator computed in any gauge. We hasten to emphasize that the reader should not conclude from the last statement that a genuinely gauge-independent mass cannot be defined. Such a mass can be (and has been) defined (see, e.g., [7]): it is the mass that emerges from the gluon self-energy studied in the Feynman gauge of the BFM, which is known to reproduce the gauge-independent answer obtained within the PT (unfortunately, no lattice studies exist for this particular gauge). However, there is no known algorithm that would connect this gauge-invariant mass with, e.g., the Landau gauge mass.

The issues discussed above are further compounded by an additional subtlety, particular to the specific approximation adopted in the present article. Specifically, the value of the gluon mass has not been determined dynamically, but rather it has been fitted to produce the best possible matching with the available lattice results. A more complete treatment of the SDE system should furnish a unique value for that mass, which would then constitute the definite SDE prediction for the gluon mass in the Landau gauge (but still would not be the gauge-independent gluon mass in the PT sense explained above). How this can be done has been presented in [58], for the case of $d = 4$. The general idea is that by employing a more sophisticated Ansatz for the three-gluon vertex one accomplishes the separation of the gluon SDE into two parts, one that determines the momentum dependence of the mass, and one that determines the dimensionless part of the gluon self-energy. In the $d = 4$ case the latter quantity is identified with the QCD effective charge; in the $d = 3$ case (trivial renormalization group) the corresponding quantity would be the $\hat{\Pi}_m^{(3)}(q)$ of Eq. (3.14). In $d = 4$ the resulting system of coupled equations was shown to furnish a unique solution. A similar analysis for $d = 3$ is expected to completely pin down the value of the gluon mass (in a given gauge). Such a study is clearly of interest, because it would eliminate one fitting parameter (the mass), thus providing a far more stringent test of the entire formalism and the basic underlying idea of gluon mass generation.

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