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Real and virtual photons in an external constant electromagnetic field of most general form

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The photon behavior in an arbitrary superposition of constant magnetic and electric fields is considered on most general grounds based on the first principles like Lorentz, gauge, charge, and parity invariance. We make model-independent and approximation-independent, but still rather informative, statements about the behavior that the requirement of causal propagation prescribes to massive and massless branches of dispersion curves, and describe the way the eigenmodes are polarized. We find, as a consequence of Hermiticity in the transparency domain, that adding a smaller electric field to a strong magnetic field in parallel to the latter causes enhancement of birefringence. We find the magnetic field produced by a point electric charge far from it—a manifestation of magnetoelectric phenomenon. We establish degeneracies of the polarization tensor that—under special kinematical conditions—occur due to space-time symmetries of the vacuum left after the external field is imposed.

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I. INTRODUCTION

In this paper we concentrate on behavior of real and virtual electromagnetic excitations of the vacuum filled with constant and homogeneous electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields that are superposed in such a way that both field invariants $\mathfrak{F} = (\mathbf{B^2} - \mathbf{E^2})/2$ and $\mathfrak{G} = (\mathbf{B} \cdot \mathbf{E})$ are different from zero. This makes the most general case of a constant in space and in time electromagnetic field, such that neither electric nor magnetic component can be eliminated from it by a Lorentz transformation, but a special reference frame always exists, where they are parallel or antiparallel.

To study this problem in the (infrared) asymptotic region of vanishing excitation momentum components $k_\mu \to 0$, $\mu = 1, 2, 3, 0$, it suffices to know the effective action as a functional on the class of constant homogenous background field. In this approximation all dispersion curves of the excitations are straight lines passing through the origin in the plane of the natural kinematical variables (we refer to the special frame) $k_0^2 - k_3^2$, $\mathbf{k}_\perp^\mathbf{2}$, where k_0 is the energy of the excitation, and k_3 and $\mathbf{k}_\perp$ are its momentum projections onto the common direction (chosen as axis **3**) of $\mathbf{E}$ and $\mathbf{B}$ and onto the transverse plane, respectively. All massive excitations, i.e. the ones with nonzero rest energy $k_0|_{\mathbf{k}=0} \neq 0$, are lost in this limit, only photons survive. Within this framework the problem of photon propagation in a constant background, where $\mathfrak{F}$ and $\mathfrak{G}$ are both nonzero, was first studied by Plebański [1]. More recently, Novello *et al.* [2] observed an interesting possibility that the constant background field may be geometrized to be represented by an equivalent metric. The present authors, too, paid attention to the infrared limit by proving recently [3] the convexity property of the effective Lagrangian as a function of both variables $\mathfrak{F}$ and $\mathfrak{G}$ in the point $\mathfrak{G} = 0$, based on the requirements of unitarity and causal propagation.

To adequately consider excitations with any momentum, one is obliged to appeal to the second-rank polarization tensor as a function of arbitrary 4-momentum k_μ, the latter being not restricted to any mass shell, i.e. with arbitrary virtuality $k^2 = \mathbf{k}^2 - k_0^2 \neq 0$. The needed polarization operator was first studied by Batalin and Shabad [4] (see also the book [5]), who found the general covariant structure and eigenvector expansion (the diagonal form) of the polarization operator and photon Green function in the constant field with both invariants $\mathfrak{F}$ and $\mathfrak{G}$, taken nonzero simultaneously, that follows exclusively from the Lorentz, gauge, and charge invariance and parity conservation of quantum electrodynamics (QED). They also calculated the polarization operator as an electron-positron loop in the external field of arbitrary strength. Such one-loop calculations were repeated by Bayer *et al.* and Urrutia [6]. The latter author also studied in more detail the useful further approximation of small external field and zero virtuality, and observed some features, which, as a matter of fact, are independent of this approximation, as well as of the one-loop approximation itself. The one-loop polarization operator was revisited in [7] and, under simplifying kinematical conditions, was separately calculated in [8].

The ardor of investigators towards the study of light propagation in the field that contains electric component in any Lorentz frame was damped by the fact that within the loop expansion such a field is, according to Schwinger [9], unstable with regard to spontaneous electron-positron pair production. A special theory for handling such fields was developed in [10]. After exploited in one-loop calculations of the polarization operator in [11] this theory indicated, with the help of the general analysis of Ref. [12], that C invariance (the Furry theorem) is violated, while PT invariance (the Onsager theorem) is preserved, hence there is no CPT. Once the Schwinger effect is unknown beyond the loop expansion, it is ignored in the analysis of the present paper, based exclusively on the general principles, to which this effect contradicts.

During the years that followed most efforts were devoted to the important special case of the one-invariant field with $\mathfrak{G} = 0$ and $\mathfrak{F} \geq 0$ that corresponds to a purely magnetic field in a special Lorentz frame. We shall refer to it as "magneticlike." The general analysis of Ref. [4] is also valid in this case, whereas the corresponding one-loop polarization operator is contained in the formulas of that work as a simple limit $\mathfrak{G} = 0$, analyzed specifically a bit later in [13]. This limiting polarization operator was recalculated separately by Tsai [14]. It must be pointed out that the one-loop polarization operator in a magneticlike field for vanishing virtuality, $k^2 = 0$, had been known earlier, after the important papers by Adler *et al.* [15]. The simplification $k^2 = 0$ is sufficient for considering small dispersion and has been permanently playing a significant role in astrophysical applications. It does not serve, however, the case when large deviations from the vacuum dispersion law take place, as is the case when cyclotron resonances of the vacuum polarization at the thresholds of creations of free [13] or mutually bound [16–18] electron-positron pairs are exploited to produce the photon capture [19] by a strong magnetic field of pulsars. Besides, the assumption that $k^2 = 0$ completely excludes massive states and, moreover, the whole of one of the three polarization modes that cannot carry massless excitations.

The reason why the magnetic field attracted so much attention was, of course, the discovery of extremely strong magnetic fields (up to $\sim 10^{14}$–10^{15} G) in the vicinity of many compact astronomical objects (soft gamma-ray repeaters, anomalous x-ray pulsars, and some radio pulsars) identified with rotating neutron stars [20]. Still stronger magnetic fields ($B \sim 10^{16}$–10^{17} G) were predicted to exist at the surface of cosmological gamma-ray bursters if they are rotation-powered neutron stars similar to radio pulsars [21]. Correspondingly, propagation of photons in a strong magnetic field has been extensively studied aimed at applications to the theory of electromagnetic radiation of strongly magnetized neutron stars (for a review, see [22]). In parallel, some more academic features of nonlinear electrodynamics in a magnetic field were clarified, such as the linear growth of dielectric constant with the magnetic field [23], dimensional reduction of the Coulomb field of a point source [24], and the upper bound to the magnetic field due to positronium collapse [25]. Also the notion of the anomalous magnetic moment of the photon was introduced [26], especially interesting in the large-field limit owing to the above linear growth [27].

In the meanwhile little attention has been paid to the admixture of the electric field. One of the reasons was that, although the electric field is generated along the magnetic field lines in the magnetospheres of rotating, strongly magnetized neutron stars [28], the component $E_\parallel$ in the vicinity of all compact astronomical objects mentioned above is sufficiently small ($E_\parallel/B \ll 1$) [28] as compared to the magnetic field. Hence, at first sight, $E_\parallel$ could result in only minor corrections [29]. However, it may not be the case at least for some processes that are forbidden without the electric field ($E_\parallel = 0$) and might be allowed at $E_\parallel \neq 0$. Splitting of photons in a strong magnetic field ($\gamma + B \rightarrow \gamma' + \gamma'' + B$) is one of the candidates to be such a process. The point is that in a magneticlike field splitting of one photon mode is allowed, while splitting of the other is strictly forbidden [15,30]. As far as we are aware, the polarization selection rules for photon splitting have never been satisfactorily considered in the case of $E_\parallel \neq 0$. Also, it is not clear beforehand how the processes that depend on the resonance will behave under inclusion of even a small electric field.

On the other hand, extremely strong electric fields with the strength as high as $\sim(10$–$10^2)E_0$ are predicted to exist at the surface of bare strange stars that are entirely made of deconfined quarks, where $E_0 = m_e^2/e \simeq 1.3 \times 10^{16}$ V/cm is the characteristic electric field value [31]. These electric fields are directed perpendicular to the stellar surface and prevent ultrarelativistic electrons of quark matter from their escape to infinity. The surface magnetic fields are expected to be more or less the same for neutron and strange stars (from $\lesssim 10^9$ G to $\sim 10^{15}$ G or even higher), and therefore, at the surface of strange stars the ratio $E_\parallel/B$ may be both $\ll 1$ and $\gg 1$, i.e., it may be practically arbitrary.

In this paper we are elaborating consequences of the general structure of the polarization operator, established in [4], for the propagation and polarization of photons and massive vector excitations of the vacuum in the presence of a constant background field with both field invariants different from zero. Also some observations are made depending on the one-loop approximation. These are exiled to Appendix B.

In Sec. II we present a kinematical orthogonal basis and the decomposition of the eigenvectors of the polarization operator over it. We also present a simpler form of this decomposition valid in the limit, where the mixing between the basic vectors is small, as for small admixture of an electric field to a large magnetic one, and under special kinematical conditions. We establish, as a consequence of Hermiticity, that this admixture leads to increasing the birefringence and strengthening Adler's kinematical bans [15] for photon splitting in a magnetic field. We discuss how Adler's CP-selection rules are modified in the case of the general field under consideration. We find in the infrared limit four invariant functions, on which the three polarization operator eigenvalues and eigenvectors depend, in terms of the field derivatives of the effective action to see explicitly that the eigenvalues disappear in the zero point of the 4-momentum as a consequence of gauge invariance.

In Sec. III this property is used to ground the statement that there always exist massless excitations to be identified with photons present in two polarization modes, whereas any number of massive branches may be present in all the three modes.

By restricting the group velocity of an excitation to be below the speed of light, we establish that in the special frame each dispersion curve is limited from above in the plane ($\sqrt{k_0^2 - k_3^2}$, $|\mathbf{k}_\perp|$) by a straight line that crosses the dispersion curve at $|\mathbf{k}_\perp| = 0$ and is inclined to the coordinate axes at the angle of 45° (see Fig. 1). Massless branches are restricted to the exterior of the light cone $\sqrt{k_0^2 - k_3^2} \leq |\mathbf{k}_\perp|$, whereas massive ones may cross it.

In Sec. IV we describe polarizations of electric and magnetic fields of the eigenmodes and find the large-distance behavior of the magnetic field produced by a point electric charge placed in the background electromagnetic field.

In Sec. V we establish degeneracies of the polarization operator that result from the symmetries of the external field.

Appendix A is technical. In Appendix B we present one-loop approximation for one of the invariant functions responsible for mixing eigenmodes in the limit of small electric admixture to the external magnetic field, $\mathfrak{G} \to 0$. It has cyclotron resonances starting with the second threshold of electron-positron pair creation by a photon in a magnetic field. Therefore, the mixing does not affect the photon capture effect at the first threshold important for radiation formation in the pulsar magnetosphere. Appendix C serves the use of the idea of group velocity in Sec. III. We

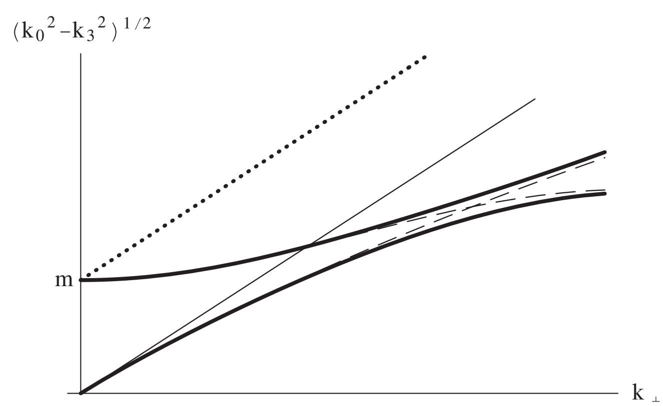

FIG. 1. Disposition of dispersion curves. The lower bold solid line is the massless (*photon*) dispersion curve restricted from above by the light cone $k^2 = 0$, presented by the thin solid line. The upper solid bold line, representing a massive branch, cannot pass higher than the straight dotted line, originating from its crossing with the vertical axis according Eq. (27). The dashed lines show a quasi-interception.

illustrate how its definition should be extended to the abnormal dispersion domain so that it might be kept below the velocity of light in that domain, too.

II. POLARIZATION OPERATOR, ITS EIGENVECTORS AND EIGENVALUES

Before starting, technical conventions are in order. There are two field invariants $\mathfrak{F} = \frac{1}{4} F_{\rho\sigma} F_{\rho\sigma}$ and $\mathfrak{G} = \frac{1}{4} F_{\rho\sigma} \tilde{F}_{\rho\sigma}$ of the background fields and two Lorentz-scalar combinations kF^2k and $k\tilde{F}^2k$ of the background field strength tensor $F_{\mu\nu}$ and momentum k_μ of the elementary excitation, subject to the relation

$$\frac{k\tilde{F}^2k}{2\mathfrak{F}} - k^2 = \frac{kF^2k}{2\mathfrak{F}}. \tag{1}$$

The dual field tensor is defined as $\tilde{F}_{\rho\sigma} = \frac{i}{2} \epsilon_{\rho\sigma\lambda\varkappa} F_{\lambda\varkappa}$, where the completely antisymmetric unit tensor is fixed in such a way that $\epsilon_{1234} = 1$. We use the notations $(\tilde{F}k)_\mu \equiv \tilde{F}_{\mu\tau} k_\tau$, $(Fk)_\mu \equiv F_{\mu\tau} k_\tau$, $F^2_{\mu\nu} \equiv F_{\mu\tau} F_{\tau\nu}$, $(F^2k)_\mu \equiv F^2_{\mu\tau} k_\tau$, $kF^2k \equiv k_\mu F^2_{\mu\tau} k_\tau$, $k^2 \equiv \mathbf{k}^2 + k_4^2 = \mathbf{k}^2 - k_0^2$, and are working in Euclidian metrics with the results analytically continued to Minkowsky space; hence, we do not distinguish covariant and contravariant indices. The scalar variables

$$\mathcal{B} = \sqrt{\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \mathfrak{G}^2}}, \qquad \mathcal{E} = \sqrt{-\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \mathfrak{G}^2}} \tag{2}$$

make the meaning, respectively, of the magnetic, $\mathcal{B} = B = |\mathbf{B}|$, and electric fields, $\mathcal{E} = E = |\mathbf{E}|$, in the (special) Lorentz frame, where $\mathbf{B}$ and $\mathbf{E}$ are directed along the same axis chosen as axis **3** in what follows. The designation "$\Leftrightarrow$" will establish correspondence between quantities relating to the general Lorentz frame and the values these take in the special frame. Referring to the fact that in the special frame the momentum-containing invariants become

$$k\tilde{F}^2k = B^2(k_3^2 - k_0^2) - E^2 k_\perp^2, \qquad kF^2k = -B^2 k_\perp^2 + E^2(k_3^2 - k_0^2), \tag{3}$$

we shall use the equivalence relations

$$\frac{k^2\mathcal{B}^2 + kF^2k}{\mathcal{B}^2 + \mathcal{E}^2} \Leftrightarrow k_3^2 - k_0^2, \qquad \frac{k^2\mathcal{E}^2 - kF^2k}{\mathcal{B}^2 + \mathcal{E}^2} \Leftrightarrow k_\perp^2 \tag{4}$$

throughout the paper.

Polarization operator $\Pi_{\mu\nu}(x, y)$ is responsible for small perturbations above the constant-field background. It follows from the translation- Lorentz, gauge, PT, and charge invariance [4,5] that its Fourier transform can be presented in a diagonal form

$$\Pi_{\mu\tau}(k,p) = \delta(k-p)\Pi_{\mu\tau}(k),$$
$$\Pi_{\mu\tau}(k) = \sum_{a=1}^{3} \varkappa_a \frac{\flat_\mu^{(a)} \flat_\tau^{(a)}}{(\flat^{(a)})^2}, \tag{5}$$

where $\flat_\tau^{(a)}$ are its eigenvectors

$$\Pi_{\mu\tau}\flat_\tau^{(a)} = \varkappa_a \flat_\mu^{(a)}, \qquad a = 1, 2, 3, 4, \tag{6}$$

while the eigenvalues $\varkappa_a$ are scalar functions of $\mathfrak{F}$, $\mathfrak{G}$, kF^2k, and $k\tilde{F}^2k$.

The fourth eigenvector is trivial, $\flat_\mu^{(4)} = k_\mu$, so the fourth eigenvalue vanishes, $\varkappa_4 = 0$, as a consequence of the 4-transverseness of the polarization operator, $\Pi_{\mu\tau}k_\tau = 0$. All eigenvectors are mutually orthogonal, $\flat_\mu^{(a)}\flat_\mu^{(b)} \sim \delta_{ab}$; this means that the first three are 4-transversal, $\flat_\mu^{(a)}k_\mu = 0$.

In the special case, where the second field invariant disappears, $\mathfrak{G} = 0$, the three meaningful eigenvectors $\flat_\mu^{(1,2,3)}$ are known [4,5,13] in the universal final form:

$$\flat_\mu^{(1)}|_{\mathfrak{G}=0} = (F^2k)_\mu k^2 - k_\mu(kF^2k),$$
$$\flat_\mu^{(2)}|_{\mathfrak{G}=0} = (\tilde{F}k)_\mu, \qquad \flat_\mu^{(3)}|_{\mathfrak{G}=0} = (Fk)_\mu. \tag{7}$$

This case implies that in the special frame only a magnetic, when $\mathfrak{F} > 0$, or only electric, when $\mathfrak{F} < 0$, field exists. In the limit $k^2 = 0$ modes 2, 3 correspond to Adler's [15] $\perp$ and $\parallel$ modes, respectively, whereas mode 1 becomes pure gauge. Vectors (7) may be used as a convenient orthogonal basis also when no external field is present. But when $\mathfrak{G} \neq 0$, they no longer diagonalize the polarization operator already because the vectors $(\tilde{F}k)_\mu$ and $(Fk)_\mu$ stop being mutually orthogonal, since their scalar product $-k\tilde{F}Fk = \mathfrak{G}k^2$ is now nonzero.

When $\mathfrak{G} \neq 0$, the first eigenvector is expressed in terms of the fields by the same formula as in (7):

$$\flat_\mu^{(1)} = (F^2k)_\mu k^2 - k_\mu(kF^2k),$$
$$(\flat^{(1)}\tilde{F}k) = (\flat^{(1)}Fk) = 0,$$
$$(\flat^{(1)})^2 = k^2(k^2\mathcal{E}^2 - kF^2k)(k^2\mathcal{B}^2 + kF^2k)$$
$$\Leftrightarrow k^2(B^2+E^2)^2k_\perp^2(k_3^2 - k_0^2) \tag{8}$$

and the first eigenvalue is given by the formula

$$\varkappa_1 = \frac{k^2(\mathcal{B}^2+\mathcal{E}^2)}{k^2\mathcal{B}^2 + kF^2k}\Lambda_1 \Leftrightarrow \frac{k^2}{k_3^2 - k_0^2}\Lambda_1, \tag{9}$$

where the scalar function of the fields and momentum Λ_1 here, as well as other Λ's below, is a *linear* superposition of the polarization tensor components $\Pi_{\mu\nu}$. The other two eigenvectors are the linear combinations

$$\flat_\mu^{(2,3)} = -2\Lambda_3 c_\mu^- + [\Lambda_2 - \Lambda_4 \pm \sqrt{(\Lambda_2-\Lambda_4)^2 + 4\Lambda_3^2}]c_\mu^+ \tag{10}$$

(where the square root is understood algebraically: $\sqrt{Z^2} = Z$, and not $|Z|$) of two orthonormalized vectors:

$$c_\mu^- = \frac{\mathcal{B}(Fk)_\mu + \mathcal{E}(\tilde{F}k)_\mu}{(\mathcal{B}^2+\mathcal{E}^2)^{1/2}(k^2\mathcal{E}^2 - kF^2k)^{1/2}}$$
$$\Leftrightarrow \frac{B(Fk)_\mu + E(\tilde{F}k)_\mu}{(B^2+E^2)|\mathbf{k}_\perp|},$$
$$c_\mu^+ = \mathrm{i}\frac{\mathcal{E}(Fk)_\mu - \mathcal{B}(\tilde{F}k)_\mu}{(\mathcal{B}^2+\mathcal{E}^2)^{1/2}(k^2\mathcal{B}^2 + kF^2k)^{1/2}} \tag{11}$$
$$\Leftrightarrow \frac{E(Fk)_\mu - B(\tilde{F}k)_\mu}{(B^2+E^2)(k_0^2 - k_3^2)^{1/2}},$$
$$(c^+c^-) = (c^\pm\flat^{(1)}) = (c^\pm k) = 0, \qquad (c^\pm)^2 = 1,$$

thereby of the former basic vectors $(\tilde{F}k)_\mu$ and $(Fk)_\mu$, too. The corresponding two eigenvalues are

$$\varkappa_{2,3} = \tfrac{1}{2}[-(\Lambda_2+\Lambda_4) \pm \sqrt{(\Lambda_2-\Lambda_4)^2 + 4\Lambda_3^2}]. \tag{12}$$

The scalar coefficients in the linear combination (10) cannot be expressed in a universal way in terms of the field and momentum, but are irrational functions of the polarization tensor components. The reason is that the polarization operator is a linear combination of four independent matrices with four scalar coefficients, whereas there may be only three eigenvalues in accordance with three polarization degrees of freedom of a vector field. (When $\mathfrak{G} = 0$, the number of independent matrices reduces to three.) The orthogonality $(b^{(2)}b^{(3)}) = 0$ is explicit in (10). The Lorentz-invariant coefficients $\Lambda_{1,2,3,4}$ are functions of the background fields and momenta. Expressions for them as simple linear superpositions of the components $\Pi_{\mu\nu}$,

$$\Lambda_1 = \frac{(kF^2)_\mu\Pi_{\mu\nu}(F^2k)_\nu}{(\mathcal{B}^2+\mathcal{E}^2)(k^2\mathcal{E}^2 - kF^2k)}, \qquad \Lambda_2 = -c_\mu^-\Pi_{\mu\nu}c_\nu^-,$$
$$\Lambda_3 = -c_\mu^-\Pi_{\mu\nu}c_\nu^+, \qquad \Lambda_4 = -c_\mu^+\Pi_{\mu\nu}c_\nu^+ \tag{13}$$

are obtained in Appendix A from a less transparent representation to be found in [4,5]; their calculations in one-loop approximation of QED are given in [4,5].

The transparency domain of momenta is such a region where absorption is absent. The electron-positron pair production by a photon is an example of absorption. The region, where it is kinematically allowed, is not the transparency domain. The absence of absorption of small perturbation of the background field is reflected in the property of Hermiticity [12] of the matrix $\Pi_{\mu\nu}$. It is symmetric when the charge conjugation invariance holds [5,12] (no charge-asymmetric plasma background, no spontaneous pair creation). Hence, in the transparency region all the components of $\Pi_{\mu\nu}$ are real in the case under consideration, once the charge conjugation invariance is assumed. Then, all Λ's defined by (13) are also real there, except the region $k_3^2 - k_0^2 > 0$ (or, in invariant terms, $k^2\mathcal{B}^2 + kF^2k > 0$) wherein Λ_3 becomes imaginary due to

(11) and (13). (We shall see later that dispersion curves cannot get into this region without violating the stability.) In this exceptional region the quantity under the square root in (10) and (12) stops being manifestly positive. Nevertheless, it should remain non-negative, since eigenvalues of a Hermitian matrix should be real.

The dispersion equations that define the mass shells of the three eigenmodes are

$$\varkappa_a(k\tilde{F}^2k, kF^2k, \mathfrak{F}, \mathfrak{G}^2) = k^2, \qquad a = 1, 2, 3. \tag{14}$$

We have explicitly indicated here that the eigenvalues should be even functions of the pseudoscalar $\mathfrak{G}$.

When, due to a certain reason, Λ_3 is small as compared to $|\Lambda_2 - \Lambda_4|$, the small mixing of eigenmodes is obtained by expanding (10) in powers of $\Lambda_3/|\Lambda_2 - \Lambda_4|$. In this way we get, with the linear accuracy in Λ_3, after normalizing out the common factors 2 and $2\Lambda_3/(\Lambda_2 - \Lambda_4)$,

$$\flat_\mu^{(2)} = -\Lambda_3 c_\mu^- + (\Lambda_2 - \Lambda_4)c_\mu^+,$$
$$\flat_\mu^{(3)} = (\Lambda_2 - \Lambda_4)c_\mu^- + \Lambda_3 c_\mu^+. \tag{15}$$

Such a situation occurs, first of all, when the electric field is small as compared to the magnetic one, which we shall discuss now, and also for two cases of special kinematical conditions considered in Sec. V.

In the limiting regime of small $\mathfrak{G} \to 0$, one has $\mathcal{E} \approx \mathfrak{G}/\mathcal{B}$. So $\mathcal{E}$ is a pseudoscalar, hence c_μ^- is a vector, and c_μ^+ a pseudovector, the same as $(Fk)_\mu$ and $(\tilde{F}k)_\mu$, respectively, are. Then, from (10) it follows that Λ_3 is a pseudoscalar vanishing linearly: $\Lambda_3 \sim \mathfrak{G} \approx (\mathcal{E}\mathcal{B}) \to 0$. [This fact is also in agreement with the infrared limit (20) below, since $\mathfrak{L}$ depends on $\mathfrak{G}^2$, and with the one-loop result in [4,5]]. The eigenvector $\flat_\mu^{(2)}$ given by Eq. (10) with the upper sign in front of the square root becomes in this limit $c_\mu^+ \sim \tilde{F}k_\mu$, as prescribed by (7). On the contrary, the eigenvector $\flat_\mu^{(3)}$ given by Eq. (10) with the lower sign becomes $c_\mu^- \sim Fk_\mu$, because the coefficient in front of c_μ^+ in (10) decreases as Λ_3^2. The coefficient Λ_3 becomes responsible for mixing eigenmodes, characteristic of an external magnetic field due to the perturbation caused by electric field. (See Appendix B for the linearly vanishing $\mathfrak{G} \to 0$ limit of Λ_3 as calculated within one-loop approximation of quantum electrodynamics.)

The difference between $\varkappa_1$ and $\varkappa_2$ in Eq. (12) is responsible for the *birefringence*, inherent to the problem of light propagation already when $\mathfrak{G} = 0$. Bearing in mind that $\Lambda_{2,4}$ are scalars and may, thus, contain the pseudoscalar $\mathfrak{G}$ only in even powers, while Λ_3 contains only odd powers, from (12) it may be concluded, prior to any dynamical calculations, that the birefringence is enhanced as soon as an extra field, small in magnitude, is added in parallel to the already existing single-invariant field to produce small $\mathfrak{G}$:

$$|\varkappa_2 - \varkappa_3| = |\sqrt{(\Lambda_2 - \Lambda_4)^2 + 4\Lambda_3^2}| \geq |\Lambda_2 - \Lambda_4|. \tag{16}$$

This is true in the domain $k_0^2 > k_3^2$, where Λ_3 is real. We discuss later why dispersion curves should lie just in this domain. Therefore, the dispersion curves of modes 2, 3 tend to repulse from each other [32]. Thereby, Adler's kinematical selection rules [15] that ban some processes of one photon splitting into two in a magnetic field are strengthened if an electric field is added. As for his *CP*-selection rules [33], those now should be applied to the eigenwaves, given as (10), and read as follows: among the three γ states involved into the reaction $\gamma \to \gamma\gamma$ there may be only two or none of mode-2 states, since $\flat_\mu^{(2)}$ is a pseudovector, while $\flat_\mu^{(1,3)}$ are vectors. However, any state, prepared as an eigenstate in the magnetic field alone may decay into two like states under the perturbation caused by the electric field disregarding the initial *CP* bans, since the electric field introduces the pseudoscalar $\mathfrak{G}$.

The infrared limit, $k_\mu \to 0$, of the polarization operator is important. To get it, it is sufficient to have at one's disposal only the effective Lagrangian $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$, from where the dependence on the time and space derivatives of the field $F_{\mu\nu}$ is disregarded [34]. In the limit of vanishing momenta the invariant coefficients $\Lambda_{1,2,3,4}$ are quadratic functions of k_μ expressed in terms of the (momentum-independent) derivatives $\mathfrak{L}_\mathfrak{F} = \partial\mathfrak{L}/\partial\mathfrak{F}$, $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} = \partial^2\mathfrak{L}/\partial\mathfrak{F}^2$, $\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = \partial^2\mathfrak{L}/\partial\mathfrak{G}^2$, and $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}} = \partial^2\mathfrak{L}/\partial\mathfrak{F}\partial\mathfrak{G}$ as follows:

$$\Lambda_1|_{k_\mu\to 0} = (k_3^2 - k_0^2)\mathfrak{L}_\mathfrak{F}, \tag{17}$$

$$\Lambda_2|_{k_\mu\to 0} = -k^2\mathfrak{L}_\mathfrak{F} - \mathbf{k}_\perp^2(B^2\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} + E^2\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} + 2\mathfrak{G}\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}), \tag{18}$$

$$\Lambda_4|_{k_\mu\to 0} = -k^2\mathfrak{L}_\mathfrak{F} + (k_3^2 - k_0^2)(E^2\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} + B^2\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} - 2\mathfrak{G}\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}), \tag{19}$$

$$\Lambda_3|_{k_\mu\to 0} = (\mathbf{k}_\perp^2)^{1/2}(k_0^2 - k_3^2)^{1/2}\{\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}(B^2 + E^2) - (\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} + \mathfrak{L}_{\mathfrak{F}\mathfrak{F}})\mathfrak{G}\}. \tag{20}$$

We have written these formulas referring to the special frame. The equivalence relations (4) allow one to immediately restore their invariant form valid in any frame. Equations (17)–(20) are obtained using the definition of the polarization operator components as the second derivatives with respect to vector-potentials components (see, e.g. [3]). Insofar as one is interested in the quantities $\Lambda_i|_{k_\mu\to 0}$ up to one-loop accuracy, one should either take the Heisenberg-Euler expression for $\mathfrak{L}$ here or pass to the infrared limit in the expressions for Λ_i calculated within one-loop approximation in [4]. The two-loop approximation for $\mathfrak{L}$ is also available [35].

From Eqs. (17)–(20) the vanishing of the eigenvalues in the zero-momentum point

$$\varkappa_a|_{k=0} = 0, \qquad a = 1, 2, 3 \tag{21}$$

follows. This property is, in the end, a consequence of the gauge invariance that requires that the effective Lagrangian should depend only on the field strengths, and not potentials.

III. DISPERSION CURVES

In the special frame dispersion equations (14) can be represented in the form

$$\varkappa_a(k_0^2 - k_3^2, k_\perp^2, B^2, E^2) = k_\perp^2 + k_3^2 - k_0^2, \qquad a = 1, 2, 3 \tag{22}$$

and their solutions that express the energy k_0 of the elementary excitation of a given mode a in terms of its spatial momentum components k_3, $\mathbf{k}_\perp$ have the following general structure, provided, in the end, by the invariance of the external field under rotation around axis **3** and the Lorentz boost along this axis,

$$k_0^2 = k_3^2 + f_a(k_\perp^2), \qquad a = 1, 2, 3, \tag{23}$$

where the dispersion functions $f_a(k_\perp^2)$ certainly depend also on the external fields.

In this point we are going to apply the causality principle in the form of the requirement that the modulus of the group velocity, $\mathbf{v}_{\rm gr} = \partial k_0/\partial \mathbf{k}$, calculated on each mass shell (23), be less than or equal to the speed of light in the free vacuum $c = 1$ to see what consequences follow for the disposition of dispersion curves. Before doing this we have to make some necessary reservations. So long as we are in the transparency domain of momentum space the group velocity is the speed of (the maximum of) a wave packet, which may be used for a signal transmission (see, e.g., [36]). Therefore, it should not exceed unity, since otherwise—in violation of causality—by an appropriate Lorentz transformation the response can be done preceding the disturbance [37]. It was demonstrated in [38] that the presence of an external agent, e.g. the background field, violating the Lorentz invariance, is not an obstacle for constructing the "time-machine," once a superluminal signal is at one's disposal—under the condition that the external agent is itself subjected to Lorentz transformation.

On the contrary, in the domain of absorption and abnormal dispersion, the wave packet is diffused and/or absorbed before it can transmit a signal. It is a common statement that then the group velocity exceeds unity without contradicting the causality, since it no longer can be interpreted as a signal speed. In Appendix C, working within the customary optic context, we introduce an extension of the notion of the group velocity from the transparency to absorption domain following the definition $\mathbf{v}_{\rm gr} = \partial k_0/\partial \mathbf{k} \Rightarrow \partial({\rm Re}\, k_0)/\partial \mathbf{k}$, whereas the standard extension, in fact, is $\mathbf{v}_{\rm gr} = \partial k_0/\partial \mathbf{k} \Rightarrow \partial k_0/\partial({\rm Re}\, \mathbf{k})$, here Re denotes real part. This enables us to keep the (so extended) group velocity below unity within the domain of abnormal dispersion, too.

Below in this section we shall establish the disposition of dispersion curves in the transparency domain as it follows from the restrictedness of the group velocity postulate. Based on Appendix C, we admit that the configuration of dispersion curves established should survive the abnormal dispersion, provided the dispersion curves are understood as giving the dependence of the *real part*, ${\rm Re}(k_0^2 - k_3^2)$, as a function of the real variable $k_\perp^2$.

The restriction imposed on the group velocity is

$$|\mathbf{v}_{\rm gr}|^2 = \left(\frac{\partial k_0}{\partial k_3}\right)^2 + \left|\frac{\partial k_0}{\partial \mathbf{k}_\perp}\right|^2 = \frac{k_3^2}{k_0^2} + \left|\frac{\mathbf{k}_\perp}{k_0} f_a'\right|^2 = \frac{k_3^2 + (f_a')^2 k_\perp^2}{k_3^2 + f_a(k_\perp^2)} \leq 1, \tag{24}$$

where $f_a' = df_a(k_\perp^2)/dk_\perp^2$. This imposes the obligatory condition on the form and location of the dispersion curves (23) in the transparency domain, i.e. on the function $f_a(k_\perp^2)$ [remember that $k_3^2 + f_a(k_\perp^2) \geq 0$ due to (23)]:

$$k_\perp^2 \left(\frac{df_a(k_\perp^2)}{dk_\perp^2}\right)^2 \leq f_a(k_\perp^2). \tag{25}$$

This inequality requires first of all that $f_a(k_\perp^2) \geq 0$, hence no branch of any dispersion curve may get into the region $k_0^2 - k_3^2 < 0$. If it might, the photon energy k_0 would be imaginary within the momentum interval $0 < k_3^2 < -f_a(k_\perp^2)$, corresponding to the vacuum excitation exponentially growing in time. This sort of ghost would signal the instability of the vacuum with a background field. Inequality (25) further requires that

$$\frac{df_a^{1/2}(k_\perp^2)}{dk_\perp} \leq 1, \qquad \text{or} \quad f_a^{1/2}(k_\perp^2) \leq f_a^{1/2}(0) + k_\perp, \tag{26}$$

where $\mathfrak{m} = f^{1/2}(0)$ is the rest energy (mass) of the elementary excitation: $\mathfrak{m}^2 = (k_0^2 - k_3^2)|_{k_\perp=0} = k_0^2|_{k_3=\mathbf{k}_\perp=0}$. The inequality

$$(k_0^2 - k_3^2)^{1/2} \leq \mathfrak{m} + k_\perp \tag{27}$$

that follows from (26) and (23) is an obligatory restriction imposed by causality principle on the dislocation of dispersion curves in the presence of constant magnetic and electric fields, at least as long as these are solutions to dispersion equations (22) with *real* energy k_0 (for real values of momentum components $k_{1,2,3}$). In the empty space the restriction that appears in the similar way is $k_0 \leq k_0|_{\mathbf{k}=0} + |\mathbf{k}|$. It is certainly obeyed by the free massive particle: $k_0 = (\mathbf{k}^2 + \mathfrak{m}^2)^{1/2} \leq \mathfrak{m} + |\mathbf{k}|$, where $\mathfrak{m} = (k_0)|_{\mathbf{k}=0}$.

The gauge invariance property (21) implies via Eq. (14) that for each mode there always exists a dispersion curve with $\mathfrak{m}^2 = f_a(0) = 0$, which passes through the origin in the $(k_0^2 - k_\parallel^2, k_\perp^2)$ plane. It is such a branch that is to be called *a photon*, since it is massless in the sense that the energy k_0 turns to zero for the excitation at rest, $k_3 = k_\perp = 0$ (although, generally, $k^2 \neq 0$ where $\mathbf{k} \neq 0$). Other branches for each polarization mode a may also appear provided that a dynamical model includes a massive excitation of the vacuum with quantum numbers of a photon, for instance the positronium atom [5,16–18] or a massive (pseudo)scalar particle (axion) in a gauge-invariant interaction with the electromagnetic field [39]. Note that, while the number of polarization modes of a vector particle is three—in correspondence with the dimension of the space and, hence, with the number of degrees of freedom—the dispersion curve for each of the three modes may have any number of branches, e.g. an infinite number of excited positronium branches. The energy on a dispersion curve should be real, since the dispersion equation (14) supplies poles to the photon propagator [4,5],

$$D_{\mu\nu}(k) = \sum_{a=1}^{3} \frac{\flat_\mu^{(a)} \flat_\nu^{(a)}}{(\flat^{(a)})^2} \frac{1}{k^2 - \varkappa_a(k)} \tag{28}$$

(defined up to arbitrary longitudinal part $\sim k_\mu k_\nu$), and these should not get into a complex plane. If the state is unstable and should therefore decay, its energy must have an imaginary part, indeed, but in this case the pole is located on a nonphysical sheet of the complex plane, whose presence must be provided by branching points in the polarization operator to be introduced within an approximation where the state is expected to be unstable. An example of such a situation is given by the cyclotron resonance [13] approximation of the polarization operator in a magnetic field. The corresponding dispersion equations are cubic with respect to energy squared. Out of its three solutions, one corresponds to a stable state and, therefore, is real, whereas the other two mutually complex conjugated branches responsible for the photon decay/capture to electron-positron pairs belong to nonphysical sheets of the complex-energy plane. Neither of these solutions can be disregarded.

On the other hand, the number of massless modes is, as a matter of fact, not three, but only two, as it should be for a photon. The point is that the massless branch of the dispersion curve for mode 1 does not correspond to any real elementary excitation, except for two special cases. It follows from (9) and (17) that

$$\varkappa_1|_{k_\mu=0} = k^2 \frac{\partial \mathfrak{L}(\mathfrak{F}, \mathfrak{G})}{\partial \mathfrak{F}}. \tag{29}$$

Hence, the light cone $k^2 = 0$ is a guaranteed solution to the dispersion equation (14) for mode 1 in the vicinity of the origin $k_\mu = 0$. Can there exist massless excitations in mode 1 other than $k^2 = 0$? The answer is "no," because from (29) it follows that $k^2 = 0$ is the only possibility for a dispersion curve of mode 1, as it approaches the origin $k_\mu = 0$. Now, from Eq. (8) it is seen that both electric and magnetic fields in mode 1 disappear at $k^2 = 0$, since on this mass shell the elementary excitation is pure gauge, unless either $k^2\mathcal{E}^2 - kF^2k \sim k_\perp^2 = 0$ or $k^2\mathcal{B}^2 + kF^2k \sim k_3^2 - k_0^2 = 0$, in which cases the common factor k^2 can be normalized out from $\flat_\mu^{(1)}$ (8). These exceptional cases propose kinematical conditions for degeneracy of the polarization tensor to be discussed in Sec. V. For the first of them, the one of parallel propagation, $k_\perp^2 = 0$, the mode-1 photon is actual, while, on the contrary, the mode-2 photon no longer exists, the mode-2 excitation becoming massive, as argued below in Sec. V. The overall number of massless degrees of freedom, therefore, is again two. Note that although $\mathbf{k}_\perp = 0$ may seem to be an isolated point, as a matter of fact this is not the case: every nonparallel propagation $\mathbf{k}_\perp \neq 0$ reduces to perpendicular propagation $k_3 = 0$ by a Lorentz boost along axis **3**, which does not lead us out of the special frame. In the second exceptional case, $k_3^2 - k_0^2 = 0$, again the mode-1 photon is actual, but the mode-3 photon does not exist according to Sec. V. So the number of massless degrees of freedom is two in this case, too.

We concluded above in this section that the causality requires that in the plane $(\sqrt{k_0^2 - k_3^2}, k_\perp)$, when $k_{0,1,2,3}$ are all real, the photon dispersion curves ($\mathfrak{m} = 0$) are located outside or coincide with the light cone: $k^2 \geq 0$. (Remember that the light cone $k^2 = \mathbf{k}^2 - k_0^2 = 0$ is the mass shell of a photon in the vacuum without an external field.) However, unlike the case indicated below Eq. (25), a violation of this ban would not lead to a complex-energy ghost or directly signalize the vacuum instability, but would mean the presence of a superluminal wave, known as tachyon. On the other hand, massive branches of the dispersion curves as restricted by the condition (27) with $\mathfrak{m} = f^{1/2}(0) \neq 0$ may well cross the light cone and pass to its exterior. They may even quasi-intercept with the massless (photon) branches or with branches possessing different $\mathfrak{m}$. The quasi-interceptions—i.e. the would-be interception of dispersion curves of two states taken as independent within a certain approximation—would result in the mutual repulsion of the dispersion curves leading to formation of mixed states, polaritons, an example of which is given by a photopositronium—a mixed state between a photon and the electron-positron bound state created by it in a strong magnetic field [5,16–18]. This situation is illustrated by Fig. 1. As noted above there are grounds to believe that the same pattern of dispersion curves holds also when at least one of the dispersion curves corresponds to an *unstable* state subject to quasi-interception with another curve.

The refraction index squared n_a^2 is defined for photons of mode a on the mass shell (23) as

$$n_a^2 \equiv \frac{|\mathbf{k}|^2}{k_0^2} = 1 + \frac{k_\perp^2 - f_a(k_\perp^2)}{k_0^2}. \tag{30}$$

It follows from (26) with $\mathfrak{m} = f^{1/2}(0) = 0$ that the refraction index, outside an abnormal dispersion band, is greater than unity—the statement common in standard optics of media [this is certainly not true for (massive) plasmon branches]. Consequently, the modulus of the phase velocity in each mode $\mathbf{v}_a^{\rm ph} = (k_0/|\mathbf{k}|^2)\mathbf{k}$ equal to $1/n_a$ is, *for the photon proper*, also smaller than the velocity of light in the vacuum $c = 1$. This is not the case for a massive—e.g. positronium—branch of the photon dispersion curve, where $|\mathbf{v}_a^{\rm ph}| > 1$ without any impact for causality.

Now that we established that for photons one has $\mathbf{k}^2 \geq k_0^2$, or $k^2 \geq 0$, we see from the dispersion equation (14) that the eigenvalues $\varkappa_a$ are non-negative in the momentum region, where the photon dispersion curves lie, i.e. the polarization operator is a non-negatively defined matrix there.

We have yet to comment on the possibility of violations of the causal propagation requirement not associated with abnormal dispersion that may result from calculations depending on certain approximations or models. Such violations are known for photon propagation in external metrics [40], also under the conditions where the Casimir effect should take place [41], and in the noncommutative electrodynamics with the external field [42]. In view of the discussion in [38] these violations should be, at least in some cases, thought of as serious trouble for the relativity theory.

Within our present context we have to note that, too, in a constant background magnetic field the group velocity (in mode 2) becomes, for magnetic fields of the Planck order of magnitude, $B > B_2^{\rm cr} = (m^2/e)\exp\{1.79 + 3\pi/\alpha\}$, where m and e are the electron mass and charge, and $\alpha = 1/137$ is the fine structure constant, greater than unity [3,43], provided we rely on one-loop calculations of the polarization operator or of the effective Lagrangian. To avoid the discrepancy with the basic principle of causality, we must be thinking of a probable mechanism that would make a production of such fields impossible—in analogy with the customary attitude towards "the perfectly rigid body" that is ruled out, already because it would produce ground for a faster-than-light sound wave. In our case we are seemingly able to indicate such a mechanism. The point is that for the magnetic field, smaller than the value $B_2^{\rm cr}$, above which the dispersion curve of mode 2 enters the interior of the light cone, forbidden for it by the causal propagation requirement, but larger than $B = B_3^{\rm cr} = (m^2/e)\exp\{0.79 + 3\pi/\alpha\} < B_2$, the magnetic field becomes unstable, since the dispersion curve is situated in the region of negative $k_0^2 - k_3^2$ in the field interval $B_3^{\rm cr} < B < B_2^{\rm cr}$.

IV. ELECTROMAGNETIC FIELDS OF SMALL PERTURBATIONS OF THE BACKGROUND FIELD

A. Polarization of eigenmodes

In the special frame some peculiarities can be revealed about orientations of electric and magnetic fields in the virtual or real eigenmodes, formed out of eigenvectors $\flat_l^{(a)}$ as of 4-potentials according to the standard rules $e_m^{(a)} = \mathrm{i}(k_4\flat_m^{(a)} - k_m\flat_4^{(a)})$, $h_m^{(a)} = \mathrm{i}\varepsilon_{mnl}\flat_n^{(a)}k_l$. To this end let us write down the eigenvector $\flat_\mu^{(1)}$ (8) (after normalizing it) and the basic vectors $c_\mu^{\pm}$ (11), in the special frame:

$$\frac{\flat_\nu^{(1)}}{\sqrt{|(\flat^{(1)})^2|}} = \frac{\sqrt{k_0^2 - k_3^2}}{\sqrt{|k^2|}}\begin{pmatrix} \frac{\mathbf{k}_\perp}{|\mathbf{k}_\perp|} \\ \frac{k_3|\mathbf{k}_\perp|}{k_0^2-k_3^2} \\ \frac{-\mathrm{i}k_0|\mathbf{k}_\perp|}{k_0^2-k_3^2}\end{pmatrix}_\nu,$$
$$c_\nu^- = \begin{pmatrix} \frac{[\mathbf{k}_\perp\times\boldsymbol{\epsilon}]}{|k_\perp|} \\ 0 \\ 0 \end{pmatrix}_\nu, \qquad c_\nu^+ = \begin{pmatrix} 0 \\ \frac{-k_0}{\sqrt{k_0^2-k_3^2}} \\ \frac{\mathrm{i}k_3}{\sqrt{k_0^2-k_3^2}}\end{pmatrix}_\nu. \tag{31}$$

The upper positions in every column are occupied by two-component vectors in the perpendicular plane (**1**, **2**); next go the third and fourth components. The normalizing factor is $|(\flat_\nu^{(1)})^2| = (E^2 + B^2)^2(k_0^2 - k_3^2)|k^2||\mathbf{k}_\perp|^2$. Here $\boldsymbol{\epsilon}$ is the unit vector along axis **3** and $[\mathbf{k}_\perp\times\boldsymbol{\epsilon}]$ stands for the vector product: $[\mathbf{k}_\perp\times\boldsymbol{\epsilon}]_m = \varepsilon_{mns}(\mathbf{k}_\perp)_n\boldsymbol{\epsilon}_s$. The difference $k_0^2 - k_3^2$ is understood to be non-negative for real excitations, but may be imaginary for virtual ones.

From the normalized expression in (31) for $\flat_\nu^{(1)}$, we find for the electric and magnetic fields in mode 1:

$$\mathbf{e}^{(1)} = k_0\frac{\mathbf{k}_\perp}{|\mathbf{k}_\perp|}\sqrt{\frac{|k^2|}{k_0^2 - k_3^2}},$$
$$\mathbf{h}^{(1)} = [\mathbf{k}_\perp\times\boldsymbol{\epsilon}]\frac{k_3}{|\mathbf{k}_\perp|}\sqrt{\frac{|k^2|}{k_0^2 - k_3^2}}. \tag{32}$$

Naturally, the electric and magnetic fields in mode 1 are oriented in the same way as in the single-invariant external field: they both lie in the plane, orthogonal to the external fields, they are mutually orthogonal; besides, the magnetic field is transverse, $(\mathbf{h}^{(1)}\mathbf{k}) = 0$, while the electric field, generally, is not: $(\mathbf{e}^{(1)}\mathbf{k}) \neq 0$; it is transverse for the special case of propagation along the external fields $k_\perp = 0$. It is seen again that a massless excitation is possible in mode 1 as long as it propagates along the external field, otherwise the fields (32) vanish if $k^2 = 0$, unless $\mathbf{k}_\perp = 0$, when the square root turns to unity.

The electric fields carried by the vectors c_μ^+ and c_μ^-, respectively, are

$$\mathbf{e}^+ = \{\mathbf{k}_\perp k_3 + \boldsymbol{\epsilon}(k_3^2 - k_0^2)\}\sqrt{\frac{(\mathcal{B}^2 + \mathcal{E}^2)}{(k^2\mathcal{B}^2 + kF^2k)}}$$
$$= \mathbf{k}_\perp\frac{k_3}{\sqrt{k_3^2 - k_0^2}} + \boldsymbol{\epsilon}\sqrt{k_3^2 - k_0^2}, \tag{33}$$
$$(\mathbf{e}^+\mathbf{e}^-) = 0, \qquad (\mathbf{e}^+\mathbf{k}) \neq 0,$$

$$\mathbf{e}^- = [\mathbf{k}_\perp \times \boldsymbol{\epsilon}]k_0\sqrt{\frac{(\mathcal{B}^2+\mathcal{E}^2)}{(k^2\mathcal{E}^2 - kF^2k)}} = [\mathbf{k}_\perp \times \boldsymbol{\epsilon}]\frac{k_0}{|k_\perp|}, \qquad (\mathbf{e}^-\mathbf{k}) = 0, \tag{34}$$

while their magnetic fields are

$$\mathbf{h}^+ = [\mathbf{k}_\perp \times \boldsymbol{\epsilon}]\frac{k_0}{\sqrt{k_3^2 - k_0^2}}, \qquad \mathbf{h}^- = -\mathbf{k}_\perp\frac{k_3}{k_\perp} + \boldsymbol{\epsilon}k_\perp, \qquad (\mathbf{h}^\pm\mathbf{k}) = (\mathbf{h}^+\mathbf{h}^-) = (\mathbf{h}^\pm\mathbf{e}^\pm) = 0. \tag{35}$$

The orthogonality of the electric fields $(\mathbf{e}^+\mathbf{e}^-) = (\mathbf{e}^{(1)}\mathbf{e}^-) = 0$, seen in Eqs. (32)–(34), originates from the fact that, in the special frame, the time component of one of their mutually orthogonal ($c^+c^- = 0$), and 4-transversal ($c^\pm k = 0$) vector potentials $c^\pm_\mu$ disappears: $c_0^- = 0$. The 3-vector $\mathbf{e}^-$ is directed along the axis (call it axis **1**), orthogonal to the plane, where the external fields and the propagation vector $\mathbf{k}$ lie [the plane (**3**, **2**)]. The vector $\mathbf{e}^+$ lies in that plane. It makes the universal angle

$$\alpha = \arctan(k_3k_\perp/(k_0^2 - k_3^2)) \tag{36}$$

with the direction of the external fields **3**. Also the magnetic field $\mathbf{h}^-$ lies in the plane spanned by the external fields and the propagation direction, while $\mathbf{h}^+$ is orthogonal to this plane.

We are in a position to conclude that electric fields in the eigenmodes 2 and 3, $\mathbf{e}^{(2,3)}$, responsible for massless excitations, lie both in the common plane, spanned by the vectors $\mathbf{e}^\pm$,

$$\mathbf{e}^{(2,3)}_\mu = -2\Lambda_3\mathbf{e}^-_\mu + [\Lambda_2 - \Lambda_4 \pm \sqrt{(\Lambda_2-\Lambda_4)^2 + 4\Lambda_3^2}]\mathbf{e}^+_\mu, \tag{37}$$

one of which ($\mathbf{e}^-$) is orthogonal to the plain, where the external fields and the propagation momentum $\mathbf{k}$ lie, while the other ($\mathbf{e}^+$) belongs to that plane and makes the angle (36), that depends only on momentum components, with the external fields. The fields (37) are not, generally, mutually orthogonal, since $\mathbf{e}^\pm$ are not unit-length vectors. Also the magnetic fields of the modes 2, 3 lie in the common plane spanned by the two vectors $\mathbf{h}^\pm$ and are linearly combined of them with the same coefficients as in (37). It can be checked that the electric and magnetic fields in each mode are mutually orthogonal, of course: $(\mathbf{h}^{(2,3)}\mathbf{e}^{(2,3)}) = 0$.

In the special case of only one invariant different from zero, $\Lambda_3 \sim \mathfrak{G} \approx (\mathcal{E}/\mathcal{B}) \to 0$, the eigenvectors $\flat^{(2,3)}_\mu$ (7) are the same as $c^\pm_\mu$ (11) owing to (10), hence $\mathbf{e}^\pm$, Eqs. (33) and (34) and $\mathbf{h}^\pm$, Eq. (35), become the electric and magnetic fields of the corresponding eigenmodes, coinciding with their expressions known from Refs. [4,13] with the particular property, that the electric field of mode 2 lies in the plane (**3**, **2**), while that of mode 3 is orthogonal to this plane, known for the special case of zero virtuality, $k^2 = 0$, from Ref. [15].

B. Magnetoelectric effect

In the magneticlike field it is known that virtual photons of mode 2 are carriers of electrostatic [3,24] forces, whereas those of modes 1 and 3 are responsible for magnetostatic interaction [3]. These statements follow from the representation (31) for the basic vectors, that become eigenvectors in that special case, and from the diagonal representation of the Green function (28) that allows one to write electric and magnetic fields created by various (static included) configurations of small—as compared to the background field—charges and currents. The mixing of the basic vectors (10) in eigenmodes 2 and 3 for the general external field with $\mathfrak{G} \neq 0$ makes these statements no longer true in what concerns these modes, mode 3 remaining as it was. Moreover, thanks to the mixing, a static electric charge, if placed in the external field with the both invariants different from zero, gives rise not only to an electric field, as usual, but also to a magnetic field of its own, like a magnetic charge or moment. Also stationary currents produce some electric admixture to their customary magnetic fields.

Here we consider this analogue to the magnetoelectric effect known in crystals [44] using the field of a pointlike static charge q taken at rest in the special frame as an example. We set the 4-current corresponding to this source in the coordinate space x as $j_\mu(x) = q\delta_{\mu 0}\delta^3(\mathbf{x})$, where $\delta_{\mu 0}$ is the Kronecker symbol and $\delta^3(\mathbf{x})$ is the Dirac delta function. Integrating this current with the Green function (28) we obtain for the vector-potential produced by the point charge (see [24] for a more detailed explanation if needed)

$$A_\mu(\mathbf{x}) = \frac{q}{(2\pi)^3}\int D_{\mu 0}(0,\mathbf{k})\exp(-\mathrm{i}\mathbf{k}\mathbf{x})d^3k. \tag{38}$$

Here the argument 0 of the Green function stands for k_0. Among the basic vectors (31) there is only one whose fourth component remains nonzero in the static limit $k_0 = 0$. It is c^+_μ. It participates in the eigenvectors $\flat^{(2,3)}_\mu$ in accord with (10). Hence, only these two eigenvectors will remain in the decomposition (28) of D after it is substituted into (38). On the other hand, the contribution of the basic vector c^-_μ may only supply the spatial components to the vector potential (38), whereas c^+_μ cannot. Bearing in mind that Eqs. (31) imply in the static limit, $k_0 = 0$, that

$$c_i^+ = c_0^- = 0, \qquad c_0^+ = 1, \qquad c_i^- = \frac{[\mathbf{k}_\perp \times \boldsymbol{\epsilon}]_i}{|\mathbf{k}_\perp|}, \qquad i = 1, 2, \qquad c_3^\pm = 0, \tag{39}$$

the spatial part of the latter is

$$A_i(\mathbf{x}) = \frac{q}{(2\pi)^3}\int\sum_{a=2,3}\frac{\flat_i^{(a)}\flat_0^{(a)}}{(\flat^{(a)})^2}\frac{\exp(-\mathrm{i}\mathbf{k}\mathbf{x})d^3k}{\mathbf{k}^2-\varkappa_a} = \frac{q}{(2\pi)^3}\int c_i^-\frac{\Lambda_3\exp(-\mathrm{i}\mathbf{k}\mathbf{x})d^3k}{(\mathbf{k}^2-\varkappa_2)(\mathbf{k}^2-\varkappa_3)}, \quad i=1,2 \qquad A_3(\mathbf{x})=0. \tag{40}$$

To find the large-distance behavior of this field note that it is determined by the limit $\mathbf{k}=0$ in the preexponential factor in the integrand. Therefore, we set $\varkappa_{2,3}=0$ and use Eq. (20) for Λ_3:

$$\Lambda_3|_{k_\mu\to 0} = \mathrm{i}|\mathbf{k}_\perp|k_3\mathfrak{M}, \quad \mathfrak{M} = \mathfrak{L}_{\mathfrak{F}\mathfrak{G}}(B^2+E^2) - (\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}+\mathfrak{L}_{\mathfrak{F}\mathfrak{F}})\mathfrak{G}. \tag{41}$$

Then

$$\mathbf{A}\,(\mathbf{x}_\perp, x_3)|_{|\mathbf{x}|\to\infty} \simeq \frac{q\mathfrak{M}}{(2\pi)^3}\int \mathrm{i}k_3[\mathbf{k}_\perp\times\boldsymbol{\epsilon}]\frac{\exp(-\mathrm{i}\mathbf{k}\mathbf{x})d^3k}{\mathbf{k}^4}. \tag{42}$$

This vector in the two-dimensional plane orthogonal to the external fields is directed as $[\mathbf{x}_\perp\times\boldsymbol{\epsilon}]$, since the coordinate vector $\mathbf{x}_\perp$ in that plane fixes the only direction on which the integral may depend. The length of (42)

$$|\mathbf{A}|_{|\mathbf{x}|\to\infty} \simeq \frac{q\mathfrak{M}}{8\pi}\frac{1}{|\mathbf{x}|} \tag{43}$$

decreases via the Coulomb law with the radial distance $|\mathbf{x}|$ from the charge. The vector potential and the magnetic field carried by it at large distances are

$$\mathbf{A}(\mathbf{x}_\perp, x_3)|_{|\mathbf{x}|\to\infty} \simeq \frac{[\mathbf{x}_\perp\times\boldsymbol{\epsilon}]}{|\mathbf{x}_\perp|}\frac{q\mathfrak{M}}{8\pi}\frac{1}{|\mathbf{x}|},$$
$$h_3 = \frac{q\mathfrak{M}}{8\pi}\left(\frac{1}{|\mathbf{x}||\mathbf{x}_\perp|} - \frac{|\mathbf{x}_\perp|}{|\mathbf{x}|^3}\right), \qquad \mathbf{h}_\perp = \frac{q\mathfrak{M}}{8\pi}\frac{\mathbf{x}_\perp}{|\mathbf{x}_\perp|}\frac{x_3}{|\mathbf{x}|^3}. \tag{44}$$

The last two equations together make a magnetic field oriented along the radius vector (when q is positive) in the upper half plane $x_3>0$ and opposite to the radius vector in the lower half plane:

$$\mathbf{h} = \frac{\mathbf{x}}{|\mathbf{x}|}\frac{q\mathfrak{M}}{8\pi}\frac{1}{|\mathbf{x}|^2}\frac{x_3}{|\mathbf{x}_\perp|}. \tag{45}$$

The magnetic lines of force make a pencil of straight lines passing through the origin where the charge is located and go radially from/to the charge with their density being the cotangent of the observation angle increasing towards the axis **3**, where it is singular. The magneton $q\mathfrak{M}/(8\pi)$ is proportional to the pseudoscalar $\mathfrak{G}$; in perturbation theory it has the fine structure constant α as its overall factor. The result (45) is approximation independent, but holds true only as long as the magnetic field produced by the charge may be considered as a small perturbation of the background field.

It is worth noting that an analogous magnetoelectric phenomenon should be present in a plasma with external magnetic field. The reason is again in the mixing—after plasma is added—of basic vectors [12] (see also [5]), which are electric and magnetic carriers in the magnetized vacuum alone. If the plasma is charge symmetric, e.g. consists of equal numbers of positively and negatively charged otherwise identical particles, say electrons and positrons, the vector $\flat_\mu^{(1)}|_{\mathfrak{G}=0}$ from (7) linearly combines with the pseudovector $\flat_\mu^{(2)}|_{\mathfrak{G}=0} = (\tilde{F}k)_\mu$, to become an eigenvector, unlike the situation considered above, while $\flat_\mu^{(3)}|_{\mathfrak{G}=0} = (Fk)_\mu$ remains an eigenvector. The pseudoscalar, needed for this combination, is built of the same pseudovector contracted with the vector of 4-velocity of the plasma. It plays the role of $\mathfrak{G}$. For a more general charge-nonsymmetric case, say, electron gas or a gas of ionized atoms, the situation is even more rich, because all the three basic vectors mix.

V. DEGENERACIES

We have to consider the degeneracies of the polarization matrix taking place for two special kinematical conditions owing to the symmetries of the external field.

For the real or virtual excitations directed parallel to the external field in the special frame, the polarization operator is symmetric under spatial rotations around the common field direction **3**, since the external field is invariant under them, while the excitation does not introduce an additional anisotropy in the perpendicular plane owing to the relation $\mathbf{k}_\perp=0$. The symmetry of the polarization operator should manifest itself as a degeneracy that implies that two out of its three eigenvalues should coincide, and their corresponding eigenvectors should transform through one another under the symmetry transformations, while remaining eigenvectors. By inspecting (31) we see that $\flat_\mu^{(1)}$ and c_μ^- do possess this mutual property, when $\mathbf{k}_\perp=0$, but c_μ^+ does not (recall that $|k^2| = k_0^2 - k_3^2$, once $\mathbf{k}_\perp=0$). Hence the latter cannot be admixed to c_μ^- in (10), and we conclude that, the same as in the one-invariant external field,

$$\Lambda_3 = 0, \qquad \text{and} \quad \varkappa_1 = \varkappa_3, \qquad \text{when } k_\perp^2 = 0. \tag{46}$$

(The condition $k_\perp^2=0$ in the general frame should be replaced by $k^2\mathcal{E}^2 - kF^2k=0$). Then Eqs. (10) and (12) imply that

$$\Lambda_1 = -\Lambda_2, \qquad \text{when } k_\perp^2 = 0. \tag{47}$$

Now from (32)–(35) we see that, when $k_\perp^2=0$, modes 1 and 3 carry mutually perpendicular and equal in magnitude transverse electric fields $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(3)} = \mathbf{e}^-$ polarized in the plane orthogonal to their propagation direction and to the external fields. The same is true for the magnetic fields in these modes. Therefore, in this special case mode 1 does correspond to a real photon, as explained in Sec. III. Simultaneously, mode 2 becomes a purely longitudinal wave that may exist only as long as $k_3^2 - k_0^2 \neq 0$, since (33) otherwise disappears, i.e. it may only be massive. The conclusion is that the number of massless modes remains two, as expected.

Another degeneracy of polarization operator is provided by the kinematical situation $k_0^2 - k_3^2 = 0$. This condition is invariant under Lorentz boosts along the common direction of the fields in the special frame. (Its invariant equivalent is $k^2\mathcal{B}^2 + kF^2k = 0$). On the other hand, the external field in the special frame is also invariant under this transformation: it does not lead out of this frame, since the constant electric and magnetic fields are not transformed by it. Hence, the polarization operator should be also invariant, which implies that some two of its eigenvalues must coincide, while the corresponding eigenvectors are transformed through one another by the Lorentz rotation in the $(\mathbf{3}, \mathbf{0})$ hyperplane. This is the case for the vectors $\flat_\nu^{(1)}$ and c_ν^+ in (31), since in the limit under consideration the 2-vector in the upper row of the former is negligible as compared to the other two components, and $|\mathbf{k}_\perp| = \sqrt{|k^2|}$, so that $\flat_\nu^{(1)}$ matches c_ν^+ as a Lorentz boost partner. On the contrary, the vector c_ν^- does not transform through any of them, because it is Lorentz-boost invariant. We conclude that c_ν^- and c_ν^+ can no longer mix together to form eigenvectors, but should be eigenvectors separately. The mixture becomes impossible if and only if Λ_3 disappears from (10) and (12). Then, according to (17), $\flat_\nu^{(2)}$ becomes $\sim c_\nu^+$, hence the degeneracy is expressed as the relation

$$\varkappa_1 = \varkappa_2, \quad \text{when } k_0^2 - k_3^2 = 0, \tag{48}$$

accompanied by the relations

$$\Lambda_3 = 0, \quad -\Lambda_4(k_3^2 - k_0^2) = k^2\Lambda_1, \text{ when } k_0^2 - k_3^2 = 0. \tag{49}$$

Now we use the fact that the electric fields in eigenmodes are defined up to a common factor to renormalize them all according to $\tilde{\mathbf{e}} = \mathbf{e}(k_0^2 - k_3^2)^{1/2}$. Then, under the special kinematic condition under consideration $k_0^2 - k_3^2 = 0$, the electric fields in modes 1 and 2 [see Eqs. (32) and (33)] are finite and equal in length $|\tilde{\mathbf{e}}^{(1)}| = |\tilde{\mathbf{e}}^+| = k_\perp k_3$, while that in mode 3 (34) disappears, $\tilde{\mathbf{e}}^- = 0$. Consequently, this degree of freedom is impossible. As pointed in Sec. III, in the exceptional point $k_0^2 - k_3^2 = 0$, the dispersion law $k^2 = 0$ may correspond to an actual mode-1 photon. In view of (48) it must be accompanied by a mode-2 partner. For $k^2 = 0$ the electric fields in modes 1 and 2, besides the fact that they are equal in size, become polarized in transverse directions, $(\tilde{\mathbf{e}}^+\tilde{\mathbf{e}}^{(1)}) = |\mathbf{k}_\perp|k_3k_0\sqrt{k^2} = 0$. Therefore, we have again two photon degrees of freedom.

The symmetry relations (46)–(49) are confirmed in the infrared limit by Eqs. (17)–(20) and—for any value k_μ of the momentum—by the one-loop calculations in QED of [4].

In a theory with the dual invariance, which is not QED, another degeneracy is possible in the one-invariant case $\mathfrak{G} = 0$ [3] that equates the eigenvalues $\varkappa_2$ and $\varkappa_3$, since under the continuous duality transformation the vector $\flat_\mu^{(2)}|_{\mathfrak{G}=0} = (\tilde{F}k)_\mu$ and the pseudovector $\flat_\mu^{(3)}|_{\mathfrak{G}=0} = (Fk)_\mu$ transform through each other.

VI. CONCLUSIONS

In this paper we studied the most general basis properties of small perturbations of the vacuum, filled with a constant and homogeneous background electromagnetic field with both of its invariants different from zero. To this end the eigenvector decomposition of the polarization operator with a contribution of three modes was exploited. We saw how the eigenvectors characteristic of the one-invariant (magnetic in a special frame) background field are linearly combined with the help of dynamics-dependent coefficients to form eigenvectors of the general problem under investigation.

Among the vacuum perturbations special attention was paid to the sourceless excitations that supply poles to the photon propagator and satisfy three different dispersion equations. These may be either massive or massless. In the latter case they are called photons. The massless excitations belong only to two modes, in accordance with two polarization degrees of freedom of a gauge vector particle, the photon. Massive excitations belong to all three modes, since a massive vector field has 3 degrees of freedom. These may have an unrestricted number of branches in each mode depending on the properties of the corresponding dispersion equation. We described admitted disposition of various dispersion curves (in the appropriate momentum plane) as it is restricted by the causal propagation requirement outside possible abnormal dispersion domains and argued that it may be extended to these domains as well. The eigenmodes are plane polarized, and the orientations of their electric and magnetic fields with respect to propagation direction and the direction of the background field are described. We dwelled on the impact the admixture of an electric field to a magnetic background may have on the selection rules for photon splitting. We noted that such admixture results in a larger separation between two different dispersion curves enhancing the birefringence.

Among possible perturbations of the background caused by small sources, we especially considered the magnetic (part of the) field produced by a point static electric charge and found its behavior far from the source.

We also established coincidences between eigenvalues of the polarization operator (degeneracies) that, under special relations between momenta, reflect the residual rotational and Lorentz symmetries of the vacuum left after the background field is imposed.

All the results are approximation independent, except for the statement, based on the one-loop calculations, that the mixing between modes is not resonant in the limit of the small electric field at the first threshold of electron-positron pair creation by a photon.

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APPENDIX A

Suppose the polarization operator $\Pi_{\mu\nu}$ is known in components. Then the invariant functions Λ_i, $i = 1, 2, 3, 4$ involved in its eigenvalues and eigenvectors are given following the receipts in [4,5] as

$$\Lambda_1 = d_\mu^{(1)}\Pi_{\mu\nu}(d^{(1)} + d^{(2)})_\nu, \qquad \Lambda_2 = -c_\mu^{(1)}\Pi_{\mu\nu}c_\nu^-,$$
$$\Lambda_3 = -c_\mu^-\Pi_{\mu\nu}c_\nu^{(3)}, \qquad \Lambda_4 = -c_\mu^{(3)}\Pi_{\mu\nu}c_\nu^+, \tag{A1}$$

where $c_\nu^\pm$ are given by (11), and

$$d_\mu^{(1)} = \frac{\mathcal{E}^2 k_\mu - \mathrm{i}\mathcal{B}(Fk)_\mu - (F^2k)_\mu - \mathrm{i}\mathcal{E}(\tilde{F}k)_\mu}{2^{1/2}(\mathcal{B}^2 + \mathcal{E}^2)^{1/2}(k^2\mathcal{E}^2 - kF^2k)^{1/2}}$$
$$\Leftrightarrow \frac{E^2 k_\mu - \mathrm{i}B(Fk)_\mu - (F^2k)_\mu - \mathrm{i}E(\tilde{F}k)_\mu}{2^{1/2}(B^2 + E^2)|k_\perp|}, \tag{A2}$$

$$d_\mu^{(1)} + d_\mu^{(2)} = 2^{1/2}\frac{\mathcal{E}^2 k_\mu - (F^2k)_\mu}{(\mathcal{B}^2 + \mathcal{E}^2)^{1/2}(k^2\mathcal{E}^2 - kF^2k)^{1/2}}$$
$$\Leftrightarrow 2^{1/2}\frac{E^2 k_\mu - (F^2k)_\mu}{(B^2 + E^2)|k_\perp|}, \tag{A3}$$

$$c_\mu^{(3)} = \mathrm{i}\frac{\mathcal{B}^2 k_\mu + \mathcal{E}(Fk)_\mu + (F^2k)_\mu - \mathcal{B}(\tilde{F}k)_\mu}{(\mathcal{B}^2 + \mathcal{E}^2)^{1/2}(k^2\mathcal{B}^2 + kF^2k)^{1/2}}$$
$$\Leftrightarrow \frac{B^2 k_\mu + E(Fk)_\mu + (F^2k)_\mu - B(\tilde{F}k)_\mu}{(B^2 + E^2)(k_0^2 - k_3^2)^{1/2}}. \tag{A4}$$

The notations used here are connected with those of Refs. [4,5] as follows:

$$c_\mu^{(1,3)} = \mathrm{i}\sqrt{2}d_\mu^{(1,3)}, \qquad c_\mu^- = \mathrm{i}(d_\mu^{(1)} - d_\mu^{(2)})/\sqrt{2},$$
$$c_\mu^+ = \mathrm{i}(d_\mu^{(3)} - d_\mu^{(4)})/\sqrt{2}. \tag{A5}$$

Using the orthogonality of the polarization operator to vector k_μ and the orthogonality of the vector $(F^2k)_\mu$ to the hyperplane spanned by the two vectors $(Fk)_\mu$ and $(\tilde{F}k)_\mu$, and also the diagonal representation (5), we find that many components of the vectors between which $\Pi_{\mu\nu}$ is sandwiched disappear from (A1). Then we get the simpler representations (13) for the Λ's. Equation (13) for Λ_1 agrees with Eq. (9) and with the relation $\varkappa_1 = \mathfrak{b}_\mu^{(1)}\Pi_{\mu\nu}\mathfrak{b}_\nu^{(1)}/(\mathfrak{b}^{(1)})^2$ that follows from (6), taking into account the length of the eigenvector $\mathfrak{b}_\mu^{(1)}$ given in (8). Equations (13) for $\Lambda_{2,3,4}$ agree with Eqs. (6), (10), and (12), but cannot be deduced from them, because the latter are invariant under a similarity transformation that changes the components of $\Pi_{\mu\nu}$.

The linear combinations Λ_i of the polarization operator components are calculated in [4] in one-loop approximation, the calculational details being presented in [5] on the basis of [45]. The latter reference as well as [6,7] contains also calculations of alternative sets of four scalar coefficient functions of an appropriate set of basic matrices in terms of which the polarization operator may be expressed. However, the set of functions Λ_i is preferred to them all, because the eigenvalues are given the simplest in their terms.

APPENDIX B

In this Appendix we write the linear in electric field correction into invariant function Λ_3 for the case, where an electric field, much smaller than the magnetic field, is added parallel to the latter (this wording refers to the special Lorentz frame). The linear part of Λ_3 defines the leading contribution into the mixing of photon eigenmodes in a magnetic field due to the perturbation introduced by the electric field. We present it here as a result of one-loop calculations of quantum electrodynamics in the external magnetic field.

Using (11) and notations (7), the expansion (15) of the slightly perturbed eigenvectors over the eigenvectors in a magnetic field alone becomes

$$\mathfrak{b}_\mu^{(2)} = -\frac{(\Lambda_2 - \Lambda_4)}{(k^2\mathcal{B}^2 + kF^2k)^{1/2}}\mathfrak{b}_\mu^{(2)}|_{\mathfrak{G}=0} + \left(\frac{\mathfrak{G}(\Lambda_2 - \Lambda_4)}{\mathcal{B}^2(k^2\mathcal{B}^2 + kF^2k)^{1/2}} - \frac{\Lambda_3}{(-kF^2k)^{1/2}}\right)\mathfrak{b}_\mu^{(3)}|_{\mathfrak{G}=0},$$
$$\mathfrak{b}_\mu^{(3)} = \frac{(\Lambda_2 - \Lambda_4)}{(-kF^2k)^{1/2}}\mathfrak{b}_\mu^{(3)}|_{\mathfrak{G}=0} + \left(\frac{\mathfrak{G}(\Lambda_2 - \Lambda_4)}{\mathcal{B}^2(-kF^2k)^{1/2}} - \frac{\Lambda_3}{(k^2\mathcal{B}^2 + kF^2k)^{1/2}}\right)\mathfrak{b}_\mu^{(2)}|_{\mathfrak{G}=0}. \tag{B1}$$

It is understood that $\Lambda_2 - \Lambda_4$ in the right-hand side are taken at $\mathfrak{G} = 0$. In writing these equations we took into account that $\Lambda_{2,4}$ are even, and Λ_3 is an odd function of $\mathfrak{G}$.

It is seen from (12) that at $\mathfrak{G} = 0$ the quantity $-\Lambda_4$ is the polarization operator eigenvalue $\varkappa_2$ in a magnetic field. Analogously, $-\Lambda_2 = \varkappa_3$. These quantities in the one-loop approximation are known [4,13]. Now we shall write Λ_3 in the same approximation by calculating the $\mathfrak{G} = 0$ limit of the corresponding expression from [4]:

$$\Lambda_3 = -\frac{\alpha}{4\pi}\frac{\mathfrak{G}}{\mathfrak{F}}\left(\frac{-kF^2k}{2\mathfrak{F}}\right)^{1/2}\left(\frac{-k\tilde{F}^2k}{2\mathfrak{F}}\right)^{1/2} \times \int_0^\infty \frac{\tau d\tau}{\sinh^2\tau}\int_{-1}^1 d\eta(1-\eta)^2\sinh^2\frac{\tau(1+\eta)}{2} \times \exp\left\{\frac{kF^2k}{2\mathfrak{F}}\frac{\sinh((1+\eta)/2)\tau)\sinh((1-\eta)\tau/2)}{\sinh(ef\tau)} - \frac{k\tilde{F}^2k}{2\mathfrak{F}}\frac{1-\eta^2}{4ef}\tau - \frac{m_e^2\tau}{ef}\right\}. \tag{B2}$$

Here $\alpha = 1/137$ is the fine structure constant, m_e and e are the electron mass and charge, and $f = \sqrt{2\mathfrak{F}}$. Note that this expression vanishes, indeed, either if $-\frac{kF^2k}{2\mathfrak{F}} \Leftrightarrow k_\perp^2 = 0$, or if $-\frac{k\tilde{F}^2k}{2\mathfrak{F}} \Leftrightarrow k_0^2 - k_3^2 = 0$, as it should in accordance with what the symmetry of the external field prescribes, as it was established in the body of this article, Eqs. (46) and (49). The integral (B2) has an infinite number of branching points, the same as $\Lambda_{1,2,4}$, with singular inverse square-root behavior—the cyclotronic resonances at thresholds of electron-positron pair creation by a photon [5,13]. However, the lowest-lying resonance is not in the point (in the variables, referring to the special frame) $k_0^2 - k_3^2 = 4m_e^2$, like in $\varkappa_2 = -\Lambda_4$, but in the point $k_0^2 - k_3^2 = [(m_e^2 + 2ef)^{1/2} + m_e]^2$, like in $\varkappa_3 = -\Lambda_2$, because this value borders the convergence domain of the τ integration in (B2). This means that the (small as compared with the magnetic) electric field cannot affect the phenomenon of the mode-2 photon capture with its adiabatic conversion into a free [19] or bound [16] electron-positron pair in the lowest Landau level.

APPENDIX C

Here we reproduce the simplest description of the phenomenon of abnormal dispersion [36] and modify it in such a way as to illustrate the mutual repulsion of the dispersion curves of the photon and of an unstable massive state, and to introduce the "extended group velocity" that does not exceed the speed of light.

We deal with an isotropic homogeneous medium without spatial dispersion considered in its rest frame. Let the polarization be determined by a polelike contribution of a massive unstable state with constant complex energy squared $\bar{k}_0^2 = \mathfrak{m}^2 - \mathrm{i}\Gamma$:

$$\varkappa(k_0^2) = \frac{-g}{k_0^2 - \bar{k}_0^2}, \tag{C1}$$

where $g > 0$ is a positive coupling constant, $\Gamma > 0$ is responsible for absorption. For further simplification we restrict ourselves to the one-dimensional case: $k_1 = k$, $k_{2,3} = 0$. Then the dispersion equation is

$$k^2 = k_0^2 + \frac{g}{\bar{k}_0^2 - k_0^2}. \tag{C2}$$

Equation (C2) defines the momentum k as a complex function of the real frequency variable k_0. The real part of this complex function (as laid out along the horizontal axis) is given by the zigzag-shaped curve in Fig. 2. Bearing in mind that the refraction index is defined as $n(k_0) = \mathrm{Re}\, k/k_0$, one can redraw this curve as the well-known zigzag curve, presenting the dependence of the refraction index on frequency k_0 or on the wavelength $\lambda = |k|^{-1} \sim k_0^{-1}$ in the case of abnormal dispersion. The abnormal dispersion band in Fig. 2 is the interval of k_0, where the decline of the dispersion curve is anomalous,

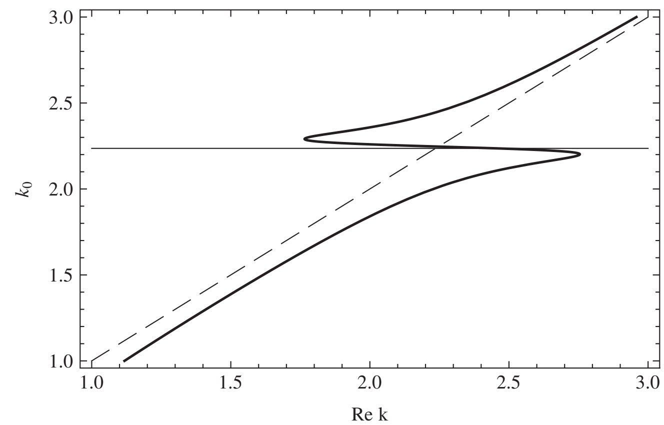

FIG. 2. Abnormal dispersion. The dotted line is the vacuum dispersion law $k_0 = k$, the horizontal line is the unstable particle dispersion law $k_0 = \mathfrak{m}$. The solid line is the frequency k_0 plotted against the real part of momentum $\mathrm{Re}\, k$ according to Eq. (C2). k_0 and $\mathrm{Re}\, k$ are taken in arbitrary mass scale M. Values of the parameters in Eq. (C2) are chosen as $\mathfrak{m} = \sqrt{5}M$, $\Gamma = 0.2M^2$, and $g = M^4$.

$(dk_0/d\,\mathrm{Re}\, k) < 0$, in other words, where the group velocity is directed opposite to the phase velocity k_0/k. The width of this band shrinks to nothing in the no-absorption limit $\Gamma \to 0$. Near the two edges of the abnormal dispersion band the modulus of the group velocity, defined as $|dk_0/d\,\mathrm{Re}\, k|$, is not restricted by unity.

Let us now solve the dispersion equation (C2) with respect to k_0^2:

$$k_0^2 = \frac{k^2 + \bar{k}_0^2}{2} \pm \sqrt{\left(\frac{k^2 - \bar{k}_0^2}{2}\right)^2 + g}. \tag{C3}$$

These are two complex functions of the real variable k^2. The real parts of their square roots give the two *disconnected* dispersion curves $\mathrm{Re}\, k_0(k)$ shown in Fig. 3. Physically, they should be viewed as the result of repulsion of the two primary dispersion curves: the one of the un-

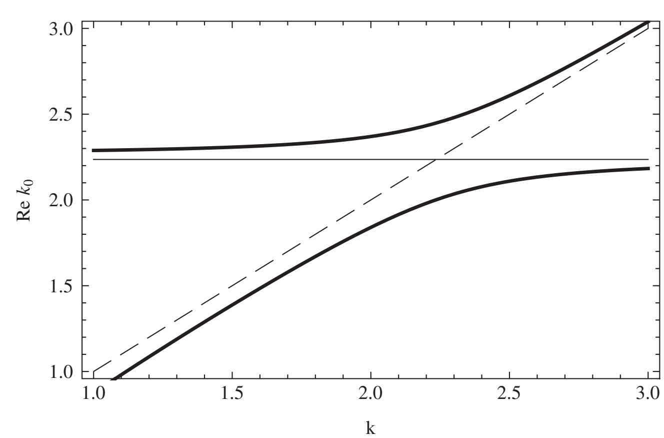

FIG. 3. Mutually repulsed dispersion curves as solid lines. The rest is the same as in Fig. 2.

stable massive state, $\mathrm{Re}\, k_0 = \mathfrak{m}$, and the other of the photon, $k_0 = k$, that quasi-intercept in the point $k_0 = k = \mathfrak{m}$. The patterns in Figs. 2 and 3 coincide in the limit $\Gamma = 0$, but otherwise are essentially different. The "extended group velocity" defined on the dispersion curves (C3), $\partial\, \mathrm{Re}\, k_0/\partial k$, is positive and less than unity.

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