

Alternative to the Palatini method: A new variational principle

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A variational principle is suggested within Riemannian geometry, in which an auxiliary metric and the Levi Civita connection are varied independently. The auxiliary metric plays the role of a Lagrange multiplier and introduces nonminimal coupling of matter to the curvature scalar. The field equations are 2nd order PDEs and easier to handle than those following from the so-called Palatini method. Moreover, in contrast to the latter method, no gradients of the matter variables appear. In cosmological modeling, the physics resulting from the *alternative* variational principle will differ from the modeling using the *standard* Palatini method.

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I. INTRODUCTION

For the derivation of the field equation of Einstein's theory of gravitation and of alternative gravitational theories sometimes a method named, alternatively, "Palatini's principle," "the Palatini method of variation" or "Palatini's device" is used. Although the starting point is Riemannian geometry and the metric is used as a variational variable, the Levi Civita connection formed from the metric is replaced by an additional second variable, i.e., an arbitrary connection Γ imagined to form the curvature tensor. In the Lagrangian, both metric and connection then are varied independently. An advantage of the method is that it leads to 2nd order field equation for Lagrangians of higher order in curvature while a variation of the metric as the only variable results in 4th-order PDEs. On the other hand, a main conceptual difficulty of the method is that it is foreign to a variational principle within Riemannian geometry: the variational procedure mixes Riemannian and metric-affine geometry. Authors either leave undetermined the space-time geometry as a frame for the *arbitrary* connection, or tacitly fix it mentally by introducing constraints (symmetric connection, no torsion etc.) which do not show up in the formalism.

For many years, warnings have been voiced that the method be working reliably only for the Hilbert-Einstein Lagrangian (plus the matter part) $\mathcal{L} = \sqrt{-g}[R(g_{ij}) + 2\kappa L_{\text{mat}}(g_{ij}, u^A)]$ with curvature scalar $R = g^{lm}R_{lm}(g_{ij})$ and matter variables u^A , but otherwise leads to under- and undeterminacies [1–3].¹ The method also has been criticized from the point of view of topological gauge theory [6]. Recently, Palatini's method has been unearthed in attempts to build cosmological models thought to explain the accelerated expansion of the universe with its consequences for dark energy [7–11]. The method also has

been applied to loop quantum cosmology [12]. Often, the starting point is a Lagrangian of the form $\mathcal{L} = \sqrt{-g}[R(g_{ij}) + \tilde{f}(R)] + \sqrt{-g}2\kappa L_{\text{mat}}(g_{ij}, u^A)$ with \tilde{f} an arbitrary smooth function.² In the following, we suggest a variant of the Palatini variational principle leading to 2nd order field equations and lacking the deficiencies of the Palatini method. Its main physical motivation is the less complicated nonminimal coupling of matter and curvature as compared to the standard Palatini method. After its introduction, it is applied to the class of $f(R)$ -theories in Sec. III and compared with the Palatini method in Sec. IV. A recent particular choice for $f(R)$ in the framework of cosmological modeling then is used as an example for the working of the *alternative variational* principle.

II. THE ALTERNATIVE VARIATIONAL PRINCIPLE

As described above, in the standard Palatini method the Levi Civita connection (represented by the Christoffel symbol) is replaced by a general affine connection. Vice versa, here we replace the (Lorentz-)metric g_{ik} by an auxiliary metric γ_{ab} but keep the Levi Civita connection formed from g_{ik} unchanged. The independent variables for the variation now are γ_{ab} and the Levi Civita connection:

$$\{ \overset{k}{i;j} \}_g = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right). \quad (1)$$

The equations following from the variation will give the dynamics of the gravitational field and link γ_{ab} with g_{ab} . We wish to emphasize that it is *not* a bi-metric theory which is aimed at.³ The auxiliary metric may be seen as playing the role of a Lagrange multiplier. This is analogous

²Recently, Lagrangians with two curvature invariants, i.e., $f(R, R_{ab}R_{lm}g^{al}g^{bm})$ have been considered.

³In bi-metric theories, one metric usually is fixed to be the flat Minkowskian metric and not varied. A formal variation of the second metric often is restricted to an infinitesimal coordinate change in order to derive conservation laws. Cf. [14].

¹For incorrectly relating Palatini's name with what is ascribed to him cf. [4], footnote on p. 40 as well as the English translation of Palatini's paper in the same volume on pp. 477–488 (1980). Cf. also [5].

to the case of scalar-tensor theories replacing $f(R)$ -theories of gravitation (cf. [15,16]). The *alternative* variational principle for Einstein gravity starts from⁴:

$$\mathcal{L} = \sqrt{-\gamma} \left[\gamma^{ab} R_{ab} \left(\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g \right) + 2\kappa L_{\text{mat}}(\gamma_{lm}, u^A) \right]. \quad (2)$$

Variation with respect to γ^{ab} leads to

$$\begin{aligned} \delta_\gamma \mathcal{L} = \sqrt{-\gamma} \left[R_{ab} \left(\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g \right) - \frac{1}{2} \gamma_{ab} R_\gamma \right. \\ \left. + 2\kappa T_{ab}(\gamma_{lm}, u^A) \right] \delta \gamma^{ab}, \end{aligned} \quad (3)$$

where

$$R_\gamma := \gamma^{lm} R_{lm} \left(\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g \right)$$

and $T_{ab} := \frac{2}{\sqrt{-\gamma}} \frac{\delta L_{\text{mat}}}{\delta \gamma^{ab}}$. Variation with respect to

$$\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g$$

gives:

$$\delta_{\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g} \mathcal{L} = [-(\sqrt{-\gamma} \gamma^{b(i)}{}_{;b} \delta_k^j) + (\sqrt{-\gamma} \gamma^{ij})_{;k}] \delta \left(\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g \right) \quad (4)$$

up to divergence terms.⁵ From

$$\delta_{\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g} \mathcal{L} = 0,$$

after a brief calculation using the trace of (4),

$$(\sqrt{-\gamma} \gamma^{ij})_{;k} = 0 \quad (5)$$

follows, where the covariant derivative is formed with the Levi Civita connection. Thus, $\gamma^{ab} = \text{const} \cdot g^{ab}$ follows. $\delta_\gamma \mathcal{L} = 0$ from (3) reduces to Einstein's field equations.

The method is particularly well suited to a calculus with differential forms. Here, the usual basic 1-forms $\theta^i = e^i_r dx^r$ and the curvature 2-form $\Omega_{ij} = \frac{1}{2} R_{ijkl}(g_{lm}) \theta^k \wedge \theta^l$ are taken as the independent variables. In place of the auxiliary metric γ_{ij} , now an auxiliary 1-form is introduced and denoted by $\bar{\theta}^i = \bar{e}^i_r dx^r$ where

⁴Latin indices a, b, i, j, \dots run from 0 to 3; the summation convention is implied.

⁵The semicolon denotes covariant differentiation with respect to the Levi Civita connection. The surface contribution $(\sqrt{-\gamma} A^k)_{;k}$ always can be written as $\sqrt{-g} (\sqrt[2]{g} A^k)_{;k}$ and thus as $(\sqrt{-\gamma} A^k)_{;k}$. Here,

$$A^k := \gamma^{ab} \delta \left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g - \gamma^{ka} \delta \left\{ \begin{smallmatrix} b \\ ab \end{smallmatrix} \right\}_g.$$

Thus, with the same assumption as in Einstein's theory, i.e., $\delta g_{ab} = 0$, $\delta(g_{ab,c}) = 0$ on the boundary, the divergence terms may be omitted.

$$\bar{e}^i_r \bar{e}^j_s \eta^{ij} = \gamma^{rs}, \quad e^i_r e^j_s \eta^{ij} = g^{rs}. \quad (6)$$

The Einstein-Hilbert Lagrangian is $\mathcal{L}_E = \Omega_{ab} \wedge *(\bar{\theta}^a \wedge \bar{\theta}^b)$ with the Hodge-star operation: $*(\bar{\theta}^a \wedge \bar{\theta}^b) := \bar{\epsilon}^{ab}$ and $\bar{\epsilon}_{ab} := \frac{1}{2!} \epsilon_{ablm} \bar{\theta}^l \wedge \bar{\theta}^m$.⁶ Variation with regard to the fundamental 1-forms and curvature form leads to the field equations:

$$D \left(\frac{\partial \mathcal{L}_E}{\partial \Omega_{ij}} \right) = 0, \quad \frac{\partial \mathcal{L}_E}{\partial \bar{\theta}^i} = 0 \quad (7)$$

with the covariant external derivative D using the Levi Civita connection (1-form). Because of $\frac{\partial \mathcal{L}_E}{\partial \Omega_{ij}} = *(\bar{\theta}^i \wedge \bar{\theta}^j)$ and of $\frac{\partial \mathcal{L}_E}{\partial \bar{\theta}^i} = \Omega_{lm} \wedge \bar{\epsilon}_{ilm}$, the field equations are

$$D \bar{\epsilon}^{ij} = 0, \quad \Omega_{lm} \wedge \bar{\epsilon}_{ilm} = 0, \quad (8)$$

where $\bar{\epsilon}_{ilm} := \epsilon_{ilm p} \theta_p$ is a 1-form; $\bar{\epsilon}^{ilm}$ is dual to $\bar{\theta}^i \wedge \bar{\theta}^l \wedge \bar{\theta}^m$. Standard manipulations with the forms show that the 1st Eq. (8) is satisfied identically due to the absence of torsion, i.e., $D\bar{\theta}^m = 0$; and that the 2nd becomes: $2G^c_a(g) \bar{\epsilon}^c = 0$ with the Einstein tensor $G^c_a(g)$ and the 3-form $\bar{\epsilon}_i := \frac{1}{3!} \epsilon_{iklm} \bar{\theta}^k \wedge \bar{\theta}^l \wedge \bar{\theta}^m$. An advantage of this formalism is that it may be adapted easily to gauge theories of gravitation or to classical unified field theories.

III. EXTENSION TO $f(R)$ -THEORIES

The *alternative* variational principle easily applies to the Lagrangian

$$\mathcal{L} = \sqrt{-\gamma} \left[f \left(\gamma^{lm} R_{lm} \left(\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g \right) \right) + 2\kappa L_{\text{mat}}(\gamma_{ij}, u^A) \right]. \quad (9)$$

The variations lead to

$$\begin{aligned} \delta_\gamma \mathcal{L} = \sqrt{-\gamma} \left[f'(R_\gamma) R_{ab} \left(\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g \right) - \frac{1}{2} \gamma_{ab} f(R_\gamma) \right. \\ \left. + 2\kappa T_{ab}(\gamma_{lm}, u^A, \partial u^A) \right] \delta \gamma^{ab}, \end{aligned} \quad (10)$$

with $f' := \frac{df}{dR}$ and to

$$\begin{aligned} \delta_{\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g} \mathcal{L} = [-(\sqrt{-\gamma} f'(R_\gamma) \gamma^{b(i)}{}_{;b} \delta_k^j) \\ + (\sqrt{-\gamma} f'(R_\gamma) \gamma^{ij})_{;k}] \delta \left(\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g \right) \end{aligned} \quad (11)$$

up to divergence terms. As in Sec. II, from (11)

⁶Notation here is somewhat ambiguous: e.g., the curvature form depends on both the Levi Civita connection and the auxiliary tetrad:

$$\Omega_{ij} = \frac{1}{2} R_{ijkl} \left(\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}_g \right) \bar{\theta}^k \wedge \bar{\theta}^l.$$

Nevertheless, no bar will be put on Ω . The notation $\Omega_{ij}(g, \bar{\theta})$ would be inconvenient.

$$(\sqrt{-\gamma}f'(R_\gamma)\gamma^{ij})_{;k} = 0, \quad (12)$$

but from which now follows:

$$\gamma^{ab} = f'(R_\gamma)g^{ab}, \quad \gamma_{ab} = (f'(R_\gamma))^{-1}g_{ab}. \quad (13)$$

From (13), $R_\gamma = f'(R_\gamma)R_g$ with

$$R_g := g^{lm}R_{lm}(\{ \overset{k}{ij} \}_g),$$

i.e., the curvature scalar in (pseudo-)Riemannian space-time. Writing

$$R_g = \frac{R_\gamma}{f'(R_\gamma)} := h(R_\gamma), \quad (14)$$

the relation $R_\gamma = h^{-1}(R_g)$ can be used to remove all entries of γ_{ab} via the curvature scalar in the field equations following from (10). Expressed by g_{ab} , they read as

$$f'(h^{-1}(R_g))R_{ab}(\{ \overset{k}{ij} \}_g) - \frac{1}{2}g_{ab} \frac{f(h^{-1}(R_g))}{f'(h^{-1}(R_g))} + 2\kappa T_{ab}((f')^{-1}(h^{-1}(R_g))g_{lm}, u^A) = 0. \quad (15)$$

Equation (15) shows that, in contrast to $f(R)$ -theories leading to 4th-order differential equations when derived by variation of only the metric g_{ab} , the new field equations are of 2nd order in the derivatives of g_{ab} . The auxiliary metric is fully determined: $\gamma^{ab} = f'(h^{-1}(R_g))g^{ab}$; it is not an absolute object. Beyond acting as a Lagrange multiplier its main function is its appearance in the matter tensor causing nonminimal coupling to the curvature scalar. No further role in the description of the gravitational field is played.⁷ For a Lagrangian of the form $\sqrt{-g}[R(g_{ij}) + \tilde{f}(R)]$, in the formalism given above f is to be replaced by $R + \tilde{f}(R)$, f' by $1 + \tilde{f}'$ while $f'' = \tilde{f}''$, $f''' = \tilde{f}'''$.

A. First, a nonvanishing trace (with respect to the auxiliary metric γ) of the matter tensor will be assumed $T_\gamma := \gamma^{lm}T_{lm}(\gamma_{rs}, u^A) \neq 0$. In this case, the curvature scalar is seen to be a functional of the trace of the matter tensor. Because of

$$T_\gamma = f'(R_\gamma)g^{lm}T_{lm}(f'(h^{-1}(R_g))g_{rs}, u^A) = f'(h^{-1}(R_g))T_g(f'(h^{-1}(R_g))g_{rs}, u^A), \quad (16)$$

with $\tilde{T}_g := g^{lm}T_{lm}(\gamma_{rs}, u^A)$ from the g -trace of (15) follows:

$$f'^2 R_g - 2f + 2\kappa f' \tilde{T}_g = 0, \quad (17)$$

or, precisely,

⁷In particular, γ^{ab} does *not* enter the Levi Civita connection, but only the matter tensor. As a metric γ_{ab} is incompatible with the Levi Civita connection; its nonmetricity tensor does not vanish.

$$(f'(h^{-1}(R_g))^2 R_g - 2f(h^{-1}(R_g)) + 2\kappa f'(h^{-1}(R_g))\tilde{T}_g((f'(R_g))^{-1}g_{lm}u^A) = 0. \quad (18)$$

With a newly defined function ω , this can be written as

$$R_g = \omega(2\kappa T_g), \quad (19)$$

where now $T_g := g^{lm}T_{lm}(g_{rs}, u^A)$. From (18) we conclude that (15) can be cast into the form of Einstein's equations with an effective matter tensor. The curvature scalar is coupled directly to the matter variables showing up in its trace; no derivatives are involved. In fact:

$$R_{ab}(\{ \overset{k}{ij} \}_g) - \frac{1}{2}g_{ab}(R_g) = -\frac{2\kappa}{f'} \left[T_{ab} \left(\frac{1}{f'} g_{lm}, u^A \right) - \frac{1}{2} T g_{ab} \right] - \frac{1}{2} g_{ab} \frac{f}{(f')^2}. \quad (20)$$

In the case of *perfect fluid* matter with energy density μ and pressure p

$$T_{ab}(\gamma_{rs}, u^A) = (\mu + p)\gamma_{al}\gamma_{bm}\bar{u}^l\bar{u}^m - p\gamma_{ab} \quad (21)$$

with $\bar{u}^l := \frac{dx^l}{ds}$ and $d\bar{s}^2 = \gamma_{lm}dx^l dx^m$. Hence, $\bar{u}^l = (f')^{1/2}u^l$, $u^l = \frac{dx^l}{ds}$ and

$$T_{ab}(\gamma_{rs}, u^A) = (f')^{-1}(R_g)T_{lm}(g_{rs}, u^A). \quad (22)$$

In this case, from (16) a simple relationship for the γ - and g -traces of the matter tensor follows:

$$T_\gamma(\gamma_{rs}, u^A) = T_g(g_{rs}, u^A) = \mu - 3p. \quad (23)$$

In place of $T^{ab}{}_{;b} = 0$ for the Einstein-Hilbert Lagrangian, in this theory a more general relationship with $T^{ab}{}_{;b} \neq 0$ follows from general covariance. This is also seen by forming the divergence of the Einstein tensor in (20).

B. For vanishing trace of the matter tensor $T_\gamma = 0$, (18) reduces to

$$f'(R_\gamma)R_\gamma - 2f(R_\gamma) = 0. \quad (24)$$

This implies two cases: i) $f = (f_0 R_\gamma)^2$, and ii) $f \neq (f_0 R_\gamma)^2$. The exceptional case i) is characterized by an additional scale invariance implying zero trace for the matter tensor. The field equations (15) become

$$2(f_0)^2(R_\gamma) \left[R_{ab}(g) - \frac{1}{4}R_g g_{ab} \right] + 2\kappa T_{ab} \left(\frac{1}{2f_0^2 R_\gamma} g_{lm}, u^A \right) = 0, \quad (25)$$

$$R_g = \frac{1}{2(f_0)^2}, \quad T_\gamma = T_g = 0.$$

If we take a sourceless Maxwell field F_{ik} as matter, then

$$T_{ab}(\gamma_{lm}, F_{lm}) = \gamma^{lm}F_{al}F_{bm} - \frac{1}{4}\gamma_{ab}\gamma^{il}\gamma^{jm}F_{il}F_{jm} = f'(R_\gamma)T_{ab}(g_{lm}, F_{lm}). \quad (26)$$

R_γ drops out and the field equations are

$$R_{ab}(g) - \frac{1}{4}R_g g_{ab} + \kappa T_{ab}(g_{lm}, u^A) = 0, \quad (27)$$

$$R_g = \frac{c_1}{f'(c_1)}, \quad T_\gamma = T_g = 0.$$

In case (ii), from (24), $R_\gamma = c_1 = \text{const}$ and we may proceed only if one real solution of $f'(c_1)c_1 - 2f(c_1) = 0$ does exist and if $f(c_1)$, $f'(c_1)$ remain finite. The field equations then are

$$f'(c_1)R_{ab}(g) - \frac{c_1}{4}g_{ab} + 2\kappa T_{ab}\left(\frac{1}{f'(c_1)}g_{lm}, u^A\right) = 0, \quad (28)$$

$$T_\gamma = T_g = 0.$$

In Einstein's theory, $R = 0$ follows if the trace of the matter tensor is vanishing. Here, the larger set of solutions $R = \text{const}$ is obtained.

Above, it has been assumed that the matter tensor does not contain covariant derivatives; this covers most cases of physical interest. Otherwise, formidable complications result even when the Einstein-Hilbert Lagrangian is taken. E.g., if the additional term in the matter Lagrangian is $\sim \sqrt{-\gamma}\gamma^{il}\gamma^{km}u_{i;k}u_{l;m}$ (5) must be replaced by $(\sqrt{-\gamma}\gamma^{ij})_{;k} = f_k^{ij}(\gamma^{lm}, u^A, \partial u^A)$ with a particular functional f_k^{ij} . Hence, the elimination of the Lagrangian multiplier will require quite an effort.

IV. COMPARISON WITH THE STANDARD PALATINI METHOD

For the Palatini method of variation with variables g_{ij} and Γ_{ij}^k , the field equations of the $f(R)$ -theory are

$$f'(R)R_{ik}(\Gamma) - \frac{1}{2}f(R)g_{ik} = -2\kappa T_{ik}, \quad (29)$$

$$(\sqrt{-g}f'(R)g^{il})_{||l} = 0, \quad (30)$$

where the covariant derivative is formed with the connection Γ and $R = g^{ik}R_{ik}(\Gamma)$. From (30) we obtain a metric \bar{g}_{ij} compatible with the connection Γ :

$$\bar{g}_{ij} = f'(R)g_{ij}, \quad \bar{g}^{ij} = (f')^{-1}(R)g^{ij}, \quad (31)$$

and the relation between Γ and the Levi Civita connection is

$$\Gamma_{ij}^k \equiv \left\{ \begin{matrix} k \\ ij \end{matrix} \right\}_{\bar{g}}$$

$$= \left\{ \begin{matrix} k \\ ij \end{matrix} \right\}_g + \frac{1}{2} \frac{d}{dR} (\ln f'(R)) [2\delta_{(i}^k R_{,j)} - g_{ij}g^{kl}R_{,l}]. \quad (32)$$

A comparison of (13) and (31) shows the difference between γ^{ij} and \bar{g}^{ij} . With the help of (32) and (31) we can rewrite the tracefree part of (29) in terms of the conformally related metric \bar{g}_{ij}

$$\bar{R}_{ab}(\bar{g}) - \frac{1}{4}\bar{R}(\bar{g})\bar{g}_{ab} = R_{ab}(g) - \frac{1}{4}R(g)g_{ab} - \frac{f''}{f'} \left[R_{,ij} - \frac{1}{4}g_{ij}\square R \right] - \left(\frac{f''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \right) \times \left[R_{,i}R_{,j} - \frac{1}{4}g_{ij}R_{,l}R_{,m}g^{lm} \right]. \quad (33)$$

When bringing the field equations into the form of Einstein's equations, the result is:

$$R_{ab}(g) - \frac{1}{2}R(g)g_{ab} = -2\kappa T_{ab}(g) - \frac{f''}{f'} [R_{,ij} - g_{ij}\square R] - \left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \right] R_{,i}R_{,j} + \left(\frac{f'''}{f'} - \frac{3}{4}\left(\frac{f''}{f'}\right)^2 \right) g_{ij}R_{,l}R_{,m}g^{lm}. \quad (34)$$

Again, the trace equation of (29), i.e.,

$$f'(R)R - 2f(R) = -2\kappa T_g, \quad (35)$$

is used to eliminate the curvature scalar in favor of the trace of the matter tensor. This means that the nonminimal coupling to the curvature scalar and its derivatives will be replaced by a coupling to the *gradients* of the matter variables contained in $g^{ik}T_{ik}$.

The remark at the end of Sec. III for the case of covariant derivatives in the matter tensor applies here as well.

V. AN EXAMPLE: EXPONENTIAL GRAVITY

A. Alternative variational principle

As an example, we now take a recent model for $f(R)$ -gravity [17] with

$$f(R) = -cr(1 - e^{-(R/r)}), \quad f' = -ce^{-(R/r)}, \quad (36)$$

where r of dimension (length)² and c , dimensionless, are constants. From (14) $R_g = -\frac{1}{c}R_\gamma e^{R_\gamma/r}$. Thus, the inverse $R_\gamma = h^{-1}(R_g)$ can be obtained only numerically. A series expansion for $\frac{R_\gamma}{r} \ll 1$ leads to

$$R_\gamma = -cR_g - \frac{c^2}{r}R_g^2 - \frac{3}{2}\frac{c^3}{r^2}R_g^3 \pm \dots, \quad (37)$$

and

$$R_g = \frac{2\kappa T_g}{c^2} \left[1 - \frac{2\kappa T_g}{rc} + \frac{2}{3} \left(\frac{2\kappa T_g}{rc} \right)^2 \pm \dots \right] \quad (38)$$

If the further calculations are restricted to the lowest order in the expansion (37), with the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}R_g g_{ab}$ the field equations (20) become

$$G_{ab} \left(\left\{ \begin{matrix} k \\ ij \end{matrix} \right\}_g \right) = -\frac{2\kappa}{c^2} \left[1 - 4\frac{\kappa T_g}{rc} \right] T_{ab} - \frac{\kappa T_g}{c^2} \frac{\kappa T_g}{rc} g_{ab}. \quad (39)$$

For a perfect fluid with pressure $p = 0$, from (21) to lowest order the equations replacing Einstein's are:

$$R_{ab}\left(\left\{\begin{matrix} k \\ ij \end{matrix}\right\}_g\right) - \frac{1}{2}R_g g_{ab} = -\frac{2\kappa}{c^2}\left(1 - 4\frac{\kappa\mu}{rc}\right)\mu u_a u_b - \frac{\kappa^2\mu^2}{rc^3}g_{ab}. \quad (40)$$

The result is a variable coupling “constant” in the effective matter tensor and a variable cosmological term both depending on the energy density of matter.

For a homogeneous and isotropic cosmological model with scale factor $a(t)$ and flat space sections, (40) leads to altered Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2\kappa\mu}{3c}\left(1 - \frac{7}{2}\frac{\kappa\mu}{cr}\right), \quad (41)$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2\mu^2}{rc^3}. \quad (42)$$

Keep in mind that r, c are free constants of the model; the velocity of light has been put equal to 1 in (42). As numerical calculations would have to be done, and the main aim of this paper is the introduction of an *alternative* variational principle, we will not comment on this particular model (exponential gravity) and the physics following from it.

B. Standard Palatini method

For exponential gravity as given by (36) and for pressureless fluid matter, the field equations according to the first equation of (29) turn out to be

$$R_{ik}(\Gamma) = \frac{2\kappa}{c}T_{ik}(g, u^A) + g_{ik}r(1 - e^{-(R/r)}). \quad (43)$$

This does not look complicated; however, the connection Γ first must be expressed by the conformally related metric \bar{g}_{ab} . To the same order of approximation, the final field equation then can be written as

$$R_{ab}(g) - \frac{1}{2}R(g)g_{ab} = -2\kappa\mu u_a u_b - \frac{2\kappa}{rc}[\mu_{,ij} - g_{ij}\square\mu] - \frac{2\kappa^2}{r^4c^2}\mu_{,i}\mu_{,j} + \frac{\kappa^2}{r^4c^2}g_{ij}\mu_{,l}\mu_{,m}g^{lm}. \quad (44)$$

The effective matter tensor in (44) depends on 1st and 2nd gradients of the energy density of matter. This shows that the physics resulting from the two variational principles may be quite different. The same can be said with regard to the Einstein-Hilbert metric variation used in [17] and e.g., in [18,19] and the *alternative* variational principle.

VI. CONCLUDING REMARKS

For physics, a significant difference between the *alternative* variational method presented here and the *standard* Palatini method is that nonminimal coupling of matter and the curvature scalar R occurs by multiplication with functions of R or the trace of the matter tensor. In the *standard* Palatini method, nonminimal coupling happens via the *gradients* of the scalar curvature (trace of the matter tensor). A conceptual advantage of the *alternative* method is that it works within (pseudo)-Riemannian geometry; metric-affine geometry never does appear.⁸ When dealing with $R + f(R)$ -Lagrangians, in both approaches a new dimensionful constant is needed whose physical meaning must be defined. Application to $f(R, R_{ab}R^{ab})$ is unproblematic; here, two new parameters will occur. In general, via the field equations both curvature invariants can be expressed as functionals of invariants of the matter tensor. The Einstein-Hilbert Lagrangian seems to be very robust: now there are at least *three* different methods for a derivation of the Einstein field equations. As the example treated shows, for more general Lagrangians the variation will lead to different physical theories. Whether the *alternative* variational principle introduced here, if applied to cosmological models, produces convincing physics will have to be shown by further studies.

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⁸For metric-affine geometry, independent variation of metric and connection is mandatory anyway. In this case, it would be misleading to mention the name Palatini.

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