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UV/IR mixing via a Seiberg-Witten map for noncommutative QED

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We consider quantum electrodynamics in noncommutative spacetime by deriving a Seiberg-Witten map, nonperturbative in θ, with fermions in the fundamental representation of the gauge group as an expansion in the coupling constant. Accordingly, we demonstrate the persistence of UV/IR mixing in noncommutative QED with charged fermions via a Seiberg-Witten map, extending the results of Schupp and You [P. Schupp and J. You, J. High Energy Phys. 08 (2008) 107.].

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I. INTRODUCTION

The construction of renormalizable quantum field theories in noncommutative spacetime endowed with canonical coordinate commutation relations $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ is a long-standing problem, the solution of which is necessary for the calculation of testable predictions for these theories. They are expected to give hints of the underlying quantum structure of spacetime, in particular, due to their appearance in string theory [1] and in semiclassical situations, where principles of quantum field theory and general relativity are combined [2]. Arguably, the most serious obstacle for the formulation of these noncommutative quantum field theories is the so-called UV/IR mixing, giving rise to nonrenormalizable divergencies, which seem to be a generic property of any quantum field theory in noncommutative spacetime due to the inherent infinite range of nonlocality induced by the noncommutativity. Various solutions have been proposed to cure the problem. (See, e.g., [3] for a review.)

In their seminal paper [1] on the connection between noncommutative geometry and string theory, Seiberg and Witten introduced a map, which relates gauge field theories in noncommutative spacetime to ordinary commutative ones, known as the Seiberg-Witten map. This map has virtues, since many aspects of gauge theories, such as observables and gauge fixing, are more easily understood and dealt with in the language of ordinary theories. On the other hand, it also has certain uniqueness ambiguities explored in [4]. Moreover, it does not seem to affect at all some problems stemming from the noncommutativity, an example of which is the no-go theorem [5,6], according to which fields can transform nontrivially under only two different gauge groups $U_\star(N)$ (see also [7]).

The appearance of UV/IR mixing in the usual formulation of noncommutative gauge theories has been confirmed, e.g., in [8]. Therefore it is interesting to study whether the Seiberg-Witten map affects the problem of UV/IR mixing. Indeed, it has been argued, for example, in [9],[1] that the mixing of UV and IR sectors of noncommutative theories is absent in the Seiberg-Witten formalism. The study has been done also for noncommutative Yang-Mills theories (see, e.g., [10,11]), up to the first order in θ, leading to the conclusion of one-loop renormalizability. However, we suspect that this may be due to the expansion in the noncommutativity parameter matrix θ in the θ-expanded Seiberg-Witten map.[2] In the θ-nonperturbative Seiberg-Witten map for noncommutative QED, the UV/IR mixing problem does appear, as we shall demonstrate. The same argument has been expressed by Schupp and You in [13], where they considered a noncommutative model with a gauge field coupled with a spinor field in the adjoint representation of the gauge group, and showed the existence of an IR-divergent term for the photon self-energy corrections. The adjoint representation, however, corresponds to a chargeless particle but with an electric dipole moment proportional to θ [14–17], and thus in their model the interaction vanishes at the commutative limit $\theta \to 0$. Therefore the model does not correspond to a noncommutative theory of electrically charged fermions, which should reduce to the ordinary QED in the commutative limit.

In this paper, our primary goal is to extend the analysis of Ref. [13] to the case of noncommutative QED with charged fermions. We first derive a Seiberg-Witten map, nonperturbative in θ, for a gauge theory with a spinor field in the fundamental representation of the gauge field, corresponding to charged fermions, as an expansion in the coupling constant, and then demonstrate the persistence of UV/IR mixing in the photon self-energy corrections.

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[1]In particular, in [9] it was shown that via a Seiberg-Witten map the photon self-energy diagram can be renormalized up to any *finite* order in θ by shifting the nonrenormalizable terms up to the next order.

[2]We should point out that even in the θ-expanded formalism with Seiberg-Witten map, NCQED appears to be nonrenormalizable in the first order in θ [12].

II. θ-NONPERTURBATIVE SEIBERG-WITTEN MAP WITH CHARGED FERMIONS

The Seiberg-Witten map, as introduced in Ref. [1], is a technique to induce a gauge orbit preserving mapping $(A_\mu, \Lambda) \mapsto (\hat{A}_\mu, \hat{\Lambda})$ between gauge fields and gauge transformation parameters in commutative and noncommutative spacetimes, respectively. The map has previously been realized either as an expansion in the noncommutativity parameters $\theta^{\mu\nu}$ or in the gauge field A_μ as established for an Abelian gauge field theory in [13,18,19] (in the respective order). However, there is a third way, namely, an expansion in the coupling constant, which is the one we shall use in the following.[3] This is particularly convenient, since for the usual perturbation theory of QED we shall perform an expansion in the coupling constant in any case. Accordingly, we are able to avoid performing multiple expansions by taking terms of the expansion series of Seiberg-Witten map appropriately into account. We will also add a spinor field in the fundamental representation of the gauge field into the picture, thus inducing a map $(\Psi, A_\mu, \Lambda) \mapsto (\hat{\Psi}, \hat{A}_\mu, \hat{\Lambda})$.

The strategy in deriving the θ-nonperturbative Seiberg-Witten map, in a nutshell, is first to relate two gauge field theories in noncommutative spacetimes with infinitesimally differing noncommutativity parameter matrices, say θ and θ', to each other in a gauge orbit preserving way, and then to integrate this relation from the origin $\theta_0 \equiv 0$ to some constant matrix θ_1 along a path in the space of 4×4 real-valued antisymmetric matrices. Thus, let us have two noncommutative gauge field theories with spinor fields, denoted by $\mathcal{T}[\theta^{\mu\nu}, A_\mu, \Psi]$ and $\mathcal{T}'[\theta'^{\mu\nu}, A'_\mu, \Psi']$, where the arguments are the noncommutativity parameters, the gauge fields and the spinor fields, respectively. Let us also introduce the notation

$$\theta'^{\mu\nu} - \theta^{\mu\nu} = \delta\theta^{\mu\nu}, \qquad A'_\mu - A_\mu = a_\mu, \qquad \Psi' - \Psi = \psi. \tag{1}$$

As prescribed, we assume that $\delta\theta^{\mu\nu}$ are infinitesimal, and that the fields depend smoothly on the noncommutativity parameters, so that a_μ, ψ and all their partial derivatives are also infinitesimal.

Let us now consider a map of the fields from $\mathcal{T}$ to $\mathcal{T}'$. We may think of the fields in $\mathcal{T}'$ as depending on the fields in $\mathcal{T}$ according to this mapping, namely[4]

$$\begin{aligned} A'_\mu &\equiv A'_\mu(A) = A_\mu + a_\mu(A) \quad \text{and} \\ \Psi' &\equiv \Psi'(\Psi, A) = \Psi + \psi(\Psi, A). \end{aligned} \tag{2}$$

Now, we apply a gauge transformation in the theory $\mathcal{T}$ with a gauge transformation parameter Λ. For a noncommutative gauge field theory a gauge transformation is given by the formulas[5]

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda + ig[\Lambda, A_\mu]_\star, \qquad \delta_\Lambda \Psi = ig\Lambda \star \Psi, \tag{3}$$

where g is the coupling constant and the noncommutative $\star$-product is the Moyal product defined as

$$f \star g = \exp\left[\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right] f(x)g(y)|_{x=y}. \tag{4}$$

The fundamental requirement for the Seiberg-Witten map is that it should preserve the gauge equivalence classes of the theory, so that the transformation Λ in $\mathcal{T}$ corresponds to a gauge transformation

$$\Lambda' \equiv \Lambda'(\Lambda, A) = \Lambda + \lambda(\Lambda, A) \tag{5}$$

in $\mathcal{T}'$:

$$A'_\mu(A + \delta_\Lambda A) = A'_\mu(A) + \delta_{\Lambda'} A'_\mu(A), \tag{6}$$

$$\Psi'(\Psi + \delta_\Lambda \Psi, A + \delta_\Lambda A) = \Psi'(\Psi, A) + \delta_{\Lambda'}\Psi'(\Psi, A). \tag{7}$$

By substituting the formulas (1) and (3) into (6) and (7), and using the relation

$$\begin{aligned} f \star' g &= f e^{(i/2)\overleftarrow{\partial}^\mu (\theta + \delta\theta)^{\mu\nu} \overrightarrow{\partial}^\nu} g \\ &= f \star g + \frac{i}{2}\delta\theta^{\mu\nu}(\partial_\mu f) \star (\partial_\nu g), \end{aligned} \tag{8}$$

we arrive at the equations

$$\begin{aligned} &a_\mu(A + \delta_\Lambda A) - a_\mu(A) - \partial_\mu \lambda(\Lambda, A) - ig[\lambda(\Lambda, A), A_\mu]_\star \\ &\quad - ig[\Lambda, a_\mu(A)]_\star = -\frac{g}{2}\delta\theta^{\alpha\beta}\{\partial_\alpha \Lambda, \partial_\beta A_\mu\}_\star \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\psi(\Psi + \delta_\Lambda \Psi, A + \delta_\Lambda A) - \psi(\Psi, A) - ig\Lambda \star \psi(\Psi, A) \\ &\quad - ig\lambda(\Lambda, A) \star \Psi = -\frac{g}{2}\delta\theta^{\alpha\beta}(\partial_\alpha \Lambda) \star (\partial_\beta \Psi) \end{aligned} \tag{10}$$

for λ, a_μ and ψ. As found by Seiberg and Witten in [1] (for $g \equiv 1$), the Eq. (9) is solved by

$$\begin{aligned} \lambda &= -\frac{g}{4}\delta\theta^{\alpha\beta}\{A_\alpha, \partial_\beta \Lambda\}_\star, \\ a_\mu &= -\frac{g}{4}\delta\theta^{\alpha\beta}\{A_\alpha, \partial_\beta A_\mu + F_{\beta\mu}\}_\star, \end{aligned} \tag{11}$$

[3] Since an expansion in $\theta^{\mu\nu}$ may obscure the possible UV/IR mixing of the noncommutative theory, a θ-nonperturbative approach is essential.

[4] Precisely which arguments are needed here depends on, and is revealed by, the solutions found below, but for clarity they are already given here. Moreover, we have dropped the Lorentz indices of the arguments for simplicity, since it is clear how they are resumed.

[5] We do not worry about gauge fixing here, since we shall ultimately fix it in the commutative QED to which we arrive in Sec. III. Therefore, the Faddeev-Popov ghost fields are not needed.

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]_\star$ is the field strength. Using (11), we find for the Eq. (10) the solution

$$\psi = -\frac{g}{2}\delta\theta^{\alpha\beta}\Big[A_\alpha \star (\partial_\beta \Psi) + \frac{1}{2}(\partial_\beta A_\alpha) \star \Psi\Big]. \tag{12}$$

As prescribed, the next step in constructing the Seiberg-Witten map, nonperturbative in θ, is to integrate these relations along a path in the space of real-valued antisymmetric matrices to obtain a relation between gauge theories in a commutative spacetime and in a noncommutative one with finite noncommutativity parameters $\theta^{\mu\nu}$. There are certain ambiguities related to choosing a particular path, following from the observation that successive Seiberg-Witten maps do not commute in general, and thus there is an infinite number of free parameters related to the path fixing. Some but not all of these correspond to gauge transformations and field redefinitions, as explored in [4]. However, for simplicity, we choose to consider a straight path[6] $\gamma: [0, 1] \to \{\theta \in \mathbb{R}^{4\times 4} | \theta \text{ antisymmetric}\}$ such that $\gamma(s) = s\theta_1$, where θ_1 is the constant matrix reached at $s = 1$. Let us denote the fields, now considered as dependent on the spacetime coordinates x^μ and the noncommutativity parameters $\theta^{\mu\nu}$, as $A_\mu(x;\theta)$ and $\Psi(x;\theta)$. Integrating the variation (9) along the straight path γ and using integration by parts, we get for the gauge field the equation

$$\begin{aligned} A_\mu(x;\theta_1) = A_\mu(x;0) + \lim_{y\to x}\Bigg\{&-\frac{g\theta_1^{\alpha\beta}}{4}\frac{e^{(i/2)\theta^{\rho\sigma}(\partial/\partial x^\rho)(\partial/\partial y^\sigma)}}{\frac{i}{2}\theta_1^{\gamma\delta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial y^\delta}} \times [A_\alpha(x;\theta)(\partial_\beta A_\mu(y;\theta) + F_{\beta\mu}(y;\theta)) + (\partial_\beta A_\mu(x;\theta) \\ &+ F_{\beta\mu}(x;\theta))A_\alpha(y;\theta)] + \frac{g\theta_1^{\alpha\beta}}{4}\sum_{n=2}^{\infty}(-1)^n \frac{e^{(i/2)\theta^{\rho\sigma}(\partial/\partial x^\rho)(\partial/\partial y^\sigma)}}{(\frac{i}{2}\theta_1^{\gamma\delta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial y^\delta})^n}\Bigg(\prod_{k=2}^{n}\theta_1^{\alpha_k\beta_k}\frac{\delta}{\delta\theta^{\alpha_k\beta_k}}\Bigg) \times [A_\alpha(x;\theta)(\partial_\beta A_\mu(y;\theta) \\ &+ F_{\beta\mu}(y;\theta)) + (\partial_\beta A_\mu(x;\theta) + F_{\beta\mu}(x;\theta))A_\alpha(y;\theta)]\Bigg\}\Bigg|_{\theta=0}^{\theta=\theta_1}, \end{aligned} \tag{13}$$

and similarly for the spinor field the equation

$$\begin{aligned} \Psi(x;\theta_1) = \Psi(x;0) + \lim_{y\to x}\Bigg\{&-\frac{g\theta_1^{\alpha\beta}}{4}\frac{e^{(i/2)\theta^{\rho\sigma}(\partial/\partial x^\rho)(\partial/\partial y^\sigma)}}{\frac{i}{2}\theta_1^{\gamma\delta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial y^\delta}} \times \Big[A_\alpha(x;\theta)(\partial_\beta\Psi(y;\theta)) + \frac{1}{2}(\partial_\beta A_\alpha(x;\theta))\Psi(y;\theta)\Big] \\ &+ \frac{g\theta_1^{\alpha\beta}}{4}\sum_{n=2}^{\infty}(-1)^n \frac{e^{(i/2)\theta^{\rho\sigma}(\partial/\partial x^\rho)(\partial/\partial y^\sigma)}}{(\frac{i}{2}\theta_1^{\gamma\delta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial y^\delta})^n}\Bigg(\prod_{k=2}^{n}\theta_1^{\alpha_k\beta_k}\frac{\delta}{\delta\theta^{\alpha_k\beta_k}}\Bigg) \times \Big[A_\alpha(x;\theta)(\partial_\beta\Psi(y;\theta)) + \frac{1}{2}(\partial_\beta A_\alpha(x;\theta))\Psi(y;\theta)\Big]\Bigg\}\Bigg|_{\theta=0}^{\theta=\theta_1}. \end{aligned} \tag{14}$$

The inverse spacetime partial derivative operators in (13) and (14) are to be understood as defined via Fourier transformation. Then, the series can be calculated iteratively in powers of the coupling constant g, since $\frac{\delta}{\delta\theta}A_\mu = \mathcal{O}(g)$ and $\frac{\delta}{\delta\theta}\Psi = \mathcal{O}(g)$, so the variations in the sums give terms of ever increasing powers in g.

III. NCQED VIA SEIBERG-WITTEN MAP

We now turn to consider exclusively the gauge group $U(1)$, i.e., quantum electrodynamics (QED). We need to study the action of noncommutative QED (NCQED),

$$\mathcal{S}_{\rm NCQED} = \int d^4x\Big[\hat{\bar{\Psi}} \star (i\not\partial - m)\hat{\Psi} - \frac{1}{4}\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} - e\hat{\bar{\Psi}} \star \hat{\not A} \star \hat{\Psi}\Big], \tag{15}$$

in terms of the ordinary fields up to the second order in the electromagnetic coupling constant e in order to catch all the second-order contributions to the photon self-energy. These arise from the diagrams drawn in Fig. 1. Denoting the noncommutative fields by hats and dropping the lower index from θ_1, we find the gauge field via the Eq. (13) up to

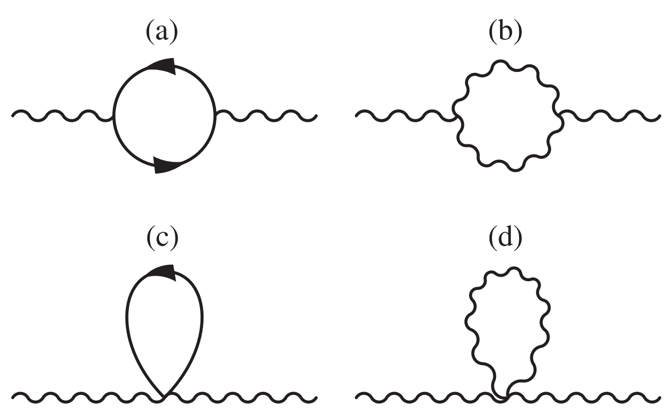

FIG. 1. Photon self-energy diagrams in the second order of e.

[6]This is the case considered also in Ref. [13].

second order in the coupling constant:

$$\hat{A}^{(0)}_\mu = A_\mu, \qquad \hat{A}^{(1)}_\mu = -e\frac{\sin(\frac{1}{2}\partial_1\wedge\partial_2)}{\frac{1}{2}\partial_1\wedge\partial_2}\Big[\eta^\beta_\mu\tilde{\partial}^\alpha_2 - \frac{1}{2}\theta^{\alpha\beta}\partial_{2\mu}\Big]A_\alpha(x_1)A_\beta(x_2)|_{x_1=x_2} = -\frac{e}{2}\theta^{\alpha\beta}A_\alpha\star^s_1(2\partial_\beta A_\mu - \partial_\mu A_\beta),$$

$$\hat{A}^{(2)}_\mu = \frac{e^2}{4}\Bigg[\frac{\sin(\frac{1}{2}\partial_1\wedge\partial_2)}{\frac{1}{2}\partial_1\wedge\partial_2}\frac{\sin(\frac{1}{2}(\partial_1+\partial_2)\wedge\partial_3)}{\frac{1}{2}(\partial_1+\partial_2)\wedge\partial_3} + \frac{\cos(\frac{1}{2}\partial_1\wedge\partial_2)\cos(\frac{1}{2}(\partial_1+\partial_2)\wedge\partial_3)-1}{[\frac{1}{2}(\partial_1+\partial_2)\wedge\partial_3]^2}\Bigg]$$
$$\times\{2[2\tilde{\partial}^\alpha_2\tilde{\partial}^\beta_3 - \theta^{\alpha\beta}\partial_2\wedge\partial_3]\eta^\rho_\mu + 2[2(\tilde{\partial}_1+\tilde{\partial}_2)^\rho\tilde{\partial}^\alpha_2 - \theta^{\rho\alpha}\partial_1\wedge\partial_2]\eta^\beta_\mu$$
$$+ [(2\theta^{\beta\rho}\tilde{\partial}^\alpha_2 - \theta^{\alpha\beta}(3\tilde{\partial}_1+\tilde{\partial}_2)^\rho - 2\theta^{\rho\alpha}\tilde{\partial}^\beta_1)\partial_{2\mu} - (\theta^{\alpha\beta}\tilde{\partial}^\rho_2 + 2\theta^{\beta\rho}\tilde{\partial}^\alpha_2)\partial_{3\mu}]\}A_\alpha(x_1)A_\beta(x_2)A_\rho(x_3)|_{x_1=x_2=x_3}. \tag{16}$$

Similarly for the spinor field, via the Eq. (14), we find

$$\hat{\Psi}^{(0)} = \Psi, \qquad \hat{\Psi}^{(1)} = -\frac{e}{2}\frac{e^{(i/2)\partial_1\wedge\partial_2}-1}{\frac{i}{2}\partial_1\wedge\partial_2}\Big(\frac{1}{2}\tilde{\partial}_1+\tilde{\partial}_2\Big)^\alpha A_\alpha(x_1)\Psi(x_2)|_{x_1=x_2} = -\frac{e}{2}\theta^{\alpha\beta}\Big[A_\alpha\star_1(\partial_\beta\Psi) + \frac{1}{2}(\partial_\beta A_\alpha)\star_1\Psi\Big],$$

$$\hat{\Psi}^{(2)} = \frac{e^2}{4}\Bigg\{\Bigg[\frac{\sin(\frac{1}{2}\partial_1\wedge\partial_2)}{\frac{1}{2}\partial_1\wedge\partial_2}\frac{e^{(i/2)(\partial_1+\partial_2)\wedge\partial_3}}{\frac{i}{2}(\partial_1+\partial_2)\wedge\partial_3} - \frac{\cos(\frac{1}{2}\partial_1\wedge\partial_2)e^{(i/2)(\partial_1+\partial_2)\wedge\partial_3}+1}{[\frac{1}{2}(\partial_1+\partial_2)\wedge\partial_3]^2}\Bigg]\times\Bigg[2\tilde{\partial}^\alpha_2\Big(\frac{1}{2}\tilde{\partial}_1+\frac{1}{2}\tilde{\partial}_2+\tilde{\partial}_3\Big)^\beta$$
$$+\theta^{\alpha\beta}\Big(\frac{1}{2}\partial_1+\partial_3\Big)\wedge\partial_2\Bigg] + \Bigg[\frac{e^{(i/2)\partial_1\wedge\partial_3}-1}{\frac{i}{2}\partial_1\wedge\partial_3}\frac{e^{(i/2)\partial_2\wedge(\partial_1+\partial_3)}}{\frac{i}{2}\partial_2\wedge(\partial_1+\partial_3)} - \frac{e^{(i/2)\partial_1\wedge\partial_3}e^{(i/2)\partial_2\wedge(\partial_1+\partial_3)}+1}{[\frac{i}{2}\partial_2\wedge(\partial_1+\partial_3)]^2}\Bigg]$$
$$\times\Big(\frac{1}{2}\tilde{\partial}_1+\tilde{\partial}_3\Big)^\alpha\Big(\tilde{\partial}_1+\frac{1}{2}\tilde{\partial}_2+\tilde{\partial}_3\Big)^\beta\Bigg\}A_\alpha(x_1)A_\beta(x_2)\Psi(x_3)|_{x_1=x_2=x_3}, \tag{17}$$

where we have introduced the notations $\tilde{u}^\mu := \theta^{\mu\nu}u_\nu$ and $u\wedge v := u_\mu\theta^{\mu\nu}v_\nu$ for any four-vectors u_μ, v_μ, and[7]

$$f\star_1 g := \frac{e^{(i/2)\partial_1\wedge\partial_2}-1}{\frac{i}{2}\partial_1\wedge\partial_2}f(x_1)g(x_2)|_{x_1=x_2},$$
$$f\star^s_1 g := \frac{1}{2}\{f,g\}_{\star_1} = \frac{\sin(\frac{1}{2}\partial_1\wedge\partial_2)}{\frac{1}{2}\partial_1\wedge\partial_2}f(x_1)g(x_2)|_{x_1=x_2}. \tag{18}$$

Clearly, the terms tend to get more complicated at each order, which makes higher order calculations via θ-nonperturbative Seiberg-Witten map highly elaborate.

Since $(f\star g)^\dagger = g^\dagger\star f^\dagger$ for any functions (or, more generally, matrices) f and g, $\hat{\bar{\Psi}} \equiv \bar{\hat{\Psi}} = \hat{\Psi}^\dagger\gamma^0$. Substituting (16) and (17) into the action (15), we find the first order fermion-photon interaction term to be

$$\mathcal{L}^{(1)}_{\bar{\Psi}A\Psi} = -e\bar{\Psi}\star\not{A}\star\Psi - \frac{e}{2}\theta^{\alpha\beta}\Big[(\partial_\beta\bar{\Psi})\star_1 A_\alpha + \frac{1}{2}\bar{\Psi}\star_1(\partial_\beta A_\alpha)\Big]\star(i\not{\partial}-m)\Psi - \frac{e}{2}\theta^{\alpha\beta}\bar{\Psi}\star(i\not{\partial}-m)\Big[A_\alpha\star_1(\partial_\beta\Psi) + \frac{1}{2}(\partial_\beta A_\alpha)\star_1\Psi\Big]. \tag{19}$$

Similarly, from (16) we find the photon-photon interaction Lagrangian up to the first order in e to be

$$\mathcal{L}^{(1)}_{A^3} = -\frac{e}{4}\Big\{\partial_\mu A_\nu - \partial_\nu A_\mu, i[A^\mu, A^\nu]_\star - \frac{1}{2}\theta^{\alpha\beta}[\partial^\mu(A_\alpha\star^s_1(2\partial_\beta A^\nu - \partial^\nu A_\beta)) - \partial^\nu(A_\alpha\star^s_1(2\partial_\beta A^\mu - \partial^\mu A_\beta))]\Big\}_\star. \tag{20}$$

The second-order contributions are considerably more complicated, but are obtained similarly by substituting the expressions (16) and (17) into the action (15) and picking up the terms with the factor e^2. The Feynman diagram vertex functions arising from the first- and second-order parts of the action are given in Appendix A.

IV. PHOTON SELF-ENERGY CORRECTIONS

Now, using the vertex function (A1) from Appendix A, we find for the second-order fermion loop correction to the photon propagator arising from the diagram (a) in Fig. 1 the form:

$$\Pi^{\alpha\beta}_{(a)}(k) = -4e^2\int\frac{d^4p}{(2\pi)^4}\times\Bigg\{T^{\alpha\beta} + \frac{i}{2}\frac{\sin(\frac{1}{4}p\wedge k)}{\frac{1}{4}p\wedge k}\times\Bigg[\Big(\tilde{p}-\frac{1}{2}\tilde{k}\Big)^\alpha k_\rho T^{\rho\beta}e^{-(i/4)p\wedge k} - \Big(\tilde{p}-\frac{1}{2}\tilde{k}\Big)^\beta k_\rho T^{\rho\alpha}e^{(i/4)p\wedge k}\Bigg] + \frac{1}{4}\frac{\sin^2(\frac{1}{4}p\wedge k)}{(\frac{1}{4}p\wedge k)^2}\times\Big(\tilde{p}-\frac{1}{2}\tilde{k}\Big)^\alpha\Big(\tilde{p}-\frac{1}{2}\tilde{k}\Big)^\beta k_\rho k_\sigma T^{\rho\sigma}\Bigg\}, \tag{21}$$

[7]Notice that our notation for these so-called "generalized $\star$-products" differs from that used in [13,18,19]. This is an attempt to make the notation more systematic. The lower index denotes the times of integration of the $\star$-product over the unit interval, and the upper index "s" denotes symmetrization of the product with respect to its arguments.

where

$$T^{\alpha\beta} := \frac{(p-k)^\alpha p^\beta + p^\alpha (p-k)^\beta + [m^2 - (p-k)\cdot p]\eta^{\alpha\beta}}{p^2 (p-k)^2}. \tag{22}$$

The first term in the integrand of (21) is the only one we get in the commutative case, whereas the second and the third terms are the contribution of noncommutativity. It is easy to see that the extra terms vanish at the limit $\theta \to 0$, and thus we obtain the same result as for the commutative QED at the commutative limit, in contrast with the result of Schupp and You [13], which vanishes completely at the commutative limit. This follows from their use of the adjoint representation for the spinor field, leading one to antisymmetrize the $\star$-products in the action, which gives rise to a sine phase factor for the vertex function. In our case of the fundamental representation for the fermions, on the other hand, one does not antisymmetrize the $\star$-products, thus obtaining an exponential phase factor. The exponential factors cancel out upon the multiplication of complex conjugates arising from the two vertices of the diagram (a), leading to the usual commutative contribution, while the sine factors arising from the vertices in the adjoint representation do not cancel out upon multiplication.

On the other hand, the integrand in (21) consists of two parts: The first term along with parts of the second and the third terms without phase factors constitute the divergent "planar" part of the correction

$$\Pi^{\alpha\beta}_{(a)p}(k) := -2e^2 \int \frac{d^4p}{(2\pi)^4} \times \bigg\{ 2T^{\alpha\beta} + \frac{1}{\frac{1}{2} p\wedge k}\bigg[\bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\alpha \times k_\rho T^{\rho\beta} + \bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\beta k_\rho T^{\rho\alpha}\bigg] + \frac{1}{(\frac{1}{2} p\wedge k)^2} \times \bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\alpha \bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\beta k_\rho k_\sigma T^{\rho\sigma}\bigg\}, \tag{23}$$

which should be regularized and renormalized in order to obtain a finite outcome for the integral. The parts of the second and the third terms with nontrivial phase factors, on the other hand, constitute the "nonplanar" part of the correction

$$\Pi^{\alpha\beta}_{(a)np}(k) := 2e^2 \int \frac{d^4p}{(2\pi)^4} \times \bigg\{ \frac{1}{\frac{1}{2} p\wedge k}\bigg[\bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\alpha k_\rho T^{\rho\beta} \times e^{-(i/2)p\wedge k} + \bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\beta k_\rho T^{\rho\alpha} e^{(i/2)p\wedge k}\bigg] + \frac{\cos(\frac{1}{2}p\wedge k)}{(\frac{1}{2} p\wedge k)^2}\bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\alpha \bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\beta k_\rho k_\sigma T^{\rho\sigma}\bigg\}, \tag{24}$$

which is finite for nonvanishing $\tilde{k}$, and therefore does not require regularization. The nonplanar part is the focus of our attention, since it is usually the origin of the UV/IR mixing problem: In the IR-limit of the external momentum $\tilde{k} \to 0$, the nonplanar part is typically divergent, since the oscillating phase factors, which otherwise dampen the integral rendering it finite, approach unity. This gives rise to the UV/IR mixing problem, since the Wilsonian renormalization scheme cannot be applied to such divergencies [20]. To confirm this expectation, we proceed to calculate the leading order contribution of the nonplanar part at the limit $\tilde{k} \to 0$ in the following.

To evaluate the first term in (24), we use the trick of Schupp and You [13] by expressing them as

$$2e^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{\frac{1}{2} p\wedge k}\bigg[\bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\mu k_\rho T^{\rho\nu} e^{-(i/2)p\wedge k} + \bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\nu k_\rho T^{\rho\mu} e^{(i/2)p\wedge k}\bigg] = 2ie^2 \sum_{\lambda=\pm 1} \int d\lambda I^{\mu\nu}(k;\lambda), \tag{25}$$

where

$$I^{\mu\nu}(k;\lambda) = \int \frac{d^4p}{(2\pi)^4}\bigg(\tilde{p} - \frac{1}{2}\tilde{k}\bigg)^\mu k_\rho T^{\rho\nu} e^{(i/2)\lambda p\wedge k}. \tag{26}$$

By performing a Wick rotation $p^\mu = e^\mu_i \bar{p}^i$, where $e^\mu_i = \mathrm{diag}(i, 1, 1, 1)$ and $\bar{p}^i$ is the Euclidean momentum, and using Schwinger parametrization

$$\frac{1}{\bar{p}^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(\bar{p}^2 + m^2)}, \tag{27}$$

we get

$$I^{\mu\nu}(k;\lambda) = ie^\mu_i e^\nu_j \iint_0^\infty d\alpha d\beta \int \frac{d^4p}{(2\pi)^2}\bigg(\bar{\tilde{p}} - \frac{1}{2}\bar{\tilde{k}}\bigg)^i \times [(\bar{k}^2 - 2\bar{k}\cdot\bar{p})\bar{p}^j + (\bar{p}^2 + m^2)\bar{k}^j] \times e^{-\alpha[(\bar{p}-\bar{k})^2 + m^2] - \beta[\bar{p}^2 + m^2] + (i/2)\lambda\bar{p}\cdot\bar{\tilde{k}}}. \tag{28}$$

We may render the momentum integral Gaussian by applying the change of variables

$$\bar{q} := \bar{p} - \frac{\alpha}{\alpha+\beta}\bar{k} - \frac{i\lambda}{4(\alpha+\beta)}\bar{\tilde{k}}, \tag{29}$$

after which we can perform the integration over $\bar{q}$. Further, multiplying the integrand by

$$1 = \int_0^\infty dc\,\delta(c - \alpha - \beta), \tag{30}$$

changing the order of integrations, and applying the change of variables $\alpha = ca$, $\beta = cb$, we get

$$I^{\mu\nu}(k;\lambda) \approx \frac{ie^\mu_i e^\nu_j \bar{\theta}^{ik}}{(4\pi)^2} \iint_0^1 da db\,\delta(1-a-b) \int_0^\infty dc c^{-3} \times \bigg[\bigg(\frac{i\lambda}{2} - \frac{i\lambda^3 \bar{\tilde{k}}^2}{64c}\bigg)\bar{\tilde{k}}_k \bar{k}^j - \frac{i\lambda}{4}\bar{k}_k \bar{\tilde{k}}^j\bigg] \times e^{-c(ab\bar{k}^2 + m^2) - (\lambda^2/16c)\bar{\tilde{k}}^2}, \tag{31}$$

where the less IR-divergent terms are dropped out.[8] The dependence on a and b drops out, and the integrals over them give unity. The integral over λ is now straightforward to perform. Moreover, the integral over c can be performed and expressed for small k using the properties of modified Bessel functions $K_r(x, y)$ [21]:

$$\int_0^\infty dc c^{-r-1} e^{-xc-y/c} = 2\left(\frac{x}{y}\right)^{r/2} K_r[2\sqrt{xy}], \quad \text{where } \mathrm{Re}[x], \quad \mathrm{Re}[y] > 0,$$
$$\text{and} \quad K_r(z) \approx \frac{\Gamma(r)}{2}\left(\frac{2}{z}\right)^r, \quad \text{when} \quad 0 < z \ll \sqrt{r+1}. \tag{32}$$

We get for small $\tilde{\tilde{k}}^2 \ll m^{-2}$ accordingly

$$i\Pi^{\mu\nu}_{(a)\mathrm{np}}(k) \approx \frac{8e^2}{\pi^2}\frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^4} + \frac{4e^2}{\pi^2}\frac{\tilde{\tilde{k}}^\mu k^\nu + k^\mu \tilde{\tilde{k}}^\nu}{\tilde{k}^4}. \tag{33}$$

The first term here is similar to the IR-divergent terms found in the usual formulation of NCQED and by Schupp and You [13]. The second term gives another quadratic IR-divergence, which is gauge variant, and therefore should be canceled, when all the second-order contributions in the coupling constant are taken into account.

Having found the gauge invariant IR-divergence in (33), we proceed to confirm the absence of canceling terms. The calculations, though more elaborate, follow precisely the same scheme as the one above. To make them manageable we only consider contributions of the form $a\tilde{k}^\mu \tilde{k}^\nu$, where a is a scalar quantity. The second term in (24) and the other nonplanar second-order contributions coming from the Fig. 1 diagrams (b), (c) and (d) also give rise to quadratic divergencies of the form $c\tilde{k}^\mu \tilde{k}^\nu / \tilde{k}^4$, where c is a constant. For all of the contributions we find $c > 0$, and thus they cannot cancel the IR-divergence of (33). Hence we conclude that the noncommutative QED formulated here via θ-nonperturbative Seiberg-Witten map suffers from the UV/IR mixing problem.

V. CONCLUSIONS AND REMARKS

We have found that UV/IR mixing is present in the photon self-energy corrections of noncommutative QED defined via θ-nonperturbative Seiberg-Witten map for a straight path in θ-space. The result further demonstrates that UV/IR mixing is a generic property of noncommutative quantum field theories, and is not cured in general by the approach via Seiberg-Witten map, contrary to some claims previously made in the literature.

[8] Here we have to take into account the following integration over c, where $c \sim \tilde{\tilde{k}}^2$. The integration over λ does not affect the relative powers of divergence.

A question remains open, though, whether the result holds generally for all possible integration paths in θ-space. It is not ruled out that by modifying the integration path one could get rid of the divergence, although on mathematical grounds this seems unlikely, at least, for paths obtainable from the straight one by smooth deformations. Of course, answering the question properly requires a rigorous analysis, which we postpone to a future study.

In the case of a scalar field theory in noncommutative spacetime, as, for example, in [20], the destruction of UV/IR mixing by θ-expansion becomes immediately obvious: The expansion in θ spoils the oscillatory behavior of the phase factors in the nonplanar contributions, which would otherwise render the nonplanar contributions finite for nonvanishing external momenta, thus giving rise to the UV/IR mixing problem. The same is true in our case for the photon self-energy corrections [e.g., Eq. (21)]. Therefore, a nonperturbative approach in θ is essential in analyzing the renormalizability of noncommutative quantum field theories, and the UV/IR mixing, in particular. The present results, as well as those obtained by Schupp and You, are in harmony with the θ-exact analysis in Ref. [8] (where NCQED is not defined by means of a Seiberg-Witten map), seeming to suggest that the NCQED models in Refs. [9–11], defined perturbatively in θ, which are renormalizable at the one-loop level and in the first order in θ, might not have a nonpertubative definition in terms of θ (in the absence of supersymmetry). This state of affairs might resemble the situation in ordinary QED, which exists as a renormalizable theory when treated perturbatively in the coupling constant, but seems not to exist as a nonperturbative theory.

In Ref. [20] a satisfactory explanation is given for the mixing as a direct result of the infinite nonlocality of $\star$-product, and thus it is deeply rooted in the very definition of noncommutativity of spacetime. It therefore seems unlikely that the resulting divergencies could be made vanish, at least, without modifying the theory itself by introducing new terms in the Lagrangian, which suppress the contributions of the IR sector. This has been done for a noncommutative scalar field theory in [22–24] and for noncommutative QED in [25,26]. For some other attempts, see [27–29]. Reducing the nonlocality of the noncommutative field theories to a finite range is also an option which has been preliminarily exploited in [30,31].

All in all, whatever the direction, more work is to be done before we are to overcome the obstacles arising from the nonlocality in noncommutative quantum field theories.

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APPENDIX: VERTEX FUNCTIONS

In the following, we consider that the momenta are incoming everywhere and are denoted by p_i's for fermions and k_i's for photons.

For the first order vertex functions we get the expressions

$$V^{\mu}_{\bar{\Psi}A\Psi}(p_1, p_2) = -ie\gamma^{\mu}e^{(i/2)p_1\wedge p_2} - \frac{ie}{2}(\tilde{p}_1 - \tilde{p}_2)^{\mu} \times (\not{p}_1 + \not{p}_2)\frac{e^{(i/2)p_1\wedge p_2} - 1}{p_1 \wedge p_2}, \tag{A1}$$

$$V^{\mu_1\mu_2\mu_3}_{A^3}(k_1, k_2, k_3) = 2e\sin\left(\frac{k_1 \wedge k_2}{2}\right)\times\bigg\{(k_1 - k_2)^{\mu_3}\eta^{\mu_1\mu_2} + \frac{1}{\frac{1}{2}k_1 \wedge k_2}[(k_1^{\mu_1}k_1^{\alpha} - k_1^2\eta^{\mu_1\alpha}) \times (2\tilde{k}_3^{\mu_2}\eta^{\mu_3}_{\alpha} - k_{3\alpha}\theta^{\mu_2\mu_3})]\bigg\} + \{\text{symm.}\}, \tag{A2}$$

where {symm} denotes terms symmetrizing the previous contributions with respect to the photons (k_i, μ_i).

For the second-order vertex functions we similarly find

$$\begin{aligned}
V^{\mu_1\mu_2}_{\bar{\Psi}A^2\Psi}(k_1, k_2, p_1, p_2) = {}& -\frac{ie^2}{4}\bigg\{\bigg[\bigg(\frac{\sin(\frac{1}{2}k_1 \wedge k_2)}{\frac{1}{2}k_1 \wedge k_2}\frac{e^{(i/2)p_1\wedge p_2}}{\frac{i}{2}p_1 \wedge p_2} - \frac{\cos(\frac{1}{2}k_1 \wedge k_2)e^{(i/2)p_1\wedge p_2} - 1}{(\frac{i}{2}p_1 \wedge p_2)^2}\bigg)\times\bigg((\tilde{p}_1 - \tilde{p}_2)^{\mu_1}\tilde{k}_1^{\mu_2} - \frac{1}{2}\theta^{\mu_1\mu_2} \\
& \times (p_1 - p_2)\wedge k_1\bigg) + \bigg(\frac{e^{(i/2)p_2\wedge k_1}}{\frac{i}{2}p_2 \wedge k_1}\frac{e^{(i/2)k_2\wedge p_1} - 1}{\frac{i}{2}k_2 \wedge p_1} - \frac{e^{(i/2)p_2\wedge k_1}e^{(i/2)k_2\wedge p_1} - 1}{(\frac{i}{2}p_2 \wedge k_1)^2}\bigg)\times\frac{1}{4}(\tilde{p}_1 - \tilde{p}_2 + \tilde{k}_2)^{\mu_1} \\
& \times (\tilde{p}_1 - \tilde{p}_2 - \tilde{k}_1)^{\mu_2}\bigg](\not{p}_2 - m) - \bigg[\bigg(\frac{\sin(\frac{1}{2}k_1 \wedge k_2)}{\frac{1}{2}k_1 \wedge k_2}\frac{e^{(i/2)p_1\wedge p_2}}{\frac{i}{2}p_1 \wedge p_2} - \frac{\cos(\frac{1}{2}k_1 \wedge k_2)e^{(i/2)p_1\wedge p_2} - 1}{(\frac{i}{2}p_1 \wedge p_2)^2}\bigg) \\
& \times\bigg((\tilde{p}_2 - \tilde{p}_1)^{\mu_2}\tilde{k}_2^{\mu_1} - \frac{1}{2}\theta^{\mu_2\mu_1}(p_2 - p_1)\wedge k_2\bigg) + \bigg(\frac{e^{(i/2)p_2\wedge k_1}}{\frac{i}{2}p_2 \wedge k_1}\frac{e^{(i/2)k_2\wedge p_1} - 1}{\frac{i}{2}k_2 \wedge p_1} \\
& - \frac{e^{(i/2)p_2\wedge k_1}e^{(i/2)k_2\wedge p_1} - 1}{(\frac{i}{2}p_2 \wedge k_1)^2}\bigg)\times\frac{1}{4}(\tilde{p}_2 - \tilde{p}_1 + \tilde{k}_1)^{\mu_2}(\tilde{p}_2 - \tilde{p}_1 - \tilde{k}_2)^{\mu_1}\bigg](\not{p}_1 + m) + \frac{1}{4}\frac{e^{(i/2)k_1\wedge p_1} - 1}{\frac{i}{2}k_1 \wedge p_1} \\
& \times\frac{e^{(i/2)k_2\wedge p_2} - 1}{\frac{i}{2}k_2 \wedge p_2}(\tilde{p}_1 - \tilde{p}_2 - \tilde{k}_2)^{\mu_1}(\tilde{p}_2 - \tilde{p}_1 - \tilde{k}_1)^{\mu_2}(\not{p}_1 + \not{k}_1 + m) + i\frac{e^{(i/2)k_1\wedge p_1} - 1}{\frac{i}{2}k_1 \wedge p_1}e^{(i/2)p_2\wedge k_2} \\
& \times (\tilde{p}_1 - \tilde{p}_2 - \tilde{k}_2)^{\mu_1}\gamma^{\mu_2} + i\frac{e^{(i/2)k_2\wedge p_2} - 1}{\frac{i}{2}k_2 \wedge p_2}e^{(i/2)p_1\wedge k_1}(\tilde{p}_2 - \tilde{p}_1 - \tilde{k}_1)^{\mu_2}\gamma^{\mu_1} + 4i\frac{\sin(\frac{1}{2}k_1 \wedge k_2)}{\frac{1}{2}k_1 \wedge k_2} \\
& \times e^{(i/2)p_1\wedge p_2}\bigg(\tilde{k}_2^{\mu_1}\gamma^{\mu_2} - \frac{1}{2}\theta^{\mu_1\mu_2}\not{k}_2\bigg)\bigg\} + \{\text{symm}\},
\end{aligned} \tag{A3}$$

$$\begin{aligned}
V^{\mu_1\mu_2\mu_3\mu_4}_{A^4}(k_1, k_2, k_3, k_4) = {}& -e^2\bigg\{\bigg[\frac{\sin(\frac{1}{2}k_1 \wedge k_2)}{\frac{1}{2}k_1 \wedge k_2}\frac{\sin(\frac{1}{2}k_3 \wedge k_4)}{\frac{1}{2}k_3 \wedge k_4} - \frac{\cos(\frac{1}{2}k_1 \wedge k_2)\cos(\frac{1}{2}k_3 \wedge k_4) - 1}{(\frac{1}{2}k_3 \wedge k_4)^2}\bigg] \\
& \times\bigg[\bigg(\tilde{k}_2^{\mu_1}\tilde{k}_3^{\mu_2} - \frac{1}{2}\theta^{\mu_1\mu_2}k_2 \wedge k_3\bigg)\eta^{\mu_3}_{\alpha} + \bigg((\tilde{k}_1 + \tilde{k}_2)^{\mu_3}\tilde{k}_2^{\mu_1} - \frac{1}{2}\theta^{\mu_3\mu_1}k_1 \wedge k_2\bigg)\eta^{\mu_2}_{\alpha} \\
& + \frac{1}{2}\bigg(\theta^{\mu_2\mu_3}\tilde{k}_2^{\mu_1} + \theta^{\mu_1\mu_3}\tilde{k}_1^{\mu_2} - \frac{1}{2}\theta^{\mu_1\mu_2}(3\tilde{k}_1 + \tilde{k}_2)^{\mu_3}\bigg)k_{2\alpha} - \frac{1}{2}\bigg(\theta^{\mu_2\mu_3}\tilde{k}_2^{\mu_1} + \frac{1}{2}\theta^{\mu_1\mu_2}\tilde{k}_2^{\mu_3}\bigg)k_{3\alpha}\bigg] \\
& \times (k_4^2\eta^{\alpha\mu_4} - k_4^{\alpha}k_4^{\mu_4}) - \frac{1}{2}\frac{\sin(\frac{1}{2}k_1 \wedge k_2)}{\frac{1}{2}k_1 \wedge k_2}\frac{\sin(\frac{1}{2}k_3 \wedge k_4)}{\frac{1}{2}k_3 \wedge k_4}\bigg(\eta^{\mu_2}_{\alpha}\tilde{k}_2^{\mu_1} - \frac{1}{2}\theta^{\mu_1\mu_2}k_{2\alpha}\bigg) \\
& \times\bigg(\eta^{\mu_4}_{\beta}\tilde{k}_4^{\mu_3} - \frac{1}{2}\theta^{\mu_3\mu_4}k_{4\beta}\bigg)((k_1 + k_2)^2\eta^{\alpha\beta} - (k_1 + k_2)^{\alpha}(k_1 + k_2)^{\beta}) \\
& + \frac{\sin(\frac{1}{2}k_1 \wedge k_2)}{\frac{1}{2}k_1 \wedge k_2}\sin\bigg(\frac{1}{2}k_3 \wedge k_4\bigg)\bigg(\eta^{\mu_2}_{\alpha}\tilde{k}_2^{\mu_1} - \frac{1}{2}\theta^{\mu_1\mu_2}k_{2\alpha}\bigg)\times(\eta^{\mu_3\mu_4}k_4^{\alpha} - \eta^{\alpha\mu_4}k_4^{\mu_3}) \\
& + \sin\bigg(\frac{1}{2}k_1 \wedge k_2\bigg)\frac{\sin(\frac{1}{2}k_3 \wedge k_4)}{\frac{1}{2}k_3 \wedge k_4}\bigg(\eta^{\mu_4}_{\alpha}\tilde{k}_4^{\mu_3} - \frac{1}{2}\theta^{\mu_3\mu_4}k_{4\alpha}\bigg)\times(\eta^{\mu_2}_{\alpha}(k_3 + k_4)^{\mu_1} - \eta^{\mu_1}_{\alpha}(k_3 + k_4)^{\mu_2}) \\
& + \sin\bigg(\frac{1}{2}k_1 \wedge k_2\bigg)\sin\bigg(\frac{1}{2}k_3 \wedge k_4\bigg)\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\bigg\} + \{\text{symm}\},
\end{aligned} \tag{A4}$$

where again {symm} denotes terms symmetrizing the previous contributions with respect to the photons (k_i, μ_i).

[1] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.

[2] S. Doplicher, K. Fredenhagen, and J. E. Roberts, Commun. Math. Phys. **172**, 187 (1995).

[3] D. N. Blaschke, E. Kronberger, A. Rofner, M. Schweda, R. I. P. Sedmik, and M. Wohlgenannt, Fortschr. Phys. **58**, 364 (2010).

[4] T. Asakawa and I. Kiskimoto, J. High Energy Phys. 11 (1999) 024.

[5] M. Chaichian, P. Prešnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Lett. B **526**, 132 (2002).

[6] M. Chaichian, P. Prešnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Lett. B **683**, 55 (2010).

[7] S. Terashima, Phys. Lett. B **482**, 276 (2000); J. M. Gracia-Bondía and C. P. Martín, Phys. Lett. B **479**, 321 (2000).

[8] A. Matusis, L. Susskind, and N. Toumbas, J. High Energy Phys. 12 (2000) 002.

[9] A. Bichl, J. Grimstrup, H. Grosse, L. Popp, M. Schweda, and R. Wulkenhaar, J. High Energy Phys. 06 (2001) 013.

[10] M. Buric, D. Latas, and V. Radovanovic, J. High Energy Phys. 02 (2006) 046.

[11] C. P. Martin and C. Tamarit, J. High Energy Phys. 12 (2009) 042.

[12] R. Wulkenhaar, J. High Energy Phys. 03 (2002) 024.

[13] P. Schupp and J. You, J. High Energy Phys. 08 (2008) 107.

[14] M. M. Sheikh-Jabbari, Phys. Lett. B **455**, 129 (1999).

[15] D. Bigatti and L. Susskind, Phys. Rev. D **62**, 066004 (2000).

[16] Z. Yin, Phys. Lett. B **466**, 234 (1999).

[17] L. Álvarez-Gaumé and J. L. F. Barbon, Int. J. Mod. Phys. A **16**, 1123 (2001).

[18] M. R. Garousi, Nucl. Phys. **B579**, 209 (2000).

[19] T. Mehen and M. B. Wise, J. High Energy Phys. 12 (2000) 008.

[20] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, J. High Energy Phys. 02 (2000) 020.

[21] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, Berlin, 1966), 3rd Ed..

[22] H. Grosse and R. Wulkenhaar, Commun. Math. Phys. **256**, 305 (2005).

[23] H. Grosse and M. Wohlgenannt, J. Phys. Conf. Ser. **53**, 764 (2006).

[24] R. Gurau, J. Magnen, V. Rivasseau, and A. Tanasa, Commun. Math. Phys. **287**, 275 (2009).

[25] D. N. Blaschke, A. Rofner, M. Schweda, and R. I. P. Sedmik, Eur. Phys. J. C **62**, 433 (2009).

[26] L. C. Q. Vilar, O. S. Ventura, D. G. Tedesco, and V. E. R. Lemes, arXiv:0902.2956.

[27] P. Fischer and V. Putz, Eur. Phys. J. C **32**, 269 (2004).

[28] S. Denk and M. Schweda, J. High Energy Phys. 09 (2003) 032.

[29] S. Denk, V. Putz, and M. Wohlgenannt, Eur. Phys. J. C **45**, 263 (2006).

[30] D. Bahns and S. Waldmann, Rev. Math. Phys. **19**, 273 (2007).

[31] M. Långvik and A. Zahabi, arXiv:1002.0956.