A simple UV completion of QED in 5 dimensions

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(Received 31 March 2010; published 4 June 2010)

We construct a Lifshitz-like version of five-dimensional (5D) QED which is UV completed and reduces at low energies to ordinary 5D QED. The UV quantum behavior of this theory is very smooth. In particular, the gauge coupling constant is finite at all energy scales and at all orders in perturbation theory. We study the IR properties of this theory, when compactified on a circle, and compare the one-loop energy dependence of the coupling in the Lifshitz theory with that coming from the standard 5D QED effective field theory. The range of validity of the 5D effective field theory is found to agree with the more conservative version of naive dimensional analysis.

I. INTRODUCTION

Quantum field theories in more than four space-time dimensions have received a lot of attention in the past ten years. They allow us to address standard well-known problems in four-dimensional (4D) physics, such as the gauge and/or flavor hierarchy problems, from a different perspective, leading to novel scenarios, such as the possibility of having a fundamental TeV-sized quantum gravity scale [1] or a TeV scale naturally generated by an extreme redshift effect from a warped extra dimension [2]. Extra dimensional (ED) field theories are nonrenormalizable and require an ultraviolet (UV) completion. There is little doubt that such UV completions exist, in particular, in string theory where ED are predicted and necessary. Constructing string theory models which reduce at low energies to the ED models considered in the literature is a difficult task. As a matter of fact, we do not know sufficiently simple and concrete UV completions of ED field theories.

The aim of this paper is to concretely provide a UV completion of ED theories. For simplicity, we will focus our attention on a specific simple model which is QED in five dimensions (5D) compactified on a circle, although our construction is more general and can allow for a possible UV completion of any 5D (or higher) gauge theory. Our model is of the Lifshitz type [3,4], where Lorentz invariance is explicitly broken at high energies. In these theories, the presence of higher derivative (in the spatial directions only) quadratic terms improves the UV behavior of the particle propagator, without introducing ghostlike degrees of freedom. This kind of UV completion does not require the introduction of extra degrees of freedom, but rather modifies the propagation of the already existing ones.

The quantum UV behavior of our Lifshitz 5D QED is incredibly simple. The photon anomalous dimension vanishes to all orders in perturbation theory and correspondingly the electric charge is completely finite. This

phenomenon is explained by the fact that the UV theory formally looks like a nonrelativistic theory for which particle–antiparticle creation is suppressed at high energies. Another marginal coupling in the theory, magneticlike, is shown to be UV free, so the theory is UV completed and perturbative at any energy scale (neglecting gravity, of course).

After having studied the UV one-loop renormalization of the theory, we turn our attention to its infrared (IR) behavior. In particular, we show in some detail that the universal IR energy dependence of the finite gauge coupling (electric charge) in the Lifshitz theory coincides with that computed in the effective field theory, as it should. En passant, we explicitly show that the decoupling of heavy massive states in 5D effective theories is less efficient than in 4D; the effect of massive particles in 5D does not vanish as $1/M^2$, like in 4D, but only as $1/M$, in agreement with the known fact that the sensitivity of an effective theory to its UV completion is higher in 5D than in 4D.

The whole one-loop energy behavior of the 4D inverse square coupling g_4^{-2} is the following: starting from the IR, for energies much smaller than the compactification scale, g_4^{-2} decreases logarithmically, for a small window above the compactification scale it decreases linearly and then for yet larger energies g_4^{-2} tends to a constant, with the one-loop correction going to zero as $1/E^{1+\gamma}$, with $\gamma = \mathcal{O}(g_4^2) > 0$.

Estimates based on naive dimensional analysis (NDA) [5] or on the unitarity of scattering amplitudes (see e.g. [6]) show that the energy range of validity of phenomenologically interesting ED theories is quite limited, sometimes at the edge of not being present at all. By using our UV-completed model, we can better quantify the cutoff Λ of the effective 5D QED, identifying it with the scale where the higher derivative Lifshitz operators become relevant. More precisely, Λ is defined as the scale where the UV-dependent one-loop photon vacuum polarization correction becomes of the same order as the calculable one in the effective 5D theory, with the asymptotic value of the

coupling still in the perturbative range. The resulting cutoff turns out to be approximately equal to the one predicted by a conservative NDA estimate:

$$\Lambda \gtrsim \frac{48\pi}{5} \frac{1}{R}\left(\frac{1}{g_4^2} - \frac{1}{g_{4,\infty}^2}\right), \tag{1.1}$$

with g_4 evaluated at the compactification scale $1/R$ and $g_{4,\infty}$ its asymptotic UV value, as computed in the Lifshitz theory. We are not taking into account here many other effects that can sensitively change the estimate of Λ, such as the number of particle species or, for $S^1/\mathbf{Z}_2$ orbifolds-interval compactifications, possible additional localized Lagrangian terms or warp factors. When these effects are considered in phenomenologically promising 5D theories, the estimate (1.1) is lowered by 1 order of magnitude or more.

Lorentz invariance is explicitly and maximally broken in the UV in Lifshitz-type theories and it is not automatically recovered in the IR, $E < \Lambda$, unless a fine-tuning is imposed [7,8] or some dynamical mechanism advocated. Aside from the above tuning, other experimental constraints, mainly of astrophysical origin, severely constrain the parameters of our theory, which should not be considered too seriously as a realistic UV completion, at the present stage at least. Yet we believe that the construction underlying it can be very useful to build renormalizable 5D theories.

The structure of the paper is as follows. In Sec. II, after a very brief review of the construction of Lifshitz-like theories, we present our model. In Sec. III the renormalization group (RG) flow in the UV regime $E > \Lambda$ is studied and it is, in particular, shown that the theory is UV completed and perturbative at any energy scale in a wide range of parameter space. In Sec. IV we study the IR behavior, $E < \Lambda$, of the gauge coupling in the Lifshitz theory.[1] We analyze the RG flow of the coupling from an effective 5D point of view and compare it with the UV model in Secs. IV A and IV B. In Sec. IV C we determine the cutoff of the 5D effective theory as determined from the Lifshitz model. In Sec. V we show how Lorentz invariance can be recovered at low energies and briefly comment on astrophysical bounds. In Sec. VI we conclude. In the Appendix an analytic formula for the energy behavior of the coupling is derived.

II. THE MODEL

The key point of the construction of Lifshitz-like theories is to break Lorentz invariance, so that one is allowed to introduce higher derivative terms in the spatial derivatives and quadratic in the fields, without necessarily introducing the dangerous higher time derivative terms that would lead to violations of unitarity. The UV behavior of the propagator is improved and theories otherwise nonrenormalizable become effectively renormalizable. In Lifshitz-like

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[1]A reader more interested in the IR properties of our model might want to skip Secs. II and III and go directly to Sec. IV.

theories an invariance under "anisotropic" scale transformations is imposed:

$$t = \lambda^z t', \qquad x^i = \lambda x^{i\prime}, \qquad \phi(x^i, t) = \lambda^{(z-d)/2}\phi'(x^{i\prime}, t'), \tag{2.1}$$

where $i = 1, \ldots, d$ parametrizes the spatial directions, ϕ denotes a generic field, and z is an integer, sometimes called a critical exponent. We will always assume to be in the preferred frame where spatial rotations and translations are unbroken symmetries. According to Eq. (2.1), we can assign to the coordinates and to the fields a "weighted" scaling dimension:

$$[t]_w = -z, \qquad [x^i]_w = -1, \qquad [\phi]_w = \frac{d-z}{2}. \tag{2.2}$$

The renormalizability properties of Lifshitz-like theories have been extensively studied for scalar, fermion [4], and gauge theories [9]. The usual power-counting argument for the renormalizability of a theory is essentially still valid, provided one substitutes the standard scaling dimensions of the operators by their "weighted scaling dimensions" [4], i.e. by the dimensions implied by the assignment (2.2).

Here we will focus on QED in 5D. The weighted dimensions of the photon and fermion fields are easily fixed by looking at their time component kinetic terms. We have

$$[\psi]_w = \frac{d}{2}, \qquad [A_0]_w = \frac{d+z-2}{2}, \qquad [A_m]_w = \frac{d-z}{2}. \tag{2.3}$$

Notice that A_0 and A_m necessarily have different weighted dimensions, since no time derivative acts on A_0. Equations (2.3) are consistent with gauge invariance, requiring

$$[g]_w = [\partial_0]_w - [A_0]_w = [\partial_m]_w - [A_m]_w = \frac{2-d+z}{2}. \tag{2.4}$$

Here and in what follows, m, n, and p are indices running over all spatial directions, including the compact directions. The Lifshitz version of QED is hence renormalizable in d spatial dimensions, provided that

$$z = d - 2. \tag{2.5}$$

In what follows we focus our attention on the case $z = 2$ and $d = 4$, i.e. 5D QED. For simplicity, we take ψ to be an ordinary Dirac field, despite the fact that at high energy the absence of $SO(5)$ invariance might allow one to construct two independent spinors related by γ^0, which is a sort of "chirality" matrix. In order to simplify the structure of the Lagrangian, we impose the conservation of a $\mathbf{Z}_2$ symmetry C, a combination of the usual charge conjugation and parity in the extra direction:

$$\psi^C(x_0, x_i, y) = C\bar\psi^T(x_0, x_i, -y),$$
$$A_0^C(x_0, x_i, y) = -A_0(x_0, x_i, -y),$$
$$A_i^C(x_0, x_i, y) = -A_i(x_0, x_i, -y),$$
$$A_y^C(x_0, x_i, y) = +A_y(x_0, x_i, -y),$$

(2.6)

where the matrix C is the usual charge conjugation matrix for spinors as defined in $D = 3 + 1$ Lorentz-invariant theories, y is the compact coordinate, and $i = 1, 2, 3$ runs over the noncompact spatial directions. The most general Lagrangian with weighted marginal and relevant couplings and invariant under (2.6) is the following:

$$\mathcal{L} = \frac{1}{2}F_{m0}^2 - \frac{c_\gamma^2}{4}F_{mn}^2 - \frac{a_\gamma^2}{4\Lambda^2}(\partial_m F_{np})^2$$
$$+ \bar\psi(i\slashed{\partial}_0 - ic_\psi\slashed{\partial} - M)\psi - \frac{a_\psi}{\Lambda}|D_m\psi|^2$$
$$- \frac{i\lambda}{\Lambda^{3/2}}F_{mn}\bar\psi\gamma^{mn}\psi,$$

(2.7)

where $\slashed{\partial}_0 = \gamma^0(\partial_0 - i(g/\sqrt\Lambda)A_0)$, $\slashed{\partial} = \gamma_m(\partial_m - i(g/\sqrt\Lambda)A_m)$, and Λ is a high-energy scale parametrizing the strength of the higher derivative operators. The symmetry (2.6), in combination with $SO(d)$ rotational invariance, is crucial to get rid of several otherwise allowed terms, such as $\bar\psi\gamma^0\psi$, $\bar\psi D_0\psi$, F_{mn}^3, etc. Below the scale Λ the theory defined by (2.7) flows to the usual 5D QED, provided that $c_\psi = c_\gamma$ to sufficient accuracy. For simplicity, in the following we will use units in which $\Lambda = 1$.

III. UV BEHAVIOR

In this section we compute the one-loop evolution of the marginal couplings in the UV regime, $E \gg 1$. We regularize the theory using dimensional regularization in the spatial directions only ($d = 4 - \epsilon$) and renormalize using a minimal subtraction scheme where only the poles in $1/\epsilon$ are subtracted, with no finite term. Being Lorentz symmetry absent in this regime, it is convenient to work in the Coulomb gauge

$$\partial_m A_m = 0. \qquad (3.1)$$

Compactification effects and all relevant couplings can be neglected at these scales, allowing us to easily perform the integration over the virtual energy exchanged in the one-loop graphs. We then set $c_\psi = c_\gamma = M = 0$. The fermion propagator can be written as

$$-iG_\psi^0(\omega, p) = \frac{P_+}{\omega - a_\psi p^2 + i\epsilon} + \frac{P_-}{-\omega - a_\psi p^2 + i\epsilon},$$

(3.2)

where ω and p denote energy and momentum, respectively, and $P_\pm = (1 \pm \gamma^0)/2$. We have explicitly written the $i\epsilon$ factors for reasons that will soon be clear. Similarly, the spatial components of the photon propagator are

$$G_{\gamma,mn}^0(\omega, p) = i\left(\delta_{mn} - \frac{p_m p_n}{p^2}\right)$$
$$\times \frac{1}{(\omega - a_\gamma p^2 + i\epsilon)(\omega + a_\gamma p^2 - i\epsilon)}.$$

(3.3)

In Eqs. (3.2) and (3.3) the superscript 0 stands for $c_\psi = c_\gamma = 0$. As can be seen from the Feynman rules reported in Fig. 1, no ω terms appear in the vertices, so that the ω integration can easily be performed using the residue theorem. By denoting q and ω_q the virtual momentum and energy running in the loop diagram, the ω_q dependence will only appear through the fermion and photon propagators. All nonvanishing loop diagrams we will consider involve one photon propagator (whose momentum we identify with q) and one or two fermion propagators. In general, for n fermion propagators, the loop graph will be proportional to

$$K_n = \int \frac{d\omega_q}{2\pi} \prod_{i=1}^n \frac{-iG_\psi^0(\omega_i + \alpha_i\omega_q, p_i + \alpha_i q)}{(\omega_q - a_\gamma q^2 + i\epsilon)(\omega_q + a_\gamma q^2 - i\epsilon)},$$

(3.4)

with ω_i and p_i external energies and momenta and α_i some constants. Since $P_+ P_- = 0$, Eq. (3.4) reduces to two terms, proportional to P_+ and P_-. The former (the latter) have all fermion poles in the lower-half (upper-half) ω plane, so that by appropriately closing the contour at infinity, we can always avoid all fermion poles. Hence we get

FIG. 1. Feynman rules of the Lifshitz 5D QED theory in the Coulomb gauge. The momenta are all incoming.

$$K_n = \frac{i}{2a_\gamma q^2} \prod_{i=1}^{n} \left(\frac{P_+}{\alpha_i a_\gamma q^2 + a_\psi (\alpha_i q + p_i)^2 - \omega_i} \right.$$

$$\left. + \frac{P_-}{\alpha_i a_\gamma q^2 + a_\psi (\alpha_i q + p_i)^2 + \omega_i} \right). \qquad (3.5)$$

A. Vacuum polarization

The vacuum polarization of the photon is easily computed in this theory and turns out to identically vanish. The fermion loop given by two trilinear vertices vanishes due to the integration over ω, since we can always choose to close the contour of integration in the upper-/lower-half ω plane with no poles, as in Eq. (3.4). The same result can be checked to hold in the Euclidean. After standard manipulations, it is easy to see that the loop is proportional to

$$\int_{-\infty}^{\infty} d\omega_E \frac{\omega_E(p_E + \omega_E) - ab}{(\omega_E^2 + a^2)[(\omega_E + p_E)^2 + b^2]} = 0, \qquad (3.6)$$

vanishing for any positive a and b. Here $p_E = i\omega$ and $\omega_E = i\omega_q$ are the Wick rotated energies. Interestingly enough, this result is not only valid at one-loop order but to all orders in perturbation theory. Consider first higher loop graphs constructed from the one-loop graph by adding photon lines only. Since, for $c_\psi = 0$, the trilinear and quartic vertices commute with $P_\pm$, any such graph with an arbitrary number of fermion propagators will split in the sum of two terms of the kind

$$\mathrm{Tr} \prod_{i=1}^{n} V_i \left[\frac{P_+}{\omega + \omega_\gamma(i) - a_\psi(p + k_\gamma(i))^2 + i\epsilon} \right.$$

$$\left. + \frac{P_-}{-\omega - \omega_\gamma(i) - a_\psi(p + k_\gamma(i))^2 + i\epsilon} \right], \qquad (3.7)$$

where ω and p are the energy and momentum of the virtual fermion running in the loop, $\omega_\gamma(i)$ and $k_\gamma(i)$ are the sum of the energies and the momenta of the photons attached to the fermion lines at the vertices V_j one encounters in arriving at the propagator i. Note that the UV vertices V_i (see Fig. 1) commute with $P_\pm$. Exactly as before, Eq. (3.7) identically vanishes when integrated over ω.

When the components of the external photon are spatial, there is in addition a tadpole graph associated with the quartic vertex (see Fig. 1). This is trivially vanishing in dimensional regularization since (after Wick rotation)

$$\int d\omega d^d k \frac{a_\psi k^2 + m}{\omega^2 + (a_\psi k^2 + m)^2} \propto \int d^d k 1 = 0. \qquad (3.8)$$

Since a single fermion loop, dressed with an arbitrary number of photons, vanishes, any graph with an arbitrary number of fermion loops clearly vanishes as well. These results also extend to the compact case, since the spatial momenta do not play any role in the argument, as long as $c_\psi = 0$.

The apparently strange absence of any quantum correction in the photon propagator when $c_\psi = 0$ has a simple physical explanation in the total decoupling, in the Lifshitz regime, of particle and antiparticles. In other words, the decomposition of the propagator as in Eq. (3.2) is telling us that electrons and positrons in the Lifshitz regime behave similarly to standard electrons and positrons in the nonrelativistic low-energy regime. In particular, there is no way to create a particle–antiparticle pair and hence conservation of the charge forbids any virtual pair production in the vacuum.

In conclusion, no radiative corrections at all (finite or infinite) are induced to the photon two-point function when $c_\psi = 0$. It then follows that in the UV the β functions associated with the coupling constant g and the parameter a_γ, as well as the anomalous dimensions of A_0 and A_m, vanish to all orders in perturbation theory:

$$\beta_g = \beta_{a_\gamma} = \gamma_{A_0} = \gamma_{A_m} = 0. \qquad (3.9)$$

B. Fermion propagator

The one-loop correction to the fermion propagator $\Sigma(p)$ is given by the two graphs in Fig. 2. The tadpole graph (b), as well as the exchange of a virtual temporal photon in graph (a), is easily shown to vanish in dimensional regularization. The only nontrivial contribution is given by the exchange of spatial photons in graph (a). We get

$$= \int \frac{d\omega_q}{2\pi} \frac{d^d q}{(2\pi)^d} G_{\gamma,ml}^0(\omega_q, q)$$

$$\times G_\psi^0(\omega_p + \omega_q, p + q)$$

$$\times i \left(g a_\psi (2p + q)^m + 2\lambda q_k \gamma^{mk} \right)$$

$$\times i \left(g a_\psi (2p + q)^l - 2\lambda q_p \gamma^{lp} \right). \qquad (3.10)$$

There is no need to introduce a Feynman parameter to compute the graph (3.10). As previously explained, one can first integrate over ω_q, using the residue theorem, and then over q by using dimensional regularization. Defining the counterterms ΔZ_ψ and ΔZ_{a_ψ} as

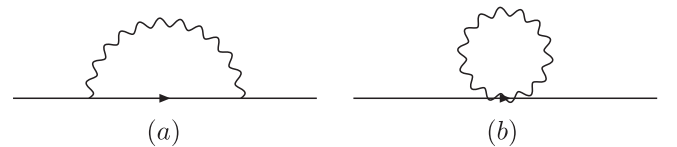$$= i\gamma^0 \omega_p \Delta Z_\psi - i a_\psi p^2 \Delta Z_{a_\psi}, \qquad (3.11)$$

FIG. 2. One-loop graphs contributing to the correction of the fermion propagator.

we get

$$\Delta Z_\psi = -\frac{1}{\epsilon} \frac{12\lambda^2}{16\pi^2 a_\gamma (a_\gamma + a_\psi)^2},$$

$$\Delta Z_{a_\psi} = \frac{1}{\epsilon} \frac{3}{16\pi^2} \left(\frac{g^2 a_\psi}{a_\gamma (a_\gamma + a_\psi)} - \frac{4\lambda^2}{(a_\gamma + a_\psi)^3} \right), \quad (3.12)$$

from which one easily derives the β function for a_ψ:

$$\beta_{a_\psi} = \epsilon a_\psi (\Delta Z_{a_\psi} - \Delta Z_\psi)$$

$$= \frac{3a_\psi^2}{16\pi^2} \left(\frac{g^2}{a_\gamma (a_\gamma + a_\psi)} + \frac{4\lambda^2}{a_\gamma (a_\gamma + a_\psi)^3} \right). \quad (3.13)$$

C. The electric vertex

In analogy to the path-integral derivation of the standard Lorentz-invariant Ward identity in scalar QED, one finds the following identity:

$$\omega_p V^0(\omega_p, p, \omega_k, k) - p^m V^m(\omega_p, p, \omega_k, k)$$
$$= g(G_\psi^f (\omega_p + \omega_k, p + k)^{-1} - G_\psi^f (\omega_k, k)^{-1}), \quad (3.14)$$

where V^0 and V^m are the full time and spatial components of the electric vertex, and G_ψ^f is the full fermion propagator, including all radiative corrections. The one-loop corrections to the electric vertex are then related to the fermion counterterms in Eq. (3.12). Denoting by ΔV_0 and ΔV^m the divergent terms of the one-loop corrections to the time and spatial components of the electric vertex, we have

$$\Delta V_0 = ig\gamma_0 \Delta Z_{V_0}, \qquad \Delta V^m = -iga_\psi (2k + p)^m \Delta Z_{V_m}. \quad (3.15)$$

The Ward identity (3.14) immediately gives

$$\Delta Z_{V_0} = \Delta Z_\psi, \qquad \Delta Z_{V_m} = \Delta Z_{a_\psi}. \quad (3.16)$$
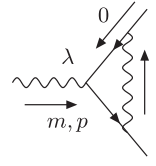
Since there is no radiative correction to the photon propagator, the coupling constant g is consistently not renormalized:

$$g_0 Z_\psi = g Z_{V_0} \rightarrow g_0 = g, \qquad g_0 Z_{a_\psi} = g Z_{V_m} \rightarrow g_0 = g. \quad (3.17)$$

We have checked, at one-loop level (for $c_\psi = c_\gamma = 0$) the validity of Eqs. (3.14), (3.15), and (3.16).
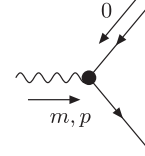
D. The magnetic vertex

The one-loop correction to the magnetic coupling λ is given by the following graph:

$$\begin{aligned}
\text{[diagram]} \quad &= (-2i\lambda) \int \frac{d\omega_q}{2\pi} \frac{d^d q}{(2\pi)^d} G_{\gamma, nl}^0 (\omega_q, q) \\
&\times G_\psi^0 (\omega_p + \omega_q, p + q) G_\psi^0 (\omega_q, q) \\
&\times i \left(-ga_\psi q^n - 2\lambda q_j \gamma^{nj} \right) p_k \\
&\times \gamma^{mk} i \left(-ga_\psi (2p + q)^l + 2\lambda q_i \gamma^{li} \right).
\end{aligned}$$

$$(3.18)$$

A straightforward computation fixes the counterterm to be

$$\text{[diagram]} \quad = -2i\lambda \Delta Z_\lambda p_k \gamma^{nk},$$

$$\Delta Z_\lambda = \frac{1}{\epsilon} \frac{4\lambda^2}{16\pi^2 a_\gamma (a_\gamma + a_\psi)^2}. \quad (3.19)$$

The $\lambda^2 g$ terms do not lead to divergences, while those of the λg^2 terms are exactly compensated by the fermion wave function counterterms in Eq. (3.12). We get

$$\beta_\lambda = \epsilon \lambda (\Delta Z_\lambda - \Delta Z_\psi) = \frac{\lambda^3}{\pi^2 a_\gamma (a_\gamma + a_\psi)^2}. \quad (3.20)$$

E. RG flow in the deep UV

The UV evolution of couplings is greatly simplified since $\beta_{a_\gamma} = \beta_g = 0$. Only a_ψ and λ undergo a quantum evolution, according to Eqs. (3.13) and (3.20). At the perturbative level no fixed points, other than the trivial $a_\psi = \lambda = 0$, may arise. For stability reasons, a_γ must be positive, whereas a_ψ can have any sign. For any choice of a_γ and a_ψ, the magnetic coupling λ grows in the UV, so that the theory does not seem to be asymptotically free. However, the effective coupling constant in the theory, the one that controls the perturbative expansion, is not λ, but a combination of λ with a_γ and a_ψ, similarly for the gauge coupling constant g. On dimensional grounds, the effective couplings are

$$\alpha \equiv \frac{g^2}{a_\gamma} f_e \left(\frac{a_\gamma}{a_\psi} \right), \qquad \beta \equiv \frac{\lambda^2}{a_\gamma a_\psi^2} f_m \left(\frac{a_\gamma}{a_\psi} \right), \quad (3.21)$$

where f_e and f_m are dimensionless functions, depending only on the ratio a_γ / a_ψ. The theory is UV completed if, for $E \rightarrow \infty$, α and β remain perturbative, and is UV free if α, $\beta \rightarrow 0$ in the limit.

Let us denote by $t = \log E/E_0$ and let us first assume that $a_\psi(0) > 0$. In this case, $\beta_{a_\psi} > 0$ and a_ψ grows in the UV. The parameter a_ψ is not a proper coupling constant and perturbativity is not necessarily lost when it grows large. It

is lost only if f_e and/or f_m in Eq. (3.21) grow with some powers of a_ψ/a_γ. On the contrary, the regime of large a_ψ is expected to be smoother and smoother, since a_ψ enters also in the fermion propagators. We now argue that f_e and f_m cannot have terms that grow like a_ψ^n, with $n > 0$. Since no fermion loops are allowed, given any graph, any additional one-particle irreducible loop is obtained by adding a photon line and two vertices to a preexisting fermion line, thus obtaining two additional fermion propagators. The photon propagator gives no factors of a_ψ, while the two fermion propagators a factor $1/a_\psi^2$. If both vertices are electric, proportional to ga_ψ, we then get $(ga_\psi)^2 \times 1/a_\psi^2 = g^2$. If they are both magnetic, we get $\lambda^2 \times 1/a_\psi^2 = \lambda^2/a_\psi^2$. Hence for large a_ψ, f_e, and f_m tend to a constant, and the effective coupling constants of the theory become

$$\alpha_0 \equiv \frac{g^2}{a_\gamma}, \qquad \beta_0 \equiv \frac{\lambda^2}{a_\gamma a_\psi^2}. \qquad (3.22)$$

Being a_γ and g constant along the RG flow,

$$\alpha_0(t) = \alpha_0(0) \qquad (3.23)$$

and α_0, if small at $t = 0$, remains so at all scales, without being asymptotically free. Setting $a_\gamma = 1$, for simplicity, the UV flow of β_0, for large a_ψ is given by

$$\dot\beta_0 = \frac{1}{2\pi^2}\beta_0^2 - \frac{3}{8\pi^2}g^2\beta_0, \qquad (3.24)$$

where a dot stands for a derivative with respect to t. Solving Eq. (3.24), we get

$$\beta_0(t) = e^{-3g^2t/8\pi^2}\beta_0(0)\left[1 + \frac{4\beta_0(0)}{3g^2}(e^{-3g^2t/8\pi^2} - 1)\right]^{-1}. \qquad (3.25)$$

Equation (3.25) shows that β_0 is asymptotically free in the UV, going to zero powerlike, provided that $4\beta_0(0) < 3g^2$.

When $a_\psi(0) < 0$, one has to distinguish the two cases $|a_\psi(0)| < a_\gamma$ and $|a_\psi(0)| > a_\gamma$. In the latter, $|a_\psi|$ grows and all the considerations made for $a_\psi > 0$ apply. In particular, β_0 is UV free. In the former case, instead, $|a_\psi|$ decreases, β_0 increases, and the coupling explodes in the UV.

Summarizing, the theory is UV completed and perturbative for $4\beta_0(0) < 3g^2$, $a_\psi > 0$, and $a_\psi < -a_\gamma$. The effective magnetic coupling β_0 goes to zero, while the electric one α_0 remains constant.

F. Relevant couplings

In this section, for completeness, we compute the UV evolution of the parameters c_ψ and c_γ in Eq. (2.7). This is easily done by considering again the one-loop fermion and photon corrections, when these parameters are nonvanish-

ing. For simplicity, we will set the magnetic coupling and the fermion mass to zero, $\lambda = M = 0$.

Let us first consider c_ψ. The counterterm (3.11) contains an extra term, so that now

$$= i\gamma^0\omega_p\Delta Z_\psi - ic_\psi p_m\gamma_m\Delta Z_{c_\psi}$$
$$- ia_\psi p^2\Delta Z_{a_\psi}. \qquad (3.26)$$

After a straightforward computation, we get

$$\Delta Z_{c_\psi} = \frac{1}{\epsilon}\frac{g^2}{32\pi^2}\frac{a_\psi a_\gamma + a_\gamma^2 - 6a_\psi^2}{a_\psi a_\gamma(a_\psi + a_\gamma)}. \qquad (3.27)$$

Since $\Delta Z_\psi = 0$ when $\lambda = 0$, Eq. (3.27) directly gives the β function for c_ψ:

$$\beta_{c_\psi} = \epsilon c_\psi\Delta Z_{c_\psi} = \frac{g^2 c_\psi}{32\pi^2}\frac{a_\psi a_\gamma + a_\gamma^2 - 6a_\psi^2}{a_\psi a_\gamma(a_\psi + a_\gamma)}. \qquad (3.28)$$

The RG evolution of c_γ is derived by computing the vacuum polarization of the spatial photon components $\Pi_{mn} = \Pi_{mn}^1 + \Pi_{mn}^2$:

$$= (-1)g^2\int\frac{d\omega_q}{2\pi}\frac{d^dq}{(2\pi)^d}$$
$$\times \mathrm{Tr}\left[i\left(c_\psi\gamma^m + a_\psi(p + 2q)^m\right)\right.$$
$$\times G_\psi(\omega_p + \omega_q, p + q)i\left(c_\psi\gamma^n\right.$$
$$\left.+ a_\psi(p + 2q)^n\right)G_\psi(\omega_q, q)\right]. \qquad (3.29)$$
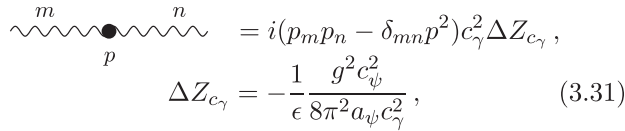
We do not write the tadpole contribution Π_{mn}^2 to Π_{mn}, given by the contraction of the fermion lines in the quartic interaction (see Fig. 1). It is energy and momentum independent and it is easily shown to ensure that $\Pi_{mn}(0, 0)$ vanishes, as dictated by gauge invariance. Spatial $SO(4)$ rotations and time reversal allow one to write Π_{mn} in the following form:

$$\Pi_{mn}(\omega, p) = i\delta_{mn}\omega^2 f_1(\omega^2, p^2)$$
$$+ i(p_m p_n - \delta_{mn}p^2)f_2(\omega^2, p^2), \qquad (3.30)$$

with f_1 and f_2 real functions. As already discussed, when $c_\psi = 0$, all photon vacuum polarization terms vanish. Hence f_1 and f_2 must be proportional to c_ψ^2. On dimensional grounds,[2] this implies that the function f_1 is finite. On the other hand, the function f_2, associated with the

---

[2]Recall that in the Lifshitz regime c_ψ and p have effectively the dimension of a mass, while ω that of a mass squared.

operator F_{mn}^2, can develop a logarithmic divergence, which will be responsible for the RG evolution of c_γ^2. We leave to the next section a detailed study of the function f_1 and focus now on the computation of the divergence of f_2. The latter is easily computed by taking the c_ψ^2 term of $f_2(0, 0)$. The counterterm ΔZ_{c_γ} canceling the above divergence is found

$$
\underset{p}{\overset{m \qquad\qquad n}{\wwwbullet}} = i(p_m p_n - \delta_{mn} p^2) c_\gamma^2 \Delta Z_{c_\gamma},
$$

$$
\Delta Z_{c_\gamma} = -\frac{1}{\epsilon} \frac{g^2 c_\psi^2}{8\pi^2 a_\psi c_\gamma^2}, \tag{3.31}
$$

from which we get

$$
\beta_{c_\gamma^2} = \epsilon c_\gamma^2 \Delta Z_{c_\gamma} = -\frac{g^2 c_\psi^2}{8\pi^2 a_\psi}. \tag{3.32}
$$

The RG Eqs. (3.13), (3.28), and (3.32) can easily be solved in the deep UV regime when $a_\psi \gg a_\gamma$, in which case we approximately have

$$
a_\psi(t) \simeq a_\psi(0) e^{(3g^2/16\pi^2)t}, \qquad c_\psi^2(t) \simeq c_\psi^2(0) e^{-(3g^2/8\pi^2)t},
$$

$$
c_\gamma^2(t) \simeq c_\gamma^2(0) + \frac{2}{9} \frac{c_\psi^2(0)}{a_\psi(0)} (e^{-(9g^2/16\pi^2)t} - 1). \tag{3.33}
$$

Notice that the UV RG behavior of c_ψ^2 and c_γ^2 does not significantly affect the physical speed $v = d\omega/dp$ of the photon and fermion, that at these energies is dominated by the classical Lifshitz regime and linearly increases with energy, $v \simeq 2\omega$.

IV. IR BEHAVIOR AND CONNECTION WITH 5D EFFECTIVE THEORIES

We now consider the behavior of the Lifshitz theory for $E \ll \Lambda$. In particular, we will focus our attention on the energy behavior of the finite function f_1, defined in Eq. (3.30). When the mass of the 5D fermion vanishes, two different regimes arise in the IR: (i) $1/R \ll E \ll 1$, where the theory is reliably described by its uncompactified 5D version and (ii) $E \lesssim 1/R \ll 1$, where compactification effects cannot be neglected. In order to distinguish the two regimes, we will denote the latter as deep IR regime. In this section we will distinguish 5D and 4D gauge couplings by a subscript. What we have so far denoted by g is replaced by g_5 and the 4D coupling is defined as

$$
g_5^2(E) \equiv 2\pi R g_4^2(E). \tag{4.1}
$$

A. RG behavior of the coupling in 5D effective field theories

The RG evolution of the inverse square gauge coupling in usual Lorentz-invariant 5D models is expected to have the usual logarithmic behavior for $E < 1/R$, when the

theory reduces to a 4D gauge theory for the zero modes, and a linear energy dependence for $E > 1/R$. Because of the presence of the infinite tower of massive Kaluza-Klein (KK) states, with increasing mass, schemes such as momentum subtraction are preferred to schemes such as minimal subtraction, in the regime $E > 1/R$.[3] The contribution due to a massive 4D fermion to the mass-dependent β function of the 4D gauge coupling g_4 is well known (see i.e. [11]):

$$
\beta_n = \frac{g_4^3}{2\pi^2} \int_0^1 dx \frac{x^2(1 - x)^2 E^2}{m_n^2 + E^2 x(1 - x)}, \tag{4.2}
$$

where E is the sliding RG (Euclidean) energy scale and $m_n^2 = M^2 + n^2/R^2$ is the mass of the KK mode n, where we keep for the moment the 5D bulk mass M. For (anti-)periodic fermions, n is an (half-)integer. The total β function is simply given by summing over all possible KK modes: $\beta_g = \sum_{n=-\infty}^{\infty} \beta_n$. Performing the sum, we get

$$
\beta_g = \frac{g_4^3 R}{2\pi} \int_0^1 dx \frac{x^2(1 - x)^2 E^2}{\sqrt{M^2 + E^2 x(1 - x)}}
$$
$$
\times \begin{cases} \coth \pi R \sqrt{M^2 + E^2 x(1 - x)}, & n \in Z, \\ \tanh \pi R \sqrt{M^2 + E^2 x(1 - x)}, & n \in Z + \frac{1}{2}. \end{cases} \tag{4.3}
$$

Equation (4.3) gives the following RG behavior for g_4^{-2}:

$$
g_4^{-2}(E) = g_4^{-2}(E_0) - \frac{1}{\pi^2} \int_0^1 dx x(1 - x)
$$
$$
\times \begin{cases} \log \frac{\sinh(\pi R \sqrt{E^2 x(1-x) + M^2})}{\sinh(\pi R \sqrt{E_0^2 x(1-x) + M^2})}, & n \in Z, \\ \log \frac{\cosh(\pi R \sqrt{E^2 x(1-x) + M^2})}{\cosh(\pi R \sqrt{E_0^2 x(1-x) + M^2})}, & n \in Z + \frac{1}{2}. \end{cases} \tag{4.4}
$$

It is useful to see in detail the regimes of the vanishing compactification radius, $R \to 0$, and of the decompactification limit, $R \to \infty$, for $M = 0$ and $M \gg E$. When $R \to 0$, independently of M, for antiperiodic fermions, the argument of the logarithmic term approaches 1, giving no running at all, as expected, being all KK modes decoupled in this limit. On the other hand, for periodic fermions, we get

$$
g_4^{-2}(E) = g_4^{-2}(E_0) - \frac{1}{6\pi^2} \log \frac{E}{E_0}, \qquad R \to 0, \qquad M = 0, \tag{4.5}
$$

---

[3]On the other hand, there are no problems in using dimensional regularization if one is interested in the evolution of the coupling below $1/R$. KK modes can be integrated out and their threshold corrections reliably computed in dimensional regularization; see e.g. [10].

$$g_4^{-2}(E) = g_4^{-2}(E_0) - \frac{1}{60\pi^2} \frac{E^2}{M^2} + \mathcal{O}\left(\frac{E^4}{M^4}\right),$$

$$R \to 0, \qquad M \gg E \gg E_0, \qquad (4.6)$$

which reproduce the usual logarithmic contribution due to a massless zero mode and the usual $1/M^2$ decoupling of a massive state in 4D. When $R \to \infty$, the hyperbolic trigonometric functions in the argument of the logarithmic term become equal and for both periodic and antiperiodic fermions we get, for $M = 0$,

$$g_5^{-2}(E) = g_5^{-2}(E_0) - \frac{3}{256\pi}(E - E_0),$$

$$R \to \infty, \qquad M = 0, \qquad (4.7)$$

where we have used Eq. (4.1). It is important to notice that Eq. (4.7) does not give a quantitatively trustable behavior of the coupling constant beyond one-loop level. Indeed, at two-loop level, by dimensional analysis and unitarity, a contribution to the right-hand side of Eq. (4.7) $\sim g_5^2(E_0)E^2 \log ER$ is expected. If we expand in $g_5^2(E_0)$, the two-loop term would be comparable or larger than the second iteration of the one-loop term. The RG flow for $1/R < E < 1$ is hence not very useful.[4] This issue is however not important for what follows, because in our UV completion of the theory at arbitrarily high energies we demand to always remain in the perturbative regime.

The uncompactified and large mass limit is taken by demanding $MR \gg ER \gg E_0R$. In this limit we get

$$g_4^{-2}(E) = g_4^{-2}(E_0) - \frac{ER}{60\pi} \frac{E}{M} + \mathcal{O}\left(\frac{E^4R}{M^3}\right),$$

$$MR \gg ER \gg E_0R. \qquad (4.8)$$

The decoupling in 5D is not as efficient as in 4D, with the heavy particle effects vanishing as $1/M$, as opposed to $1/M^2$, as in Eq. (4.8). In the 5D regime with $E \gg 1/R$, the factor of R in the numerator of Eq. (4.8) enhances the

effect. Contrary to usual 4D theories, massive particles in 5D start to give a significant contribution to the gauge coupling evolution at energies well before their masses.[5]

B. One-loop coupling behavior in the Lifshitz theory

The energy behavior of the coupling in the Lifshitz model is obtained by studying the function f_1, defined in Eq. (3.30). Gauge invariance implies the following relations among the different photon polarization terms:

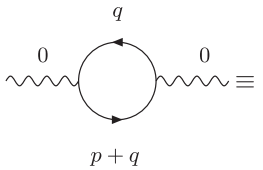$$\omega \Pi_{00} - p_m \Pi_{m0} = 0, \qquad \omega \Pi_{0n} - p_m \Pi_{mn} = 0. \quad (4.9)$$

In the parametrization (3.30) of Π_{mn}, Eq. (4.9) fixes

$$\Pi_{00} = ip^2 f_1(\omega^2, p^2), \qquad \Pi_{0m} = i\omega p_m f_1(\omega^2, p^2). \qquad (4.10)$$

The function f_1 is responsible for a finite renormalization of the photon kinetic term. The redefinition of the photon field $A \to A/\sqrt{1 + f_1}$ brings the kinetic term back in canonical form, but in so doing we get a rescaled coupling constant

$$g_5^2(\omega, p^2) = \frac{g_5^2(\omega_0, p_0^2)}{1 + f_1(\omega^2, p^2) - f_1(\omega_0^2, p_0^2)}, \qquad (4.11)$$

where ω_0 and p_0 are arbitrary. Because of the Ward identity (3.14) there is no further renormalization of the coupling. In the IR, by fine-tuning $c_\psi = c_\gamma$ with sufficient accuracy (see Sec. V), our model flows to a Lorentz-invariant 5D theory. It is then unnecessary to study g_5^2 as a function of both energy and momentum. We set, after having extracted the appropriate powers of p as given by Eq. (4.10), $p = 0$ and study the gauge coupling as a function of the energy only. We take vanishing momentum of the photon in the compact space. The function f_1 is most easily computed by looking at the Π_{00} component of the photon vacuum polarization for Euclidean energies $\omega = iE$. We have

$$\equiv \Pi_{00} = (-1)g_4^2 \sum_{n=-\infty}^{\infty} \int \frac{d\omega_q}{2\pi} \frac{d^3q}{(2\pi)^3} \text{Tr}\left[i\gamma^0 G_\psi(iE + \omega_q, p + q)i\gamma^0 G_\psi(\omega_q, q)\right]$$

$$= -4ig_4^2 \sum_{n=-\infty}^{\infty} \int \frac{d\omega_E}{2\pi} \frac{d^3q}{(2\pi)^3} \int_0^1 dx \frac{-\omega_E^2 + E^2x(1-x) + a_\psi^2 q^2(p+q)^2 + c_\psi^2(q^2 + p \cdot q)}{\left[\omega_E^2 + E^2x(1-x) + r(p,q)\right]^2}, \quad (4.12)$$

---

[4]We thank R. Rattazzi for this observation.
[5]An equation very similar to Eq. (4.4) has been computed in [12] in dimensional regularization, but the $1/M$ decoupling of heavy states was not emphasized, being decoupling not manifest in that scheme.

with

$$r(p, q) = a_\psi^2 q^4 (1 - x) + a_\psi^2 (p + q)^4 x + c_\psi^2 q^2 (1 - x)$$
$$+ c_\psi^2 (p + q)^2 x,$$
$$q^2 = q_i q_i + \frac{n^2}{R^2}. \tag{4.13}$$

Expanding up to quadratic terms in the spatial momentum p, and by integrating over ω_E and afterward over x, we get the desired form of the function $f_1(iE, 0)$:

$$f_4(E, 1/R) \equiv g_4^{-2} f_1(iE, 0)$$
$$= \frac{1}{3\pi^2 c_\psi} \sum_{n=-\infty}^{\infty} \int_0^\infty$$
$$\times \frac{s^2 (3\tilde{s}^4 + 3\tilde{s}^2 - s^2) ds}{\tilde{s}^3 (1 + \tilde{s}^2)^{3/2} (4\tilde{s}^2 + 4\tilde{s}^4 + \mu^2)}, \tag{4.14}$$

with

$$\mu = \frac{a_\psi E}{c_\psi^2}, \qquad s = \frac{q a_\psi}{c_\psi}, \qquad \tilde{s}^2 = s^2 + \frac{a_\psi^2 n^2}{c_\psi^2 R^2}. \tag{4.15}$$

At *any* energy scale, $f_4(E, 1/R)$ gives the one-loop energy behavior of the gauge coupling:

$$g_4^{-2}(E) = g_4^{-2}(E_0) + f_4(E, 1/R) - f_4(E_0, 1/R). \tag{4.16}$$

When $E \gg 1/R$, the sum over the KK modes in Eq. (4.14) is reliably approximated by a continuos integration and we get

$$f_5(E) \equiv g_5^{-2} f_1(iE, 0)$$
$$\simeq \frac{1}{16\pi^2 a_\psi} \int_0^\infty \frac{s^2 (3 + 4s^2) ds}{(1 + s^2)^{3/2} (4s^2 + 4s^4 + \mu^2)},$$
$$E \gg 1/R. \tag{4.17}$$

Independently of the UV completion of the theory, the energy dependence of the coupling constant at low energies should be dictated by its RG evolution. Indeed, the logarithmic and linear terms in Eqs. (4.5) and (4.7) are nonanalytic in E^2, due to the IR behavior of the integrand, and thus calculable. We did not find an explicit formula for $f_4(E, 1/R)$, but its deep IR behavior can be extracted by considering the $n = 0$ R-independent KK mode only and noticing that the IR divergence of the integrand comes from small values of s so that we can simplify the integrand. We get

$$f_4(E) - f_4(E_0) \simeq \frac{2}{3\pi^2 c_\psi} \int_0^\infty \frac{s(\mu_0^2 - \mu^2) ds}{(4s^2 + \mu^2)(4s^2 + \mu_0^2)}$$
$$= -\frac{1}{6\pi^2 c_\psi} \log \frac{E}{E_0},$$
$$E, E_0 \ll \frac{c_\psi}{R}, \tag{4.18}$$

reproducing Eq. (4.5) for $c_\psi = 1$. In the uncompactified

limit, it is possible to write an approximate analytic formula for $f_5(E)$ (which gives the exact limit in both the UV and the IR) from which the IR behavior (4.7) is computed:

$$f_5(E) \simeq \frac{1}{16\pi^2 a_\psi} \left[\int_0^1 ds \frac{3s^2}{(4s^2 + \mu^2)} \right.$$
$$\left. + \int_1^\infty ds \frac{4s}{(4s^4 + \mu^2)} \right]$$
$$= \frac{4\pi + 6\mu - (8 + 3\mu^2) \arctan(\frac{2}{\mu})}{128\pi^2 a_\psi \mu}. \tag{4.19}$$

Expanding Eq. (4.19) for small μ, one finds

$$f_5(E) - f_5(E_0) = -\frac{3}{256\pi c_\psi^2} (E - E_0) + \mathcal{O}(E^2),$$
$$\tag{4.20}$$
$$E \ll \frac{c_\psi}{a_\psi},$$

reproducing the coefficient in Eq. (4.7), with $c_\psi = 1$.

In the deep UV regime, $E \gg 1$, Eq. (4.19) gives

$$f_5(E) = \frac{c_\psi^2}{32\pi a_\psi^2} \frac{1}{E} + \mathcal{O}\left(\frac{1}{E^2}\right), \tag{4.21}$$

showing that the correction to the coupling vanishes like $1/E$ for $E \to \infty$.[6]

At a more quantitative level, Eq. (4.19) is not a very accurate approximation of $f_5(E)$ in the whole E range. A more reliable, but more complicated, analytic expression is reported in the Appendix; see Eq. (A2). We plot, for illustration, $g_5^{-2}(E)$ in the 5D uncompactified limit, as given by Eqs. (4.7) and (4.17) (Fig. 3) and $g_4^{-2}(E)$ in the 4D compact case, as given by Eqs. (4.7) and (4.14) (Fig. 4).

C. Cutoff of the effective theory and comparison with NDA estimates

The range of validity of 5D theories as perturbative and calculable effective field theories is typically estimated, in absence of a concrete UV completion, by using naive dimensional analysis [5]. A possible definition of the maximum energy scale Λ above which the theory breaks down is derived from the photon vacuum polarization term.[7] When the one-loop correction becomes of the same order as the tree-level term, calculability is certainly lost. A naive often used estimate takes into account the phase space of the loop integration, taken as in 5D uncompactified space, giving

---

[6]The correction actually vanishes as $1/E^{1+3g^2/(4\pi^2)}$, due to the UV RG evolution of c_ψ and a_ψ, as given by Eq. (3.33).
[7]Strictly speaking, one should use physical observables to identify the breakdown of the theory, yet we believe that the photon propagator is a reliable quantity to look at.
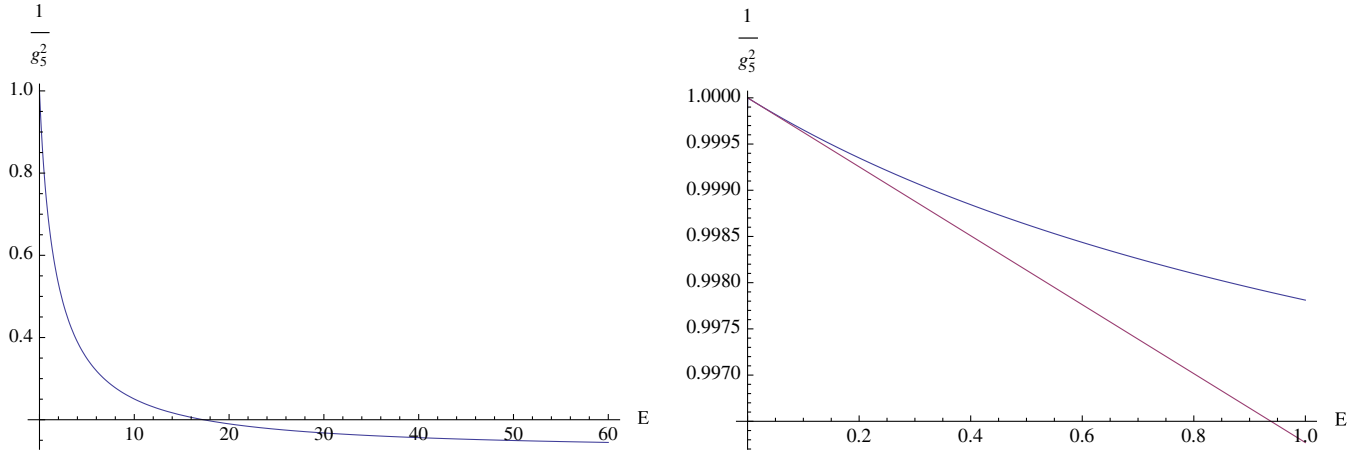
FIG. 3 (color online).   Left panel: g_5^{-2} as a function of the energy scale E in the uncompactified Lifshitz theory. Right panel: Comparison between the Lifshitz behavior (blue line) with the 5D linear regime (4.7) (straight red line). The energy is in units of Λ. We have taken $E_0 = 0$, $g_5^{-2}(0) = 1$, and $a_\psi = 1$.

$$\frac{g_5^2 \Lambda^{(1)}}{24\pi^3} = 1 \Rightarrow \Lambda^{(1)} = \frac{12\pi^2}{g_4^2}\frac{1}{R}, \qquad (4.22)$$

where $24\pi^3$ is the 5D loop factor, and g_4^2 is computed, say, at the compactification scale $1/R$. A closer inspection of the 5D vacuum polarization diagram shows that further factors of π arise from the momentum integration. A more careful and conservative estimate would use the standard 4D loop factor to get

$$\frac{g_5^2 \Lambda^{(2)}}{16\pi^2} = 1 \Rightarrow \Lambda^{(2)} = \frac{8\pi}{g_4^2}\frac{1}{R}, \qquad (4.23)$$

roughly a factor of 5 smaller than (4.22). Another similar estimate can be given by comparing the one-loop term in Eq. (4.7) to the classical one, $g_5^{-2}(E_0)$. In this way, we get
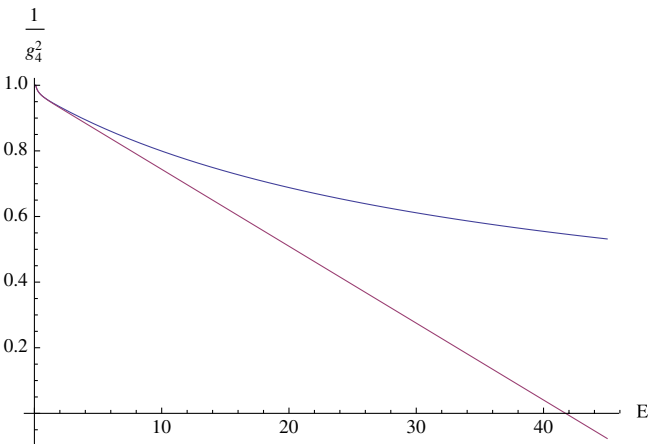

FIG. 4 (color online).   Comparison between the exact Lifshitz behavior as given by Eq. (4.14) (blue line) with the effective coupling (4.4) for periodic fermions with $M = 0$ (straight red line). The energy is in units of $1/R$. We have taken $E_0 = 1/(10R)$, $g_4^{-2}(1/(10R)) = 1$, $a_\psi = 1$, and $\Lambda = 25/R$.

$$\Lambda^{(3)} = \frac{128}{3g_4^2}\frac{1}{R}. \qquad (4.24)$$

All these estimates unambiguously show that there is not a parametrically large range in energies (when $g_4 \sim 1$) where 5D theories are calculable and reliable effective field theories. In this situation a factor of a few in the cutoff estimate can make the difference in defining a model reliable or not and it is hence very important to improve by any means in discerning between the above estimates or adding new ones.

Our Lifshitz-like UV completion can be quite useful in this sense. The cutoff scale Λ should be here identified with the Lifshitz cutoff scale (so far set to 1) which we now make explicit as in Sec. II by setting $a_\psi \to a_\psi/\Lambda$ in our previous formulas. The impossibility of having a too large window between $1/R$ and Λ is clearly visible in our theory. If $1/R$ is too small with respect to Λ, the 5D linear regime (4.7) of g^{-2} is too long before the Lifshitz operators come to the rescue and perturbation theory breaks down. We can compute which is the maximum allowed value for ΛR by demanding that $g_4^{-2}(E)$ is definite positive for arbitrarily high-energy scales. Since for $E \gg \Lambda$, $f_4 \simeq 2\pi R f_5$ goes to zero [see Eq. (4.21)], taking $E = \infty$ and $E_0 = 1/R$ in Eq. (4.16), we get

$$0 \le g_{4,\infty}^{-2} = g_4^{-2} - f_4\left(\frac{1}{\Lambda R}, \frac{1}{\Lambda R}\right), \qquad (4.25)$$

where $g_4 = g_4(E_0 = 1/R)$ and $g_{4,\infty} = g_4(E = \infty)$. Inverting Eq. (4.25) to get Λ as a function of R and g_4 is complicated. However, for $\Lambda R \gg 1$, $\mathcal{O}(\Lambda R)$ KK modes significantly contribute to f_4 and we can safely replace f_4 by its noncompact version f_5, $f_4(1/\Lambda R, 1/\Lambda R) \simeq 2\pi R f_5(1/\Lambda R)$. We can still approximate this expression
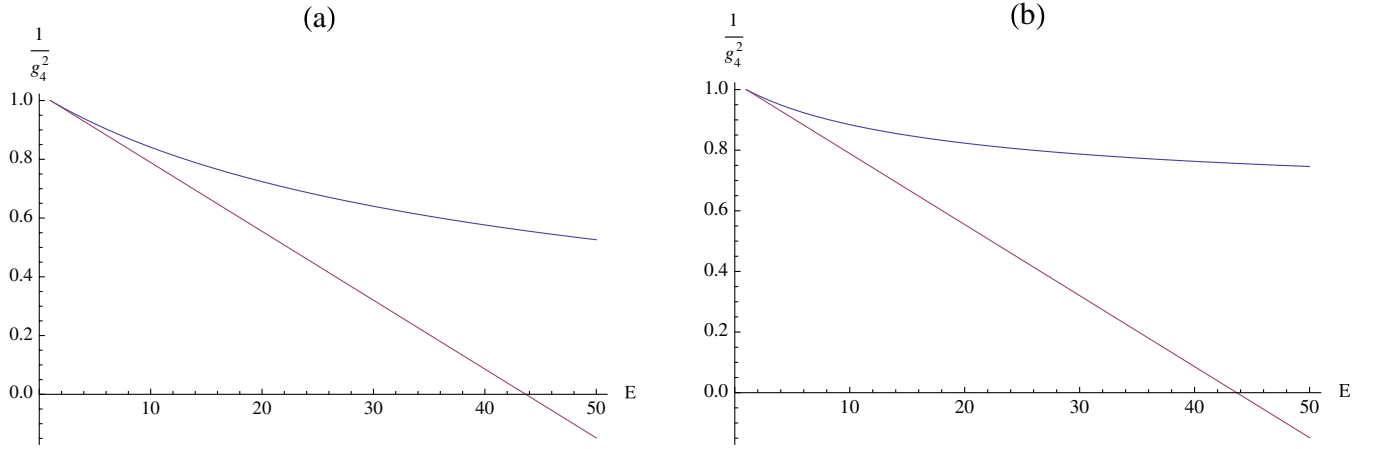
FIG. 5 (color online). Comparison between g_4^{-2} as a function of the energy scale E in the Lifshitz theory (blue lines), as given by Eq. (A2), and the effective 5D coupling (4.7) (straight red lines). The energy is in units of $1/R$. We have taken $E_0 = 1/R$ and $g_4^{-2}(1/R) = 1$. (a) $a_\psi = 1$ and $\Lambda = 25/R$. (b) $a_\psi = 10$ and $\Lambda = 100/R$. Despite the higher cutoff, the range of validity of the 5D effective theory is lower in (b) than in (a).

by evaluating f_5 at zero[8]:

$$f_5(0) = \frac{5\Lambda}{96a_\psi\pi^2},\qquad(4.26)$$

from which we finally find (writing here $\Lambda \to \Lambda^{\mathrm{Lif}}$ to emphasize that it is the cutoff of the Lifshitz completion of the theory)

$$\Lambda^{\mathrm{Lif}} \gtrsim \frac{48\pi a_\psi}{5}\frac{1}{R}\left(\frac{1}{g_4^2} - \frac{1}{g_{4,\infty}^2}\right),\qquad(4.27)$$

where $a_\psi = a_\psi(\Lambda)$, essentially constant below Λ [7]. It is reasonable that Λ^{Lif} depends on a_ψ, since the effective scale regulating the low-energy fermion propagators is Λ/a_ψ and not Λ. Increasing a_ψ, however, implies that the Lifshitz regime takes over the usual 5D regime earlier, and before reaching the scale Λ the 5D theory receives sizable UV-sensitive corrections. This is illustrated in Fig. 5, where we show how increasing a_ψ allows a higher cutoff Λ, but does not change the range of validity of the effective field theory, which is always given by the scale (4.27) with $a_\psi = 1$.

Comparing Eq. (4.27) with Eqs. (4.22), (4.23), and (4.24), we see that for $a_\psi = 1$ the Lifshitz cutoff computation is closer to the most conservative of the three NDA cutoff estimates, Eq. (4.23), provided that $g_{4,\infty}$ is not too small. The more perturbative the UV completion is, the lower the cutoff becomes.

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[8]We have numerically verified that the above two approximations lead to less than 10% deviations from the exact value for $\Lambda R \sim 10$–30.

V. IR EVOLUTION OF c_γ AND c_ψ, FINE-TUNING AND ASTROPHYSICAL CONSTRAINTS

The recovery of Lorentz invariance at low energy from the Lifshitz 5D QED is not automatic, since there is no mechanism enforcing $c_\gamma = c_\psi$. These parameters evolve in the UV according to Eq. (3.33), but their IR evolution is radically different. For simplicity, we focus only on the 5D IR regime, $1/R < E < \Lambda$. As explained before, in this regime it makes sense only a perturbative expansion in the coupling and thus the RG technique is not very useful. We will nevertheless continue to use this language for convenience and for homogeneity with the deep IR and UV regimes, in which the RG flow is useful.

When $E \ll \Lambda$, we can safely neglect the higher-derivative Lifshitz operators in the Lagrangian (2.7), and we end up with the usual 5D QED, with $c_\gamma \neq c_\psi$. When $c_\gamma \neq c_\psi$, Lorentz invariance is broken and both parameters run. The IR β functions $\beta_{c_\psi}^{\mathrm{IR}}$ and $\beta_{c_\gamma}^{\mathrm{IR}}$ are easily determined. We take $c_\gamma = 1$ as the initial condition, define $\delta c = c_\psi - c_\gamma$, and assume $\delta c \ll 1$, so that we keep only up to linear terms in δc. Let us first consider $\beta_{c_\gamma}^{\mathrm{IR}}$ that can be determined by looking at the spatial components Π_{mn} of the photon propagator corrections. In the one-loop vacuum polarization photon diagram only one particle, the fermion, enters. Modulo a rescaling the graph is Lorentz invariant and correspondingly $\beta_{c_\gamma}^{\mathrm{IR}}$ is proportional to the gauge coupling β function. At linear order in δc, we find

$$\beta_{c_\gamma}^{\mathrm{IR}} = k_\gamma g_5^2 E\delta c,\qquad(5.1)$$

where $k_\gamma = -3/(256\pi)$. The β function $\beta_{c_\psi}^{\mathrm{IR}}$ is determined by looking at the one-loop fermion propagator correction Σ. We define the functions $f_\psi^{0,1}$ as

$$\Sigma = i\gamma^0\omega f_\psi^0 - ic_\psi\gamma_m p_m f_\psi^1.\qquad(5.2)$$

In this way,

$$\beta_{c_\psi}^{\text{IR}} = c_\psi E \frac{\partial}{\partial E}(f_\psi^1 - f_\psi^0). \tag{5.3}$$

Although the functions $f_\psi^{0,1}$ are gauge dependent, the latter cancels in the difference so that β_{c_ψ} is gauge invariant. We get

$$\beta_{c_\psi}^{\text{IR}} = k_\psi g_5^2 E \delta c, \tag{5.4}$$

where $k_\psi = 25/(1024\pi)$. Equations (5.1) and (5.4) are easily integrated giving

$$\delta c(E) = \delta c(E_0)[1 + g_5^2(E_0)k_\gamma(E - E_0)]^{(k_\psi - k_\gamma)/k_\gamma},$$

$$c_\psi(E) = c_\psi(E_0) + \delta c(E_0)\frac{k_\psi}{k_\psi - k_\gamma}$$

$$\times [1 + g_5^2(E_0)k_g(E - E_0)]^{(k_\psi - k_\gamma)/k_\gamma}. \tag{5.5}$$

Plugging the values found for k_γ and k_ψ in Eq. (5.5), we get

$$\delta c(E) \simeq \frac{\delta c(E_0)}{[1 + g_5^2(E_0)k_\gamma(E - E_0)]^3}. \tag{5.6}$$

The factor $\delta c(E)$ decreases toward the IR, as desired. Unfortunately, the decrease one gets is not very efficient to avoid the need of fine-tuning. Even if we pretend that Eq. (5.6) is reliable beyond one-loop level, and take, for example, $E = 1/R$, $E_0 \simeq 25/R$, $g_4^2(E_0) \simeq 4\pi$ (strong coupling), δc decreases along the flow by at most 3 orders of magnitude, whereas experimental bounds require $\delta c \lesssim 10^{-21}$ for ordinary particles [13] (for electrons and photons the bounds are slightly less severe, see e.g. [14]). Nevertheless, by appropriately tuning $\delta c(E_0)$, Lorentz invariance can always be achieved with the desired accuracy. In the deep IR regime, the evolution of c_ψ and c_γ will change from a linear to a logarithmic behavior below $1/R$. No new qualitative features emerge and we will not report the corresponding results.

Even if $c_\gamma = c_\psi = 1$ to a sufficient precision in the deep IR, the dispersion relations of photon and electrons in our theory are modified:

$$v_i(p) = \frac{d\omega_i}{dp} = \frac{2a_i^2 p^3 + c_i^2 p}{\sqrt{a_i^2 p^4 + c_i^2 p^2}} = 1 + \frac{3a_i^2 p^2}{2} + \mathcal{O}(p^{-4}),$$

$$i = \psi, \gamma, \tag{5.7}$$

leading to an energy-dependent maximum allowed speed for the two particles. Astrophysical bounds, particularly coming from cosmic ray observations, generally constrain the size of the $a_i p^2$ corrections above, pushing Λ to very high scales for $a_i \sim 1$ (see e.g. [15] for an overview). We have not systematically studied the bounds on Λ coming from these experiments, but considered only a specific one, which is quite likely not the most stringent one. It arises from the time delay measured by the Fermi experiment in the gamma ray burst GRB 080916C at redshift $z = 4.35$

[16] and has the advantage of being purely kinematical. This bound can be roughly cast in the following way:

$$|v_\gamma^2(1 \text{ MeV}) - v_\gamma^2(10 \text{ GeV})| \lesssim 10^{-17}. \tag{5.8}$$

Reinserting the scale Λ in Eq. (5.7), the bound (5.8) gives

$$\Lambda \gtrsim 5 \times 10^9 \text{ GeV} a_\gamma. \tag{5.9}$$

VI. CONCLUSIONS

We have constructed a renormalizable, UV-completed, Lifshitz-like theory that reduces at low energies to the standard QED in 5D. This is the simplest and most concrete UV completion of a ED theory we are aware of, with excellent UV properties. In particular, the gauge coupling constant is finite to all orders in perturbation theory. The one-loop behavior of the coupling is described, at *all* energy scales, by Eq. (4.14). We have shown in detail how Eq. (4.14) reproduces, as it should, the energy behavior of a coupling constant in 4 and 5 dimensions at lower energies. We have then derived a bound on the size of the cutoff in the 5D QED theory, based on our UV completion. Our results show that the often used NDA estimate (4.22) is too optimistic, while the more conservative estimate (4.23) is more reliable.

Admittedly, our UV completion is not very well motivated. One has to impose a severe fine-tuning to recover Lorentz invariance at low energies [7,8]. Moreover, the Lifshitz cutoff is severely constrained by astrophysical data, as shown e.g. in Eq. (5.9). Nevertheless, we think our model can be useful, at least seen as a toy UV-completion mechanism of effective ED theories. Several issues related to the calculability of higher dimensional nonrenormalizable theories and the UV sensitivity of observables can concretely be addressed using generalizations of our QED Lifshitz construction. Given the simplicity of the theory, we think that all the necessary generalizations needed to construct a UV completion of phenomenologically interesting models (interval compactifications, localized brane terms, warp factors, etc.) should not represent a too complicated task.

ACKNOWLEDGMENTS

APPENDIX: AN APPROXIMATE ANALYTIC EXPRESSION FOR $f_5(E)$

This approximation is found by decomposing the integral over s appearing in Eq. (4.17) in two: $\int_0^\infty ds = \int_0^1 ds + \int_1^\infty ds$, and simplifying the integrand in the two regimes as follows:

$$f_5(E) \simeq \frac{1}{16\pi^2 a_\psi} \left[\int_0^1 ds \frac{3s^2}{(4s^2 + 4s^4 + \mu^2)} + \int_1^\infty ds \frac{4s}{(4s^2 + 4s^4 + \mu^2)} \right]. \tag{A1}$$

The s integration can now be performed and we obtain the following expression for $f_5(E)$:

$$f_5(E) = \frac{1}{128\pi^2 a_\psi} \left\{ \frac{3\sqrt{2}[(-1 + \sqrt{1-\mu^2}) \arctan(\frac{\sqrt{2}}{\sqrt{1-\sqrt{1-\mu^2}}}) + \mu \arctan(\frac{\sqrt{2}}{\sqrt{1+\sqrt{1-\mu^2}}})]}{\sqrt{1-\mu^2}\sqrt{1-\sqrt{1-\mu^2}}} \right.$$
$$\left. - \frac{4[(\pi + 2\arctan(\frac{3}{\sqrt{-1+\mu^2}}))\theta(1-\mu) - (\pi - 2\arctan(\frac{3}{\sqrt{-1+\mu^2}}))\theta(\mu-1)]}{\sqrt{\mu^2 - 1}} \right\}, \tag{A2}$$

with μ as in Eq. (4.15). Despite the appearance of negative square roots for any μ, the function $f_5(E)$ is real. Equation (A2) turns out to be a very good approximation of Eq. (4.17).

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