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Black hole solutions in 5D Horava-Lifshitz gravity

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We study the full spectrum of spherically symmetric solutions in the five-dimensional nonprojectable Horava-Lifshitz type gravity theories. For appropriate ranges of the coupling parameters, we have found several classes of solutions which are characterized by an AdS_5, dS_5, or flat large distance asymptotic behavior, plus the standard $1/r^2$ tail of the usual five-dimensional Schwarzschild black holes. In addition we have found solutions with an unconventional short or large distance behavior, and, for a special range of the coupling parameters, solutions which coincide with black hole solutions of conventional relativistic five-dimensional Gauss-Bonnet gravity.

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I. INTRODUCTION

A novel quantum gravity model, which claims power-counting renormalizability, has been formulated recently by Horava [1]. This scenario is based on an anisotropy between space and time coordinates, which is expressed via the scalings $t \longrightarrow b^z t$ and $x \longrightarrow bx$, where z is a dynamical critical exponent. For $z \neq 1$ the UV behavior of the model is governed by a nonstandard Lifshitz fixed point, while for $z = 1$ we recover the well-known free Gaussian fixed point. In the Horava model, for three spatial dimensions, the suitable choice is $z = 3$.

It is worth noting that in the Horava-Lifshitz (HL) gravity the four-dimensional diffeomorphism invariance of general relativity is sacrificed in order to achieve power-counting renormalizability. The action of the model can be split into a kinetic plus a potential term, which both respect a restricted (3 + 1) diffeomorphism invariance. The interesting feature is that the kinetic term contains only second order time derivatives, while the potential term consists of higher order spatial derivatives of the metric components. This particular structure improves significantly the UV properties of the graviton propagator, and renders the model power-counting renormalizable. Moreover, in this way we avoid ghost modes which are usual in conventional higher order gravity models.

For the construction of the potential term, there has been proposed [1] the so-called "detailed balance principle," which is inspired by condensed matter physics. The main advantage of this approach is the restriction of the large number of arbitrary couplings that appear in the bare action of the model. However, the physical motivation for a consideration such as the "detailed balance" is not clear [2,3]. An alternative way for constructing an action is to include all possible operators which are compatible with the renormalizability of the model; this implies that all operators with dimension less or equal to six are allowed in the action (for the exact form of the action see [3]).

As it has been already mentioned, the HL gravity violates local Lorentz invariance in the UV, however it is expected that general relativity is recovered in the IR limit. This implies a very special renormalization group flow for the couplings of the model. In particular, the parameter λ in the kinetic term of the action (which measures the departure from the Lorentz invariance) should flow to unity, while the higher order couplings should vanish, or they should become appropriately small, in the IR. Note that, even though phenomenology suggests a particular IR limit there is no theoretical study supporting this behavior.

In addition, there are several other possible inconsistencies in HL gravity which have been discussed in several works. In particular, the absence of full diffeomorphism invariance introduces an additional scalar mode which can lead to strong coupling problems or instabilities (see for example [4–9] and references therein). However, these problems will not be addressed in the present paper.

Apart from these problems, the HL gravity is an interesting quantum gravity theory, which has stimulated an extended research on cosmology and black hole solutions [2,10–28]. In addition to general relativity studies, quantum field theory models in flat space-time with anisotropy have also been considered [29–39].

Before proceeding it is important to mention that the HL gravity can be separated into two versions which are known as projectable and nonprojectable. In the projectable version the lapse function N^2 depends only on the time coordinate, while in the nonprojectable version N^2 is a function of both the space and time coordinates. Although in general relativity the projectable and nonprojectable ansatz for the metric are equivalent, since they are connected via a diffeomorphism transformation; in the HL gravity the full diffeomorphism invariance is broken and they lead to two distinct theories.

Issues connected with broken Lorentz invariance were studied in the spherically symmetric solutions of 4D HL

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gravity. In the case of detailed balance, such spherically symmetric solutions were found [11], but they exhibited an unconventional large distance asymptotic behavior. The correct Schwarzchild-flat asymptotic behavior can be recovered if the detailed balance action is modified in the IR by a term proportional to Ricci scalar, and the cosmological constant term is considered to be zero [12]. A similar study, in the case of a nonvanishing cosmological constant, has also been carried out [13]. A generalization to topological black holes was obtained in [14]. Finally, a systematic study of static spherically symmetric solutions of 4D HL gravity was presented in [15] where the most general spherically symmetric solution for $\lambda \neq 1$ and general coupling parameters was obtained.

In this work we present a full study of spherically symmetric solutions in the nonprojectable version of the five-dimensional Horava-Lifshitz gravity, for $z = 4$. For the construction of the 5D action we do not use the "detailed balance principle," but we include all the terms which are compatible with the renormalizability of the model. In particular, we can include all spatial curvature terms with dimension less than or equal to eight. However, the large number of possible terms, which are allowed in the action, leads to an equation of motion of great complexity. For this reason we restrict our study only to terms of up to second order in the curvature. Also, we suppose that in the IR limit 5D the HL gravity reduces to the 5D general relativity plus a bulk cosmological constant. A class of spherically symmetric solutions of the 5D HL gravity has been considered previously [17], but only for a very specific choice of the couplings.

Our main motivation in considering static solutions in 5D HL gravity is to investigate whether the rich spectrum of black hole solutions found in 4D (see Ref. [15]) also persists in 5D. It seems that the known static solutions of the HL gravity in 4D with $\lambda \neq 1$ do not have any obvious relation with the corresponding static solutions of the relativistic 4D gravity. In 5D however we found that there is a class of spherically symmetric solutions which after a proper identification of coupling parameters coincide with the known black hole solutions of conventional relativistic 5D Gauss-Bonnet (GB) gravity.

Static solutions of 5D Gauss-Bonnet theory are well known [40]. Among them there is a black hole solution which has two branches (for a review see [41]). The first branch is referred to as the Einstein branch while the second as the Gauss-Bonnet branch. Both branches coincide in the Chern-Simons limit for zero gravitational mass. As we will discuss in the following, we find both branches of solutions and, in addition, these solutions can also be obtained for a different combination of coupling parameters of the quadratic curvature terms than the usual combination that appears in the relativistic Gauss-Bonnet theory. We also find the black hole solution corresponding to the Chern-Simons limit with a particular choice of coupling parameters. This solution has also been found in [17] using the detailed balance principle.

The paper is organized as follows. In Sec. II we write down the action of 5D HL gravity. In Sec. III we derive the equations of motion. In Sec. IV we analyze the static spherically symmetric solutions for a special choice of coupling parameters. In Sec. V we study the most general static spherically symmetric solutions of 5D Horava-Lifshitz gravity and finally Sec. VI contains our conclusions.

II. 5D HORAVA-LIFSHITZ GRAVITY MODELS

In this section we introduce the notation for the so-called Horava gravity models in the case of four spatial dimensions ($d = 4$). These models are characterized by an anisotropy between space and time dimensions

$$[t] = -z, \qquad [x] = -1, \tag{2.1}$$

where z is an integer dynamical exponent. In order to derive the action of the model, it is useful to express the space-time metric in the Arnowitt, Deser, and Misner form

$$ds^2 = -c^2N^2dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \tag{2.2}$$

where c is the velocity of light, with dimension $[c] = z - 1$, and spatial components dx^i/dt ($i = 1, 2, 3, 4$). In addition, N and N_i are the "lapse" and "shift" functions which are used in general relativity in order to split the space-time dimensions, and g_{ij} is the spatial metric of signature $(+, +, +, +)$. For the dimensions of lapse and shift functions we obtain

$$[N] = 0, \qquad [N_i] = z - 1. \tag{2.3}$$

In this paper the dynamical exponent z is set equal to 4. The 5D action of the model is constructed from a kinetic plus a potential term according to the equation

$$S = \frac{1}{16\pi G_5 c} \int dt d^d x \sqrt{|g|} N\{\mathcal{L}_K + \mathcal{L}_V\}, \tag{2.4}$$

in which d ($D = d + 1 = 5$) is the spatial dimension and G_5 is the five-dimensional Newton constant.

We stress that the main motivation for considering models of this type is the construction of a power-counting renormalizable gravity model. However, in order to achieve renormalizibility, and simultaneously keep the time derivatives up to second order, we have to sacrifice the standard 5D diffeomorphism invariance of general relativity, which is now restricted to the transformation

$$\tilde{x}^i = \tilde{x}(x^j, t), \qquad \tilde{t} = \tilde{t}(t). \tag{2.5}$$

The kinetic part in the above Lagrangian of Eq. (2.4) can be expressed via the extrinsic curvature as

$$\mathcal{L}_K = (K^{ij}K_{ij} - \lambda K^2),$$
$$K_{ij} = \frac{1}{2N}\{-\partial_t g_{ij} + \nabla_i N_j + \nabla_j N_i\}, \qquad (2.6)$$
$$i, j = 1, 2, 3, 4,$$

which is invariant under the transformations of Eq. (2.5). For the construction of the potential term we will not follow the standard detailed balance principle, but we will use the more general approach [2,3], according to which the potential term is constructed by including all possible renormalizable operators,[1] that have dimension smaller or equal to eight, hence we write

$$\mathcal{L}_V = \mathcal{L}_R + \mathcal{L}_{R^2} + \mathcal{L}_{R^3} + \mathcal{L}_{\Delta R^2} + \mathcal{L}_{R^4} + \mathcal{L}_{\Delta R^3} + \mathcal{L}_{\Delta^2 R^2}. \qquad (2.7)$$

where the symbol Δ is defined as $\Delta = \partial_i \partial_i$ $(i = 1, 2, 3, 4)$.

The dimensions of the various terms in the Lagrangian read

$$[R] = 2, \qquad [R^2] = 4, \qquad [R^3] = [\Delta R^2] = 6,$$
$$[R^4] = [\Delta R^3] = [\Delta^2 R^2] = 8. \qquad (2.8)$$

In this work we are mainly interested in the lowest order operator $\mathcal{L}_R$ and the operator $\mathcal{L}_{R^2}$, which contains contributions of second order in the curvature:

$$\mathcal{L}_R = \eta_{0a} + \eta_{1a} R,$$
$$\mathcal{L}_{R^2} = \eta_{2a} R^2 + \eta_{2b} R^{ij} R_{ij} + \eta_{2c} R^{ijkl} R_{ijkl}, \qquad (2.9)$$

where we have used the notations R, R_{ij}, and R_{ijkl} for the Ricci scalar, the Ricci, and the Riemann tensors $(i, j = 1, 2, 3, 4)$, which correspond to the spatial 4D metric g_{ij}. Note that in the case of three spatial dimensions the term $R^{ijkl}R_{ijkl}$ is absent, as the Weyl tensor in three dimensions automatically vanishes. However, in four spatial dimensions this term cannot be omitted from the action.

The first term $\mathcal{L}_R$ is necessary in order to recover 5D general relativity with a cosmological constant in the IR limit. The second term $\mathcal{L}_{R^2}$, includes all possible quadratic terms in curvature, and becomes important in the short distance regime of the theory. Moreover, η_{0a} plays the role of the cosmological constant, while η_{1a}, η_{2a}, η_{2b}, and η_{2c} are dimensionful coupling constants with dimensions

$$[\eta_{1a}] = 6, \qquad [\eta_{2a}] = [\eta_{2b}] = [\eta_{2c}] = 4. \qquad (2.10)$$

In the present analysis we ignore higher order Lagrangian terms of dimension six and eight. Although we have not derived the detailed expression for the Lagrangian, it is worth writing some of the higher order curvature terms here,

$$\mathcal{L}_{R^3} = R^3 + R_{ij}R^{ij}R + \ldots,$$
$$\mathcal{L}_{\Delta R^2} = R \triangle R + R^{ij}\Delta R_{ij} + \ldots, \qquad (2.11)$$

$$\mathcal{L}_{R^4} = R^4 + (R_{ij}R^{ij})^2 + \ldots,$$
$$\mathcal{L}_{\Delta R^3} = R^2 \triangle R + \ldots,$$
$$\mathcal{L}_{\Delta^2 R^2} = R\Delta^2 R + \ldots, \qquad (2.12)$$

where $\Delta = \nabla_i \nabla^i$. The short distance effects of these terms may be important, but this topic is left for a future investigation.

If this model is to make sense, it is necessary that the 5D general relativity (with a cosmological constant in our case) is recovered in the IR limit. Although there is no theoretical proof for this difficult question, we will assume that the renormalization group flow towards the IR leads the parameter λ to the value one $(\lambda = 1)$, hence 5D general relativity is recovered. Also, to obtain the Einstein-Hilbert action

$$S_{\rm EH} = \frac{1}{16\pi G_5}\int dx^0 d^4x \sqrt{g} N(\tilde{K}_{ij}\tilde{K}^{ij} - \tilde{K}^2 + R + \eta_{0a}), \qquad (2.13)$$

we have to set $\eta_{1a} = c^2$, and

$$\tilde{K}_{ij} = \frac{1}{2N}\left\{-\partial_0 g_{ij} + \nabla_i\left(\frac{N_j}{c}\right) + \nabla_j\left(\frac{N_i}{c}\right)\right\}, \qquad (2.14)$$

where the timelike coordinate x_0 is defined as $x_0 = ct$.

III. EQUATIONS OF MOTION

We are looking for 5D spherically symmetric solutions of the Horava-type gravity model we constructed in the previous section. We use the following ansatz[2] for the metric

$$ds^2 = -N(r)^2 dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_k^2, \qquad (3.1)$$

in which r is a radius coordinate that corresponds to the extra dimension, and $d\Omega_k^2$ is the metric of a 3D maximally symmetric space, where k is the spatial curvature of 3D hypersurfaces and for $k = 1, -1, 0$ we have a sphere, hyperboloid or 3D torus topology correspondingly. In what follows it is convenient to perform the transformation

$$f(r) = k + r^2 Z(r). \qquad (3.2)$$

Then the action of the model to second order in curvature terms is

[1]We have ignored the terms which violate parity, see also [3].

[2]There is a more general ansatz for the metric, of the form $ds^2 = -N(r)^2dt^2 + f^{-1}(r)(dr + N^r(r)dt)^2 + r^2 d\Sigma_k^2$ with nonzero shift $N^r(r)$, but we have set $N^r(r) = 0$ [15] to simplify the equations.

$$S = \frac{1}{16\pi G_5}\int dt d^d x\sqrt{|g|}N(K^{ij}K_{ij} - \lambda K^2 + \eta_{0a} + \eta_{1a}R + \eta_{2a}R^2 + \eta_{2b}R^{ij}R_{ij} + \eta_{2c}R^{ijkl}R_{ijkl}), \quad (3.3)$$

which can be put into the form

$$S\left[N(r), Z(r), \frac{dZ(r)}{dr}\right] = \int_0^{+\infty} drL\left[N(r), Z(r), \frac{dZ(r)}{dr}\right], \quad (3.4)$$

after we integrate out the angular coordinates, where

$$L\left[N, Z, \frac{dZ}{dr}\right] \sim r^3\sqrt{\frac{N^2}{f}}\left(P\left(r\frac{dZ}{dr}\right)^2 + M(Z)\left(r\frac{dZ}{dr}\right) + Q(Z)\right), \quad (3.5)$$

and the coefficients P, $M(Z)$, and $Q(Z)$ are defined by the equations

$$\begin{aligned} P &= 3(3\eta_{2a} + \eta_{2b} + \eta_{2c}),\\ M(Z) &= 6(12\eta_{2a} + 3\eta_{2b} + 2\eta_{2c})Z - 3\eta_{1a},\\ Q(Z) &= 12(12\eta_{2a} + 3\eta_{2b} + 2\eta_{2c})Z^2 - 12\eta_{1a}Z + \eta_{0a}. \end{aligned} \quad (3.6)$$

If we set

$$\begin{aligned} \eta &= (3\eta_{2a} + \eta_{2b} + \eta_{2c}),\\ \varrho &= (12\eta_{2a} + 3\eta_{2b} + 2\eta_{2c}), \end{aligned} \quad (3.7)$$

we can reduce the number of the free parameters of the model. This is possible because of the spherical ansatz for the metric, as it is given by Eq. (3.1). In we set $\eta_{1a} = 1$, by choosing a coordinate system in which $c = 1$, we obtain the simplified expressions

$$\begin{aligned} P &= 3\eta, \qquad M(Z) = 6\varrho Z - 3,\\ Q(Z) &= 12\varrho Z^2 - 12Z + \eta_{0a}. \end{aligned} \quad (3.8)$$

We note that the coefficient P is independent from r, while the functions $M(Z)$ and $Q(Z)$ do not depend explicitly on the radius r. The Euler-Lagrange equations for the action (3.4) are

$$\frac{d}{dr}\left(\frac{\partial L}{\partial N'}\right) - \frac{\partial L}{\partial N} = 0, \qquad \frac{d}{dr}\left(\frac{\partial L}{\partial Z'}\right) - \frac{\partial L}{\partial Z} = 0. \quad (3.9)$$

The first equation of motion, the one for the function $Z(r)$, reads

$$P\left(r\frac{dZ}{dr}\right)^2 + M(Z)\left(r\frac{dZ}{dr}\right) + Q(Z) = 0. \quad (3.10)$$

If we algebraically solve the above equation we obtain the first order differential equations

$$r\frac{dZ}{dr} = H(Z), \quad (3.11)$$

where $H(Z)$ are the solutions of second order algebraic equation (3.10). For $P \neq 0$:

$$H(Z) = \frac{-M(Z) + \sigma\sqrt{M(Z)^2 - 4PQ(Z)}}{2P}, \quad (3.12)$$

where σ is a sign. For $P = 0$ we have only one solution:

$$H(Z) = -\frac{Q(Z)}{M(Z)}. \quad (3.13)$$

Now we can derive the equation of motion for the function $N(r)$. If we set

$$\bar{N}(r) = \sqrt{\frac{N(r)^2}{f(r)}}, \quad (3.14)$$

we obtain from the second Euler-Lagrange equation in (3.9):

$$\begin{aligned} &\frac{d\bar{N}(r)}{dr} + \bar{C}(r)\bar{N}(r) = 0,\\ &\bar{C}(r) = \left[\frac{1}{r^4 G_1}\frac{d(r^4 G_1)}{dr} - \frac{G_2}{rG_1}\right], \end{aligned} \quad (3.15)$$

where

$$G_1 = 2P\left(r\frac{dZ}{dr}\right) + M(Z), \quad (3.16)$$

$$G_2 = M'(Z)\left(r\frac{dZ}{dr}\right) + Q'(Z). \quad (3.17)$$

Changing variables from r to Z, Eq. (3.15) becomes

$$\begin{aligned} &\frac{d\tilde{N}(Z)}{dZ} + \tilde{C}(Z)\tilde{N}(Z) = 0,\\ &\tilde{C}(Z) = \frac{1}{H(Z)}\left[4 - \frac{\tilde{G}_2}{\tilde{G}_1}\right] + \frac{1}{\tilde{G}_1}\frac{d\tilde{G}_1}{dZ}, \end{aligned} \quad (3.18)$$

where

$$\begin{aligned} \tilde{G}_1(Z) &= 2PH(Z) + M(Z),\\ \tilde{G}_2(Z) &= M'(Z)H(Z) + Q'(Z). \end{aligned} \quad (3.19)$$

Finally, we emphasize that the parameter λ does not appear in the equations of motion, so they only depend on three parameters: η, ϱ, and the cosmological constant η_{0a}. This is similar to the 4D case [15]. The reason is that we are looking for static solutions, hence extrinsic curvature terms do not contribute to the equations of motion, as they contain only time derivatives of the metric components. In addition, note that the parameter λ appears only in the extrinsic curvature part of the action. Therefore, the solutions we will obtain in the following sections will be valid for arbitrary values of λ.

IV. SPHERICALLY SYMMETRIC SOLUTIONS, SPECIAL CASES

In this section we study static spherically symmetric solutions for three special cases of the free coupling parameters η and ϱ: (a) $\eta = 0$ and $\varrho = 0$, (b) $\eta = 0$ and $\varrho \neq 0$, (c) $\varrho = 0$ and $\eta \neq 0$.

A. No quadratic terms, $\eta = 0$ and $\varrho = 0$

If we set $\eta = \varrho = 0$ in Eq. (3.10) we find

$$r\frac{dZ}{dr} = \frac{\eta_{0a}}{3} - 4Z, \tag{4.1}$$

from which it follows that

$$-3Z + \frac{\eta_{0a}}{4} + \frac{\tilde{C}_\mu}{r^4} = 0, \tag{4.2}$$

or equivalently we take the simple solution

$$f(r) = k + r^2 Z = k + \frac{\eta_{0a}}{12} r^2 + \frac{\tilde{C}_\mu}{3r^2}, \tag{4.3}$$

where $\tilde{C}_\mu$ is a constant of integration. If we set

$$\Lambda_{\rm eff} = -\eta_{0a}, \qquad \mu = -\frac{\tilde{C}_\mu}{3}, \tag{4.4}$$

the above equation takes the well-known form

$$f(r) = k - \frac{\Lambda_{\rm eff}}{12} r^2 - \frac{\mu}{r^2}, \tag{4.5}$$

which is the standard AdS_5 (for $\Lambda_{\rm eff} < 0$) or dS_5 (for $\Lambda_{\rm eff} > 0$) or the asymptotically flat (for $\Lambda_{\rm eff} = 0$) Schwarzschild black hole solution of 5D general relativity with a cosmological constant.

B. $\eta = 0$ and $\varrho \neq 0$

If $\eta = 0$ Eq. (3.8) implies $P = 0$, hence Eq. (3.10) can be written as

$$r\frac{dZ}{dr} = -\frac{Q(Z)}{M(Z)}. \tag{4.6}$$

In this case the function $Q(Z)$ and $M(Z)$ can be written as

$$M(Z) = -3 + 6\varrho Z, \tag{4.7}$$

$$Q(Z) = \eta_{0a} - 12Z + 12\varrho Z^2, \tag{4.8}$$

and Eq. (4.6) becomes:

$$r\frac{dZ}{dr} = -\frac{\eta_{0a} - 12Z + 12\varrho Z^2}{-3 + 6\varrho Z}. \tag{4.9}$$

Integration of this equation yields:

$$3\varrho Z^2 - 3Z + \frac{\eta_{0a}}{4} + \frac{\tilde{C}_\mu}{r^4} = 0, \tag{4.10}$$

where $\tilde{C}_\mu$ is an integration constant which is related to the mass of the black hole. The algebraic equation (4.10) gives two solutions

$$Z(r) = \frac{1}{2\varrho} + \sigma\frac{\sqrt{3(3 - \varrho\eta_{0a})r^4 - 12\varrho\tilde{C}_\mu}}{6\varrho r^2}, \tag{4.11}$$

where σ is a sign ($\sigma = \pm 1$), so for the function $f(r) = k + r^2 Z$ we obtain

$$f(r) = k + \frac{r^2}{2\varrho}\left[1 + \sigma\sqrt{\left(1 - \frac{\varrho\eta_{0a}}{3}\right) - \frac{4\varrho\tilde{C}_\mu}{3r^4}}\right]. \tag{4.12}$$

In what follows we will assume that $\varrho\tilde{C}_\mu < 0$, because for $\varrho\tilde{C}_\mu > 0$ the range of radius r has a lower bound ($r > r_{\rm min}$). This case will be discussed further in Sec. V B.

From the above equation (4.12) we can extract the large distance asymptotic behavior for $f(r)$, which reads

$$f(r) = k + \frac{1}{2\varrho}\left(1 + \frac{\sigma}{\sqrt{3}}\sqrt{3 - \varrho n_{0a}}\right)r^2 - \frac{\sigma}{\sqrt{3}} \times \frac{\tilde{C}_\mu}{\sqrt{3 - \varrho n_{0a}}r^2} + O\left(\frac{1}{r^6}\right). \tag{4.13}$$

Note that $f(r)$ has a large distance limit only if $3 > \varrho n_{0a}$, or else there is an upper bound for the radius r. The asymptotic formula (4.13) is of the form

$$f(r) \simeq k - \frac{\Lambda_{\rm eff}}{12} r^2 - \frac{\mu}{r^2}, \tag{4.14}$$

with

$$\Lambda_{\rm eff} = -\frac{6}{\varrho}\left[1 + \frac{\sigma}{\sqrt{3}}\sqrt{3 - \varrho n_{0a}}\right], \qquad \mu = \frac{\sigma}{\sqrt{3}}\frac{\tilde{C}_\mu}{\sqrt{3 - \varrho n_{0a}}}. \tag{4.15}$$

Depending on the values of the free parameters ϱ and η_{0a}, the asymptotic behavior is either AdS_5 (for $\Lambda_{\rm eff} < 0$), or dS_5 (for $\Lambda_{\rm eff} > 0$), or flat (for $\Lambda_{\rm eff} = 0$). Also, note that $[\mu] = 2.$ and $\Lambda_{\rm eff}$ is an effective 5D cosmological constant. The mass parameter of the black hole, when $\eta = \varrho = 0$, is $m = (8\pi G_5)^{-1}\Lambda_{\rm eff}^{-2}\mu$.

The Euler-Lagrange equations for $N(r)$ yield

$$\frac{d\tilde{N}(Z)}{dZ} + \tilde{C}(Z)\tilde{N}(Z) = 0, \qquad \tilde{C}(Z) = \frac{1}{H(Z)}\left[4 - \frac{\tilde{G}_2}{\tilde{G}_1}\right] + \frac{1}{\tilde{G}_1}\frac{d\tilde{G}_1}{dZ}, \tag{4.16}$$

where

$$H(Z) = -\frac{Q(Z)}{M(Z)}, \qquad \tilde{G}_1 = M(Z), \qquad \tilde{G}_2 = -\frac{Q(Z)M'(Z)}{M(Z)} + Q'(Z).$$

Then we obtain

$$\tilde{C}(Z) = \frac{Q'(Z) - 4M(Z)}{Q(Z)} = 0,$$

where we have taken into account Eqs. (4.7) and (4.8) for $M(Z)$ and $Q(Z)$. Finally, we find

$$N(r)^2 = f(r). \tag{4.17}$$

C. Comparing with the 5D Gauss-Bonnet gravity

It is worth noting that the spherically symmetric solutions we obtained in the previous section, for $\eta = 0$ and $\varrho \neq 0$, are identical with the corresponding solutions of the 5D relativistic GB gravity. The action of the GB gravity is given by:

$$S = \frac{1}{16\pi G_5}\int d^D x\sqrt{|g^{(D)}|}\{R^{(D)} - 2\Lambda + \hat{a}\,\hat{G}\}, \quad D = 5, \tag{4.18}$$

where the GB density $\hat{G}$ is

$$\hat{G} = R^{(5)abcd}R^{(5)}_{abcd} - 4R^{(5)ab}R^{(5)}_{ab} + (R^{(5)})^2, \quad a, b, c, d = 0, 1, \ldots 4. \tag{4.19}$$

Note that in the GB gravity, the definition of the symbols $R^{(5)}$, $R^{(5)ab}$, and $R^{(5)abcd}$ is based on the relativistic 5D metric $g^{(5)}_{ab}$, where a, b, c, $d = 0, 1, \ldots 4$. In addition, $\hat{a}$ is the Gauss-Bonnet coupling and Λ is the 5D cosmological constant. The static spherically symmetric solutions of the GB gravity in AdS space, for $D = 5$, are of the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_k^2, \tag{4.20}$$

and

$$f(r) = k + \frac{r^2}{4\hat{a}}\left[1 + \sigma\sqrt{1 - 8\hat{a}n^2 + 8\frac{a\mu_g}{r^4}}\right], \quad \sigma = \pm 1, \tag{4.21}$$

in which μ_g is a constant of integration which is related with the mass of the black hole, and the parameter $n^2 = -2\Lambda$ corresponds to a negative bulk cosmological constant.

If we replace

$$\hat{a} \to \frac{\varrho}{2}, \qquad n^2 \to \frac{\eta_{0a}}{12}, \qquad \mu_g \to -\frac{C_\mu}{3}, \tag{4.22}$$

in Eq. (4.21), we recover the black hole solution of Eq. (4.12) for the specific case $\eta = 0$ and $\varrho \neq 0$ of the previous section.

Note, that the condition $\eta = 3\eta_{2a} + \eta_{2b} + \eta_{2c} = 0$ is satisfied in the case of GB coefficients $\eta_{2a} = \hat{a}$, $\eta_{2b} = -4\hat{a}$, and $\eta_{2c} = \hat{a}$, but there are other different combinations of the coupling parameters η_{2a}, η_{2b}, η_{2c} which give $\eta = 0$ and $\varrho \neq 0$. This is a very interesting result which merits further investigation. Note also that the relation $1 - \frac{\varrho\eta_{0a}}{3} = 0$ corresponds to the Chern-Simons limit of GB gravity.

D. $\varrho = 0$ and $\eta \neq 0$

If we assume that $\varrho = 0$, Eqs. (3.8) yield

$$P = 3\eta, \qquad M(Z) = -3, \qquad Q(Z) = -12Z + \eta_{0a}. \tag{4.23}$$

By solving Eq. (3.10) we obtain the following two solutions:

$$r\frac{dZ}{dr} = \frac{3 - \sigma\sqrt{9 - 12\eta\eta_{0a} + 144\eta Z}}{6\eta}, \tag{4.24}$$

with $\sigma = \pm 1$. The above differential equation can be integrated analytically, to give

$$\left(\frac{\sigma}{3}\sqrt{9 - 12\eta\eta_{0a} + 144\eta Z} - 1\right)e^{[(\sigma/3)\sqrt{9-12\eta\eta_{0a}+144\eta Z}-1]} = \frac{\tilde{C}_\mu}{r^4}, \tag{4.25}$$

where $\tilde{C}_\mu$ is an integration constant.

In this section we are interested only in solutions with $\sigma = 1$ and $\tilde{C}_\mu > 0$. The other cases lack a short distance limit (for details see Appendix A) and hence their interpretation is problematic, as we explain in Sec. V B below.

From Eq. (4.25) we obtain

$$\frac{1}{3}\sqrt{9 - 12\eta\eta_{0a} + 144\eta Z} - 1 = W_L\left(\frac{\tilde{C}_\mu}{r^4}\right), \tag{4.26}$$

Note that $W_L(x)$ is the Lambert function, which is defined as the real solution of the equation $e^{W_L(x)}W_L(x) = x$. We recall some of the properties of this equation: (a) for $x < -1/e$ it has no real solutions, (b) for $-1/e \leq x < 0$ it has two real solutions and, (c) for $x \geq 0$ it has a unique real solution. In the case (b) we define the function $W_L(x)$ by demanding that $-1 \leq W_L(x) < 0$ [the other set of solutions, which lies in the range $(-\infty, -1)$, is not considered].

Now, from Eq. (4.26) we find that

$$Z(r) = \frac{\eta_{0a}}{12} + \frac{1}{16\eta}\left(W_L^2\left(\frac{\tilde{C}_\mu}{r^4}\right) + 2W_L\left(\frac{\tilde{C}_\mu}{r^4}\right)\right), \tag{4.27}$$

so we find for the function $f(r) = k + r^2 Z(r)$:

$$f(r) = k + \frac{\eta_{0a}}{12}r^2 + \frac{r^2}{16\eta}\left(W_L^2\left(\frac{\tilde{C}_\mu}{r^4}\right) + 2W_L\left(\frac{\tilde{C}_\mu}{r^4}\right)\right). \tag{4.28}$$

The large r asymptotic behavior of Eq. (4.28) is found to be

$$f(r) = k + \frac{\eta_{0a}}{12}r^2 + \frac{\tilde{C}_\mu}{8\eta r^2} + O\left(\frac{1}{r^6}\right), \tag{4.29}$$

which is of the standard form

$$f(r) \simeq k - \frac{\Lambda_{\rm eff}}{12} r^2 - \frac{\mu}{r^2}, \qquad \Lambda_{\rm eff} = -\eta_{0a}, \qquad \mu = -\frac{\tilde{C}_\mu}{8\eta}. \tag{4.30}$$

Now, Eqs. (3.8), (3.18), and (3.19) yield

$$\tilde{C}(Z) = \frac{72\eta}{9 - 12\eta\eta_{0a} + 144\eta Z} - \sigma \frac{24\eta}{\sqrt{9 - 12\eta\eta_{0a} + 144\eta Z}}, \tag{4.31}$$

where the function $\tilde{N}(Z)$ satisfies the equation

$$\frac{d\tilde{N}(Z)}{dZ} + \tilde{C}(Z)\tilde{N}(Z) = 0. \tag{4.32}$$

Solving this equation and choosing the constant of integration appropriately, we get

$$\tilde{N}(Z) = \frac{3e^{[(\sigma/3)\sqrt{9-12\eta\eta_{0a}+144\eta Z}-1]}}{\sqrt{9 - 12\eta\eta_{0a} + 144\eta Z}}. \tag{4.33}$$

Note that for $\sigma = 1$ and $\tilde{C}_\mu > 0$, if we take into account Eqs. (4.26) and (4.33), the function $N(r)^2$ can be expressed in the following closed form:

$$N(r)^2 = f(r)\tilde{N}(Z(r))^2 = \frac{\tilde{C}_\mu^2 f(r)}{r^8 (W_L^2(\frac{\tilde{C}_\mu}{r^4}) + W_L(\frac{\tilde{C}_\mu}{r^4}))^2}. \tag{4.34}$$

In the large r regime we find, from the above equation, that

$$N(r)^2 = f(r)\left(1 + \frac{\tilde{C}_\mu^2}{r^8} + O\left(\frac{\tilde{C}_\mu^3}{r^{12}}\right)\right), \tag{4.35}$$

hence in the large distance limit we recover the standard asymptotic behavior $N(r)^2 \simeq f(r)$.

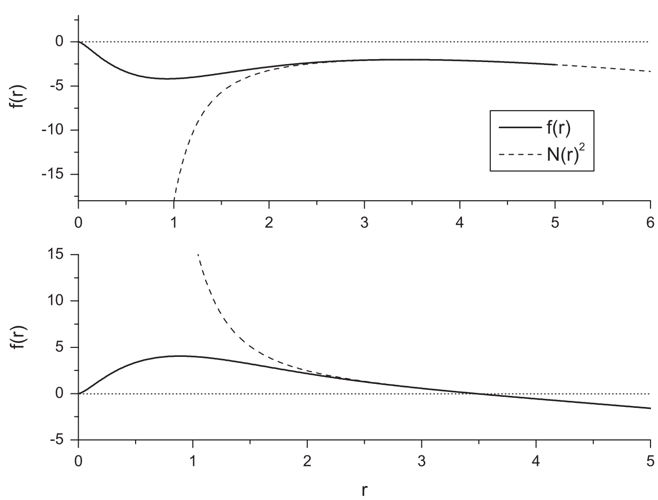

FIG. 1. A typical plot of $f(r)$ and $N(r)^2$ versus r, in the case of $\varrho = 0$ and $\eta \neq 0$, for $k = 0$, $\eta_{0a} = -1$ (positive cosmological constant), $\tilde{C}_\mu = 10$, $\eta = -0.1$ (top) and $\eta = +0.1$ (bottom).

In Fig. 1 we give a typical plot of the functions $f(r)$ and $N(r)^2$ for zero spatial curvature $k = 0$ and positive effective cosmological constant $\Lambda_{\rm eff} = 1$. We see that $f(r)$ is finite [in particular $f(0) = 0$] but $N(r)^2$ blows up at the singularity $r_s = 0$. We also see that only when the coupling η is positive there exists a horizon, while for negative η there is a naked singularity. In the case of nonzero spatial curvature the horizon r_h is determined by solving equation $f(r_h) = -k$, ($k = \pm 1$). It is possible to see by inspection of Fig. 1, that for $k = \pm 1$, we have the same situation which holds for zero spatial curvature. Finally, we observe that $f(r)$ and $N(r)^2$ tend rapidly to their common asymptotic behavior (dS_5), as it is expected from the analysis above [see Eqs. (4.29) and (4.35)].

V. STATIC SOLUTIONS IN THE GENERIC CASE ($\eta \neq 0$ AND $\varrho \neq 0$)

In the generic case, $\eta \neq 0$ and $\varrho \neq 0$, we obtain from Eqs. (3.8), (3.11), and (3.12)

$$r\frac{dZ}{dr} = \frac{3 - 6\varrho Z + \sigma\sqrt{9 - 12\eta_{0a}\eta + 36(\varrho - 4\eta)(\varrho Z^2 - Z)}}{6\eta}. \tag{5.1}$$

If we make the replacement

$$Z = \frac{1}{2\varrho} - \frac{y}{3} \tag{5.2}$$

in Eq. (5.1), we find

$$r\frac{dy}{dr} = \tilde{H}(y), \qquad \tilde{H}(y) = -\frac{1}{\eta}(\varrho y + \sigma\sqrt{A + By^2}), \tag{5.3}$$

where the new parameters A and B are defined through

$$A \equiv -3\eta_{0a}\eta + \frac{9\eta}{\varrho}, \qquad B \equiv \varrho(\varrho - 4\eta). \tag{5.4}$$

For the computation of the function $N(r)$, Eqs. (3.8), (3.18), and (3.19) yield

$$\frac{d\tilde{N}(y)}{dy} - \tilde{C}(y)\tilde{N}(y) = 0, \tag{5.5}$$

where $\tilde{C}(y)$ is given by the equation

$$\tilde{C}(y) = -\frac{B}{\varrho}\frac{\varrho y + \sigma\sqrt{A + By^2}}{A + By^2}. \tag{5.6}$$

In the following sections we analyze two cases for the parameter B: (1) $B \geq 0$, (2) $B < 0$. The mathematical details for the derivation of the final formulas for $f(r)$ and $N(r)$ are given in Appendix B.

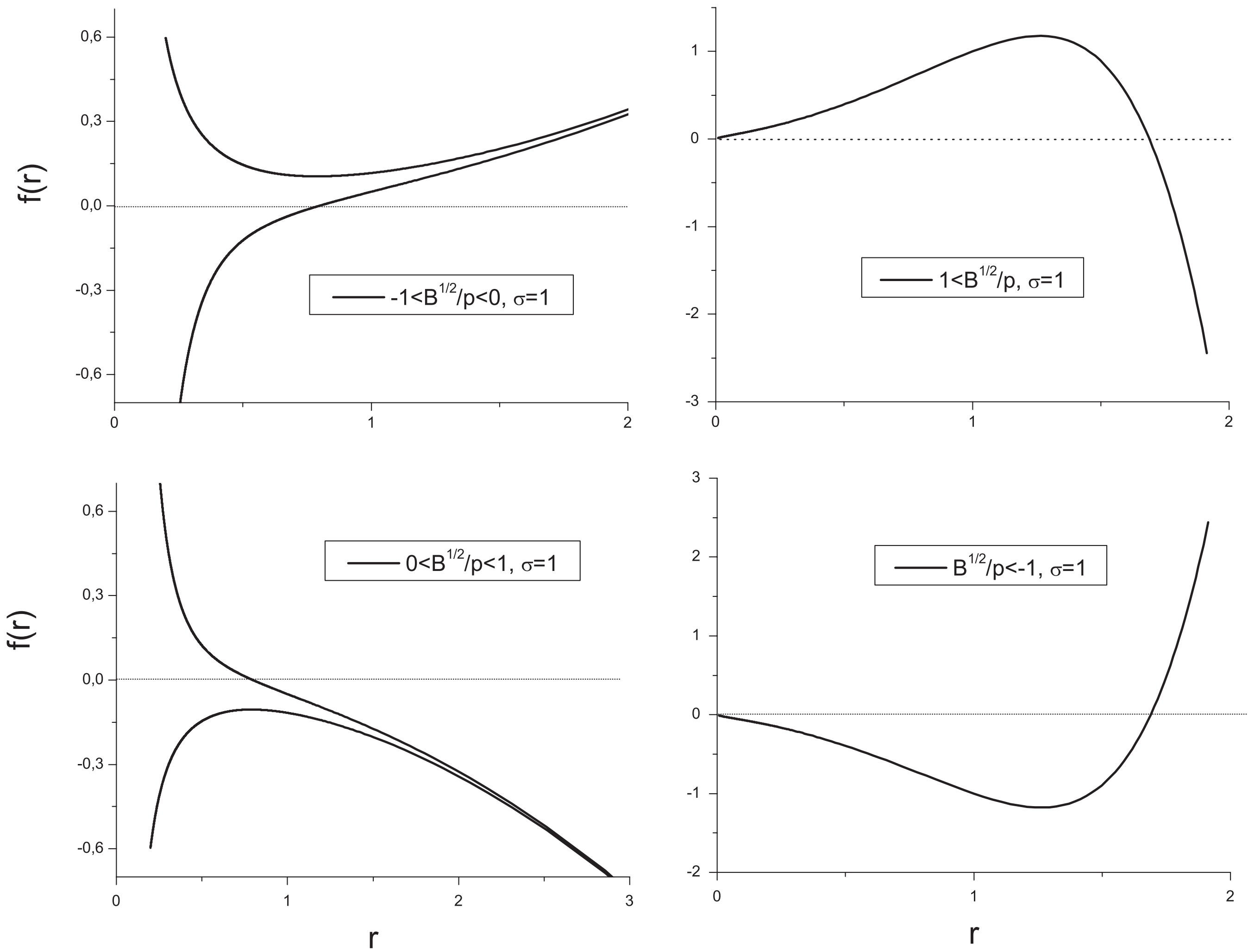

FIG. 2. A typical plot of $f(r)$ versus r, in the case of $k = 0$, $A = 1$, $B = 1$, $\tilde{C}_\mu = 1$, $\sigma = 1$. In the left part of the figure the parameter ϱ equals -10.0 (top), or $+10.0$ (bottom), while for the right part of the figure $\varrho = +0.5$ (top) and $\varrho = -0.5$ (bottom).

A. $B \geq 0$

By integrating Eq. (5.3) for $B \geq 0$ we obtain

$$|\varrho y + \sigma\sqrt{A + By^2}||\sqrt{B}y + \sigma\sqrt{A + By^2}|^{-(\sqrt{B}/\varrho)} = \frac{\tilde{C}_\mu}{r^4}. \quad (5.7)$$

Note, that in this case the parameter A can be positive, negative, or zero. For the function $\tilde{N}$, we obtain from Eqs. (5.6) and (5.5)

$$\tilde{N}(y) = \frac{\tilde{C}_N}{\sqrt{A + By^2}|\sqrt{B}y + \sigma\sqrt{A + By^2}|^{\sqrt{B}/\varrho}}, \quad (5.8)$$

where $\tilde{C}_\mu$ and $\tilde{C}_N$ are constants of integration. Note, that $\tilde{C}_\mu$ must be always positive ($\tilde{C}_\mu > 0$). Also, the constant $\tilde{C}_N$ is fixed if we demand $\tilde{N} \to 1$ for $r \to +\infty$, as we will discuss later in this section. An alternative expression for $\tilde{N}$ can be found if we take into account Eqs. (5.7) and (5.8). In particular, we obtain that

$$\tilde{N}(y) = \frac{\bar{C}_N}{r^4\sqrt{A + By^2}|\varrho y + \sigma\sqrt{A + By^2}|}, \quad (5.9)$$

$$\bar{C}_N = \tilde{C}_N\tilde{C}_\mu.$$

For $B > 0$, two cases for the ratio $\sqrt{B}/\varrho$ will be discussed[3]: (a) $|\sqrt{B}/\varrho| < 1$ and (b) $|\sqrt{B}/\varrho| > 1$. The case $B = 0$ will also be examined separately in Sec. V C.

1. $|\sqrt{B}/\varrho| < 1$ *and* $A > 0$

For $|\sqrt{B}/\varrho| < 1$ and $A > 0$, we can verify that Eq. (5.7) above has two solutions for y (y_1 and y_2) for a given value of the radius r in the range $[0, +\infty)$. Hence, the function $f(r)$ has two branches $f_1(r)$ and $f_2(r)$, which are exhibited in the left part of Fig. 2, when $\sigma = 1$ for two typical values of ϱ ($\varrho = \pm 10$). However, only one of them has a horizon for zero spatial curvature ($k = 0$), while the other repre-

[3]Note that if $|\sqrt{B}/\varrho| = 1$ we have no solution at all.

sents a spherically symmetric solution with a naked singularity. In addition, as we see in Fig. 2, for $\varrho > 0$ the black hole solution is AdS_5 asymptotically, while for $\varrho < 0$ it is dS_5 (note that ϱ should be nonzero in the case we examine here).

In what follows, we will assume that $A > 0$, as for negative A the solutions have no large distance limit[4] when $|\sqrt{B}/\varrho| < 1$, hence they will not be examined here.

In particular for $|\sqrt{B}/\varrho| < 1$, if we take into account Eq. (5.7), we obtain

(i) $r \to +\infty \Rightarrow y \to y_0$ (large distance asymptotic behavior)

(ii) $r \to 0 \Rightarrow y \to \pm\infty$ (short distance asymptotic behavior)

where the plus and minus signs above correspond to the two branches of the function $f(r)$. In addition,

$$y_0 = -\mathrm{sgn}\left(\frac{\sigma}{\varrho}\right)\sqrt{\frac{A}{\varrho^2 - B}} \tag{5.10}$$

is the unique solution of the following equation:

$$\varrho y_0 + \sigma\sqrt{A + By_0^2} = 0. \tag{5.11}$$

If we expand Eq. (5.7) around y_0 we can find the large distance asymptotic behavior for $y(r)$ which reads

$$y(r) \simeq y_0 \mp \frac{3\mu}{r^4} + O\left(\frac{1}{r^8}\right), \quad \mu = |\varrho|\frac{|y_0(\sqrt{B} - \varrho)|^{\sqrt{B}/\varrho}}{3(\varrho^2 - B)}\tilde{C}_\mu, \tag{5.12}$$

hence

$$f(r) = k + r^2 Z = k + r^2\left(\frac{1}{2\varrho} - \frac{y}{3}\right) \simeq f(r) \simeq k + \left(\frac{1}{2\varrho} - \frac{y_0}{3}\right)r^2 \pm \frac{\mu}{r^2} + O\left(\frac{1}{r^6}\right), \tag{5.13}$$

which has the standard asymptotic behavior, AdS_5, dS_5, or flat, depending on the values of the free parameters of the model. The plus and minus signs in the above asymptotic formulas give rise to the two branches of the function $f(r)$.

Now, for the function $\tilde{N}(y)$, if take into account Eqs. (5.9) and (5.12), we obtain the following large distance asymptotic behavior:

$$\tilde{N}(y(r)) \simeq 1 + O\left(\frac{1}{r^8}\right), \tag{5.14}$$

where the constant of integration $\tilde{C}_N$ has been set to

$$\tilde{C}_N = -\sigma\varrho y_0|y_0(\sqrt{B} - \varrho)|^{\sqrt{B}/\varrho}, \tag{5.15}$$

in order to satisfy the condition $\tilde{N}(y_0) = 1$.

[4]For $A < 0$ and $|\sqrt{B}/\varrho| < 1$ the left-hand side of Eq. (5.7) is never zero, so the radius r has an upper bound.

The asymptotic behavior of Eq. (5.14) is verified graphically in Fig. 3. We observe that the function $N(r)^2$ tends rapidly to its asymptotic behavior which is identical with that of $f(r)$, as it is given by Eq. (5.13). Also, we would like to note, that the above analysis is valid only when $A > 0$. For negative A we see that the left -hand side of Eq. (5.7) cannot vanish [see Eqs. (5.10) and (5.11)], so the radius r has an upper bound. We conclude that this class of solutions lacks physical interest, since the function $f(r)$ has no large distance limit, so it will not be examined here. The case $A = 0$ is also examined separately in Sec. VA 4.

2. $|\sqrt{B}/\varrho| > 1$ *and* $A > 0$

In the right part of Fig. 2 we have plotted the function $f(r)$ when $|\sqrt{B}/\varrho| > 1$ ($\sigma = 1$, $\varrho = \pm 0.5$). As we see, in contrast with the case $|\sqrt{B}/\varrho| < 1$, now there is only one branch for the function $f(r)$. However, there is an even more significant difference, as this class of solutions does not exhibit the standard AdS_5, dS_5, or flat asymptotic behavior, as it is shown in the following analysis. For negative σ ($\sigma = -1$) the behavior of $f(r)$ does not change significantly, so we do not give a figure for reasons of space. The asymptotic behavior of $f(r)$, for negative and positive σ, is described in Eq. (5.17) below, and $f(r)$ obeys the same power law for both values of σ.

Although we will assume that $A > 0$, there are also solutions for negative A in the case $|\sqrt{B}/\varrho| > 1$. These solutions do not have any new features, as we comment in Sec. VA 3.

For $|\sqrt{B}/\varrho| > 1$ and $A > 0$, from Eq. (5.7) above, we obtain

(i) $r \to +\infty \Rightarrow y \to \epsilon(+\infty)$ (large distance asymptotic behavior)

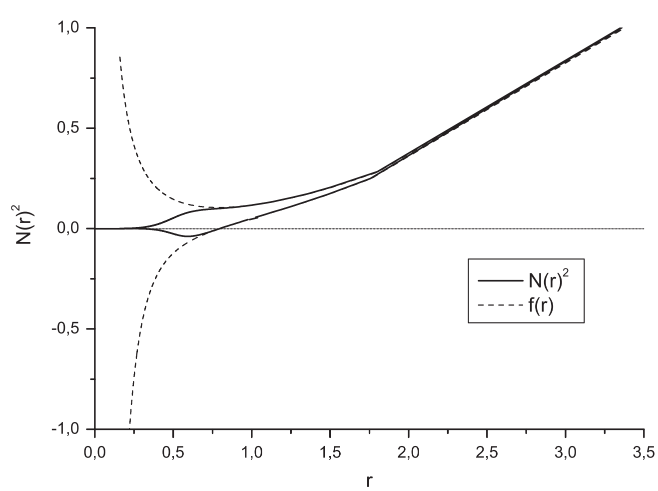

FIG. 3. A typical plot of the two branches of $N(r)^2$ versus r, in the case of $B > 0$ and $|\sqrt{B}/\varrho| < 1$, for $k = 0$, $A = 1$, $B = 1$, $\tilde{C}_\mu = 1$, $\sigma = 1$, and $\varrho = 10$. We observe that $N(r)^2 \simeq f(r)$ when $r \to +\infty$, as expected.

(ii) $r \rightarrow 0 \Rightarrow y \rightarrow -\epsilon(+\infty)$ (short distance asymptotic behavior)

where ϵ is a sign defined as $\epsilon = \mathrm{sgn}(\varrho\sigma)$. Note, that the left hand side of Eq. (5.7) does not vanish for a finite value of y ($y = y_0$), the way it did in the previous section.

For the large distance asymptotic behavior of $y(r)$, if we take into account Eq. (5.7), we find

$$y \simeq C_\epsilon r^{(4|\varrho|)/(\sqrt{B}-|\varrho|)}, \tag{5.16}$$

where the form of the coefficient C_ϵ depends on the sign of the parameter $\epsilon = \mathrm{sgn}(\varrho\sigma)$, according to the equations:

(i) $C_\epsilon = (\frac{1}{2\sqrt{B}})^{\sqrt{B}/(\sqrt{B}-|\varrho|)}(\frac{\sqrt{B}+|\varrho|}{\tilde{C}_\mu})^{|\varrho|/(\sqrt{B}-|\varrho|)}$, if $\epsilon = 1$

(ii) $C_\epsilon = -(\frac{2\sqrt{B}}{A})^{\sqrt{B}/(\sqrt{B}-|\varrho|)}(\frac{\sqrt{B}+|\varrho|}{\tilde{C}_\mu})^{|\varrho|/(\sqrt{B}-|\varrho|)}$, if $\epsilon = -1$.

The leading terms for $f(r)$ are

$$f(r) \simeq k + \frac{r^2}{2\varrho} - \frac{c_\epsilon r^2}{3} r^{(4|\varrho|)/(\sqrt{B}-|\varrho|)}, \tag{5.17}$$

which is not the standard asymptotic behavior for a 5D black hole solution, since it is proportional to $r^{2+\delta}$, $\delta \equiv \frac{4|\varrho|}{\sqrt{B}-|\varrho|} > 0$, rather than r^2. The large distance asymptotic behavior for the modified lapse function $\tilde{N}(r)$, if we take into account Eqs. (5.9) and (5.16), is

$$\tilde{N}(r) \simeq \frac{\bar{C}_N}{C_\epsilon\sqrt{B}|\varrho + \mathrm{sgn}(\varrho)\sqrt{B}|} r^{-4(\sqrt{B}/(\sqrt{B}-|\varrho|))}. \tag{5.18}$$

We see that $\tilde{N}(r) \rightarrow 0$ for large r, which implies a violation of the common large distance asymptotic behavior $N(r)^2 \simeq f(r)$. Finally, from the above equations (5.17) and (5.18) we can determine the leading term for the lapse function $N(r)$, which reads

$$N(r) \simeq -\frac{\bar{C}_N}{3\sqrt{B}|\varrho + \mathrm{sgn}(\varrho)\sqrt{B}|} r^{(-6\sqrt{B}+2|\varrho|)/(\sqrt{B}-|\varrho|)}. \tag{5.19}$$

We observe that also the lapse function $N(r)$ vanishes for large values of r.

3. $|\sqrt{B}/\varrho| > 1$ *and* $A < 0$

In fact, the solutions for $|\sqrt{B}/\varrho| > 1$ and $A < 0$, are a mixture of solutions that were presented is the previous two sections. More specifically every solution consists of two branches. One of them has exactly the same short and large distance asymptotic behavior with the solutions of Sec. VA 1. This is mainly due to the fact that the left hand side of Eq. (5.7) vanishes for a finite value of y. The other branch is similar to the solutions of Sec. VA 2 but it lacks a short distance limit. We will not discuss these cases further since their behavior is similar to the solutions we have already discussed.

4. $B > 0$ *and* $A = 0$

If we set[5] $A = 0$ (or $\eta_{0a}\varrho = 3$) in Eq. (5.7), we find

$$y(r) = C_0 r^{(4\varrho)/(\sqrt{B}-\varrho)},$$
$$C_0 = \sigma\left(\frac{1}{2\sqrt{B}}\right)^{\sqrt{B}/(\sqrt{B}-\varrho)}\left(\frac{\varrho + \sqrt{B}}{\tilde{C}_\mu}\right)^{\varrho/(\sqrt{B}-\varrho)}, \tag{5.20}$$

so we obtain for the functions $f(r)$ and $\hat{N}(r)$:

$$f(r) = k + \frac{r^2}{2\varrho} - \frac{C_0}{3} r^{2((\sqrt{B}+\varrho)/(\sqrt{B}-\varrho))}, \tag{5.21}$$

and

$$\hat{N}(r) = \frac{\bar{C}_N}{C_0^2\sqrt{B}|\varrho + \sqrt{B}|} r^{-4((\sqrt{B}+\varrho)/(\sqrt{B}-\varrho))}. \tag{5.22}$$

As we see the above solutions exhibit an unconventional asymptotic behavior which is not of the type AdS_5, dS_5, or flat 5D Schwarzschild form.

B. $B < 0$

For $B < 0$, if we integrate Eq. (5.3), we find

$$|\varrho y + \sigma\sqrt{A - |B|y^2}|e^{((\sigma\sqrt{|B|})/\varrho)\tan^{-1}((\sqrt{|B|}y)/(\sqrt{A-|B|y^2}))} = \frac{\tilde{C}_\mu}{r^4}. \tag{5.23}$$

Note that in this case the parameter A should be always positive ($A > 0$). Equations (5.5) and (5.6) yield

$$\tilde{N}(y) = \frac{\tilde{C}_N}{\sqrt{A - |B|y^2}} e^{((\sigma\sqrt{|B|})/\varrho)\tan^{-1}((\sqrt{|B|}y)/(\sqrt{A-|B|y^2}))}. \tag{5.24}$$

Now, if we use Eqs. (5.23) and (5.24) we can get an alternative expression for $\tilde{N}$ according to the equation

$$\tilde{N}(y) = \frac{\bar{C}_N}{r^4\sqrt{A - |B|y^2}|\varrho y + \sigma\sqrt{A - |B|y^2}|}. \tag{5.25}$$

If we take into account the restriction from the square root in Eq. (5.23), we find that y varies in the finite interval $|y| < \sqrt{\frac{A}{|B|}}$.

In Fig. 4 we have plotted the function $f(r)$, for positive σ ($\sigma = 1$), two typical values of ϱ ($\varrho = 5 > 0$ and $\varrho = -5 < 0$) and zero spatial curvature $k = 0$. As we see $f(r)$ consists of two distinct branches $f_1(r)$ and $f_2(r)$, with a dS_5 (for $\varrho < 0$) and AdS_5 (for $\varrho > 0$) large distance asymptotic behaviors, correspondingly. In contrast with the previous case ($B > 0$) the range of radius r terminates at a lower nonzero bound $r_{\min1}$ [for $f_1(r)$] and $r_{\min2}$ [for $f_2(r)$].

[5] We remind the reader, that we can set $A = 0$ only when B is positive or zero.

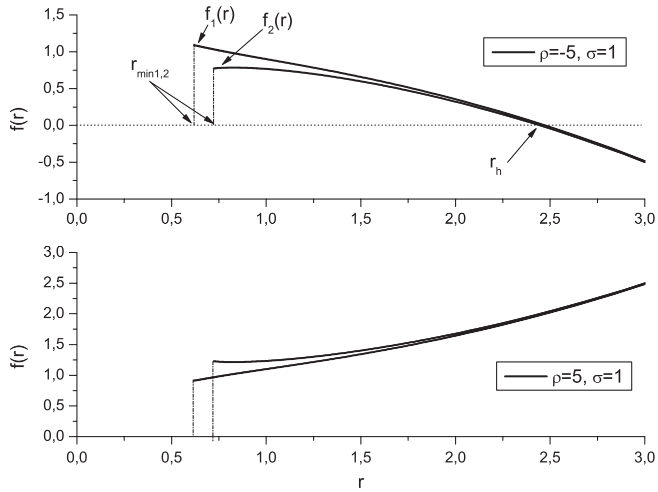

FIG. 4. $f(r)$ versus r, in the case of $B < 0$, for $k = 0$, $A = 1$, $B = -1$, $\tilde{C}_\mu = 1$, $\sigma = 1$, and $\varrho = -5$ (top), $+5$ (bottom).

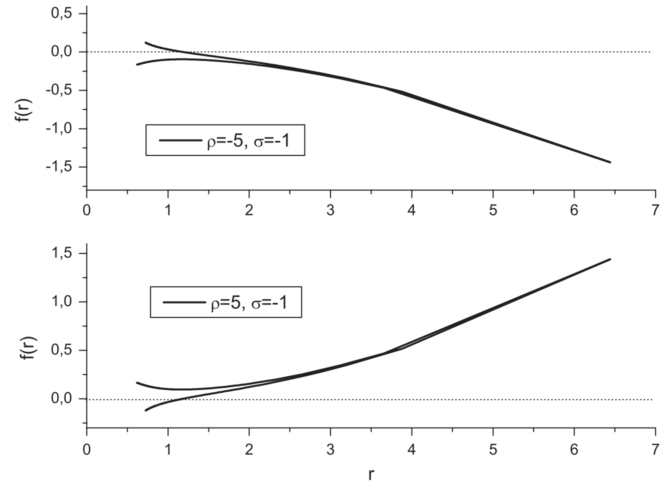

FIG. 6. $f(r)$ versus r, in the case of $B < 0$, for $k = 0$, $A = 1$, $B = -1$, $\tilde{C}_\mu = 1$, $\sigma = -1$, and $\varrho = -5.5$.

For zero spatial curvature, $k = 0$, we observe that for $\varrho < 0$ both the two branches have a horizon, while for $\varrho > 0$ we do not obtain horizons at all. Notice that the lower branch in the bottom of Fig. 4 may have a horizon if $k = -1$. In Fig. 5 we see that the function $N(r)^2/f(r)$ tends to unity as r tends to infinity, but $N(r)^2/f(r)$ blows up when $r \rightarrow r_{\min1,2}$. We observe that in this case there is an infinite set of singularity points which lie on a 4D hypersurface of constant radius $r_{\min1,2}$. (This situation is to be compared with the standard case of a black hole solution where the singularity point is located at the origin.) These solutions may have a physical relevance in the case where the singular shell is protected by a horizon, for example, the branches of $f(r)$ with negative ϱ develop a horizon as we pointed out above.

Also, in Fig. 6, we have plotted the function $f(r)$ for negative σ ($\sigma = -1$) for the same two typical values of ϱ

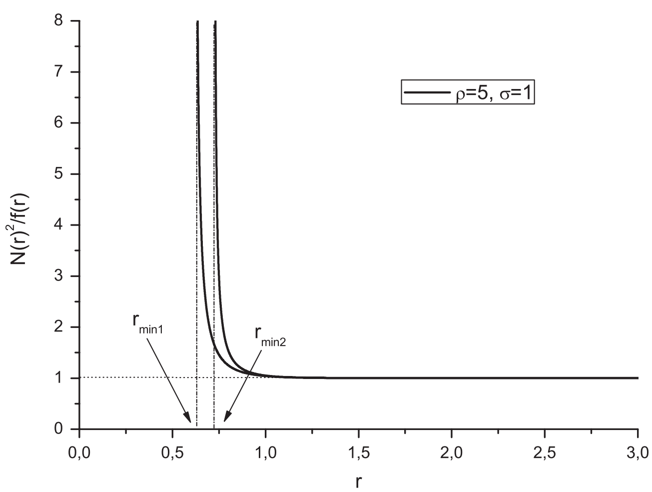

FIG. 5. $N(r)^2/f(r)$ versus r, in the case of $B < 0$, for $k = 0$, $A = 1$, $B = -1$, $\tilde{C}_\mu = 1$, $\sigma = 1$, and $\varrho = +5$.

and $k = 0$. Note that there are no significant differences if we compare with the corresponding figure for $\sigma = 1$ (Fig. 4). However, we see that for negative σ only one of the two branches of $f(r)$ possesses a horizon, while the other exhibits a naked singularity (in fact we have a naked singular shell).

Although in the above mentioned figures we have used specific values for the free parameters, our conclusions are quite general as the values we chose represent a wide range of the parameter space.

In what follows we try to better understand some of the properties of the solutions by using the analytical formulas of Eqs. (5.23) and (5.24) above. First, we would like to stress that the transcendental equation (5.23) gives two solutions for y for a fixed value of the radius r, as it may also be seen in Figs. 4 and 5. In particular the two branches $f_1(r)$ and $f_2(r)$ correspond to the two intervals of the parameter y: (1) $[-\frac{A}{|B|}, y_0]$ and (2) $[y_0, \frac{A}{|B|}]$. Now, if we take into account Eq. (5.23) we find that for large r the function $y(r)$ tends to a constant value, or equivalently that

$$r \longrightarrow +\infty \Rightarrow y \longrightarrow y_0. \tag{5.26}$$

We note that

$$y_0 = -\mathrm{sgn}\left(\frac{\sigma}{\varrho}\right)\sqrt{\frac{A}{\varrho^2 + |B|}} \tag{5.27}$$

is the unique solution of the equation

$$\varrho y_0 + \sigma\sqrt{A - |B| y_0^2} = 0. \tag{5.28}$$

Note, that for negative B ($B < 0$) there is no short distance limit with $r \rightarrow 0$. We find that $r > r_{\min}$, where $r_{\min}$ is a nonzero lower bound for the radius r, which is given by the equation

$$r_{\min} = \left(\tilde{C}_\mu^{-1}|\varrho|\sqrt{\frac{A}{|B|}}\right)^{-1/4} e^{\mp((\sigma\sqrt{|B|}\pi)/(8\varrho))}. \quad (5.29)$$

It turns out that

$$\lim_{r\to r_{\min}} y = \pm\sqrt{\frac{A}{|B|}}. \quad (5.30)$$

The large distance asymptotic behavior for $y(r)$ is obtained by expanding Eq. (5.23) around y_0:

$$y(r) \simeq y_0 \mp \frac{3\mu}{r^4} + O\left(\frac{1}{r^8}\right), \qquad \mu = \frac{|\varrho|\tilde{C}_\mu}{3(\varrho^2 + |B|)} e^{(\sqrt{|B|}/\varrho)\tan^{-1}(\sqrt{|B|}/\varrho)}, \quad (5.31)$$

hence for the function $f(r)$ we find the corresponding asymptotic behavior

$$f(r) \simeq k + \left(\frac{1}{2\varrho} - \frac{y_0}{3}\right)r^2 \pm \frac{\mu}{r^2} + O\left(\frac{1}{r^6}\right), \quad (5.32)$$

which has the standard AdS_5, dS_5, or flat asymptotic behavior, depending on the values of the free parameters of the model.

For the function $\tilde{N}(y(r))$, when the radius tends to infinity, we find

$$\tilde{N}(y(r)) \simeq 1 + O\left(\frac{1}{r^8}\right), \quad (5.33)$$

where the constant of integration $\tilde{C}_N$ has been set to

$$\tilde{C}_N = -\sigma\varrho y_0 e^{-(\sqrt{|B|}/\varrho)\tan^{-1}(|B|/\varrho)}, \quad (5.34)$$

to satisfy the condition $\tilde{N}(y_0) = 1$.

C. $B = 0$

Although the special case of $B = 0$ is included in Sec. VA 1, we present the relevant results separately here, since $f(r)$ can be expressed explicitly as a function of r. If $B = 0$ and $A > 0$, from Eq. (5.1) we obtain

$$r\frac{dy}{dr} = -\frac{1}{\eta}(\varrho y + \sigma\sqrt{A}). \quad (5.35)$$

From $B = \varrho(\varrho - 4\eta) = 0$ and $\varrho \neq 0$, we conclude that $\varrho = 4\eta$, hence the above equation (5.35) can be written as

$$r\frac{dy}{dr} = -\frac{4}{\varrho}(\varrho y + \sigma\sqrt{A}), \quad (5.36)$$

which yields

$$y = -\frac{\sigma\sqrt{A}}{\varrho} \mp \frac{\tilde{C}_\mu}{\varrho r^4}, \quad (5.37)$$

while for the function $f(r)$ we find

$$f(r) = k + \left(\frac{1}{2\varrho} + \frac{\sigma\sqrt{A}}{3\varrho}\right)r^2 \pm \frac{\tilde{C}_\mu}{3\varrho r^2}. \quad (5.38)$$

This has the standard form of a AdS_5, dS_5, or flat solution. The above formulas can also be obtained if we set $B = 0$ in Eqs. (5.10), (5.12), and (5.13), of Sec. VA 1. Finally, from Eqs. (5.5) and (5.6), choosing suitably the integration constant, we find that

$$N(r)^2 = \frac{1}{f(r)}. \quad (5.39)$$

It is interesting to note that (at least the positive branch of) the solution (5.38) coincides, after a proper identification of the parameters, with the black hole solutions found in Lanczos-Lovelock theories [42]. The Lanczos-Lovelock action is a polynomial of degree [D/2] (the integer part of D/2) in the curvature and in 5D it has solutions of the Schwarzschild-AdS type.

VI. CONCLUSIONS AND DISCUSSION

We studied static spherically symmetric solutions in the framework of the 5D Horava-Lifshitz gravity. We considered an action consisting of terms of up to second order in the curvature and we solve the theory with a nonprojectable spherically symmetric ansatz for the metric. The black hole spectrum we found is controlled by three parameters η, ϱ, and η_{0a}, where η_{0a} is a cosmological constant. The black hole solutions we found do not depend on the parameter λ which measures the departure from the Lorentz invariance as it appears only in the extrinsic curvature part of the action.

We presented a full analysis of 5D Horava-Lifshitz static solutions scanning the values of the free parameters η, ϱ, and η_{0a}, which can be positive, negative, or zero. This analysis comes as a generalization and extension of the 4D case studied in [15]. More specifically, we obtained three main sets of solutions: the two special cases ($\eta = 0$, $\varrho \neq 0$) and ($\eta \neq 0$, $\varrho = 0$), and the generic case ($\eta \neq 0$, $\varrho \neq 0$). In all cases we obtained analytic black hole solutions which have the standard AdS_5, dS_5 of flat asymptotic behavior, plus the well-known $1/r^2$ tail. However, we also obtained solutions with an unconventional short and large distance asymptotic behavior. For example, in the generic case, we found solutions with an asymptotic fall-off which is stronger than the AdS_5 or dS_5 asymptotic behavior. We have also found solutions (in the cases with $\eta \neq 0$) in which the asymptotic behavior is the usual one, but the radius has a lower bound $r_{\min}$ different from zero. Also, in many cases we obtained solutions with a naked singularity.

We also found static solutions which, after a proper identification of coupling parameters, coincide with static black hole solutions of relativistic gravity theories with quadratic curvature correction terms. One class of these solutions consists of the Schwarzschild-AdS black hole solutions of five-dimensional Lanczos-Lovelock gravity theories. Another class of solutions contains the well-known Gauss-Bonnet black hole solutions. The interesting result we obtained in our investigation is that the non-relativistic solutions of the HL gravity corresponding to the Gauss-Bonnet solutions can be obtained for various combinations of the coupling parameters η and ϱ and not just the standard Gauss-Bonnet combination. This may be attributed to the fact that the HL static solutions are insensitive to the coupling parameter λ, so they hold even if $\lambda \neq 1$ (the value which signals the breaking of Lorentz invariance).

We do not have a full understanding of the quantum UV structure of the Horava-Lifshitz gravity. The β functions for the UV marginal couplings have not yet been calculated, so the claim that in the IR the $\lambda = 1$ is a fixed point is an assumption. The fact that a class of our solutions coincides with relativistic Gauss-Bonnet black hole solutions is suggesting that in the running of couplings towards the IR, the Gauss-Bonnet regime is reached before the 5D gravity is attained. After all, the Gauss-Bonnet theory is an UV correction of 5D gravity.

Important issues remain to be investigated. One of them is the stability of our solutions. For example, the stability of the class of solutions we found which coincides with the Gauss-Bonnet solutions is an interesting issue to be studied. The stability of the Gauss-Bonnet static solutions has been extensively studied [43]. It was found in [44] that one branch of these solutions suffers from ghostlike instability up to the strongly coupled Chern-Simons limit where linear perturbation theory breaks down. In our case because the Lorentz invariance is broken in the UV this behavior could be different. Therefore, a careful stability analysis is required.

Other interesting issues are the thermodynamic properties of our solutions, or the contribution of terms of higher order in the curvature, which have been omitted in the present work. It would also be interesting to generalize the solutions we presented here in the presence of electric charge.

Finally, one field that our findings can be applied is the extra-dimensional gravity theories, in particular, the brane world models. Note that in contrast to the standard vacuum of Randall-Sundrum [45], the five-dimensional AdS black hole vacuum does not preserve 4D Lorentz invariance on the brane, which may have interesting phenomenological implications [46,47]. For appropriate ranges of the coupling parameters, we have obtained solutions with an AdS_5 large distance asymptotic behavior, plus the standard $1/r^2$ tail of a usual 5D Schwarzschild black hole. These solutions may serve as backgrounds in the framework of brane worlds models, but now there is additional advantage: the starting point is a renormalizable theory such as the 5D HL gravity, in contrast with the 5D General Relativity, which is nonrenormalizable and requires a UV complement.

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APPENDIX A: ASYMPTOTIC BEHAVIOUR ANALYSIS IN THE CASE $\varrho = 0$ AND $\eta \neq 0$

In this appendix we will examine the short and large distance asymptotic behavior of the solution (4.25) in the case $\varrho = 0$ and $\eta \neq 0$. In particular we will examine two cases (I) $\sigma = 1$, and (II) $\sigma = -1$ for the equation

$$\left(\frac{\sigma}{3}\sqrt{9 - 12\eta\eta_{0a} + 144\eta Z} - 1\right)e^{[(\sigma/3)\sqrt{9-12\eta\eta_{0a}+144\eta Z}-1]} = \frac{\tilde{C}_\mu}{r^4}. \tag{A1}$$

As we will see, the only interesting case is that for $\sigma = 1$ and $\tilde{C}_\mu > 0$. The problem in other cases is that $Z(r)$ is not defined on the whole interval $[0, +\infty)$.

1. Case I ($\sigma = 1$)

Although, the case $\sigma = 1$ and $\tilde{C}_\mu > 0$ has been examined in detail in Sec. IV D, we summarize our results here

(i) $r \to +\infty \Rightarrow Z \to \frac{\eta_{0a}}{12}$ (and $Z > \frac{\eta_{0a}}{12}$)

(ii) $r \to 0 \Rightarrow Z \to \mathrm{sgn}(\eta)(+\infty)$.

For $\tilde{C}_\mu > 0$, we see that $Z(r)$ is well defined in the range $[0, +\infty)$.

For $\sigma = 1$ and $\tilde{C}_\mu < 0$ we find

(i) $r \to +\infty \Rightarrow Z \to \frac{\eta_{0a}}{12}$ (and $Z < \frac{\eta_{0a}}{12}$)

(ii) $r \to 0 \Rightarrow$ the limit does not exist, as there is a lower bound $r > r_{\min}$.

For $\tilde{C}_\mu < 0$, the function $Z(r)$ is defined in a range of the form $[r_{\min}, +\infty)$, where $r_{\min} > 0$. In both cases the large distance asymptotic behavior is given by Eq. (4.29) in Sec. IV D, which is of the standard form

$$f(r) \simeq k - \frac{\Lambda_{\rm eff}}{12}r^2 - \frac{\mu}{r^2}. \tag{A2}$$

In addition, for $\tilde{C}_\mu < 0$, the $r_{\min}$ can be determined if we take into account that the Lambert function has a lower bound $W_L(x) \geq -1/e$, we then obtain

$$r_{\min} = \sqrt[4]{-e\tilde{C}_\mu}, \tag{A3}$$

and

$$Z(r_{\min}) = \frac{12\eta\eta_{0a} - 9}{144\eta}. \tag{A4}$$

In order to derive the above equation we used that $W_L(-1) = -1/e$.

2. Case II ($\sigma = -1$)

For $\sigma = -1$, from Eq. (A1) we see that the parameter $\tilde{C}_\mu$ should be negative, hence we examine only the case $\tilde{C}_\mu < 0$

(i) $r \to +\infty \Rightarrow Z \to \mathrm{sgn}(\eta)(+\infty)$

(ii) $r \to 0 \Rightarrow$ the limit does not exist, as there is a lower bound $r > r_{\min}$.

We see that for $\sigma = -1$, the function $Z(r)$ is defined in a range of the form $[r_{\min}, +\infty)$, where $r_{\min}$ and $Z(r_{\min})$ are given by Eqs. (A3) and (A4) above. In addition, the large distance asymptotic behavior in this case can be estimated from Eq. (A1) if we keep only the exponential term

$$Z(r) \simeq \frac{1}{16\eta} \ln\left(\frac{-\tilde{C}_\mu}{r^4}\right), \tag{A5}$$

hence

$$f(r) \simeq k + r^2 \frac{1}{16\eta} \ln\left(\frac{-e\tilde{C}_\mu}{r^4}\right), \tag{A6}$$

while for $\tilde{N}$, from Eq. (4.33), we obtain

$$\tilde{N}(Z(r)) \simeq \frac{\tilde{C}_\mu}{3r^4 \ln(\frac{-e\tilde{C}_\mu}{r^4})}. \tag{A7}$$

We conclude that for $\sigma = -1$ (and $\varrho = 0$, $\eta \neq 0$) the large distance asymptotic behavior is not of the standard form (A2), so we will not study this class of solutions further.

APPENDIX B: TECHNICAL DETAILS IN THE GENERIC CASE ($\eta \neq 0$ AND $\varrho \neq 0$)

In this section we present the same mathematical details for the derivation of the formulas of Eqs. (5.7) and (5.23). In particular we have to integrate Eq. (5.1)

$$r\frac{dZ}{dr} = \frac{3 - 6\varrho Z + \sigma\sqrt{9 - 12\eta_{0a}\eta + 36(\varrho - 4\eta)(\varrho Z^2 - Z)}}{6\eta}. \tag{B1}$$

In order to perform the above integral we make the replacement $Z = \frac{1}{2\varrho} - \frac{y}{3}$, then we obtain

$$r\frac{dy}{dr} = \tilde{H}(y), \qquad \tilde{H}(y) = -\frac{1}{\eta}(\varrho y + \sigma\sqrt{A + By^2}), \tag{B2}$$

where

$$A = -3\eta_{0a}\eta + \frac{9\eta}{\varrho}, \qquad B = \varrho(\varrho - 4\eta). \tag{B3}$$

Note that

$$\frac{\eta\varrho}{2(\varrho^2 - B)} = \frac{1}{8},$$

so we have $\varrho^2 - B = 4\varrho\eta \neq 0$. For $B \neq 0$, we obtain

$$c + \ln(r) = -\eta \int \frac{dy}{\varrho y + \sigma\sqrt{A + By^2}} = -\eta \int dy \frac{\varrho y - \sigma\sqrt{A + By^2}}{(\varrho^2 - B)y^2 - A} \tag{B4}$$

$$= -\frac{1}{8} \ln|(\varrho^2 - B)y^2 - A| + \sigma\eta \int dy \frac{\sqrt{A + By^2}}{(\varrho^2 - B)y^2 - A}, \tag{B5}$$

where c is an integration constant. In order to calculate the second integral in Eq. (B4) separately, we write the corresponding integrand in the form

$$\frac{\sqrt{A + By^2}}{(\varrho^2 - B)y^2 - A} = \frac{1}{\varrho^2 - B}\left(\frac{B}{\sqrt{A + By^2}} + \frac{A\varrho^2}{\sqrt{A + By^2}} \times \frac{1}{((\varrho^2 - B)y^2 - A)}\right). \tag{B6}$$

For $B > 0$ we obtain

$$\int \frac{\sqrt{A + By^2}}{(\varrho^2 - B)y^2 - A} = \frac{\sqrt{B}}{\varrho^2 - B} \ln|\sqrt{B}y + \sqrt{A + By^2}| - \frac{\varrho}{2(\varrho^2 - B)} \ln\left|\frac{\varrho y - \sqrt{A + By^2}}{\varrho y + \sqrt{A + By^2}}\right|, \tag{B7}$$

while for $B < 0$ we get

$$\int dy \frac{\sqrt{A - |B|y^2}}{(\varrho^2 + |B|)y^2 - A} = \frac{1}{\varrho^2 + |B|} \times \sqrt{|B|}\tan^{-1}\left(\frac{\sqrt{|B|}y}{\sqrt{A - |B|y^2}}\right) - \frac{\varrho}{2(\varrho^2 + |B|)} \times \ln\left|\frac{\varrho y - \sqrt{A + By^2}}{\varrho y + \sqrt{A + By^2}}\right|, \tag{B8}$$

where we have taken into account that

$$\frac{A\varrho^2}{\sqrt{A + By^2}} \frac{1}{((\varrho^2 - B)y^2 - A)} = -\frac{A\varrho^2}{(A + By^2)^{3/2}} \frac{1}{(1 - \frac{\varrho^2 y^2}{A + By^2})}, \tag{B9}$$

and the well-known relations

$$\frac{d}{dy}\left(\frac{y}{\sqrt{A+By^2}}\right)=\frac{A}{(A+By^2)^{3/2}},$$
$$\int\frac{dx}{1-x^2}=\frac{1}{2}\ln\left|\frac{1-x}{1+x}\right|,\qquad x=\frac{\varrho y}{\sqrt{A+By^2}}. \tag{B10}$$

Moreover, we will use the identity

$$\frac{1}{2}\ln\left|\frac{\varrho y-\sqrt{A+By^2}}{\varrho y+\sqrt{A+By^2}}\right|=-\frac{1}{2}\ln|(\varrho^2-B)y^2-A| + \ln|\varrho y+\sqrt{A+By^2}|. \tag{B11}$$

For $B>0$ from Eqs. (B4), (B7), and (B11), if we take into account that $\varrho^2-B=4\varrho\eta$, we obtain

$$c+\ln(r)=\frac{\sigma-1}{8}\ln|(\varrho^2-B)y^2-A|+\frac{\sigma\sqrt{B}}{4\varrho}\ln|\sqrt{B}y + \sqrt{A+By^2}|-\frac{\sigma}{4}\ln|\varrho y+\sqrt{A+By^2}|, \tag{B12}$$

or equivalently

$$c'+\ln(r)=-\frac{1}{4}\ln|\varrho y+\sigma\sqrt{A+By^2}| + \frac{\sqrt{B}}{4\varrho}\ln|\sqrt{B}y+\sigma\sqrt{A+By^2}|, \tag{B13}$$

where c' is a new constant which is defined appropriately. Finally, we obtain

$$|\varrho y+\sigma\sqrt{A+By^2}||\sqrt{B}y+\sigma\sqrt{A+By^2}|^{-(\sqrt{B}/\varrho)}=\frac{\tilde{C}_\mu}{r^4}, \tag{B14}$$

where $\tilde{C}_\mu=e^{c'}$ is the final integration constant which is related to the mass of the black hole. For $B<0$, in a similar way, if we use Eq. (B8) instead of (B7), we obtain

$$|\varrho y+\sigma\sqrt{A-|B|y^2}|e^{(\sigma\sqrt{|B|}/\varrho)\tan^{-1}(\varrho y/(\sqrt{A-|B|y^2}))}=\frac{\tilde{C}_\mu}{r^4}. \tag{B15}$$

Finally, for the computation of $N(r)$ we use the following equations:

$$\frac{d\tilde{N}(y)}{dy}-\tilde{C}(y)\tilde{N}(y)=0, \tag{B16}$$

where $\tilde{C}(y)$ is given by the equation

$$\tilde{C}(y)=-\frac{B}{\varrho}\frac{\varrho y+\sigma\sqrt{A+By^2}}{A+By^2}, \tag{B17}$$

and we find in a similar way, as it is described above, that

$$\tilde{N}(y)=\frac{\tilde{C}_N}{\sqrt{A+By^2}|\sqrt{B}y+\sigma\sqrt{A+By^2}|^{\sqrt{B}/\varrho}} \tag{B18}$$

for $B>0$, and

$$\tilde{N}(y)=\frac{\tilde{C}_N}{\sqrt{A-|B|y^2}}e^{(\sigma\sqrt{|B|}/\varrho)\tan^{-1}(\sqrt{|B|}y/(\sqrt{A-|B|y^2}))} \tag{B19}$$

for $B<0$, where $\tilde{C}_N$ is an integration constant.

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