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Conserved charges in 3D gravity

M. Blagojević* and B. Cvetković†

University of Belgrade, Institute of Physics, P. O. Box 57, 11001 Belgrade, Serbia

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The covariant canonical expression for the conserved charges, proposed by Nester, is tested on several solutions in three-dimensional gravity with or without torsion and topologically massive gravity. In each of these cases, the calculated values of energy momentum and angular momentum are found to satisfy the first law of black hole thermodynamics.

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I. INTRODUCTION

Topologically massive gravity with a cosmological constant (TMG_Λ) is an extension of the usual, Einstein's three-dimensional (3D) gravity, obtained by adding the gravitational Chern-Simons term to the action [1]. This step leads to an essential modification of Einstein's (topological) theory, giving rise to a propagating degree of freedom, the massive graviton. However, for generic values of the coupling constants, the physical interpretation of TMG_Λ in the anti–de Sitter (AdS) sector suffers from serious difficulties, related to the instability of the vacuum [1–3]. To avoid the problem, Li *et al.* [3] introduced the so-called chiral version of TMG_Λ and argued that it might be free from these difficulties, but there are also opposite opinions [4]. Another option would be to choose a new vacuum, such as the spacelike stretched AdS_3, which could be a stable ground state of the theory [5].

It is clear that all these issues are closely related to the concept of conserved charges, which is essential for a clear understanding of the stability problem. The conserved charges were calculated for a number of different solutions of TMG_Λ [6–9], but, as noted in [10], the methods used in these papers are not best suited for studying the general structure of the boundary dynamics. An attempt to improve the situation was made by Mišković and Olea [10], who proposed a background-independent Noether construction of the conserved charges in TMG_Λ. They found correct conserved charges for the Bañados-Teitelboim-Zanelli (BTZ) black hole and the logarithmic solution; however, when applied to the spacelike stretched black hole, their method yields the value of energy with an extra factor $1/2$, as compared to [7,9], the result that is not supported by the first law of black hole thermodynamics [5,7]. In the present paper, we focus our attention on another idea—the *covariant canonical formalism* developed by Nester [11] and collaborators [12–15], which has had excellent results for a large class of the gravitational solutions in 4D. We intend to show that it leads to a rather general expression for the conserved charges in 3D gravity, valid not only in TMG_Λ but also in 3D gravity with or without torsion.

The paper is organized as follows. In Sec. II, we give a brief account of the general first order Lagrangian formalism in 3D gravity, with a focus on (a) 3D gravity with or without torsion and (b) TMG_Λ. In Sec. III, we introduce Nester's covariant canonical formalism and define general covariant expressions for the conserved charges. In Sec. IV, we use these results to evaluate the conserved charges for several solutions in 3D gravity, including not only the BTZ black hole and the logarithmic solution, but also the spacelike stretched black hole. The conserved charges found here satisfy the first law of thermodynamics, and moreover, they are in agreement with the results obtained earlier in [7–9,16,17]. Section V is devoted to concluding remarks, and appendices contain some technical details.

Our conventions are as follows: the Latin indices $(i, j, k, \ldots)$ refer to the local Lorentz frame, the Greek indices $(\mu, \nu, \lambda, \ldots)$ refer to the coordinate frame, and both run over $0, 1, 2$; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor ε^{ijk} is normalized to $\varepsilon^{012} = 1$. Our notation follows the Poincaré gauge theory (PGT) framework in 3D [8,9]: fundamental dynamical variables are the triad field b^i and the Lorentz connection ω^i (1-forms), $T^i = db^i + \varepsilon^{ijk}\omega_j b_k$ and $R^i = d\omega^i + \frac{1}{2}\varepsilon^i{}_{jk}\omega^j\omega^k$ are the corresponding field strengths, the torsion and the curvature (2-forms), the wedge product signs ($\wedge$) are omitted for simplicity, and the relation to the standard 4D notation is given by $\omega^{ij} = -\varepsilon^{ij}{}_k\omega^k$, $R^{ij} = -\varepsilon^{ij}{}_k R^k$.

II. FIRST ORDER LAGRANGIANS IN 3D GRAVITY

Using Nester's ideas [11], we show in this section that the following gravitational models in three dimensions:

(a) 3D gravity with torsion [16,18], including also Einstein's 3D gravity, and
(b) topologically massive gravity [1],

can be described by a generic *first order* Lagrangian (density), with

$$\mathcal{L} = \tau_i T^i + \rho_i R^i - V(b^i, \omega^i, \tau_i, \rho_i). \tag{2.1}$$

*mb@ipb.ac.rs
†cbranislav@ipb.ac.rs

Here, not only b^i and ω^i, but also τ^i and ρ^i are *independent* dynamical variables (1-forms), and V is a conveniently chosen term for each of the above two cases, as described below.

(a) Three-dimensional gravity with torsion is a topological theory in *Riemann-Cartan spacetime,* defined by the Mielke-Baekler Lagrangian [16,18]:

$$\mathcal{L}_{\rm MB} = 2ab^iR_i - \frac{\Lambda}{3}\varepsilon_{ijk}b^ib^jb^k + \alpha_3 L_{\rm CS}(\omega) + \alpha_4 b^iT_i, \tag{2.2a}$$

where $a = 1/16\pi G$, and $L_{\rm CS}(\omega) = \omega^i d\omega_i + \frac{1}{3}\varepsilon_{ijk}\omega^i\omega^j\omega^k$ is the Chern-Simons Lagrangian for the Lorentz connection. The Lagrangian can be rewritten in the form

$$\mathcal{L}_{\rm MB} = \tau_iT^i + \rho_iR^i - \frac{\Lambda}{3}\varepsilon_{ijk}b^ib^jb^k - \frac{\alpha_3}{6}\varepsilon_{ijk}\omega^i\omega^j\omega^k, \tag{2.2b}$$

where τ_i and ρ_i are defined as the covariant field momenta, conjugate to the field strengths T^i and R^i:

$$\tau_i := \frac{\partial L_{\rm MB}}{\partial T^i} = \alpha_4 b^i, \qquad \rho_i := \frac{\partial L_{\rm MB}}{\partial R^i} = 2ab_i + \alpha_3\omega_i. \tag{2.2c}$$

Now, instead of using the original Lagrangian (d2.2), it is more convenient to reformulate it so that τ_i and ρ_i become *independent* dynamical variables. This goal is achieved by the first order Lagrangian (2.1), where the constraints (2.2c) are enforced via Lagrange multipliers in V [12,14].

Note that for $\alpha_3 = \alpha_4 = 0$, the Mielke-Baekler Lagrangian reduces to Einstein's 3D gravity (with a cosmological constant) in Riemannian spacetime. Thus, the first order Lagrangian (2.1) is suitable also for Einstein's 3D gravity.

(b) A simple description of TMG_Λ is given by the Lagrangian [8]

$$\mathcal{L}_{\rm TMG} = 2ab^iR_i - \frac{\Lambda}{3}\varepsilon_{ijk}b^ib^jb^k + a\mu^{-1}L_{\rm CS}(\omega) + \lambda^iT_i, \tag{2.3a}$$

where λ^i (1-form) is a Lagrange multiplier that ensures the vanishing of torsion. This theory can also be transformed into the form (2.1), with a choice of V that ensures the correct on-shell values of τ_i, ρ_i:

$$\tau_i \approx \frac{\partial L_{\rm TMG}}{\partial T^i} = \lambda_i, \qquad \rho_i \approx \frac{\partial L_{\rm TMG}}{\partial R^i} = 2ab_i + a\mu^{-1}\omega_i. \tag{2.3b}$$

The explicit form of V in the generic Lagrangian (2.1) is not important for our discussion of the conserved charges; all we need to remember is that *V does not contain derivatives of dynamical variables,* $V = V(b^i, \omega^i, \tau_i, \rho_i)$.

III. COVARIANT CANONICAL FORMALISM FOR CONSERVED CHARGES

Starting with the first order Lagrangian (2.1) and using the field-theoretic analogue of the classical mechanics relation $Ldt = (p\dot q - H)dt$, one can introduce a timelike vector field ξ and define the Hamiltonian 2-form $\mathcal{H}(\xi)$ on the spatial hypersurface Σ of spacetime [11]. In analogy with old Dirac's ideas [19], these considerations can be generalized by allowing ξ to be either timelike *or* spacelike, whereupon $\mathcal{H}(\xi)$ becomes the *generalized Hamiltonian* density, associated with the dynamical evolution along ξ.

The Hamiltonian density $\mathcal{H}(\xi)$ contains a boundary term dB, but the requirement that $\mathcal{H}(\xi)$ generates the correct equations of motion allows a freedom in the choice of B. As shown by Regge and Teitelboim [20], the proper form of B is determined by the requirement that the functional derivatives of $H(\xi) = \int_\Sigma \mathcal{H}(\xi)$ are well defined or, equivalently, that the boundary term in $\delta H(\xi)$ vanishes. The verification of this criterion is closely related to the specific form of the asymptotic conditions.

If ξ is asymptotically a Killing vector, the related conserved charge is naturally identified as the on-shell value of $H(\xi)$. Since the generalized Hamiltonian has the form

$$H(\xi) = \int_\Sigma \xi^\mu \mathcal{H}_\mu + \int_{\partial\Sigma} B(\xi),$$

where $\mathcal{H}_\mu$ is found to be proportional to the field equations, the value of $H(\xi)$ on a solution of the field equations reduces to the boundary integral $\int_{\partial\Sigma} B(\xi)$. Hence, the corresponding conserved charge is defined as the value of $\int_{\partial\Sigma} B(\xi)$.

The construction of the correct boundary term for specific boundary conditions can be a rather complicated task. However, to find a *unique* expression for B that is compatible with a number of *different* boundary conditions is much more difficult. Ideally, we would like to have a universal expression for B that holds for all (physically acceptable) boundary conditions. Nester [11] started a search for an ideal B in the context of PGT by proposing an expression for it, compatible with solutions having either flat or constant curvature asymptotic behavior. Later modifications of this boundary term were intended to make it valid in more general gravitational theories and for a larger set of boundary conditions, and also to improve its covariance properties [12–15].

In order to properly define the boundary term, it is necessary to choose a reference dynamical configuration. This configuration is most naturally linked to the minimal value of the conserved charge [20]. Let us denote the difference between any variable X and its reference value $\bar X$ by $\Delta X = X - \bar X$. In 3D, the boundary term B is a 1-form. With a suitable set of boundary conditions for the fields, the proper boundary term reads [13]:

$$B = (\xi\rfloor b^i)\Delta\tau_i + \Delta b^i(\xi\rfloor\bar{\tau}_i) + (\tilde{\nabla}_j\xi^i)\Delta\rho_i^j + \Delta\omega^i_j(\xi\rfloor\bar{\rho}_i^j), \tag{3.1a}$$

where $\tilde{\nabla}\xi^i := d\xi^i + (\omega^i_j + e_j\rfloor T^i)\xi^j$, and the relation to our 3D notation is defined by $\omega^{ij} = -\varepsilon^{ijk}\omega_k$, $\rho_{ij} = -\varepsilon_{ijk}\rho^k$. If ξ is asymptotically a Killing vector, this formula can be simplified by choosing b^i so that $\pounds_\xi b^i$ vanishes on the boundary. In that case, the identity $\pounds_\xi b^i \equiv \tilde{\nabla}\xi^i - (\xi\rfloor\omega^i_j)b^j$ leads to

$$B = (\xi\rfloor b^i)\Delta\tau_i + \Delta b^i(\xi\rfloor\bar{\tau}_i) + (\xi\rfloor\omega^i{}_j)\Delta\rho_i{}^j + \Delta\omega^i{}_j(\xi\rfloor\bar{\rho}_i{}^j). \tag{3.1b}$$

One should note that when triad and connection are independent dynamical variables, the concept of a Killing vector differs from the usual one, as defined in GR; see, for instance, [21]. By our convention, since ω^{ij} and ρ_{ij} are antisymmetric objects, the summation over i, j in the last two terms goes over $i < j$. After returning to our 3D notation, we obtain

$$B = (\xi\rfloor b^i)\Delta\tau_i + \Delta b^i(\xi\rfloor\bar{\tau}_i) + (\xi\rfloor\omega^i)\Delta\rho_i + \Delta\omega^i(\xi\rfloor\bar{\rho}_i). \tag{3.2}$$

In our calculations of the boundary integrals, we shall use the Schwarzschild-like coordinates $x^\mu = (t, r, \varphi)$. For solutions with Killing vectors ∂_t and ∂_φ, the conserved charges are energy and angular momentum, respectively:

$$E = \int_{\partial\Sigma} B(\partial_t) = \int_{\partial\Sigma} b^i_0\Delta\tau_i + \Delta b^i\bar{\tau}_{i0} + \omega^i_0\Delta\rho_i + \Delta\omega^i\bar{\rho}_{i0}, \tag{3.3a}$$

$$M = \int_{\partial\Sigma} B(\partial_\varphi) = \int_{\partial\Sigma} b^i_2\Delta\tau_i + \Delta b^i\bar{\tau}_{i2} + \omega^i_2\Delta\rho_i + \Delta\omega^i\bar{\rho}_{i2}, \tag{3.3b}$$

where $\partial\Sigma$ is a circle (which may be located at infinity), described by coordinate φ.

IV. CONSERVED CHARGES IN 3D GRAVITY

In 4D, the expression (3.1a) for B was shown to be valid for general gauge theories of gravity, such as PGT or metric-affine gravity, and for a large set of known solutions with different boundary conditions [11–15]. Here, we wish to verify the correctness of B in 3D gravity by evaluating the conserved charges for several solutions in either 3D gravity with torsion or topologically massive gravity.

A. BTZ black hole in 3D gravity with or without torsion

The BTZ black hole [22], a well-known solution of Einstein's 3D gravity in the AdS sector (with $\Lambda = -1/\ell^2$), is also a solution of 3D gravity with torsion [16,21]. In the Schwarzschild-like coordinates, the black hole metric is given by

$$ds^2 = N^2dt^2 - N^{-2}dr^2 - r^2(d\varphi + N_\varphi dt)^2,$$
$$N^2 = \left(-8Gm + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}\right), \qquad N_\varphi = \frac{4GJ}{r^2}.$$

Our approach to 3D gravity with torsion is based on the PGT formalism; see [16]. Given the metric, we choose the triad field to have the simple, "diagonal" form:

$$b^0 = Ndt, \qquad b^1 = N^{-1}dr, \qquad b^2 = r(d\varphi + N_\varphi dt), \tag{4.1a}$$

while the solution for Cartan's connection reads

$$\omega^i = \tilde{\omega}^i + \frac{p}{2}b^i, \tag{4.1b}$$

where $\tilde{\omega}^i$ is Riemannian connection,

$$\tilde{\omega}^0 = -Nd\varphi, \qquad \tilde{\omega}^1 = N^{-1}N_\varphi dr, \qquad \tilde{\omega}^2 = -\frac{r}{\ell^2}dt - rN_\varphi d\varphi, \tag{4.1c}$$

and $p = (\alpha_3\Lambda + \alpha_4 a)/(\alpha_3\alpha_4 - a^2)$. Equations (4.1) define the analogue of the BTZ black hole in Riemann-Cartan spacetime.

The black hole solution (4.1) possesses two Killing vectors, $\xi_{(0)} = \partial_t$ and $\xi_{(2)} = \partial_\varphi$, which leave the triad field b^i form invariant, $\pounds_\xi b^i = 0$. As the reference configuration, we take the solution (4.1) with $m = J = 0$ (the black hole vacuum). The conserved charges are defined in (3.3), with the field momenta τ_i and ρ_i given in (2.2c). Using the asymptotic behavior of b^i and ω^i (Appendix A), we obtain the following expressions for the energy and angular momentum, respectively:

$$E = m + \frac{\alpha_3}{a}\left(\frac{mp}{2} - \frac{J}{\ell^2}\right), \qquad M = J + \frac{\alpha_3}{a}\left(\frac{pJ}{2} - m\right). \tag{4.2}$$

The result was derived in [21(a)] using Nester's covariant method, and rederived in [16] using Dirac's canonical formalism. Recalling the expressions for the black hole entropy and angular velocity [16], one can verify the validity of the first law of black hole thermodynamics.

For $\alpha_3 = \alpha_4 = 0$, 3D gravity with torsion reduces to Einstein's 3D gravity, and we have $E = m$ and $M = J$, as expected.

B. BTZ black hole in TMG_Λ

The BTZ black hole is a trivial solution of TMG_Λ, since the related Cotton tensor identically vanishes. The solution for b^i and ω^i has the same form as in (4.1) but with $p = 0$,

$$b^0 = Ndt, \qquad b^1 = N^{-1}dr, \qquad b^2 = r(d\varphi + N_\varphi dt), \tag{4.3a}$$

$$\omega^i = \tilde{\omega}^i, \tag{4.3b}$$

while the Lagrange multiplier is given by [8]

$$\lambda^i = \frac{a}{\mu\ell^2} b^i. \tag{4.3c}$$

To find the conserved charges, we combine (2.3b) with (4.3c) to obtain

$$\tau_i = \frac{a}{\mu\ell^2} b^i, \qquad \rho_i = 2ab_i + \frac{a}{\mu}\omega^i. \tag{4.4}$$

Note that the solution (4.3) and the covariant field momenta (4.4) can be obtained from the corresponding expressions (4.1) and (2.2c) in 3D gravity with torsion, by implementing the following limit on parameters:

$$p \to 0, \qquad \alpha_3 \to \frac{a}{\mu}, \qquad \alpha_4 \to \frac{a}{\mu\ell^2}.$$

Hence, the same limit yields the conserved charges in TMG_Λ:

$$E = m - \frac{J}{\mu\ell^2}, \qquad M = J - \frac{m}{\mu}, \tag{4.5}$$

in complete agreement with the results found in [2,8,10]. The first law of black hole thermodynamics is a direct consequence of the arguments given in the previous subsection.

At the chiral points $\mu\ell = \mp 1$, the BTZ charges (4.5) satisfy the chirality relations $\ell E = \pm M$, independently of the values of m and J. Specifically, for the extreme black holes $J = \mp m\ell$ at the chiral points $\mu\ell = \mp 1$, both energy and angular momentum vanish:

$$\mu\ell = \mp 1 \quad \text{and} \quad J = \mp m\ell \Rightarrow E = M = 0. \tag{4.6}$$

C. Spacelike stretched black hole in TMG_Λ

After introducing a convenient notation, $\Lambda = -1/\ell^2$ and $\nu = \mu\ell/3$, the metric of the spacelike stretched black hole can be written in the form [5,7]

$$ds^2 = N^2dt^2 - B^{-2}dr^2 - K^2(d\varphi + N_\varphi dt)^2, \tag{4.7a}$$

where

$$\begin{aligned} N^2 &= \frac{(\nu^2+3)(r-r_+)(r-r_-)}{4K^2}, \qquad B^2 = \frac{4N^2K^2}{\ell^2},\\ K^2 &= \frac{r}{4}[3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-)\\ &\quad - 4\nu\sqrt{r_+r_-(\nu^2+3)}],\\ N_\varphi &= \frac{2\nu r - \sqrt{r_+r_-(\nu^2+3)}}{2K^2}. \end{aligned} \tag{4.7b}$$

The solution is not of constant curvature.

Going over to the PGT formalism [9], we choose the triad field as

$$b^0 = Ndt, \qquad b^1 = B^{-1}dr, \qquad b^2 = K(d\varphi + N_\varphi dt), \tag{4.8a}$$

and calculate the corresponding Riemannian $\tilde\omega^i$:

$$\begin{aligned} \tilde\omega^0 &= -\frac{N\nu}{\ell}dt - \frac{2NKK'}{\ell}d\varphi, \qquad \tilde\omega^1 = -\frac{KN'_\varphi}{2N}dr,\\ \tilde\omega^2 &= -\frac{KN_\varphi\nu}{\ell}dt + \frac{K^3N'_\varphi}{\ell}d\varphi. \end{aligned} \tag{4.8b}$$

Finally, the solution for λ_m has the form

$$\begin{aligned} \lambda_0 &= \frac{2a}{\mu\ell^2}\Big[\Big(-\frac{3}{2} + 2\nu^2\Big)Ndt + 3(\nu^2-1)NK^2N_\varphi d\varphi\Big],\\ \lambda_1 &= \frac{2a}{\mu\ell^2}\Big(-\frac{3}{2} + \nu^2\Big)B^{-1}dr,\\ \lambda_2 &= \frac{2a}{\mu\ell^2}\Big[\Big(\frac{3}{2} - 2\nu^2\Big)KN_\varphi dt\\ &\quad + \Big(\frac{3}{2} - 2\nu^2 - 3(\nu^2-1)N^2\Big)Kd\varphi\Big]. \end{aligned} \tag{4.8c}$$

The result follows from the general formula:

$$\lambda_m = 2a\mu^{-1}[(Ric)_{mn} - \tfrac{1}{4}\eta_{mn}R]b^n, \tag{4.9}$$

where $(Ric)_{mn} = -\varepsilon^{ij}{}_m R_{ijn}$ is the Ricci tensor, and $R = -\varepsilon^{ijm}R_{ijm}$ the scalar curvature; see Eqs. (2.5c) and (B.2b) of Ref. [9].

Now, using the expressions (2.3b) for τ_i and ρ_i and the asymptotic behavior of N, B, K, and N_φ (Appendix B), with reference configuration defined by $r_- = r_+ = 0$, Eq. (3.3) yields the energy and angular momentum of the spacelike stretched black hole (4.8):

$$\begin{aligned} E &= \frac{(\nu^2+3)}{24G\ell}\Big[r_+ + r_- - \frac{1}{\nu}\sqrt{r_+r_-(3+\nu^2)}\Big],\\ M &= \frac{\nu(\nu^2+3)}{96G\ell}\Big[\Big(r_+ + r_- - \frac{1}{\nu}\sqrt{r_+r_-(3+\nu^2)}\Big)^2\\ &\quad - \frac{5\nu^2+3}{4\nu^2}(r_+ - r_-)^2\Big]. \end{aligned} \tag{4.10}$$

These values coincide with those found in [5,7,9], and moreover, [5,7]. On the other hand, a comparison with the results of Ref. [10] shows a discrepancy by a factor of 1/2 in energy.

D. Logarithmic solution of TMG_Λ

Let us now consider another genuine solution to TMG_Λ, the logarithmic solution at the chiral point $\mu\ell = -1$ [23]. The metric of the solution is stationary and spherically symmetric:

$$ds^2 = A^2dt^2 - B^{-2}r^2 - K^2(d\varphi + Cdt)^2, \tag{4.11a}$$

where

$$A = \frac{r}{K}B, \qquad B = \frac{r}{\ell} - \frac{4Gm\ell}{r}, \qquad K^2 = r^2 + \ell^2 L_k,$$
$$C = -\frac{4G\ell m + \ell L_k}{K^2}, \qquad L_k = k\ln\frac{r^2 - 4Gm\ell^2}{r_0^2}. \tag{4.11b}$$

The metric depends on two constants m and k, whereas r_0 is a normalization factor that does not influence the values of the conserved charges, but does have an impact on the geometric properties of the solution. For $k = 0$, (4.11) reduces to the extreme BTZ black hole with $J = -m\ell$, at the chiral point $\mu\ell = -1$.

In the PGT formalism, we start with

$$b^0 = Adt, \qquad b^1 = B^{-1}dr, \qquad b^2 = K(d\varphi + Cdt), \tag{4.12a}$$

the corresponding Riemannian connection $\omega^i = \tilde{\omega}^i$ takes the form

$$\tilde{\omega}^0 = \frac{k}{K}dt - \frac{r^2 + \ell^2(k - 4Gm)}{\ell K}d\varphi,$$
$$\tilde{\omega}^1 = -\frac{\ell(4Gm - k + L_k)}{BK^2}dr, \tag{4.12b}$$
$$\tilde{\omega}^2 = \left(\frac{k}{K} - \frac{K}{\ell^2}\right)dt + \frac{\ell(4Gm - k + L_k)}{K}d\varphi,$$

and the solution for λ^m reads:

$$\lambda^0 = -\left[\frac{a}{\ell}A + \frac{4a\ell k}{K^2}(A - KC)\right]dt + \frac{4a\ell k}{K}d\varphi,$$
$$\lambda^1 = -\frac{a}{\ell B}dr,$$
$$\lambda^2 = -\left[\frac{a}{\ell}KC + \frac{4a\ell k}{K^2}(A - KC)\right]dt - \left(\frac{a}{\ell}K - \frac{4a\ell k}{K}\right)d\varphi. \tag{4.12c}$$

For a detailed derivation of (4.12), see Appendix C.

Using our basic formula (3.3) for the conserved charges, the expressions (2.3b) for the field momenta τ_i, ρ_i, the asymptotic behavior as described in Appendix D and taking the reference configuration with $m = k = 0$, we find that the energy and angular momentum of the logarithmic solution (4.12) are

$$E = \frac{k}{2G}, \qquad M = -\frac{k\ell}{2G}. \tag{4.13}$$

To verify (4.13), we first used the standard canonical approach and found that the energy and angular momentum of the logarithmic solution (4.12) are given by the formulas

$$E_c = \int_{\partial\Sigma} 2ab_0^0\left(\omega^0 - \frac{1}{2\ell}b^0 + \frac{1}{2a}\lambda^0 + \frac{1}{\ell}b^2 - \omega^2\right),$$
$$M_c = -\int_{\partial\Sigma} 2ab_2^2\left(\omega^2 - \frac{1}{2\ell}b^2 + \frac{1}{2a}\lambda^2 + \frac{1}{\ell}b^0 - \omega^0\right),$$

which are the same as those describing the AdS sector of TMG_Λ, taken at $\mu\ell = -1$ [8]. The evaluation of these expressions produces the same result as in (4.13). Moreover, since the Hawking temperature of the logarithmic solution vanishes, $4\pi T = (A^2)'|_{r_+} = 0$, and the angular velocity is $\Omega = C|_{r_+} = -1/\ell$ (with $r_+^2 = 4Gm\ell^2$), it follows that the charges (4.13) satisfy the first law of black hole thermodynamics:

$$\delta E - \Omega\delta M = \delta E + \frac{1}{\ell}\delta M = 0.$$

The results (4.13) essentially agree with those obtained by Clement [17], up to some differences in conventions. Indeed, his angular momentum and Ω have opposite signs as compared to ours, but they still satisfy the first law. Our conserved charges also agree with the results found in [10], but they differ by an overall factor $3/2$ when compared to [23].

Let us observe that in parallel with the logarithmic solution at the chiral point $\mu\ell = -1$, there exists another logarithmic solution at $\mu\ell = 1$, obtained from (4.12) by changing the sign of C. For $k = 0$, the new solution reduces to the extreme BTZ black hole with $J = m\ell$. An analogous evaluation of its conserved charges leads to

$$\bar{E} = \frac{k}{2G}, \qquad \bar{M} = \frac{k\ell}{2G}.$$

Since now $\bar{\Omega} = 1/\ell$, the first law is again satisfied. Note that this "antichiral" solution is *different* from the solution displayed in Eq. (25) of Ref. [23].

V. CONCLUDING REMARKS

In this paper, we examined Nester's covariant canonical expression (3.1) for the conserved charges in 3D gravity. It is shown that the evaluated energy and angular momentum of the BTZ black hole (in both 3D gravity with torsion and TMG_Λ), spacelike stretched black hole, and the logarithmic solution, are in complete agreement with the results obtained by different methods in [2,7–9,16,17]. Moreover, for each of these solutions, the calculated conserved charges are seen to satisfy the first law of black hole thermodynamics.

On the other hand, the authors of [10], working in the context of TMG_Λ, obtained the same results as ours in the case of the BTZ black hole and the logarithmic solution, but their treatment of the spacelike stretched black hole features certain difficulties. Thus, our approach based on (3.1) exhibits a wider range of validity.

The Lagrangians considered in the present paper are linear in torsion and/or curvature. As a further test of universality of the formula (3.1) in 3D gravity, it would be interesting to apply it to the Lagrangians quadratic in torsion and/or curvature. After this paper has been completed, we started studying the conserved charges in new massive gravity [24]. As a first result in this direction, we found that the formula (3.3) gives the correct conserved charges for the BTZ solution [25].

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APPENDIX A: ASYMPTOTICS OF THE BTZ BLACK HOLE

In this appendix, we display the asymptotic form of the BTZ black hole solution in 3D gravity with torsion and in TMG_Λ. The subleading terms are given up to the order that contributes to the values of conserved charges.

BTZ black hole in 3D gravity with torsion.—Starting with (4.1a), we find the asymptotic form of the triad field:

$$b^0 \sim \left(\frac{r}{\ell} - \frac{4Gm\ell}{r}\right)dt, \qquad b^1 \sim \left(\frac{\ell}{r} + \frac{4Gm\ell^3}{r^3}\right)dr, \qquad b^2 \sim \frac{4GJ}{r}dt + rd\varphi. \tag{A1a}$$

Recalling that Cartan's connection has the form (4.1b),

$$\omega^i = \tilde\omega^i + \frac{p}{2}b^i, \tag{A1b}$$

it follows that its asymptotics is determined by (A1a) and

$$\tilde\omega^0 \sim \left(-\frac{r}{\ell} + \frac{4Gm\ell}{r}\right)d\varphi, \qquad \tilde\omega^1 \sim \frac{4GJ\ell}{r^3}dr, \qquad \tilde\omega^2 \sim -\frac{r}{\ell^2}dt - \frac{4GJ}{r}d\varphi. \tag{A1c}$$

BTZ black hole in TMG_Λ.—Looking at (4.3), we see that the asymptotic form of the basic dynamical variables in TMG_Λ is determined by the formulas (A1) taken at $p = 0$, where the torsion vanishes.

APPENDIX B: ASYMPTOTICS OF THE SPACELIKE STRETCHED BLACK HOLE

Dynamical variables of the spacelike stretched black hole solution (4.8) are expressed in terms of the functions N, K, and N_φ. Consequently, the asymptotic behavior of the solution is determined by the following asymptotic relations:

$$N \sim \sqrt{\frac{\nu^2+3}{3(\nu^2-1)}} - \frac{2\nu\sqrt{\nu^2+3}(\nu(r_+ + r_-) - \sqrt{(\nu^2+3)r_+r_-})}{[3(\nu^2-1)]^{3/2}r},$$

$$K \sim \frac{\sqrt{3(\nu^2-1)}}{2}r + \frac{(\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{(\nu^2+3)r_+r_-}}{4\sqrt{3(\nu^2-1)}} - \frac{((\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{(\nu^2+3)r_+r_-})^2}{16[3(\nu^2-1)]^{3/2}r},$$

$$N_\varphi \sim \frac{4\nu}{3(\nu^2-1)r} - \frac{2(2\nu(\nu^2+3)(r_+ + r_-) - (5\nu^2+3)\sqrt{(\nu^2+3)r_+r_-})}{[3(\nu^2-1)]^2r^2}.$$

APPENDIX C: DERIVATION OF THE LOGARITHMIC SOLUTION

To derive the logarithmic solution (4.12), we start with the triad field (4.12a), and calculate the Riemannian connection $\omega^i = \tilde\omega^i$:

$$\tilde\omega^0 = -\beta b^0 - \gamma b^2, \qquad \tilde\omega^1 = -\beta b^1, \qquad \tilde\omega^2 = -\alpha b^0 + \beta b^2,$$

where

$$\alpha := \frac{BA'}{A} = \frac{r^2 + \ell^2(4Gm - k + 2L_k)}{\ell K^2},$$

$$\beta := \frac{BKC'}{2A} = \frac{\ell(4Gm - k + L_k)}{K^2},$$

$$\gamma := \frac{BK'}{K} = \frac{r^2 + \ell^2(k - 4Gm)}{\ell K^2}.$$

This result implies (4.12b). Note that the coefficients α, β, and γ are not independent:

$$\alpha - \beta = \frac{1}{\ell}, \qquad \beta + \gamma = \frac{1}{\ell}.$$

In the next step, we calculate the Riemann curvature:

$$\tilde R_0 = -\left(\frac{1}{\ell^2} - \frac{2k}{K^2}\right)b^1b^2 + \frac{2k}{K^2}b^0b^1, \qquad \tilde R_1 = -\frac{1}{\ell^2}b^2b^0,$$

$$\tilde R_2 = -\left(\frac{1}{\ell^2} + \frac{2k}{K^2}\right)b^0b^1 - \frac{2k}{K^2}b^1b^2.$$

The solution for λ^m, based on Eq. (4.9), takes the form

$$\lambda^0 = -2a\ell\left[\frac{1}{2\ell^2}b^0 + \frac{2k}{K^2}(b^0 - b^2)\right],$$

$$\lambda^1 = -2a\ell\left(\frac{1}{2\ell^2}b^1\right),$$

$$\lambda^2 = -2a\ell\left[\frac{1}{2\ell^2}b^2 + \frac{2k}{K^2}(b^0 - b^2)\right],$$

which implies (4.12c).

In order to show that (4.12) is indeed a solution of TMG_Λ, we calculated the Cotton 2-form $C_i = (\mu/2a)\nabla\lambda_i$,

$$C_0 = \frac{2k}{\ell K^2}(b^0 b^1 + b^1 b^2), \qquad C_1 = 0, \qquad C_2 = -C_0,$$

and verified the basic field equation of TMG_Λ:

$$2\tilde{R}_i + \frac{1}{\ell^2}\varepsilon_{ijk} b^j b^k + 2\mu^{-1} C_i = 0.$$

APPENDIX D: ASYMPTOTICS OF THE LOGARITHMIC SOLUTION

The logarithmic solution (4.12) at $\mu\ell = -1$ is determined by the functions A, B, K, and C. Using their asymptotic behavior,

$$L_k \sim 2k \ln\frac{r}{r_0}, \qquad B \sim \frac{r}{\ell} - \frac{4Gm\ell}{r}, \qquad K \sim r + \frac{\ell^2 L_k}{2r},$$
$$C \sim -\frac{4Gm\ell + \ell L_k}{r^2}, \qquad A \sim \frac{r}{\ell} - \frac{4Gm\ell + \ell L_k/2}{r}.$$

we can find the asymptotic form of all the fields needed to calculate the conserved charges:

$$b^0 \sim \left(\frac{r}{\ell} - \frac{4Gm\ell + \ell L_k/2}{r}\right)dt, \qquad b^2 \sim \left(r + \frac{\ell^2 L_k}{2r}\right)d\varphi - \frac{4Gm\ell + \ell L_k}{r}dt, \tag{D1a}$$
$$\tilde{\omega}^0 = \frac{k}{r}dt - \frac{r^2 + \ell^2(k - 4Gm - L_k/2)}{\ell r}d\varphi, \qquad \tilde{\omega}^2 = \left(-\frac{r}{\ell^2} + \frac{2k - L_k}{2r}\right)dt + \frac{\ell(4Gm - k + L_k)}{r}d\varphi, \tag{D1b}$$
$$\lambda^0 = -a\left(\frac{r}{\ell^2} - \frac{4Gm + L_k/2}{r} + \frac{4k}{r}\right)dt + \frac{4a\ell k}{r}d\varphi, \qquad \lambda^2 = a\left(\frac{4Gm + L_k}{r} + \frac{4k}{r}\right)dt - a\ell\left(\frac{r}{\ell^2} + \frac{L_k}{2r} - \frac{4k}{r}\right)d\varphi. \tag{D1c}$$

Similarly, one can find the asymptotic form of the logarithmic solution at $\mu\ell = 1$.

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