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Quantum field theory without divergences

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It is shown that loop divergences emerging in the Green functions in quantum field theory originate from correspondence of the Green functions to *unmeasurable* (and hence unphysical) quantities. This is because no physical quantity can be measured in a point, but in a region, the size of which is constrained by the resolution of measuring equipment. The incorporation of the resolution into the definition of quantum fields $\phi(x) \rightarrow \phi^{(A)}(x)$ and appropriate change of Feynman rules results in finite values of the Green functions. The Euclidean ϕ^4-field theory is taken as an example.

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I. INTRODUCTION

The fundamental problem of quantum field theory is the problem of divergences of Feynman integrals. The formal infinities appearing in perturbation expansion of Feynman integrals are tackled with different regularization methods, from Pauli-Villars regularization to renormalization methods for gauge theories, see, e.g., [1] for a review. Let us consider the quantum field theory in its Euclidean formulation. The widely known example which fairly illustrates the problem is the ϕ^4 interaction model in $\mathbb{R}^d$, see e.g. [1,2], determined by the generating functional

$$W[J] = \mathcal{N} \int e^{-\int d^d x [(1/2)(\partial\phi)^2 + ((m^2)/2)\phi^2 + (\lambda/(4!))\phi^4 - J\phi]} \mathcal{D}\phi, \tag{1}$$

where $\mathcal{N}$ is a formal normalization constant. The connected Green functions are given by variational derivatives of the generating functional:

$$\Delta^{(n)} \equiv \langle \phi(x_1) \dots \phi(x_n) \rangle_c = \frac{\delta^n \ln W[J]}{\delta J(x_1) \dots \delta J(x_n)} \Bigg|_{J=0}. \tag{2}$$

In a statistical sense these functions have the meaning of the n-point correlation functions [3]. The divergences of Feynman graphs in the perturbation expansion of the Green functions (2) with respect to the small coupling constant λ emerge at coinciding arguments $x_i = x_k$. For instance, the bare two-point correlation function

$$\Delta_0^{(2)}(x-y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{\imath p(x-y)}}{p^2 + m^2} \tag{3}$$

is divergent at $x = y$ for $d \geq 2$.

Since in their correspondence to the c-valued fields the products $\psi^*(x)\psi(x)\Delta x$ have the probability meaning, it is quite obvious physically that neither of the joint probabilities of the measured quantities can be infinite. The infinities seem to be caused by an inadequate choice of the functional space the fields belong to.

This standard approach inherited from quantum mechanics disregards two important notes:

(1) To localize a particle in an interval Δx, the measuring device requests a momentum transfer of order $\Delta p \sim \hbar/\Delta x$. If the value of this momentum is too large we may get out of the applicability range of the initial model, in the sense that $\phi(x)$ at a fixed point x has no experimentally verifiable meaning. What is meaningful is the vacuum expectation of the product of fields in a certain region centered around x, the width of which (Δx) is constrained by the experimental conditions of the measurement.

(2) Even if the particle, described by the field $\phi(x)$, has been initially prepared on the interval $(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2})$, the probability of registering this particle on this interval is generally less than unity: for the probability of registration depends on the strength of interaction and the ratio of typical scales of the measured particle and the measuring equipment. The maximum probability of registering an object of typical scale Δx by the equipment with typical resolution a is achieved when these two parameters are comparable. For this reason, the probability of registering an electron by visual range photon scattering is much higher than by that of long radio-frequency waves. As a mathematical generalization, we should say that if a measuring equipment with a given spatial resolution a fails to register an object, prepared on spatial interval of width Δx with certainty, then tuning the equipment to *all* possible resolutions a' would lead to the registration. This certifies the fact of the existence of the measured object.

Most of the regularization methods applied to make the Green functions finite imply a certain type of self-similarity—the independence of physical observables on the scale transformation of an arbitrary parameter of the theory—the cutoff length or the normalization scale.

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Covariance with respect to scale transformations is expressed by the renormalization group equation [1]. Another regularization idea based on self-similarity and widely used in lattice gauge theories is the Kadanoff blocking procedure, which averages the small-scale fluctuations up to a certain scale into a kind of effective interaction for a larger blocks, assuming the larger blocks interact with each other in the same way as their sub-blocks [4,5]. However, the theory based on the Fourier transform of fields leaves no place for such self-similarity: the product of fields $\prod_i \int_{|k|<\Lambda} e^{-\imath k_i x}\tilde\phi(k_i)[(d^dk)/(2\pi)^d]$ describes the strength of the interaction of all fluctuations *up to* the scale $1/\Lambda$, but says nothing about the interaction strength *at* a given scale. An abstract harmonic analysis based on a group G, which is wider than the group of translations G: $x \to x + b$, should be used to account for self-similarity.

The present paper aims to show how the quantum field theory of the scale-dependent fields can be constructed using the continuous wavelet transform, i.e. using the decomposition of fields with respect to the representations of the affine group G: $x \to ax + b$.

In Sec. II we present a theory of the fields $\phi_{\Delta x}(x)$, which explicitly depend on the resolution Δx rather than on the point x alone. The finiteness of the Green functions is shown on the simplest example of the scalar field theory with the ϕ^4 interaction. In Sec. III we present the commutation relations for the operator-valued scale-dependent fields and apply the region causality relations [6] to establish a causal ordering for scale-dependent fields. Further possible applications of the proposed method, including that to gauge theories, and its existing discrete counterparts are mentioned in Sec. IV.

II. QUANTUM FIELD THEORY BASED ON THE CONTINUOUS WAVELET TRANSFORM

To observe the two notes above, we need to modify the definition of the field function. If the ordinary quantum field theory defines the field function $\phi(x)$ as a scalar product of the state vector of the system and the state vector corresponding to the localization at the point x,

$$\phi(x) \equiv \langle x|\phi\rangle, \tag{4}$$

the modified theory should respect the resolution of the measuring equipment. Namely, we define the *resolution-dependent fields*,

$$\phi_a(x) \equiv \langle x, a; g|\phi\rangle, \tag{5}$$

also referred to as scale components of ϕ, where $\langle x, a; g|$ is the bra-vector corresponding to localization of the measuring device around the point x with the spatial resolution a; g labels the apparatus function of the equipment, an *aperture* [7]. In terms of the resolution-dependent field (5), the unit probability of registering the object ϕ anywhere in space at any resolution is expressed by normalization

$$\int |\phi_a(x)|^2 d\mu_g(a, x) = 1, \tag{6}$$

where $d\mu_g(a, x)$ is a translational-invariant measure, which depends on the position x, the resolution a, and the aperture g.

Similarly to representation of a vector $|\phi\rangle$ in a Hilbert space of states $\mathcal{H}$ as a linear combination of an eigenvector of momentum operator $|\phi\rangle = \int |p\rangle dp\langle p|\phi\rangle$, any $|\phi\rangle \in \mathcal{H}$ can be represented as a linear combination of different scale components:

$$|\phi\rangle = \int_G |g; a, b\rangle d\mu(a, b)\langle g; a, b|\phi\rangle. \tag{7}$$

Here, according to [8,9], $|g; a, b\rangle = U(a, b)|g\rangle$; $d\mu(a, b)$ is the left-invariant measure on the affine group G; $U(a, b)$ is a representation of the affine group G: $x' = ax + b$; $|g\rangle \in \mathcal{H}$ is a *admissible vector*, satisfying the condition

$$C_g = \frac{1}{\|g\|_2}\int_G |\langle g|U(a, b)|g\rangle|^2 d\mu(a, b) < \infty.$$

If the measuring equipment has the resolution A, i.e. all states $\langle g; a \geq A, x|\phi\rangle$ are registered with significant probability, but those with $a < A$ are not, the regularization of the model (1) in momentum space, with the cutoff momentum $\Lambda = 2\pi/A$ corresponds to the UV-regularized functions

$$\phi^{(A)}(x) = \frac{1}{C_g}\int_{a\geq A} \langle x|g; a, b\rangle d\mu(a, b)\langle g; a, b|\phi\rangle. \tag{8}$$

The regularized n-point Green functions are $\mathcal{G}^{(A)}(x_1, \ldots, x_n) \equiv \langle \phi^{(A)}(x_1), \ldots, \phi^{(A)}(x_n)\rangle_c$.

However, the momentum cutoff is merely a technical trick: the physical analysis, performed by renormalization group method [1,10], demands the independence of physical results from the cutoff at $\Lambda \to \infty$.

In the present paper we give an alternative, geometrical, interpretation to the cutoff. We assert that if for a given physical system ϕ and given measuring equipment there exist the finest resolution scale A, so that it is impossible to measure any physical quantity related to ϕ with a resolution $a < A$, then any description of ϕ should comprise only such functions, the typical variation scales of which are not less than A. This looks like we observe the system ϕ *from outside the scale* A. The Feynman functional integrations in this approach are performed only over the functions with typical scales $a \geq A$. Our method does not apply any direct cutoff to the momenta—the arguments of the Fourier transform. The momentum conservation in each vertex remains intact. The calculations can be performed either for the scale-component Green functions $\langle \phi_{a_1}(x_1)\ldots\phi_{a_n}(x_n)\rangle$, or for the integrals of those over the scales $\langle \phi^{(A_1)}(x_1)\ldots\phi^{(A_n)}(x_n)\rangle$.

The technical realization of our scheme is based on the substitution of the fields $\langle x|\phi\rangle$ with $|\phi\rangle$ given by (7) into the generating functional (1). In coordinate representation this is known as the continuous wavelet transform (see e.g. [11]). To keep the scale-dependent fields $\phi_a(x)$ the same physical dimension as the ordinary fields $\phi(x)$, we write the coordinate representation of wavelet transform in L^1-norm [7,12,13]:

$$\phi(x) = \frac{1}{C_g}\int \frac{1}{a^d} g\left(\frac{x-b}{a}\right)\phi_a(b)\frac{da d^d b}{a}, \qquad (9)$$

$$\phi_a(b) = \int \frac{1}{a^d}\overline{g\left(\frac{x-b}{a}\right)}\phi(x) d^d x. \qquad (10)$$

In the latter equations the field $\phi_a(b)$ has a physical meaning of the amplitude of the field ϕ measured at point b using a device with an aperture g and a tunable spatial resolution a. For isotropic wavelets g the normalization constant C_ψ is readily evaluated using Fourier transform,

$$C_g = \int_0^\infty |\tilde{g}(ak)|^2 \frac{da}{a} = \int |\tilde{g}(k)|^2 \frac{d^d k}{S_d |k|} < \infty, \qquad (11)$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the area of unit sphere in $\mathbb{R}^d$.

Substitution of the continuous wavelet transform (9) into field theory (1) gives the generating functional for the scale-dependent fields $\phi_a(x)$ [14]:

$$\begin{aligned} W_W[J_a] = \mathcal{N}\int \mathcal{D}\phi_a(x)\exp\Big[-\frac{1}{2}\int \phi_{a_1}(x_1) \\ \times D(a_1, a_2, x_1 - x_2)\phi_{a_2}(x_2)\frac{da_1 d^d x_1}{a_1}\frac{da_2 d^d x_2}{a_2} \\ - \frac{\lambda}{4!}\int V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4}\phi_{a_1}(x_1)\cdots\phi_{a_4}(x_4)\frac{da_1 d^d x_1}{a_1} \\ \times \frac{da_2 d^d x_2}{a_2}\frac{da_3 d^d x_3}{a_3}\frac{da_4 d^d x_4}{a_4} \\ + \int J_a(x)\phi_a(x)\frac{da d^d x}{a}\Big], \end{aligned} \qquad (12)$$

with $D(a_1, a_2, x_1 - x_2)$ and $V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4}$ denoting the wavelet images of the inverse propagator and that of the interaction potential. The Green functions for scale-component fields are given by functional derivatives

$$\langle \phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle_c = \frac{\delta^n \ln W_W[J_a]}{\delta J_{a_1}(x_1)\ldots\delta J_{a_n}(x_n)}\Big|_{J=0}.$$

Surely the integration in (12) over all scale variables $\int_0^\infty [(da_i)/a_i]$ turns us back to the divergent theory (1).

This is the point to restrict the functional integration in (12) only to the field configurations $\{\phi_a(x)\}_{a\ge A}$. The restriction is imposed at the level of the Feynman diagram technique. Indeed, applying the Fourier transform to the right-hand side of (9) and (10), one yields

$$\phi(x) = \frac{1}{C_g}\int_0^\infty \frac{da}{a}\int \frac{d^d k}{(2\pi)^d} e^{-\imath kx}\tilde{g}(ak)\tilde{\phi}_a(k),$$
$$\tilde{\phi}_a(k) = \overline{\tilde{g}(ak)}\,\tilde{\phi}(k).$$

Doing so, we have the following modification of the Feynman diagram technique [15]:

(i) each field $\tilde{\phi}(k)$ will be substituted by the scale component $\tilde{\phi}(k) \to \tilde{\phi}_a(k) = \overline{\tilde{g}(ak)}\,\tilde{\phi}(k)$.

(ii) each integration in momentum variable is accompanied by a corresponding scale integration:

$$\frac{d^d k}{(2\pi)^d} \to \frac{d^d k}{(2\pi)^d}\frac{da}{a}.$$

(iii) each interaction vertex is substituted by its wavelet transform; for the Nth power interaction vertex this gives multiplication by factor $\prod_{i=1}^N \overline{\tilde{g}(a_i k_i)}$.

The finiteness of the loop integrals is provided by the following rule: there should be no scales a_i in internal lines smaller than the minimal scale of all external lines. Therefore the integration in a_i variables is performed from the minimal scale of all external lines up to the infinity.

To illustrate the method we present the calculation of the one-loop contribution to the two- and the four-point Green functions in the ϕ^4 model in $\mathbb{R}^4$. The best choice of the wavelet function $g(x)$ would be the apparatus function of the measuring device; however, a simple choice,

$$g(x) = -xe^{-x^2/2}, \qquad \tilde{g}(k) = (-\imath k)e^{-k^2/2}, \qquad (13)$$

demonstrates the method qualitatively. The function (13) is well localized in both the coordinate and the momentum spaces, it satisfies the admissibility condition with $C_g = 1$. Because of the property $\int_{-\infty}^{\infty} g(x)dx = 0$ the detector with such aperture is insensitive to constant fields, but detects the gradients of the fields.

Let us consider the contribution of the tadpole diagram to the two-point Green function $G^{(2)}(a_1, a_2, p)$ shown in Fig. 1(a). The bare Green function is

$$G_0^{(2)}(a_1, a_2, p) = \frac{\tilde{g}(a_1 p)\tilde{g}(-a_2 p)}{p^2 + m^2}. \qquad (14)$$

The tadpole integral, to keep with the notation of [14], is

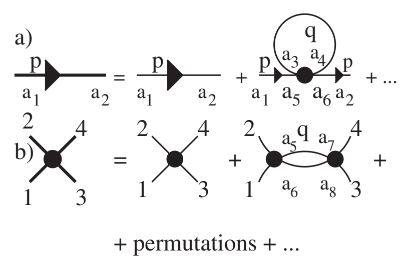

FIG. 1. Feynman diagrams for the Green functions $G^{(2)}$ and $G^{(4)}$ for the resolution-dependent fields.

written as

$$T_1^d(Am) = \frac{1}{C_g^2}\int_{a_3,a_4\geq A}\frac{d^dq}{(2\pi)^d}\frac{|\tilde{g}(a_3q)|^2|\tilde{g}(-a_4q)|^2}{q^2+m^2}\frac{da_3}{a_3}\frac{da_4}{a_4}$$
$$= \frac{S_d m^{d-2}}{(2\pi)^d}\int_0^\infty f^2(Amx)\frac{x^{d-1}dx}{x^2+1}$$
$$f(x) \equiv \frac{1}{C_g}\int_x^\infty |\tilde{g}(a)|^2\frac{da}{a}.$$

For our simple model aperture (13) the filtering factor is $f(x) = e^{-x^2}$.

In $d = 4$ dimension we get

$$T_1^4(\alpha) = \frac{-4\alpha^4 e^{2\alpha^2}\mathrm{Ei}(1,2\alpha^2)+2\alpha^2}{64\pi^2\alpha^4}m^2, \quad (15)$$

where $\alpha \equiv Am$ is dimensionless scale factor, $A = \min(a_1, a_2)$, and

$$\mathrm{Ei}\,(1,z) = \int_1^\infty \frac{e^{-xz}}{x}dx$$

denotes the exponential integral. Finally, the $O(\lambda)$ contribution to the two-point Green function in $\mathbb{R}^d$, shown in Fig. 1(a), is

$$G^{(2)}(a_1,a_2,p) = \frac{\tilde{g}(a_1p)\tilde{g}(-a_2p)}{p^2+m^2} - \frac{\lambda}{2} \times \frac{\tilde{g}(a_1p)\tilde{g}(-a_2p)f^2(Ap)T_1^d(Am)}{(p^2+m^2)^2} + \cdots. \quad (16)$$

In the one-loop contribution to the vertex, shown in Fig. 1(b), the value of the loop integral is

$$X_d = \frac{\lambda^2}{2}\frac{1}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}\frac{f^2(qA)f^2((q-s)A)}{[q^2+m^2][(q-s)^2+m^2]}, \quad (17)$$

where $s = p_1 + p_2$, $A = \min(a_1, a_2, a_3, a_4)$. The integral (17) can be calculated by symmetrization of loop momenta $q \to q + \frac{s}{2}$ in Fig. 1(b); doing so after a simple algebra we yield

$$X_d = \frac{\lambda^2}{2}\frac{S_{d-1}}{(2\pi)^{2d}}s^{d-4}e^{-A^2s^2}\int_0^\infty e^{-4A^2s^2y^2}I_d(y)y^{d-3}dy,$$
$$I_d(y) = \int_0^\pi \frac{\sin^{d-2}\theta d\theta}{\beta^2(y)-\cos^2\theta}, \qquad \beta(y) = \frac{y^2+\frac{1}{4}+\frac{m^2}{s^2}}{y}, \quad (18)$$

where θ is the angle between the loop momentum q and the total momentum s.

In critical dimension $d = 4$,

$$X_4 = \frac{\lambda^2}{256\pi^6}e^{-A^2s^2}\int_0^\infty e^{-4A^2s^2y^2}(1-\sqrt{1-\beta^{-2}(y)})dy^2. \quad (19)$$

In Fig. 2 below we present the graph of large momentum asymptotics of (19)

$$\lim_{s^2\gg 4m^2} X_4(\alpha^2) = \frac{\lambda^2}{256\pi^6}\frac{e^{-2\alpha^2}}{2\alpha^2}[e^{\alpha^2} - 1 - \alpha^2 e^{2\alpha^2}\mathrm{Ei}(1,\alpha^2) + 2\alpha^2 e^{2\alpha^2}\mathrm{Ei}(1,2\alpha^2)], \quad (20)$$

where $\alpha \equiv As$, compared to the (15) factor of the two-point Green function. Other diagrams contributing to the vertex shown in Fig. 1(b) give similar factors with appropriate substitution of s to $s = p_i + p_j$.

Turning back to the coordinate representation of the Green functions for the fields $\phi_a(x)$, we can see there no divergences at coinciding spatial arguments. Say, the bare two-point Green function

$$G_0^{(2)}(a_1,a_2,b_1-b_2) = \int\frac{d^dp}{(2\pi)^d}e^{\imath p(b_1-b_2)}\frac{\tilde{g}(a_1p)\tilde{g}(-a_2p)}{p^2+m^2}$$

gives at our model choice (13) in $d = 4$ dimension

$$G_0^{(2)}(a_1,a_2,b_1-b_2=0) = \pi^2m^2\alpha_1\alpha_2\left[\frac{4}{(\alpha_1^2+\alpha_2^2)^2} - \frac{2}{\alpha_1^2+\alpha_2^2} + e^{((\alpha_1^2+\alpha_2^2)/(2))} \times \mathrm{Ei}\left(1,\frac{\alpha_1^2+\alpha_2^2}{2}\right)\right],$$

where $\alpha_i = a_im$ are dimensionless scale parameters.

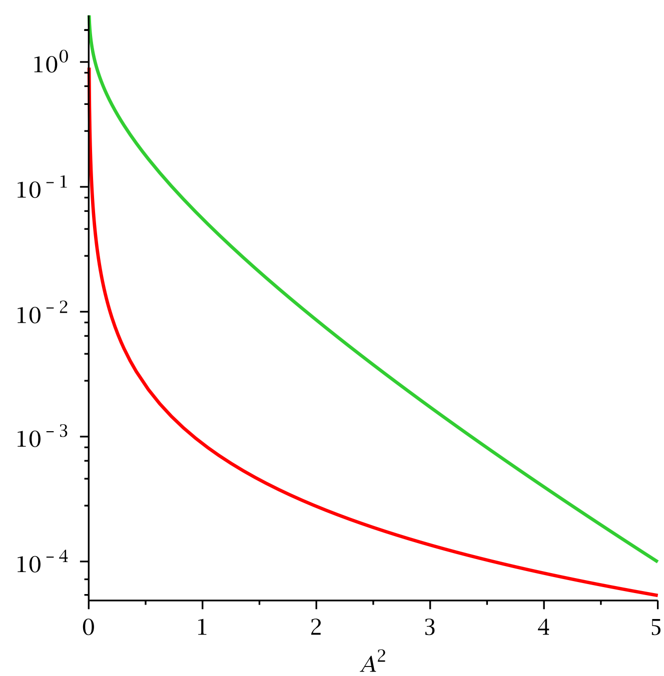

FIG. 2 (color online). Scale-decay factors for the two-point and four-point Green functions. The bottom curve is the graph of (15) as a function of A^2; the top curve is the graph of (20) divided by $\lambda^2/(256\pi^6)$ as a function of A^2. $m = s^2 = 1$ is set for both curves.

We would like to emphasize that, in spite of the fact that application of wavelets to quantum field theory is not new, the interpretation of the fields $\phi_a(x)$ (or their integrals $\phi^{(A)}$) as physical fields yields a finite theory with no need for renormalization. Indeed, the t'Hooft and Veltman dimensional regularization scheme [16] works perfectly well in the presence of the scale factor A. The difference is that the integrated function f in

$$I_A = \int d^4\underline{p} \int_0^\infty d\omega \omega^{n-5} \frac{2\pi^{(n/2)-1}}{\Gamma(\frac{n}{2}-2)} f(A, \underline{p}, \omega^2), \quad (21)$$

where n is the formal integration dimension, in our case contains the exponential factor $f(A, \underline{p}, \omega^2) \sim \exp(-A^2(\underline{p}^2 + \omega^2))$, which suppresses all ultraviolet divergences. In the limit $A \to 0$ the integration by parts in (21) over the ω^2 argument recovers the well-known poles at the physical dimension $n = 4$.

III. CAUSALITY AND COMMUTATION RELATIONS

We have considered a multiscale scalar field theory determined by the generating functional (12). Such a theory is used if the field $\phi_a(x)$ is a *c-valued* function. In quantum field theory adjusted to high energy physics applications, the fields $\phi_a(x)$ are operator-valued functions. So, as it was already emphasized in the context of the wavelet application to quantum chromodynamics [17], the operator ordering and the commutation relations are to be defined.

The commutation relations $[\phi_a(x), \phi_{a'}(x')]$ can be imposed in such a way that they recover ordinary commutation relations after integration over the scale arguments. This was already done in [18]. The decomposition of the operator-valued field $\hat{\phi}(x)$ into the positive and negative frequency scale components is

$$\hat{\phi}(x) = \int \frac{da}{a} \frac{d^dk}{(2\pi)^d} \frac{\tilde{g}(ak)}{C_g} [e^{\imath kx} u_a^+(k) + (-1)^d e^{-\imath kx} u_a^-(k)], \quad (22)$$

where $u_a^\pm(k) = u_a(\pm k)\theta(k_0)|_{k_0>0}$. Since

$$u^\pm(k) = \frac{1}{C_g} \int \frac{da}{a} \tilde{g}(ak) u_a^\pm(k),$$

the standard commutation relations can be satisfied if we set

$$[u_{a_1}^+(k_1), u_{a_2}^-(k_2)] = C_g a_1 \delta(a_1 - a_2)[u^+(k_1), u^-(k_2)]. \quad (23)$$

As was shown in [19], the nonlocal field theory with the propagator cutoff $V(l^2k^2)$ satisfies the *microcausality condition* for the S matrix [20],

$$\frac{\delta}{\delta\phi(x)}\left(\frac{\delta S}{\delta\phi(y)} S^+\right) = 0 \quad \text{for } x \lesssim y, \quad (24)$$

in each order of the perturbation theory. For the theory of scale-dependent fields, a stronger microcausality condition,

$$\frac{\delta}{\delta\phi_a(x)}\left(\frac{\delta S}{\delta\phi_b(y)} S^+\right) = 0 \quad \text{for } x <^T y \quad \text{or} \quad x \sim y, \quad (25)$$

may be suggested if the derivation is performed with the generalized causal T ordering (''the coarse acts first'') defined in [18] according to the region causality rules [6]. The definition of the generalized causal ordering given in [18], is the following:

$$T(A_{\Delta x}(x) B_{\Delta y}(y)) = \begin{cases} A_{\Delta x}(x) B_{\Delta y}(y), & y_0 < x_0, \\ \pm B_{\Delta y}(y) A_{\Delta x}(x), & x_0 < y_0, \\ A_{\Delta x}(x) B_{\Delta y}(y), & \Delta x \subset \Delta y, \\ \pm B_{\Delta y}(y) A_{\Delta x}(x), & \Delta y \subset \Delta x, \end{cases} \quad (26)$$

i.e., if the region Δx is inside the region Δy, the operator related to the larger region Δy acts on vacuum first. If the regions Δx and Δy (the vicinities of two distinct points $x \neq y$) have zero intersection $\Delta x \cap \Delta y = \emptyset$, the causal ordering (26) coincides with usual T ordering.

IV. CONCLUSION

In this paper we presented a regularization method for quantum field theory based on the continuous wavelet transform. The idea of substituting wavelet decomposition of the fields into the action functional is not new. It was used by many authors, but using the *discrete* wavelet transform. This efficiently works for the Monte Carlo simulations [21,22], and provides a frame for renormalization [23,24], including the regularization of gauge theories [17]. In many aspects, the discrete wavelet transform works as a lattice regularization [25]. The novelty of the approach presented in this paper consists in using the *continuous* wavelet transform of the fields (along with the region causality assumptions [6]) with the operator ordering rules given in [18].

An attempt to apply the continuous wavelet transform to the ϕ^4 field theory was undertaken in [26] based on the general ideas of the wavelet transform on the Poincare group [27]. However, a physical interpretation of the wavelet transform scale argument as a physical parameter of observation was given much later in [14,18] in the context of quantum electrodynamics. The key issue of the quantum field theory is gauge invariance. In our wavelet framework, this problem was addressed in [28], where the Ward-Takahashi identities for $U(1)$ gauge theory were derived. Later, we are going to consider this problem in more detail.

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