Quasiblack holes with pressure: Relativistic charged spheres as the frozen stars

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In general relativity coupled to Maxwell's electromagnetism and charged matter, when the gravitational potential W^2 and the electric potential field ϕ obey a relation of the form $W^2 = a(-\epsilon\phi + b)^2 + c$, where a, b, and c are arbitrary constants, and $\epsilon = \pm 1$ (the speed of light c and Newton's constant G are put to one), a class of very interesting electrically charged systems with pressure arises. We call the relation above between W and ϕ, the Weyl-Guilfoyle relation, and it generalizes the usual Weyl relation, for which $a = 1$. For both, Weyl and Weyl-Guilfoyle relations, the electrically charged fluid, if present, may have nonzero pressure. Fluids obeying the Weyl-Guilfoyle relation are called Weyl-Guilfoyle fluids. These fluids, under the assumption of spherical symmetry, exhibit solutions which can be matched to the electrovacuum Reissner-Nordström spacetime to yield global asymptotically flat cold charged stars. We show that a particular spherically symmetric class of stars found by Guilfoyle has a well-behaved limit which corresponds to an extremal Reissner-Nordström quasiblack hole with pressure, i.e., in which the fluid inside the quasihorizon has electric charge and pressure, and the geometry outside the quasihorizon is given by the extremal Reissner-Nordström metric. The main physical properties of such charged stars and quasiblack holes with pressure are analyzed. An important development provided by these stars and quasiblack holes is that without pressure the solutions, Majumdar-Papapetrou solutions, are unstable to kinetic perturbations. Solutions with pressure may avoid this instability. If stable, these cold quasiblack holes with pressure, i.e., these compact relativistic charged spheres, are really frozen stars.

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I. INTRODUCTION

Frozen star was a name advocated in the 1960s by Zel'dovich [1] to give to an object which is now called, after Wheeler's suggestion in 1968, a black hole (see e.g. [2]; see also [3] for the historical evolution of the concept). As we will see, a frozen star is actually a quasiblack hole.

Quasiblack holes are objects on the verge of becoming black holes but avoid it; their boundary approaches their own gravitational radius as closely as one likes. They appeared in Einstein-Maxwell systems with special matter, as had been found not so explicitly in [4] (see also [5]) and then thoroughly discussed in [6–9]. They also arise, and indeed were first discussed as such, in the context of Einstein–Yang-Mills–Higgs systems [10,11]. These objects have well-defined properties [12–16]; for instance, in general, regular quasiblack holes are extremal. Rotating quasiblack holes are also known; see [17] with hindsight and [18,19] for the properties of such objects.

There are several ways of solving Einstein's equations when the gravitational field is coupled to matter. Since we will focus on Einstein-Maxwell static systems with matter we can perhaps distinguish two ways. One we can call Weyl's way; the other is the Tolman-Oppenheimer-Volkoff (TOV) way.

Let us deal with Weyl's way now. Weyl [20], while studying stationary electric fields in vacuum Einstein-Maxwell theory, perceived that it is interesting to consider a functional relation between the metric potential $g_{tt} \equiv W^2(x^i)$ and the electric potential $\phi(x^i)$ (where x^i represent the spatial coordinates, $i = 1, 2, 3$) given by the ansatz $W = W(\phi)$. By assuming the system is vacuum and axisymmetric, Weyl found that such a relation must be quadratic in ϕ. One can go beyond vacuum solutions, and consider fluids which obey the Weyl relation, obtaining their properties. These fluid systems were explored later by Majumdar and Papapetrou [21,22] who in going beyond vacuum found equilibrium configurations for charged extremal matter where the electric repulsion is balanced by the gravitational attraction, and moreover for this particular case they showed there is a perfect square relationship between W and ϕ; see also [23] for the extension of these results to d dimensions. An interesting development on Weyl's work was performed much later by Guilfoyle [24] who considered charged fluid distributions with the hy-

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pothesis that the functional relation between the gravitational and the electric potential, $W = W(\phi)$, is given by $W^2 = a(-\epsilon\phi + b)^2 + c$, where a, b, and c are arbitrary constants, and $\epsilon = \pm 1$ (the speed of light c and Newton's constant G are put to one). This relation generalizes the usual Weyl relation, for which $a = 1$, and it allows a further beautiful relationship between the various field and matter quantities [25], which in turn generalizes the Gautreau and Hoffman results [26] for fluids obeying a pure Weyl relation. For both, Weyl and Weyl-Guilfoyle relations, the electrically charged fluid, if present, may have nonzero pressure, and this turns out to be important in our context. Fluids obeying the Weyl-Guilfoyle relation are called Weyl-Guilfoyle fluids.

Up to now we have only mentioned properties of the fluid itself, be it a Weyl-Guilfoyle fluid or, in the particular case of zero pressure, a Majumdar-Papapetrou fluid. But matter solutions can be matched into vacuum asymptotically flat solutions, yielding star solutions which, besides having the local properties associated to the fluid itself, have global properties for the spacetime as a whole. Thus, Majumdar-Papapetrou matter solutions, when joined to vacuum solutions yield star solutions, the Bonnor stars [4,5]. These Bonnor stars when sufficiently compact show quasiblack hole behavior [6–9]. These stars and quasiblack holes have no matter pressure, only electromagnetic pressure. But now, Weyl-Guilfoyle fluids have, besides electromagnetic pressure, matter pressure. Under the assumption of spherical symmetry, Guilfoyle [24] exhibited solutions which can be matched to the electrovacuum Reissner-Nordström spacetime to yield global asymptotically flat stars, i.e., charged stars with pressure.

Here we explore one particular class of those spherically symmetric cold charged fluid stars and show that these stars display quasiblack hole behavior, i.e., the matter boundary approaches its own gravitational radius (or horizon) in a well-behaved manner. Although the cold charged stars have a nonextremal outer metric, the quasiblack hole regime is extremal always. We analyze in which cases the energy conditions are obeyed, and also study a subclass for which the speed of sound in the matter is less than the speed of light. So a quasiblack hole with pressure obeying the necessary physical requirements is presented here. Quasiblack holes purely supported by electrical charge have been known [6–8] (see also [4,5] with hindsight, and [9] for further results and references). The presence of pressure in quasiblack hole solutions is important, since it tends to stabilize the system. Indeed, an important development provided by these solutions is that without pressure the matter, Majumdar-Papapetrou matter, is unstable to kinetic perturbations. So the whole solution is unstable to these perturbations. Charged solutions with pressure, and along with them quasiblack holes with pressure, avoid this instability. Thus, if indeed stable, these compact relativistic charged spheres in the form of cold quasiblack holes with pressure, are really frozen stars [1,2]. As a black hole, a quasiblack hole freezes to observers outside, but unlike a black hole, at the horizon limit, the star is still intact, it does not collapse. In the quasiblack hole context, unlike in the black hole case, the star is not irrelevant. Thus the name frozen star is appropriate. From a well-motivated chain of works, applying Weyl's leading method of solving Einstein's equations we thus arrive at the concept of frozen stars, solutions which exhibit the behavior of quasiblack holes with pressure. Of course, quasiblack holes considered as frozen stars are more akin to the dark stars of Michell and Laplace (see [3]) than black holes themselves.

The other way to solve Einstein's equations with charged matter with pressure is through the integration of the TOV equation for the gradient of the pressure, in conjunction with the other equations for the other fields. There are many works that use this method, and we only quote the most relevant to our work. In an important work, de Felice and collaborators [27,28] found relativistic charged sphere solutions with pressure by the TOV method. Moreover, they took the limit to the black hole regime, and through numerical integration still found in this limit a solution. With hindsight this solution is an extremal quasiblack hole with pressure solution. The equation of state for the fluid considered is one for an incompressible fluid, with constant energy density, and in such a case the speed of sound is infinite, which might be considered a drawback. Bonnor [5] took notice of this solution and made comments and comparisons showing that his previous solution [4] had indeed the same gravitational overall properties. The search of quasiblack hole solutions through the TOV method is nontrivial. In fact, charged star solutions with pressure were explored in [29,30] without finding the quasiblack hole regime. There are yet other ways to solve Einstein's equations. For instance in [10,11] an improved method, analytical and numerical, was devised to solve the Einstein–Yang-Mills–Higgs system of equations. The quasiblack holes that arose from this system are indeed also extremal quasiblack holes with pressure since the Yang-Mills–Higgs system has an effective built-in pressure on its own. The generic properties of all types of extremal quasiblack holes with pressure are discussed in [16].

The existence or not of solutions for compact objects in general relativity is connected with the Buchdahl limit. Buchdahl [31] found for perfect fluids that if the radius r_0 and the mass m of the star is such that $m \geq \frac{4}{9} r_0$, then there is no equilibrium; see also [32] for more on this limit. For instance the Schwarzschild interior solution with constant energy density for the matter obeys this inequality. If one adds electric charge to the system the Buchdahl limit is changed, the mass to radius of the star ratio $\frac{m}{r_0}$ can certainly increase. Indeed the existence of quasiblack holes shows that the radius of an electrically charged star can approach its own gravitational radius,

$m = r_0$. Buchdahl limits for charged stars have been worked out in [33–36] and in particular [37] gives a sharp limit, $m \geq (\frac{\sqrt{r_0}}{3} + \sqrt{\frac{r_0}{9} + \frac{q^2}{3r_0}})^2$, where q is the total charge of the star. For $q = r_0$, this result admits the extremal case $m = r_0$ which, as we shall see below, corresponds to the quasiblack hole limit satisfied by the particular solutions studied here. Incidentally, the $a \to \infty$ limit of the solutions we consider yield the Schwarzschild interior solutions. These uncharged Schwarzschild interior solutions contain no quasiblack hole, of course.

Another related issue is concerned with regular black holes. Regular black holes are black holes devoid of singularities. There are two types of regular black holes. In one type there is a magnetic core with an event horizon which joins into a magnetically charged vacuum solution different from the Reissner-Nordström solution [38–40] (see also [41] for electrically charged regular solutions in nonminimal theories). In the other type, the bulk inside the horizon is formed of a portion of the de Sitter space which joins into a Schwarzschild vacuum solution [42–47]. Curiously, the $a \to \infty$ limit of the solutions we consider here yields a branch of solutions, other than the Schwarzschild interior solutions, which are regular electrically charged black holes.

The present paper is organized as follows. In Sec. II the basic equations describing a charged fluid of Weyl-Guilfoyle type are written. A particular spherically symmetric solution to the equations representing relativistic charged stars, found by Guilfoyle, is shown in Sec. III. Then, Sec. IV is devoted to the study of the main properties of these stars. We first review the definition of a quasiblack hole in Sec. IV A. Then we obtain analytically the quasiblack hole limit of the solution in Sec. IV B. In passing we take the $a \to \infty$ limit and show that there are two branches of solutions; one corresponds to the Schwarzschild interior solutions, the other to regular charged black holes. In Sec. IV C we plot, for three distinct cases, the relevant curves for each one of the metric and electric potentials, and for the fluid quantities. All three cases contain relativistic star solutions and quasiblack hole (frozen star) solutions. In Sec. V we conclude.

II. WEYL-GUILFOYLE CHARGED FLUIDS: BASIC EQUATIONS

The cold charged fluids considered in the present work are described by Einstein-Maxwell equations, which can be written as

$$G_{\mu\nu} = 8\pi(T_{\mu\nu} + E_{\mu\nu}), \tag{1}$$

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu, \tag{2}$$

where Greek indices μ, ν, etc., run from 0 to 3. $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, with $R_{\mu\nu}$ being the Ricci tensor, $g_{\mu\nu}$ the metric tensor, and R the Ricci scalar. We have put both the speed of light c and the gravitational constant G equal to unity throughout. $E_{\mu\nu}$ is the electromagnetic energy-momentum tensor, which can be written in the form

$$4\pi E_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}, \tag{3}$$

where the Maxwell tensor is

$$F_{\mu\nu} = \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu, \tag{4}$$

∇_μ representing the covariant derivative, and $\mathcal{A}_\mu$ the electromagnetic gauge field. In addition,

$$J_\mu = \rho_e U_\mu \tag{5}$$

is the current density, ρ_e is the electric charge density, and U_μ is the fluid velocity. $T_{\mu\nu}$ is the material energy-momentum tensor given by

$$T_{\mu\nu} = (\rho_m + p) U_\mu U_\nu + p g_{\mu\nu}, \tag{6}$$

where ρ_m is the fluid matter-energy density, and p is the fluid pressure.

We assume the spacetime is static and the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{7}$$

can be written in the form

$$ds^2 = -W^2 dt^2 + h_{ij} dx^i dx^j, \qquad i, j = 1, 2, 3. \tag{8}$$

The gauge field $\mathcal{A}_\mu$ and four-velocity U_μ are then given by

$$\mathcal{A}_\mu = -\phi \delta^0_\mu, \tag{9}$$

$$U_\mu = -W \delta^0_\mu. \tag{10}$$

The spatial metric tensor h_{ij}, the metric potential B, and the electrostatic potential ϕ are functions of the spatial coordinates x^i alone.

The particular relativistic cold charged stars we are going to study here belong to a special class of systems in which the metric potential W and the electric potential ϕ are functionally related through a Weyl-Guilfoyle relation (see Guilfoyle [24]; see also Lemos and Zanchin [25])

$$W^2 = a(-\epsilon\phi + b)^2 + c, \tag{11}$$

with a, b, and c being arbitrary constants and $\epsilon = \pm 1$, the parameter a being called the Guilfoyle parameter. Then, the charged pressure fluid quantities ρ_m, p, and ρ_e satisfy the constraint [25]

$$ab\rho_e = \epsilon[\rho_m + 3p + (1 - a)\rho_{em}]W + \epsilon a \rho_e \phi, \tag{12}$$

or

$$a\rho_e(-\epsilon\phi + b) = \epsilon[\rho_m + 3p + \epsilon(1 - a)\rho_{em}]W, \tag{13}$$

with ρ_{em} standing for the electromagnetic energy density defined by

$$\rho_{\rm em} = \frac{1}{8\pi}\frac{(\nabla_i\phi)^2}{W^2}. \quad (14)$$

Such matter systems we call Weyl-Guilfoyle fluids.

III. SPHERICAL SOLUTIONS: GENERAL ANALYSIS

We repeat in this section one class of the spherically symmetric solutions found by Guilfoyle [24] in order to study its properties. The metric (7) instead of being generically written as (8), is more conveniently written in a spherically symmetric form, namely,

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega, \quad (15)$$

where r is the radial coordinate, A and B are functions of r only, and $d\Omega$ is the metric of the unit sphere S^2. The gauge field $\mathcal{A}_\mu$ and the four-velocity U_μ are now

$$\mathcal{A}_\mu = -\phi(r)\delta^0_\mu, \quad (16)$$

$$U_\mu = -\sqrt{B(r)}\delta^0_\mu. \quad (17)$$

The cold charged pressure fluid is bounded by a spherical surface of radius $r = r_0$, and in the electrovacuum region, for $r > r_0$, the metric and the electric potentials are given by the Reissner-Nordström solution

$$B(r) = \frac{1}{A(r)} = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \quad (18)$$

$$\phi(r) = \frac{q}{r} + \phi_0, \quad (19)$$

ϕ_0 being an arbitrary constant which defines the zero of the electric potential, and that, in asymptotically Reissner-Nordström spacetimes as we consider here, can be set to zero.

Now we review the general properties of the spherically symmetric solutions with the boundary condition given above. The first integral of the only nonzero component of Maxwell's equations (2) furnishes

$$Q(r) = r^2\frac{\phi'(r)}{\sqrt{B(r)A(r)}}, \quad (20)$$

where the prime denotes the derivative with respect to the radial coordinate r and an integration constant was set to zero. $Q(r)$ is the total electric charge inside the surface of radius r. The class of solutions we are interested in here has $c = 0$ in Eq. (11), and is classified as class Ia in [24], and so we can write $\phi(r)$ in terms of $B(r)$ as

$$\epsilon\phi(r) = b - \sqrt{\frac{B(r)}{a}}. \quad (21)$$

With this result the amount of electric charge inside a spherical surface of radius r, cf. Equation (20), is now given by

$$Q(r) = \frac{-\epsilon r^2 B'(r)}{2\sqrt{aA(r)}B(r)}. \quad (22)$$

The continuity of the metric functions on the surface $r = r_0$ yields $B(r_0) = 1/A(r_0) = 1 - \frac{2m}{r_0} + \frac{q^2}{r_0^2}$. Using such a boundary condition and Eq. (22) we get

$$\frac{q}{m} = \sqrt{\frac{r_0}{2(a-1)m}}\left[2(a-1) - \frac{r_0}{m} + \sqrt{4a(a-1)\left(1-\frac{r_0}{m}\right) + a^2\frac{r_0^2}{m^2}}\right]^{1/2}, \quad (23)$$

or inverting,

$$\frac{m}{q} = (1-a)\frac{q}{r_0} + \sqrt{a\left[1 + (a-1)\frac{q^2}{r_0^2}\right]}, \quad (24)$$

where the fact that $Q(r = r_0) = q$ was taken into account. The equivalent constraints (23) or (24) hold for all spherically symmetric charged pressure fluid distributions whose boundaries are given by a spherical surface of radius $r = r_0$, and whose potentials are related as in Eq. (21). The extremal relation $q = m$ holds when $a = 1$, in which case the relation among B and ϕ is always a perfect square and the $r_0 = m$ limit represents a quasiblack hole. (In four dimensions, these Majumdar-Papapetrou charged fluids were studied in [21,22]and their corresponding Bonnor stars and quasiblack holes in [4–8] as well as in [12–15], and in $d > 4$ dimensions these fluids were studied in [23] and their corresponding Bonnor stars and quasi black holes in [9]. Now, for $a \neq 1$, we are still considering $B = B(\phi)$ as a perfect square [because $c = 0$, see Eq. (11)] but, unlike a Bonnor star, the relation $q = m$ does not hold in general. However, from Eq. (23) or (24) one sees that, for $a \neq 1$, there is also the possibility in which the equality $mr_0 = q^2$ holds, which may yield back, eventually, the relation $q = m$. As we will see below, this is the quasiblack hole limit. Since the inequality $a \neq 1$ holds in this case, it means the corresponding quasiblack hole is made of a fluid with nonzero pressure

Guilfoyle's solutions are found under the assumptions that the effective energy density $\rho_{\rm eff}(r) = \rho_{\rm m}(r) + \rho_{\rm em}(r) = \rho_{\rm m}(r) + \frac{Q^2(r)}{8\pi r^4}$ is a constant, and that the metric potential $A(r)$ is a particularly simple function of the radial coordinate. Namely [24],

$$8\pi\rho_{\rm m}(r) + \frac{Q^2(r)}{r^4} = \frac{3}{R^2}, \quad (25)$$

$$A(r) = \left(1 - \frac{r^2}{R^2}\right)^{-1}, \quad (26)$$

where R is a constant to be determined by the junction conditions of the metric at the surface $r = r_0$. In fact, by joining the function $A(r)$ in Eq. (26) with the g_{rr} coefficient of the exterior metric (18), the constant R in Eq. (25) is found in terms of the parameters r_0, m, and q, through the equation

$$\frac{1}{R^2} = \frac{1}{r_0^3}\left(2m - \frac{q^2}{r_0}\right). \tag{27}$$

With these simplifying hypotheses, Guilfoyle [24] was able to write some interesting exact solutions. We shall show that the class Ia of such solutions admits an extremal case such that $q = m = r_0$, which represents a quasiblack hole. Let us then write here this Ia class of solutions:

$$B(r) = \left[\frac{2-a}{a^{1+1/a}}F(r)\right]^{2a/(a-2)}, \tag{28}$$

$$8\pi\rho_{\rm m}(r) = \frac{3}{R^2} - \frac{a}{(2-a)^2}\frac{k_0^2 r^2}{F^2(r)}, \tag{29}$$

$$Q(r) = \frac{\epsilon\sqrt{a}}{2-a}\frac{k_0 r^3}{F(r)}, \tag{30}$$

$$8\pi p(r) = -\frac{1}{R^2} + \frac{a}{(2-a)^2}\frac{k_0^2 r^2}{F^2(r)} + \frac{2k_0 a}{2-a}\frac{\sqrt{1-r^2/R^2}}{F(r)}, \tag{31}$$

where k_0 is an integration constant, and $F(r)$ and $Q(r)$ are defined, respectively, by

$$F(r) = k_0 R^2\sqrt{1-\frac{r^2}{R^2}} - k_1, \tag{32}$$

$$Q(r) = 4\pi\int_0^r \rho_{\rm e}(r)\frac{r^2 dr}{\sqrt{1-\frac{r^2}{R^2}}} = \frac{r^2}{\sqrt{B(r)}}\sqrt{1-\frac{r^2}{R^2}}\frac{d\phi(r)}{dr}, \tag{33}$$

with k_1 being another integration constant. The integration constants k_0 and k_1 are determined by using the continuity of the metric potentials $A(r)$ and $B(r)$ and the first derivative of $B(r)$ with respect to r at the boundary $r = r_0$. The result is

$$k_0 = \frac{|q|a^{2/a}}{r_0^3}\left(\frac{m}{q} - \frac{q}{r_0}\right)^{1-2/a}, \tag{34}$$

$$k_1 = \sqrt{1-\frac{r_0^2}{R^2}}\left[k_0 R^2 - \frac{a^{1+1/a}}{2-a}\left(1-\frac{r_0^2}{R^2}\right)^{-1/a}\right]. \tag{35}$$

From Eqs. (30) and (33) we get both the electric charge density $\rho_{\rm e}$ and the electromagnetic energy density $\rho_{\rm em}$,

$$8\pi\rho_{\rm e}(r) = \frac{\epsilon\sqrt{a}}{2-a}\frac{k_0^2 r^2}{F^2(r)}\left(\frac{3F(r)}{k_0}\sqrt{1-\frac{r^2}{R^2}} - 1\right), \tag{36}$$

$$8\pi\rho_{\rm em}(r) = \frac{a}{(2-a)^2}\frac{k_0^2 r^2}{F^2(r)}. \tag{37}$$

Since $c = 0$ in the solutions we are considering [see Eq. (11)], Eq. (12) with Eq. (11) turns into

$$\sqrt{a}\rho_{\rm e} = \epsilon[\rho_{\rm m} + 3p + (1-a)\rho_{\rm em}], \tag{38}$$

which can be interpreted as an equation of state which, as one can check, is obeyed by the charged fluid represented by the solution presented above. Furthermore, the above solution is valid for all $a > 0$, the case $a = \infty$ yielding the uncharged ($q = 0$) Schwarzschild interior solution.

Another important quantity to determine is the speed of sound within the fluid. We take the usual definition for the speed of sound $c_{\rm s}$,

$$c_{\rm s}^2 = \frac{\delta p}{\delta\rho_{\rm m}}, \tag{39}$$

and consider variations of the pressure p and of the energy density $\rho_{\rm m}$ in terms of the radial coordinate r, i.e., $\delta p = p'(r)\delta r$, etc., where the prime denotes derivative with respect to r. Hence, we may write $c_{\rm s}^2 = \frac{p'}{\rho_{\rm m}'}$. The result is the speed of sound as a function of the radial coordinate,

$$c_{\rm s}^2 = |2-a|\frac{k_2^2 - k_2\sqrt{1-r^2/R^2}}{1-k_2\sqrt{1-r^2/R^2}} - 1, \tag{40}$$

where we have defined $k_2 = \frac{k_1}{k_0 R^2}$.

In the following we analyze in some detail the physical properties of this charged solution, exploiting, in particular, the dependence of the metric and electric potentials, and of the fluid quantities on the free Guilfoyle parameter a.

IV. RELATIVISTIC CHARGED STARS AND QUASIBLACK HOLES WITH PRESSURE (OR FROZEN STARS)

A. Definition of generic relativistic charged stars and definition of quasiblack holes with pressure (or frozen stars)

A generic relativistic cold charged star, or sphere, is here defined as a smooth ball in which the gravitational, electromagnetic, and matter fields have nonsingular behavior throughout the matter, which in turn is matched smoothly to the Reissner-Nordström exterior solution.

A quasiblack hole or frozen star can also be properly defined. For such a task we follow Ref. [12] and consider the metric in the form (15) with solution given in Eqs. (26) and (28) for the interior which matches continuously to the exterior metric in the form (18). Then, consider the following properties: (a) the function $1/A(r)$ attains a minimum at some $r^* \neq 0$, such that $1/A(r^*) = \varepsilon$, with $\varepsilon \ll 1$, this minimum being achieved either from both sides of r^* or from $r > r^*$ alone; (b) for such a small but nonzero ε the configuration is regular everywhere with a nonvanishing metric potential $B(r)$; and (c) in the limit $\varepsilon \to 0$ the potential $B(r) \to 0$ for all $r \leq r^*$. These three features define a quasiblack hole and, in turn, entail some nontrivial consequences: (i) there are 3-volume regions, rather than 2-surface regions (as in the black hole case), for which the redshift is infinite; (ii) when $\varepsilon \to 0$, a free-falling observer finds in his own frame infinitely large tidal forces in the whole inner region, although the spacetime curvature invariants remain perfectly regular everywhere; (iii) in the limit, outer and inner regions become mutually impenetrable and disjoint; and one can also show that (iv) for external far away observers the spacetime is virtually indistinguishable from that of an extremal black hole. In addition, well-behaved (with no infinite surface stresses) quasiblack holes must be extremal. The quasiblack hole is on the verge of forming an event horizon, but it never forms one; instead, a quasihorizon appears. For a quasiblack hole the metric is well defined and everywhere regular. Nonetheless, the properties that arise when $\varepsilon = 0$ have to be examined with care for each model in question.

B. Analytical study of quasiblack holes with pressure (or frozen stars)

An interesting property of the solution presented in Sec. III [cf. Eqs. (26)–(31)] is that all the physical quantities are well behaved even in the quasiblack hole limit. In what follows we first verify that the quasiblack hole limit can be attained, and then we check that in such a limit the physical quantities related to the charged fluid are well-defined.

1. The metric potentials and the electric potential

Let us show here that the quasiblack hole really exists as a special limit of the relativistic charged star solution presented above. According to the definition of the preceding section, the quasiblack hole should be extremal, so that the mass m approaches the charge from above, $m \to q^+$. In such a limit, the exterior metric (18) tends to the quasiextremal Reissner-Nordström metric, for which the two horizons $r_\pm = m \pm \sqrt{m^2 - q^2}$ are very close to each other, $r_+ \sim m \sim r_-$. Moreover, there must be a quasihorizon r^*, and then the radius of the star r_0 must coincide with r^*. Hence, the quasiblack hole limit of the star corresponds to the limit $r_0 \to r_+$ from above. We can then assume a relation of the form $q \sim (1 - \sqrt{\varepsilon}) r_0$, for a small nonnegative ε. This means that the quasiblack hole limit corresponds to the most compact charged star we can have, with r_0/m close to unity. In the present case, the mass m and the charge q are related by Eq. (24), which implies in $m/q \sim 1 + (a-1)\varepsilon/2a$, and then $m/r_0 \sim 1 - \sqrt{\varepsilon}$. From these results we may obtain the quasiblack hole limit for all other quantities. For instance, the difference $m/q - q/r_0$ which appears in the constants k_0, k_1, etc., is of the order of $\sqrt{\varepsilon}$. Taking these considerations into the corresponding equations we find, for instance, $R^{-2} \sim (1 - 2\sqrt{\varepsilon})/r_0^2$, $k_0 \sim a^{2/a} \varepsilon^{(a-2)/2a}/r_0^2$, $B(r_0) \sim \varepsilon$, $k_1 \sim a^{(a+4)/2a} \varepsilon^{(a-2)/2a}/(2-a)$. The function $F(r)$ goes as k_1 for all r inside the star, and then Eq. (28) implies that $B(r)$ is of the order of ε for all r in the interval $0 \le r \le r_0$. Moreover, we find $1/A(r_0) \sim \varepsilon$, satisfying the properties of a quasiblack hole as defined in Sec. IV A. In the quasiblack hole limit, the metric potentials become

$$B(r) = \begin{cases} \left(1 + \frac{2-a}{\sqrt{a}} \sqrt{1 - \frac{r^2}{r_0^2}}\right)^{2a/(a-2)} \frac{\varepsilon}{a}, & r \le r_0, \\ (1 - \frac{r_0}{r}[1 - \frac{\sqrt{\varepsilon}}{\sqrt{a}}])^2, & r \ge r_0, \end{cases} \tag{41}$$

$$A(r)^{-1} = \begin{cases} (1 - \frac{r^2}{r_0^2}[1 - \frac{\varepsilon}{a}]), & r \le r_0, \\ (1 - \frac{r_0}{r}[1 - \frac{\sqrt{\varepsilon}}{\sqrt{a}}])^2, & r \ge r_0, \end{cases} \tag{42}$$

where we have used the fact that $|q| \simeq m = (1 - \sqrt{\varepsilon}) r_0$ and have written $|q|/r_0^3 = 1/r_0^2$. The electric potential $\phi(r)$ at the quasiblack hole limit is obtained from the above result for $B(r)$ and from Eq. (21).

2. The fluid quantities

The study of the physical properties satisfied by the energy density and by the pressure for the present solution was in part done in Ref. [24]. There are, however, further interesting aspects that should be considered.

We can for instance obtain the analytical expressions at the quasiblack hole limit for the fluid quantities like mass density, charge, and pressure,

$$8\pi\rho_{\rm m}(r) = \frac{3|q|}{r_0^3} - \frac{q^2 r^2}{r_0^6}\left(1 + \frac{2-a}{\sqrt{a}}\sqrt{1 - \frac{r^2}{r_0^2}}\right)^{-2}, \tag{43}$$

$$Q(r) = \frac{q r^3}{r_0^3}\left(1 + \frac{2-a}{\sqrt{a}}\sqrt{1 - \frac{r^2}{r_0^2}}\right)^{-1}, \tag{44}$$

$$8\pi p(r) = -\frac{|q|}{r_0^3} + \frac{q^2 r^2}{r_0^6}\left(1 + \frac{2-a}{\sqrt{a}}\sqrt{1 - \frac{r^2}{r_0^2}}\right)^{-2} + \frac{2a|q|}{r_0^3}\left(2 - a + \sqrt{a}\Big/\sqrt{1 - \frac{r^2}{r_0^2}}\right)^{-1}. \tag{45}$$

The quasiblack hole limit of the other fluid quantities, namely, of the charged density $\rho_{\rm e}$ and of the electromagnetic density $\rho_{\rm em}$ are easily obtained, respectively, from Eq. (38) and from the identity $\rho_{\rm em} = Q^2(r)/8\pi r^4$, and then we do not write them here.

Let us now analyze the speed of sound $c_{\rm s}$ within these relativistic charged stars in the quasiblack hole limit. For that we use Eq. (40) and get the limiting values of the constants R^2, k_0, and k_1, as done in Sec. IV B 1. Then we find

$$c_{\rm s}^2(r) = \frac{a + (2-a)\sqrt{a}\sqrt{1 - r^2/r_0^2}}{2 - a + \sqrt{a}\sqrt{1 - r^2/r_0^2}} - 1. \tag{46}$$

As expected the speed of sound of the quasiblack hole is zero if $a = 1$. At the surface, c_s^2 tends to $-1 + a/(2 - a)$ which reaches unity for $a = 4/3$. The function $c_s^2(r)$ at the center ($r = 0$) of the quasiblack hole is such that $c_s^2(0)$ is bounded to -1 from above as a tends to zero. In fact we see that $c_s^2(0)$ is zero for $a = 1$, and tends monotonically to -1 as a goes to zero. Hence, c_s is undefined for all $a < 1$. One can further impose that the speed of sound is smaller than the speed of light to yield a further interesting class of solutions.

3. The mass to radius relation and the $a \to \infty$ limit (the Schwarzschild interior and the regular black hole branches)

The mass m, the charge q, and the radius r_0 of the relativistic stars analyzed here are related by Eq. (23), or equivalently, by Eq. (24). Hence, besides the parameter a, out of the three quantities m, q, and r_0, we are left with two more free parameters. One can consider various possibilities. One possibility is to normalize the quantities in terms of the mass m. This choice is interesting, since the uncharged limit (the Schwarzschild interior solution) is easily obtained by taking the limit $a \to \infty$. Therefore, in order to observe this limit to the (uncharged) Schwarzschild stars, we will in general normalize the quantities in terms of the mass m of the star. Another possibility is to normalize the quantities in terms of the charge q. This choice is also interesting since from it one can read directly the star's mass to radius relation, $m \times r_0$. Of course, other possibilities could be considered.

In order to see the difference between the two normalizations just mentioned we plot Fig. 1. The plot on the left-hand side of Fig. 1 shows the behavior of q/m as a function of r_0/m. From the curves, obtained from Eq. (23), we read that, for a fixed mass, the electric charge of the star decreases with its radius, the maximum value being $q/m = 1$ for $r_0/m = 1$, the quasiblack hole limit. The plot on the right-hand side of Fig. 1 gives the mass to charge ratio m/q for a few values of the parameter a as a function of the normalized radius of the star r_0/q, whose curves are obtained from Eq. (24). It shows that, for a fixed charge, the radius of the star decreases as the mass decreases. This mass to radius relation behavior is analogous to the behavior of the mass to radius relation in main sequence stars, and contrary to the mass to radius relation in white dwarfs.

Another point to be noted is that these kind of stars bear a relatively large charge to mass ratio. The ratio m/q runs from unity, the minimum value, at the quasiblack hole limit, to $\sqrt{a}$ for an extremely sparse star, with $r_0 \to \infty$. In fact, we see from Eq. (24) that, for fixed a, the maximum value of m/q is found in the limit $r_0 \to \infty$, and it is given by $\frac{m}{q}|_{\max} = \sqrt{a}$.

Now we show that in the limit of $a \to \infty$ one recovers the Schwarzschild interior solution $q = 0$ and $\rho_m = \text{const}$, and also picks up another branch. Indeed, taking $a \to \infty$ in Eq. (23) we find

$$\lim_{a\to\infty} \frac{q^2}{m^2} = \frac{1}{2}\left(2 + \left|\frac{r_0}{m} - 2\right| - \frac{r_0}{m}\right)\frac{r_0}{m}. \tag{47}$$

Because of the absolute value $|r_0/m - 2|$ one concludes, remarkably, there are two branches, $r_0/m > 2$ and $r_0/m < 2$.

For $r_0/m > 2$ one finds

$$\lim_{a\to\infty} \frac{q^2}{m^2} = 0, \tag{48}$$

which gives the uncharged Schwarzschild interior solution.

For $r_0/m < 2$ one finds

$$\lim_{a\to\infty} \frac{q^2}{m^2} = \left(2 - \frac{r_0}{m}\right)\frac{r_0}{m}. \tag{49}$$

This solution is weird in this context. Equation (49) is

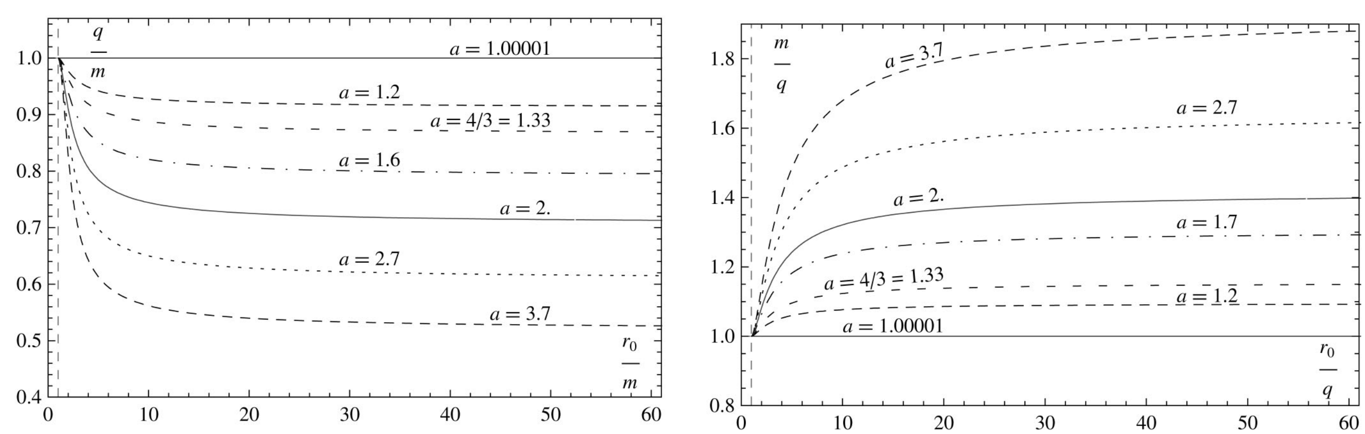

FIG. 1. Left: the charge of the star (normalized to the mass) as a function of the radius of the star (normalized to the mass) for a few values of a in the interval of interest, $1 \leq a \leq 4$. Right: the mass of the star (normalized to the charge) as a function of the radius of the star (normalized to the charge) for a few values of a in the interval of interest, $1 \leq a \leq 4$.

equivalent to

$$1 - \frac{2m}{r_0} + \frac{q^2}{r_0^2} = 0, \tag{50}$$

which shows that, for this branch, when $a \to \infty$ for given m and q there are charged solutions in which the radius of the star r_0 is the horizon radius. For $1 < r_0/m < 2$ these are nonextremal solutions which represent nonextremal regular black holes with tension matter. For $r_0/m = 1$ these are extremal solutions which represent extremal regular black holes with tension matter. Note that these extremal solutions can be obtained from Eq. (41) when one takes the limit $a \to \infty$ and absorbs the ε term into the time coordinate so that $B(r)$ is continuous at r_0. This $a \to \infty$ branch, representing regular black holes rather than quasiblack holes, should be carefully handled and we do not explore it further here. Regular black holes either with a charged core [38–41] or with a de Sitter core [42–47] are known, but with charge and a de Sitter core together, as found here, seem to have not been explored.

C. Numerical study of relativistic charged stars and quasiblack holes (or frozen stars): Three typical cases

1. Intervals of a

Although valid for all $a > 0$, there are some special intervals in the domain of the parameter a for which the solutions can be considered more physical. It is known that the Schwarzschild interior solution, $a = \infty$, having an incompressible fluid as matter source, yields violation of the dominant energy condition as well as an infinite speed of sound c_s, i.e., a speed greater than the speed of light, bringing into question causality issues. Thus, when discussing the interesting intervals of the Guilfoyle parameter a, which yield what might be considered physical solutions, it is important to take into account the energy conditions and the behavior of the speed of sound. It is a straightforward task verifying that, within a certain range of parameters, the fluid quantities satisfy the energy conditions for the whole star, even in the quasiblack hole limit. Let us then find such a interval.

First we note that the most restrictive conditions arise in the quasiblack hole limit, and then we study the energy conditions for this kind of frozen stars. In a charged static fluid, besides the fluid energy density and pressure, there are the electromagnetic energy density $\rho_{\rm em}$, the radial electric pressure $-\rho_{\rm em}$, and the tangential electromagnetic pressures $\rho_{\rm em}$. It is then useful to define the effective energy density and pressures of the charged fluid by

$$\rho_{\rm eff}(r) \equiv \rho_{\rm m}(r) + \rho_{\rm em}(r) = \rho_{\rm m}(r) + \frac{Q(r)^2}{8\pi r^4}, \tag{51}$$

$$p^r_{\rm eff}(r) \equiv p(r) - \rho_{\rm em}(r) = p(r) - \frac{Q(r)^2}{8\pi r^4}, \tag{52}$$

$$p^t_{\rm eff}(r) \equiv p(r) + \rho_{\rm em}(r) = p(r) + \frac{Q(r)^2}{8\pi r^4}, \tag{53}$$

where "r" and "t" in the above definitions stand for radial and tangential pressures, respectively. Therefore, testing the energy conditions in the present case leads us to check inequalities such as

(a) $\rho_{\rm eff}(r) \geq 0$,
(b) $\rho_{\rm eff}(r) + p^r_{\rm eff}(r) \geq 0$, or equivalently, $\rho_{\rm m}(r) + p(r) \geq 0$,
(c) $\rho_{\rm eff}(r) + p^t_{\rm eff}(r) \geq 0$,
(d) $\rho_{\rm eff}(r) + p^r_{\rm eff}(r) + 2p^t_{\rm eff}(r) \geq 0$,
(e) $\rho_{\rm eff}(r) \geq |p^r_{\rm eff}(r)|$,
(f) $\rho_{\rm eff}(r) \geq |p^t_{\rm eff}(r)|$, or equivalently, $\rho_{\rm m}(r) \geq |p(r)|$.

These are requirements for the weak energy condition (WEC) [(a), (b), and (c)], null energy condition (NEC) [(b) and (c)], strong energy condition (SEC) [(b), (c), and (d)], and dominant energy condition (DEC) [(e) and (f)].

Condition (a) is promptly verified after Eq. (25). Condition (b) in the form $\rho_{\rm m}(r) + p(r) \geq 0$ implies the charged fluid satisfies the inequality conditions (c) and (d). In addition, when the charged fluid satisfies the second inequality of (f) it also satisfies (e). Therefore, it is convenient to check first the conditions on the matter quantities $\rho_{\rm m}(r)$ and $p(r)$ and some relations between them.

Starting with the matter-energy density $\rho_{\rm m}(r)$, a numerical analysis shows that in the case $r_0/m = 1.000\,01$ (the quasiblack hole limit) one has $\rho_{\rm m}(r) > 0$ for all r if a is in the interval $0 < a \lesssim 3.535\,23$. For $3.535\,23 \lesssim a \leq 4$, $\rho_{\rm m}(r)$ is negative in some intervals of r inside the star, the corresponding (negative) minimum of $\rho_{\rm m}(r)$ moves toward the center of the star while a increases. For $a = 4$, the minimum value of $\rho_{\rm m}(r)$ shifts to $r \simeq 0.017$ and becomes very large (negative) but finite. For $a > 4$, at the quasiblack hole limit $\rho_{\rm m}(r)$ behaves wildly with r. The extremal negative value of $\rho_{\rm m}(r)$ becomes arbitrarily large, moves toward the surface $r = r_0$ as a grows, and eventually disappears for very large a.

Now we turn to the behavior of the matter pressure $p(r)$ at the quasiblack hole limit $r_0/m = 1.000\,01$. It is negative for all a belonging to the interval $0 < a < 1$, vanishing for $a = 1$ in which case the matter is composed of charged dust. The pressure $p(r)$ is positive in the interval $1 < a \leq 4$, reaching a very large finite positive value at $a = 4$. In the interval $4 < a < \infty$, the behavior of the pressure is as wild as the energy density $\rho_{\rm m}(r)$, the difference being that $p(r)$ reaches arbitrarily large positive values at points where the energy density gets arbitrarily large negative values.

It is also useful to compare $\rho_{\rm m}(r)$ with $p(r)$. For small a the absolute value of $p(r)$ is smaller than $\rho_{\rm m}(r)$. In particular, the central pressure is smaller than the central energy density, the equality $p(r = 0) = \rho_{\rm m}(r = 0)$ being reached at $a = (1 + \sqrt{13})^2/9 \simeq 2.356\,79$. For a larger than that value, the central pressure is always larger than the central energy density. A more detailed analysis valid for any r, not just $r = 0$, shows that at the quasiblack hole limit $r_0/m = 1.000\,01$, the equality $p = \rho_{\rm m}$ is reached at $a \simeq 2.326\,65$, above this value one has $p(r) > \rho_{\rm m}(r)$ at some interval of values of r inside the quasiblack hole or frozen star. Another quantity of importance involving $\rho_{\rm m}(r)$ and $p(r)$ together is $\rho_{\rm m}(r) + p(r)$ for which we obtain that it is positive for all stars and for the frozen star if a is in the interval $0 < a \leq 4$.

Finally, let us stress that the analysis was performed for $r_0/m = 1.000\,01$, i.e., in the quasiblack hole limit. On the other hand, sufficiently sparse stars, $r_0/m > 1$, satisfy all of the energy conditions for all a.

In summary, after a careful numerical analysis of the quasiblack hole limit, we can state the following:

(i) For $0 < a \leq 4$ conditions (b) and (c) are satisfied for all r_0/m including the quasiblack hole. Therefore, the charged fluid satisfies the NEC as long as the Guilfoyle parameter a is in the interval $0 < a \leq 4$.

(ii) The WEC requires that conditions (a), (b), and (c) are satisfied. Hence it is satisfied by all of the stars and by the quasiblack hole or frozen star if $0 < a \leq 4$.

(iii) The SEC requires conditions (b), (c), and (d) to be satisfied, and so the charged fluid satisfies the SEC for a in the interval $0 < a \leq 4$.

(iv) Conditions (e) and (f) are required by the DEC. A numerical analysis yields $\rho_{\rm m}(r) \geq |p(r)|$ for $0 < a \lesssim 2.326\,65$ and, within this interval of a, condition (e) is satisfied too. Hence the DEC is satisfied in the interval $0 < a \lesssim 2.33$, where we put $2.326\,65 \simeq 2.33$ from here onward.

Now, we analyze the speed of sound. From Eq. (40) we see that $c_{\rm s}^2(r)$ is a monotonically increasing function of the radial coordinate r. Hence, even if at the center of the star, the parameters are fixed so that the speed of sound is smaller than the speed of light, it may happen that $c_{\rm s}(r)$ reaches values greater than unity at some point r inside the star. As the numerical analysis has shown, the restriction that the speed of sound is at most as large as the speed of light imposes the strongest bounds on the range of values of a. Indeed, such an imposition restricts the allowed values of the parameter a to be in the interval $1 \leq a \leq 4/3$. This is confirmed by the curves shown in Fig. 2, where we plot $c_{\rm s}(r)$ as a function of $a \geq 1$, at the center and at the surface of the star, for several different values of the parameter r_0/m, which represents the compactness of the star. The curves terminate at $a = 1$ because $dp(r)/d\rho_{\rm m}(r)$ is negative in the interval $0 < a < 1$, and so the speed of sound is not defined in that interval. The value $a = 4/3$, shown as a vertical dashed line in Fig. 2, is a critical value in the sense that for a very sparse star, i.e., in the limit $r_0 \to \infty$, the speed of sound at the center of the star reaches unity for $a = a_c = 4/3$. It is also seen from Fig. 2 that, for each a, the speed of sound at the center of the star strongly depends on the compactness parameter β. Interestingly, more compact stars have smaller speeds of sound at the center. On the other hand, as seen from the right-hand plots in Fig. 2, for a in the interval $1 \leq a \leq 4/3$, the speed of sound close to the surface of the star is practically the same for all r_0/m.

For each value of the Guilfoyle parameter a there is an infinity of star solutions, functions of the compactness of the stars themselves, i.e., functions of r_0/m. The parameter a is related to the pressure: for $a < 1$, the stars are supported by tension; for $a = 1$, the stars have no pressure and they are Bonnor stars [4–9]; and for $a > 1$, the stars are

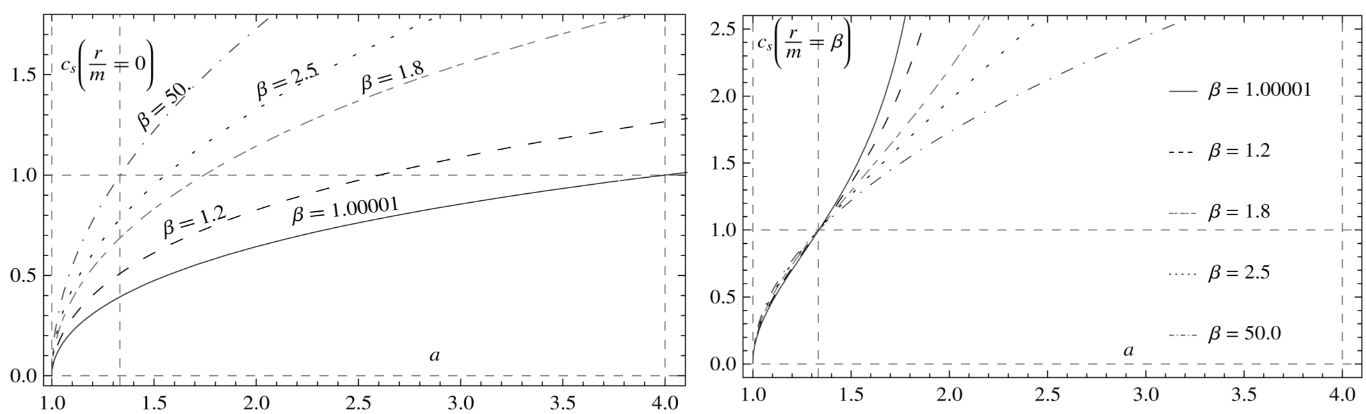

FIG. 2. On the left, the speed of sound at the center of the star, $c_{\rm s}(0)$, as a function of the parameter $a \geq 1$, for a few different values of $\beta = r_0/m$. For a in the interval $0 < a < 1$, the speed of sound is not defined, since $dp/d\rho_{\rm m} < 0$. The value $a_c = 4/3 = 1.33$ is shown as a vertical dashed line. Notice that for a very sparse star, i.e., in the limit $r_0 \to \infty$, the speed of sound at the center of the star reaches unity for $a = a_c = 4/3$. For $a \to \infty$, the Schwarzschild interior solution $c_{\rm s}(0)$ is infinite at the center. On the right, the speed of sound at the surface of the star $c_{\rm s}(r_0)$ as a function of the parameter $a \geq 1$. The value $a_c = 4/3$ is shown as a vertical dashed line. In all the cases, i.e., for all values of β, the speed of sound close to the surface of the star tends to unity at $a = a_c$. For $a \to \infty$, the Schwarzschild interior solution, the speed of sound is infinite at the surface (in fact the speed of sound in the Schwarzschild interior solution is infinite throughout the whole fluid region).

supported by pressure. In detail, a can be divided into the following intervals: $0 < a < 1$ yields tension stars; $a = 1$ yields charged dust stars; $1 < a \leq 4/3$ yields stars and quasiblack holes which obey the energy conditions and have a speed of sound less than the speed of light; $4/3 < a \lesssim 2.33$ yields stars and quasiblack holes which obey the energy conditions and have a speed of sound greater than the speed of light; $2.33 < a \leq 4$ yields stars and quasiblack holes which obey all the energy conditions but the DEC and have a speed of sound greater than the speed of light; and $4 \leq a \leq \infty$ yields normal stars which obey all the energy conditions but the DEC, but yields no quasiblack holes. In this latter interval of a the matter behaves as in the Schwarzschild solution ($a \to \infty$), in the sense that before the gravitational radius is reached gravitational collapse ensues.

The interval $0 < a < 1$ does not interest us here because it gives tension stars, the value $a = 1$ yields charged dust Bonnor stars studied previously, and the interval $4 < a \leq \infty$ does not yield quasiblack holes and so again is of no interest here, although very interesting in other contexts. The intervals of a that yield systems that can be pushed into quasiblack holes is $1 \leq a \leq 4$. So, since $a = 1$ has been studied in previous works, we will study the interval

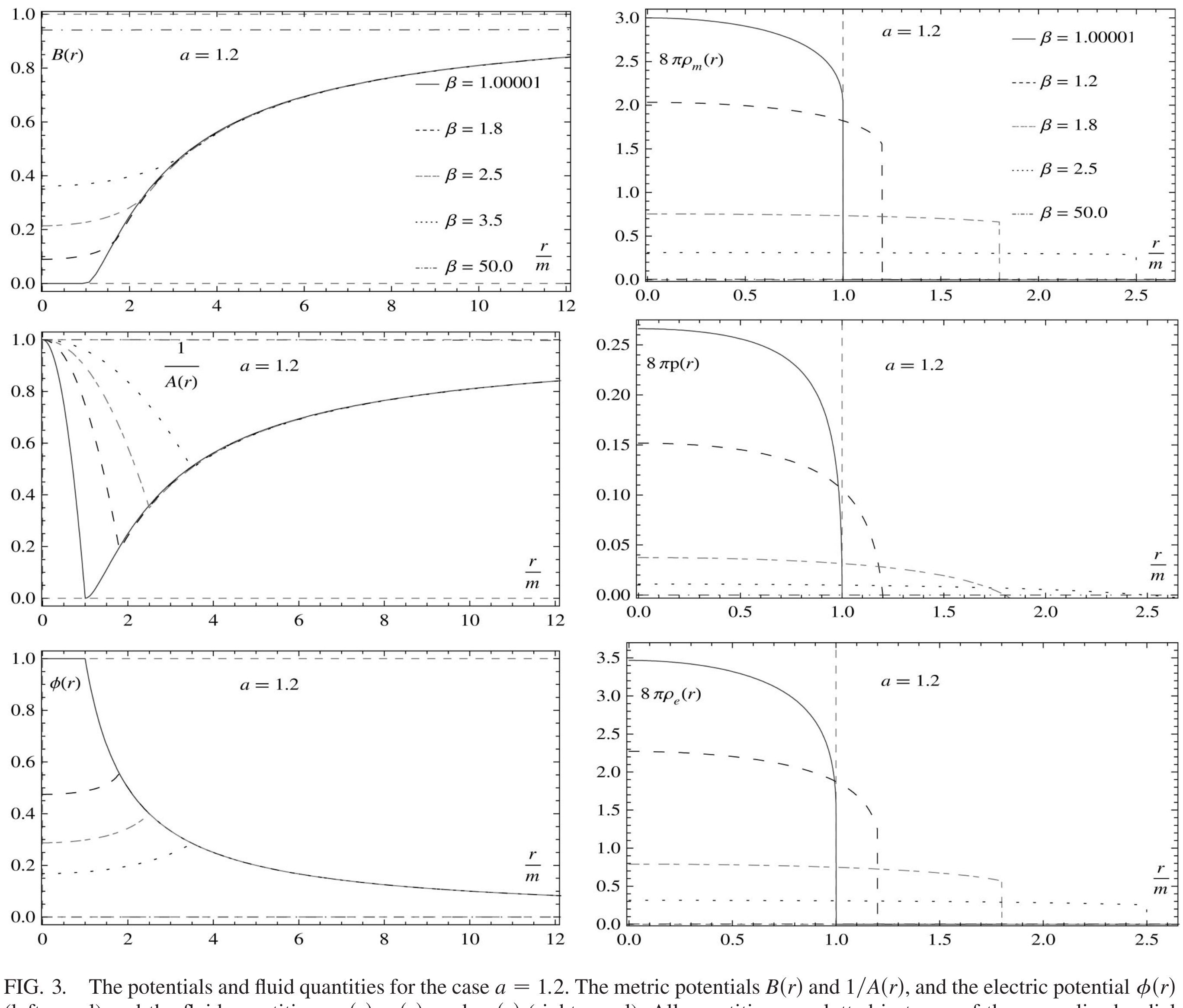

FIG. 3. The potentials and fluid quantities for the case $a = 1.2$. The metric potentials $B(r)$ and $1/A(r)$, and the electric potential $\phi(r)$ (left panel) and the fluid quantities $\rho_{\rm m}(r)$, $p(r)$, and $\rho_{\rm e}(r)$ (right panel). All quantities are plotted in terms of the normalized radial coordinate r/m, where m is the mass of the star, which is kept fixed, for five values of $\beta = r_0/m$ in each graph: $\beta = 1.000\,01$ (solid line) $\beta = 1.2$ (space-dashed line), $\beta = 1.8$ (dashed line), $\beta = 2.5$ (dotted line), and $\beta = 50.0$ (dot-dashed line). Notice that the case $\beta = 1.2$ is not shown for the potentials, the case $\beta = 3.5$ being shown instead, because the curves of the potentials for $\beta = 1.2$ practically coincide with the curves for $\beta = 1.000\,01$. The horizontal straight dotted line represents the asymptotic limit of the metric potentials for large r, $B(r) = 1/A(r) = 1$, and $\phi(r) = 0$, and is plotted for comparison. The case $\beta = 1$, which gives $q = m = r_0$, is a quasiblack hole.

$1 < a \leq 4/3$, as well as the intervals $4/3 < a \lesssim 2.33$, and $2.33 < a \leq 4$. In order to see the main features of these charged stars and, in particular, to see the quasiblack hole limit, we have plotted some important curves for three typical cases within each interval, namely, $a = 1.2$, $a = 1.7$, and $a = 3$. We first verify that the quasiblack hole limit can be attained, and then we check that in such a limit the physical quantities related to the charged fluid are well defined.

2. The interval $1 < a \leq 4/3$. *Typical case:* $a = 1.2$

The metric and electromagnetic fields, and the fluid quantities, such as mass density $\rho_m(r)$, charge density $\rho_e(r)$, pressure $p(r)$, and speed of sound $c_s(r)$, are well-behaved functions of the radial coordinate for $1 \leq a \leq 4/3$, even in the quasiblack hole limit. In fact, we have numerically analyzed each one of these functions for several values of r_0/m as functions of the normalized radial coordinate r/m. Within this interval we analyze the typical case $a = 1.2$.

Metric and the electric potentials: The metric potentials $B(r)$ and $1/A(r)$, and the electric potential $\phi(r)$ as a function of the normalized radial coordinate r/m, for the case $a = 1.2$, are shown in the left panel of Fig. 3. The metric potentials $B(r)$ and $1/A(r)$ are obtained from Eqs. (28) and (26), respectively, while $\phi(r)$ is obtained using relation (21) with $c = 0$, and by choosing the constant b so that the potential $\phi(r)$ is a continuous function at the surface $r = r_0$. The exterior metric is the Reissner-Nordström metric, and then the curves for the potential $B(r)$ and $1/A(r)$ tend to unity for large values of the radial coordinate r/m. We plot the curves for each one of the potentials for five different values of the normalized radius of the star $\beta = r_0/m$, namely, $\beta = 1.00001$ (solid line), $\beta = 1.8$ (space-dashed line), $\beta = 2.5$ (dashed line), $\beta = 3.5$ (dotted line), and $\beta = 50$ (dash-dotted line). The case $\beta = 50$ represents a very sparse star and the metric potentials are nearly constant close to unity in the hole region inside the star. As a consequence, the curves of the potentials in this case are horizontal lines, with $\phi(r)$ very close to zero. On the other hand, the electric potential is close to zero inside the star. It is also seen that for the quasiextremal case where $r_0/m = 1.00001$, the quasiblack hole features show up. Namely, $B(r) \to \varepsilon$ for the whole interior region, $0 \leq r < r_0$ and $1/A(r) \to \varepsilon$ at $r = r_0$.

The fluid quantities: The mass density $\rho_m(r)$, the electric charge density $\rho_e(r)$, and the pressure $p(r)$ as a function of the normalized radial coordinate r/m are shown in the right panel of Fig. 3, for the case $a = 1.2$. As in the case of the potential functions (see the left panel of Fig. 3), we plot the curves in terms of the radial coordinate r/m for five different values of the normalized radius of the star $\beta = r_0/m$, namely, $\beta = 1.00001$ (solid line), $\beta = 1.2$ (space-dashed line), $\beta = 1.8$ (dashed line), $\beta = 2.5$ (dotted line), and $\beta = 50$ (dash-dotted line).

The case $\beta = 50$ represents a very sparse star and the fluid quantities are very small and are nearly constant in the hole region inside the star, with the corresponding curves being the lowest horizontal lines, almost coinciding with the horizontal axes line (the bottom line) of the plot. In fact, such curves do not appear in the graphs. It is also seen that for the quasiblack hole case, represented in the figures by the case $r_0/m = 1.00001$, the fluid quantities assume the largest possible values. With the chosen value for the parameter a ($1 \leq a \leq 4/3$), the fluid quantities are continuous decreasing functions of the radial coordinate inside the star. The exterior region is given by the Reissner-Nordström electrovacuum metric. The functions $\rho_m(r)$ and $\rho_e(r)$ are truncated at $r = r_0$, resulting in two discontinuous functions at that point which signal the jump into vacuum. On the other hand, the function $p(r)$ is a monotonically decreasing function of r, starting at $p(0)$ with the highest value and reaching $p = 0$ at $r = r_0$. We also see that the central pressure $p(0)$ increases with the compactness parameter β of the star, analogous to the behavior of the central pressure in white dwarfs, and contrary to the behavior of the central pressure in main sequence stars. One can also verify numerically that the energy conditions listed in the previous section are satisfied within the parameter space considered here.

Another final physical quantity worthy of numerical analysis is the speed of sound. The results confirm that the interesting solutions, for which c_s is well defined and the system preserves causality, are those in which the parameter a belongs to the interval $1 \leq a \leq 4/3$. The curves for $c_s(r)$ as a function of r/m for $a = 1.2$ are shown in Fig. 4. The chosen values of the normalized radius of the star are the same as for the other fluid quantities, as in the right panel of Fig. 3. The horizontal dot-dashed line is for a very sparse star, with $\beta = 50$. The radial dependence of the speed of sound is not seen because in this case the radius of the star is too large. The other horizontal line, the

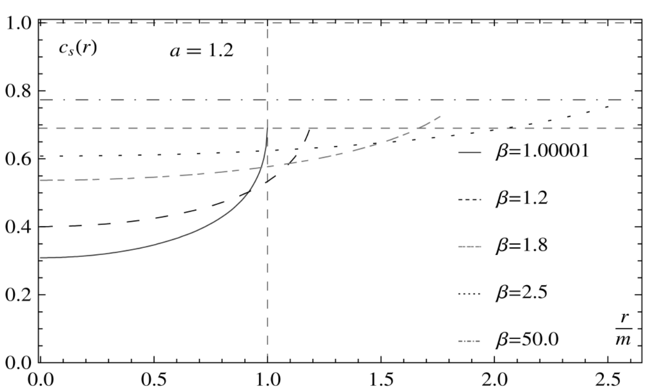

FIG. 4. The speed of sound c_s for $a = 1.2$, as a function of the normalized coordinate r/m, for five values of $\beta = r_0/m$ (from bottom to top: $\beta = 1.00001$, $\beta = 1.2$, $\beta = 1.8$, $\beta = 2.5$, and $\beta = 50.0$). Notice that the speed of sound is smaller than the speed of light throughout the star.

dashed line, indicates the maximum value of $c_s(r)$ close to the surface of the star in the quasiblack hole case, which is approximately 0.7. It also can be verified that for $a = 4/3$ such a maximum value is $c_s \simeq 1$, for r very close to r_0, independently of how compressed or how sparse the star is. In fact, the value of $c_s(r_0)$ is the maximum value of the speed of sound for all the stars within the parameter space considered in this section. In turn, the minimum value of $c_s(r)$ occurs at the center of the star.

3. The interval $4/3 < a \lesssim 2.33$*. Typical case:* $a = 1.7$

In this section we show the main features of the metric and electric functions and of the fluid quantities for a particular case of a in the interval $4/3 < a < 2$. In this interval, all the potentials are well-behaved functions of the radial coordinate for all the stars. The fluid quantities are also smooth inside the star, and satisfy the energy conditions (see Sec. IV B 2). However, the speed of sound may reach values larger than the speed of light. Within this interval we analyze the typical case $a = 1.7$.

The metric and the electric potentials: The general form of the curves for the metric potentials for $a = 1.7$ is nearly the same as for $a = 1.2$ (see the left panels of Figs. 3 and 5). This is true for all finite values of a in the intervals we investigated. As seen from Fig. 5 the metric potentials are more sensible to the radius to mass relation, r_0/m, than to the parameter a. The metric potential $B(r)$ in the interior region decreases slightly with a, indicating that the gravitational field strength increases with a. Similar small changes are observed also in the metric function $A(r)$.

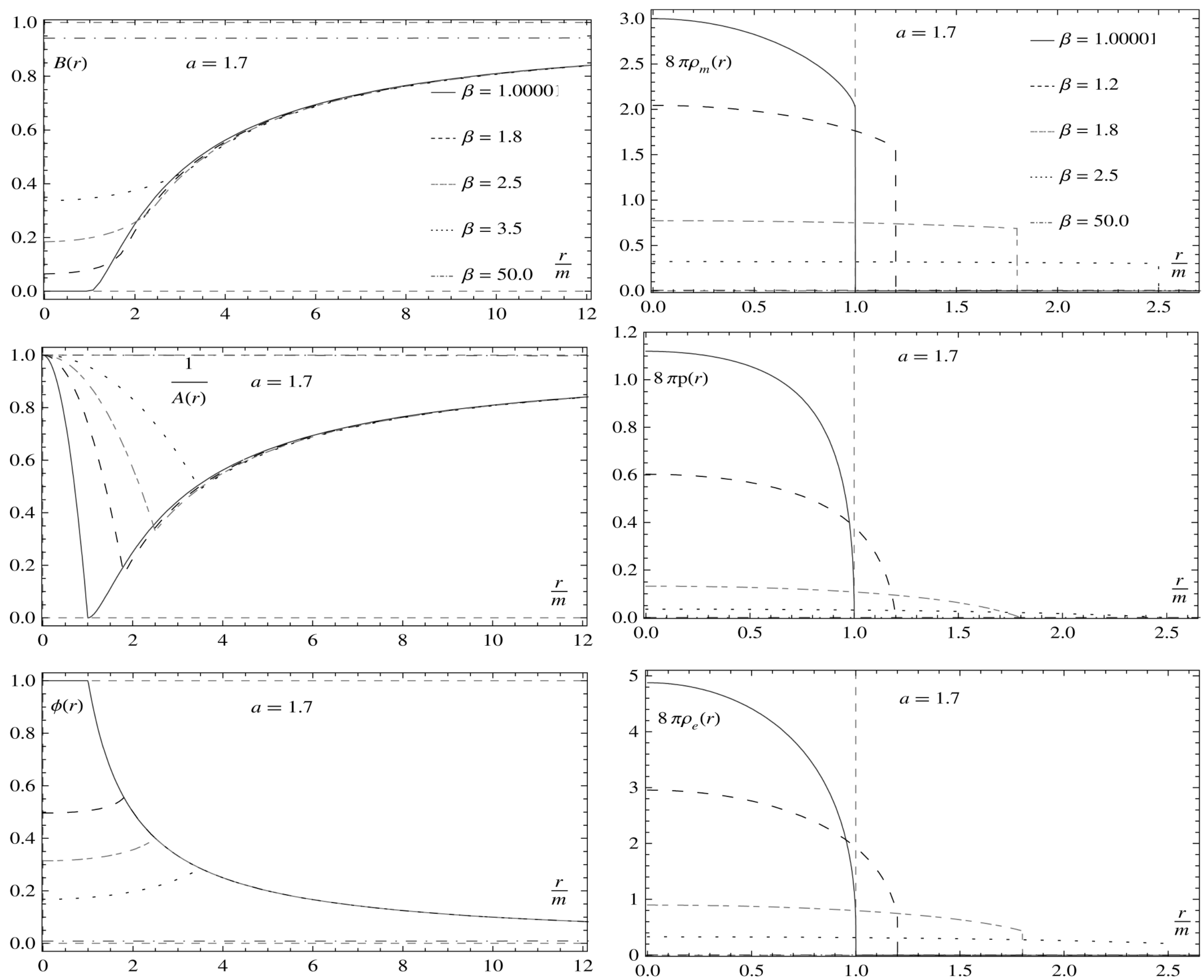

FIG. 5. The same potentials (left panel) and the fluid quantities (right panel) as in Fig. 3, but here for the case $a = 1.7$. As above, all quantities are plotted in terms of the normalized radial coordinate r/m, for five values of $\beta = r_0/m$ in each graph, and the same conventions for the lines are used. Again, the case $\beta = 1$, which gives $q = m = r_0$, is a quasiblack hole. Notice that the pressure is larger than in Fig. 3, but it is smaller than the energy density.

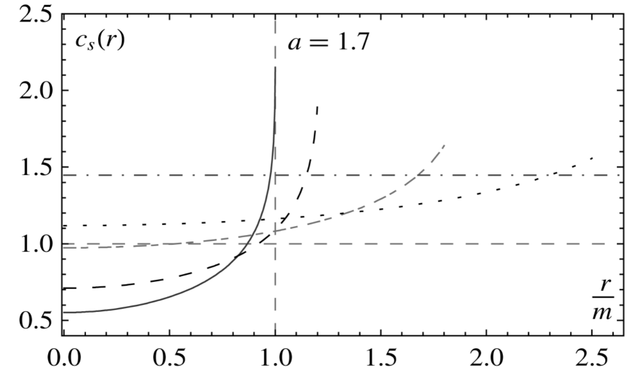

FIG. 6. The speed of sound c_s, for $a = 1.7$, as a function of the normalized coordinate r/m, for five values of $\beta = r_0/m$ in each graph (from bottom to top: $\beta = 1.000\,01$, $\beta = 1.2$, $\beta = 1.8$, $\beta = 2.5$, and $\beta = 50.0$). The case $\beta = 1$ is a quasiblack hole.

The main changes, even though small too, are in the electric potential $\phi(r)$ which, with our choice of positive charge, increases with a. This indicates a noticeable change in the central electric charge density.

The fluid quantities: A comparison between the right panels of Figs. 3 and 5 indicates that even though the fluid quantities strongly depend on the parameter a, for the values of a in the interval $4/3 < a < 2$, the overall behavior of the density and pressure is similar to the case $1 \leq a \leq 4/3$. Namely, $\rho_{\rm m}(r)$, $\rho_{\rm e}(r)$, and $p(r)$ are monotonically decreasing functions of the radial coordinate. However, while the central value of the mass density remains almost the same, the values of the pressure and of the charge density increase substantially from $a = 1.2$ to $a = 1.7$. In particular, the central pressure $p(r = 0)$ increases by a factor of approximately 4. Interestingly, even though the central charge density increases with a,

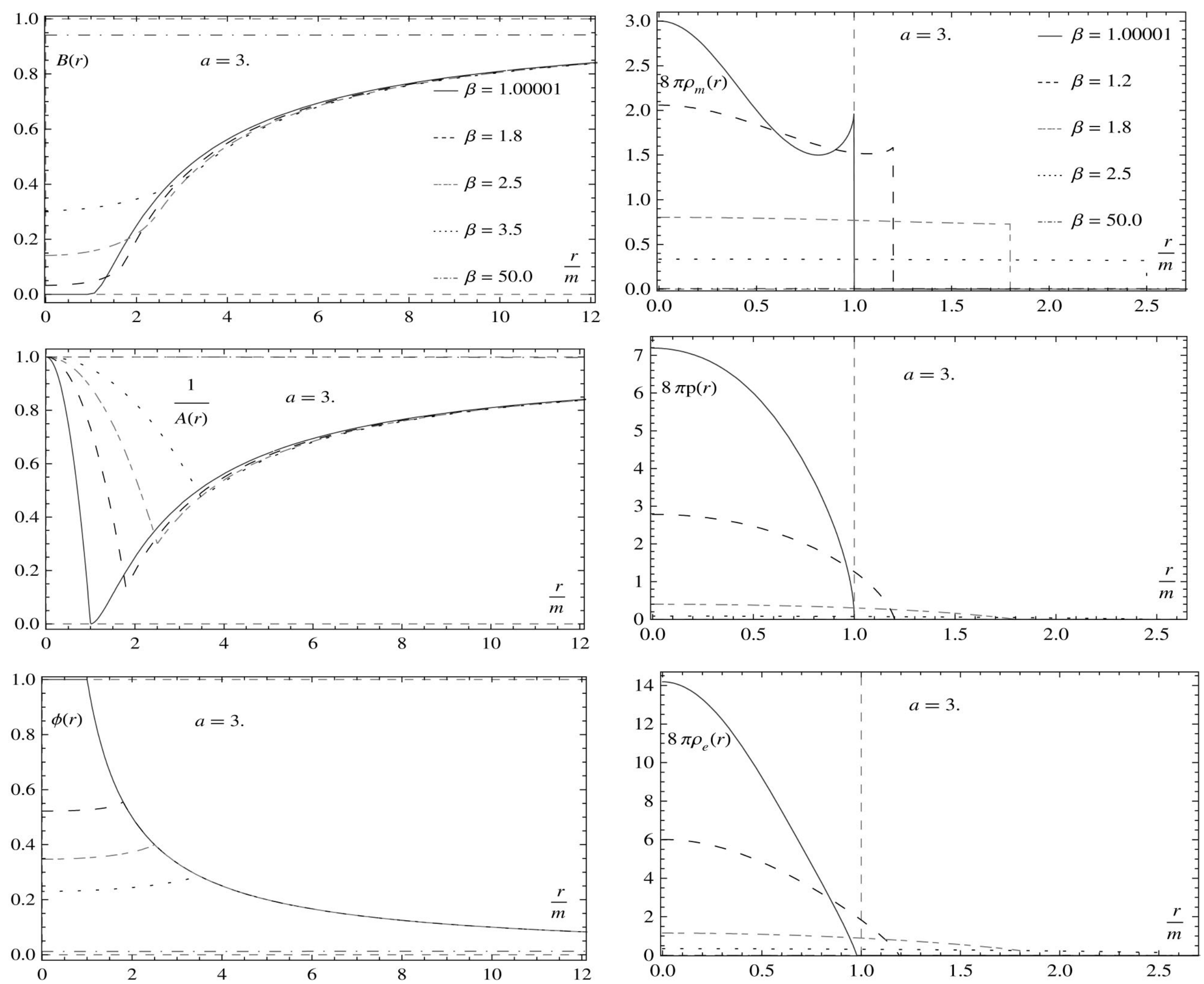

FIG. 7. The same quantities as in Figs. 3 and 5 are plotted as a function of the normalized coordinate r/m, but here for the case $a = 3.0$. The same five values of $\beta = r_0/m$, and the same conventions are used too. As above, the case $\beta = 1.2$ is not shown in the plots for the potentials, the case $\beta = 3.5$ being shown instead. Notice that $\rho_{\rm m}(r)$ does not decrease monotonically toward the surface $r = r_0$. Note also that the central pressure is larger than the energy density.

the total charge inside the star decreases slowly with a, reaching zero as $a \to \infty$ for $r_0/m \geq 2$ as discussed above. Moreover, in this interval, the energy conditions are satisfied by all of the stars. On the other hand, the speed of sound may be greater than the speed of light in some cases. This fact is illustrated in Fig. 6, where we see that the curves representing $c_s(r)$ for very compact stars, i.e., for $\beta = r_0/m$ small, reach unity at some r well inside the star. It is also seen that, for sufficiently sparse stars, i.e., for $\beta = r_0/m$ large enough compared to unity, the speed of sound is larger than the speed of light everywhere inside the star.

4. The interval $2.33 < a \leq 4$. *Typical case:* $a = 3.0$

Here we show the main features of the metric and electric functions and of the fluid quantities for a particular case of a in the interval $2.33 \leq a \leq 4$. In this interval, all the potentials are well-behaved functions of the radial coordinate for all the stars. The fluid also is smooth inside the star, and satisfies the energy conditions (see Sec. IV B 2). However, the speed of sound reaches values larger than the speed of light and, for the most compact stars, it diverges at some point inside the star. Within this interval we analyze the typical case $a = 3.0$.

Metric and electric potentials: The metric potentials $B(r)$ and $1/A(r)$, and the electric potential $\phi(r)$ as a function of the normalized radial coordinate r/m, for the case $a = 3$, are shown in the left panel of Fig. 7. The general features of the curves are the same as in the case of the left panel of Figs. 3 and 5, for which $a = 1.2$ and $a = 1.7$, respectively. By comparing the three cases, the dependence of these potentials upon the Guilfoyle parameter a is now more clearly seen. As a grows, the central values of the metric potentials $B(r)$ and $1/A(r)$ diminish by a small amount, and the electric potential $\phi(r)$ grows substantially when compared to the case $a = 1.2$. Of course, because of the normalization used, these changes are not observed in the quasiblack hole limit.

The fluid quantities: The fact that the fluid quantities strongly depend on the parameter a is especially noticed for values of a in the interval $2.33 \leq a < 4$, as seen in the right panel of Fig. 7. An interesting particularity is that, for sufficiently large a and small β, i.e., more compact stars, the energy density $\rho_m(r)$ is not a monotonically decreasing function of r anymore. As seen in that figure, the curve of $\rho_e(r)$ for the quasiblack hole case oscillates, attaining a minimum value for some $r = r_m$ inside the star. This happens for all the sufficiently compact stars. Also worthy of note is the fact that the central value of $\rho_m(r)$ does not depend on a. Meanwhile, the pressure $p(r)$ starts from a relatively high value at the center of the star, and decreases very fast to zero at the surface. As a consequence, even though the pressure is always positive and a monotonically decreasing function of r, the speed of sound increases very rapidly from the center, diverges at $r = r_m$, and becomes undefined in the region $r > r_m$. For large a, the central

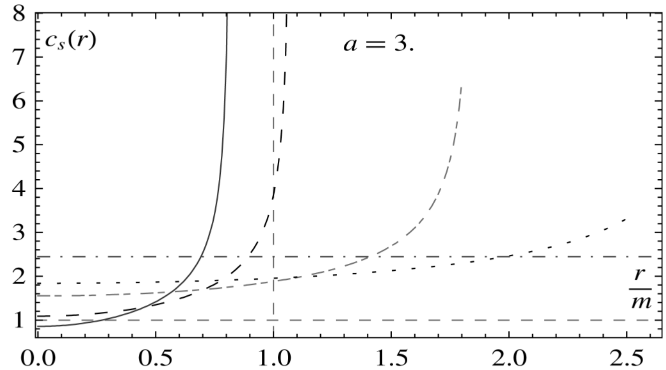

FIG. 8. The speed of sound c_s, for $a = 3.0$, as a function of the normalized coordinate r/m, for five values of $\beta = r_0/m$ in each graph (from bottom to top: $\beta = 1.00001$, $\beta = 1.2$, $\beta = 1.8$, $\beta = 2.5$, and $\beta = 50.0$). The case $\beta = 1$ is a quasiblack hole.

value of the electric charge density also becomes much larger than the central energy density. Moreover, for $2.33 \leq a < 4$, the dominant energy condition is not satisfied by any one of the stars. In fact, excluding the central region of very compact stars, the speed of sound is greater than the speed of light for all the stars. This fact is illustrated in Fig. 8, where we see that the curves for $c_s(r)$ are larger than unity for all r inside the stars.

V. CONCLUSIONS

We have analyzed the class Ia of solutions provided by Guilfoyle [24]. Such spherically symmetric relativistic charged fluid distributions are bounded by a surface of radius r_0. The interior region is filled with a fluid characterized by its mass and charge densities and by a nonzero pressure. The spacetime in the exterior region is represented by the Reissner-Nordström metric. These global solutions represent relativistic stars, i.e., relativistic cold charged spheres with pressure. Besides the mass m (or charge q, which are related to each other) and the radius of the star r_0, this class of solutions is characterized by another free parameter, the Guilfoyle parameter a. This parameter is related to the pressure: for $a < 1$, the stars are supported by tension; for $a = 1$, the stars have no pressure (they are Bonnor stars); and for $a > 1$, the stars are supported by pressure. The interval of the free parameter a can be fixed in such a way that the fluid satisfies the energy conditions, and other physical requirements for a relativistic cold star. We have then studied relativistic stars within the interval $1 < a \leq 4$. We have found that these cold stars show a mixed behavior; in one instance they behave as main sequence stars, in another instance as white dwarfs. Indeed, the mass to radius relation of these cold stars is analogous to the behavior of the mass to radius relation in main sequence stars and contrary to the mass to radius relation in white dwarfs, whereas the central pressure of these cold stars has an analogous behavior to the central pressure of white dwarfs, and a contrary behavior to the central pressure of main sequence stars. We have also shown that, in the interval $1 < a \leq 4$, the most compact configuration is a quasiblack hole with pressure. Thus, quasiblack holes without pressure ($a = 1$) studied previously, as well as quasiblack holes with pressure ($1 < a \leq 4$) studied here, can be found within the context of general relativity. As the most compact configuration is a cold star, it can be called a frozen star.

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