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Brane $f(R)$ gravity cosmologies

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By the application of the generalized Israel junction conditions we derive cosmological equations for the fourth-order $f(R)$ brane gravity and study their cosmological solutions. We show that there exists a nonstatic solution which describes a four-dimensional de Sitter (dS_4) brane embedded in a five-dimensional anti-de Sitter (AdS_5) bulk for a vanishing Weyl tensor contribution. On the other hand, for the case of a nonvanishing Weyl tensor contribution, there exists a static brane solution only. We claim that in order to get some more general nonstatic $f(R)$ brane configurations, one needs to admit a dynamical matter energy-momentum tensor in the bulk rather than just a bulk cosmological constant.

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I. INTRODUCTION

Similarly as in the standard general relativity, it is interesting to consider generalizations of the brane universes [1–3]. Some of such generalizations are the higher-order brane gravity theories of which the simplest is $f(R)$ gravity [4] (for the most recent reviews see Refs. [5,6]). However, the combination of brane models with higher-order theories is nontrivial, since, except for the Lovelock (or, in the lowest order, the Gauss-Bonnet) densities [7,8], one faces ambiguities of the quadratic delta function contributions to the field equations. This problem was first challenged successfully in our earlier works [9,10], in which we found the ways to avoid ambiguities not only for $f(R)$ brane theories (see, e.g., [11,12]), but also for more general actions which depend arbitrarily on the three curvature invariants $f(R, R_{ab}R^{ab}, R_{abcd}R^{abcd})$, although the linear combination of these invariants was studied in Ref. [13]. One of the methods applied, was the reduction of the fourth-order brane gravity to the second-order theory by introducing an extra degree of freedom—the scalaron [9,10]. Such a procedure leads to the second-order gravity which is just the scalar-tensor Brans-Dicke gravity [14] with a Brans-Dicke parameter $\omega = 0$, and an appropriate scalaron potential (with the scalaron playing the role of the Brans-Dicke field). We then obtained the Israel junction conditions [15] which generalized both the conditions obtained in Refs. [16,17] for the Brans-Dicke field without a scalar field potential, and also the conditions derived in Refs. [18,19] for $f(R)$ brane gravity. The junction conditions which did not assume scalaron continuity, but for a static brane, were presented in Ref. [20].

In this paper we apply the Israel junction conditions for $f(R)$ Friedmann-Robertson-Walker metric brane configurations and study the set of their admissible cosmological solutions. In Sec. II, we present $f(R)$ brane models and derive the set of field equations. In Sec. III, we apply the field equations to cosmology. In Sec. IV, we give our conclusions.

II. $f(R)$ GRAVITY ON THE BRANE

Let us consider the $f(R)$ gravity on the brane described by the action [9]

$$S_p = \frac{1}{2\kappa_5^2}\int_{M_p} d^5x\sqrt{-g}f(R) + S_{\mathrm{bulk},p}, \tag{2.1}$$

where R is the Ricci scalar, κ_5^2 is a five-dimensional Einstein constant, $S_{\mathrm{bulk},p}$ is the bulk matter action ($p = 1, 2$), and M_p is the spacetime volume. The action (2.1) gives fourth-order field equations. It is then advisable to use an equivalent action

$$\bar{S}_p = \int_{M_p} d^5x\sqrt{-g}\{f'(Q)(R-Q) + f(Q)\}, \tag{2.2}$$

where Q is an extra field (Lagrange multiplier), and $f'(Q) = df(Q)/dQ$. The equation of motion which comes from (2.2) is just $Q = R$, provided that $f''(Q) \neq 0$, so that $f'(Q)$ may be interpreted as an extra scalar field (called the scalaron)

$$\phi = f'(Q) = f'(R), \tag{2.3}$$

and the action can be rewritten as

$$\bar{S}_p = \int_{M_p} d^5x\sqrt{-g}\{\phi R - V(\phi)\} + S_{\mathrm{bulk},p}, \tag{2.4}$$

where $V(\phi) = -\phi R(\phi) + f(R(\phi))$ [10]. The action (2.4) is equivalent to a scalar-tensor Brans-Dicke gravity with a Brans-Dicke parameter $\omega = 0$. One of the ways to derive the junction conditions for the theory described by the action (2.4) is to append it with an appropriate Hawking-Lutrell boundary term, which reads as [21]

$$S_{\mathrm{HL}_p} = -2(-1)^p\epsilon\int_{\partial M_p}\sqrt{-h}\phi K d^4x, \tag{2.5}$$

where K is the trace of the extrinsic curvature tensor K_{ab}, h

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is the determinant of the induced metric $h_{ab} = g_{ab} - \epsilon n_a n_b$, n^a is a unit normal vector to a boundary ∂M_p, and $\epsilon = 1$ ($\epsilon = -1$) for a timelike (a spacelike) brane, respectively. The total action of the theory is then

$$\bar{S}_{\mathrm{tot}_p} = \bar{S}_p + S_{\mathrm{LH}_p}. \tag{2.6}$$

The variation of the action (2.6) leads to the following junction conditions for $f(R)$ gravity theory [10] ($a, b, \ldots = 0, 1, 2, 3, 5$):

$$[K] = 0, \tag{2.7}$$

$$S^{ab} n_a n_b = 0, \tag{2.8}$$

$$S^{ab} h_{ac} n_b = 0, \tag{2.9}$$

$$-h_{ab}[\phi_{,c} n^c] - [\phi] K h_{ab} + [\phi K_{ab}] = \epsilon \kappa_5^2 S^{cd} h_{ca} h_{db}, \tag{2.10}$$

where for an arbitrary quantity A we have defined a discontinuity (a jump) of A as: $[A] \equiv A^+ - A^-$. Here S_{ab} is the brane energy-momentum tensor. In particular, the condition (2.7) comes from the requirement that the variation of the Hawking-Lutrell boundary term (2.5) should vanish.

These junction conditions can be compared with those previously obtained in Ref. [18] [see their Eqs. (12) and (13)] and in Ref. [19] [see their Eq. (3.11)]. The difference is that we have not assumed the continuity of the scalaron on the brane. If we do so, i.e., assume that $[\phi] = 0$ which due to the definition (2.3) implies $[R] = 0$, and additionally impose the mirror symmetry $g_{ab} = g_{ab}(|n|)$, where n is a normal Gaussian coordinate originating at the brane, then the junction conditions (2.7), (2.8), (2.9), and (2.10) take the form

$$K = 0, \tag{2.11}$$

$$\phi_{,c} n^c = -\frac{\kappa_5^2}{8} S^{ab} h_{ab}, \tag{2.12}$$

$$K_{ab} = \frac{\kappa_5^2}{2\phi}\left\{S^{cd} h_{ca} h_{db} - \frac{h_{ab}}{4} S^{cd} h_{cd}\right\}. \tag{2.13}$$

We can now express the equations above in terms of the Ricci scalar R instead of the scalaron ϕ. Using the Gaussian coordinate system we obtain

$$R = 0, \tag{2.14}$$

$$K = 0, \tag{2.15}$$

$$f''(R) R_{,n} = -\frac{\kappa_5^2}{8} \tilde{S}, \tag{2.16}$$

$$f'(R) \tilde{K}_{ab} = \frac{\kappa_5^2}{2} \tilde{S}^{ab}, \tag{2.17}$$

where we have used the definition of a traceless part of the brane energy-momentum tensor $\tilde{S}^{ab} = S^{cd} h_{ca} h_{db} - (1/4) h_{ab} S^{cd} h_{cd}$, the definition of the traceless part of the extrinsic curvature tensor $\tilde{K}^{ab} = K^{cd} h_{ca} h_{db} - (1/4) h_{ab} K^{cd} h_{cd}$ and the definition of the trace of the brane energy-momentum tensor $\tilde{S} = S^{ab} h_{ab}$. The condition (2.14) is a consequence of the definition (2.3). The junction conditions (2.14), (2.15), (2.16), and (2.17) coincide with those obtained in Ref. [19] [Eq. (3.11)].

Now we can apply the junction conditions (2.7), (2.8), (2.9), and (2.10) for $f(R)$ gravity in order to obtain the effective Einstein equations on the brane. The manipulation of the Gauss-Codazzi equation [22]

$${}^{(5)}R_{cd} h^d{}_a n^c = -D_c K^c{}_a + D_a K, \tag{2.18}$$

where D_a means a four-dimensional covariant derivative on the brane, leads to a standard decomposition of a four-dimensional Einstein tensor ${}^{(4)}G_{ab}$ in the form [23]

$$\begin{aligned} {}^{(4)}G_{ab} = {}& K K_{ab} - K_a{}^c K_{bc} - \tfrac{1}{2} h_{ab}(K^2 - K^{cd} K_{cd}) \\ & - {}^{(5)}E_{ab} + \tfrac{2}{3}[{}^{(5)}G_{cd} h^c{}_a h^d{}_b \\ & + h_{ab}({}^{(5)}G_{cd} n^c n^d - \tfrac{1}{4}{}^{(5)}G)], \end{aligned} \tag{2.19}$$

where E_{ab} is an electric part of the bulk Weyl tensor projected onto the brane. We also assume that in the neighborhood of the brane, the normal vector field n^a fulfills the geodesic equations $n^a \nabla_a n^b = 0$ (geodesic gauge). Using this last assumption, the following relations are derived (see Appendix A):

$${}^{(5)}\Box\phi = {}^{(4)}\Box\phi + K n^a \nabla_a \phi + (n^a \nabla_a)^2 \phi, \tag{2.20}$$

$$h^c{}_a h^d{}_b \nabla_c \nabla_d \phi = D_a D_b \phi + K_{ab} n^c \nabla_c \phi. \tag{2.21}$$

Here ∇_c means the five-dimensional (bulk) covariant derivative. Assuming that the matter in the bulk has the form of the five-dimensional cosmological constant $T_{ab} = -g_{ab}{}^{(5)}\Lambda$, the variation of the action (2.6) gives the following field equations in the bulk:

$$\begin{aligned} {}^{(5)}G_{ab} = {}& -\frac{1}{2\phi} g_{ab} V(\phi) + \frac{1}{\phi} g_{ab}{}^{(5)}\Box\phi \\ & - \frac{1}{\phi}\phi_{;ab} + \frac{\kappa_5^2}{\phi} g_{ab}{}^{(5)}\Lambda, \end{aligned} \tag{2.22}$$

$${}^{(5)}R = -\frac{\partial V(\phi)}{\partial\phi} \equiv -W(\phi). \tag{2.23}$$

Substituting (2.11), (2.12), (2.13), (2.20), (2.21), and (2.22) to (2.19), one obtains the effective Einstein equations on the brane as

$$
{}^{(4)}G_{ab} = \left(\frac{\kappa_5^2}{2\phi}\right)^2 Q_{ab} - \frac{1}{4}h_{ab}\frac{V(\phi)}{\phi} + \frac{2}{3}\frac{1}{\phi}h_{ab}{}^{(4)}\Box\phi - \frac{2}{3}\frac{1}{\phi}D_aD_b\phi + \frac{\kappa_5^2}{2\phi}{}^{(5)}\Lambda h_{ab} - {}^{(5)}E_{ab}, \tag{2.24}
$$

where

$$
Q_{ab} = -\frac{2}{3}\lambda\left(\tilde{T}_{ab} - \frac{1}{4}\tilde{T}h_{ab}\right) + \frac{2}{3}\tilde{T}\tilde{T}_{ab} - \tilde{T}^c{}_b\tilde{T}_{ca} + \frac{1}{2}h_{ab}\left(\tilde{T}_{cd}\tilde{T}^{cd} - \frac{11}{24}\tilde{T}^2\right) \tag{2.25}
$$

and

$$
\tilde{T}_{ab} := \lambda h_{ab} + S_{ab}. \tag{2.26}
$$

This should be appended by the conservation law for the matter energy-momentum tensor on the brane (see Appendix B)

$$
D_aS^a{}_b = 0. \tag{2.27}
$$

III. $f(R)$ FRIEDMANN COSMOLOGY ON THE BRANE

We restrict ourselves to the case of the matter in the bulk in the form of the cosmological constant. This allows to assume that the bulk spacetime is an Einstein space

$$
{}^{(5)}G_{ab} = -{}^{(5)}\Lambda_{\mathrm{eff}}g_{ab}, \tag{3.1}
$$

where ${}^{(5)}\Lambda_{\mathrm{eff}} > 0$ is an effective cosmological constant. The five-dimensional line element reads as

$$
ds^2 = -b^2(n,t)dt^2 + a^2(n,t)\left[\frac{dr^2}{1-kr^2} + r^2d\Omega^2\right] + dn^2, \tag{3.2}
$$

where $k = 0, \pm 1$. The electric part of Weyl tensor E_{ab} can be expressed in the following form [23]:

$$
{}^{(5)}E_{ab} = \mathcal{F}[u_au_b + \tfrac{1}{3}(h_{ab} + u_au_b)]. \tag{3.3}
$$

In the case with vanishing $\mathcal{F}$, we deal with a nonstatic Friedmann-Robertson-Walker brane (3.2) embedded in an AdS_5 bulk [24]. Moreover, the junction condition (2.12) requires that the trace of the brane energy-momentum tensor vanishes $S = \tilde{T} + 4\lambda = 0$ [the second bulk Eq. (2.23) sets the scalaron to be a constant because of the constancy of the curvature in the bulk]. The assumption that the energy-momentum tensor of the matter on the brane is a perfect fluid

$$
S_{ab} = (\rho + p)u_au_b + pg_{ab}, \tag{3.4}
$$

which fulfills the barotropic equation of state $p = w\rho$ with the four-velocity vector $u^a = \delta^a_0$, gives the effective $f(R)$ gravity Friedmann equations on the brane in the form

$$
-\mathcal{F} - \frac{\kappa_5^2}{2\phi}{}^{(5)}\Lambda + \frac{V(\phi)}{4\phi} - 3\frac{k+\dot{a}^2}{a^2} + 2\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} = -\left(\frac{\kappa_5^2}{8\phi}\right)^2(p+\rho)(9p+\rho-8\lambda), \tag{3.5}
$$

$$
\frac{\mathcal{F}}{3} - \frac{\kappa_5^2}{2\phi}\Lambda + \frac{V(\phi)}{4\phi} - \frac{k+\dot{a}^2}{a^2} + \frac{4}{3}\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - 2\frac{\ddot{a}}{a} + \frac{2}{3}\frac{\ddot{\phi}}{\phi}(5) = \frac{1}{3}\left(\frac{\kappa_5^2}{8\phi}\right)^2(p+\rho)(21p+13\rho-8\lambda). \tag{3.6}
$$

Now the brane energy-momentum tensor conservation law (2.27)

$$
\frac{\dot{\rho}}{\rho} = -3(w+1)\frac{\dot{a}}{a} \tag{3.7}
$$

can be integrated to give

$$
\rho = \rho_0 a^{-3(w+1)}. \tag{3.8}
$$

The requirement that the trace of the brane energy-momentum tensor vanishes imposes a condition that the energy density of the matter on the brane is constant, i.e.,

$$
\rho = \rho_0 = -\frac{4\lambda}{(1+3w)} = \mathrm{const.} \tag{3.9}
$$

Multiplying (3.6) by three, and adding it to (3.5), we get one cosmological equation to solve (for simplicity we consider flat $k = 0$ models only)

$$
-2\kappa_5^2\frac{\Lambda}{\phi} + \frac{V(\phi)}{\phi} + 6\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} + 2\frac{\ddot{\phi}}{\phi} - 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) = \frac{3}{4}\left(\frac{\kappa_5^2}{2}\frac{\rho}{\phi}\right)^2(w+1)^2. \tag{3.10}
$$

The Eq. (2.23) forces the scalaron ϕ to be constant $\phi = \phi_0$, as well. On the other hand, the Eq. (2.22) leads to the following relation:

$$
\frac{V(\phi_0)}{2\phi_0} - \frac{\kappa_5^{2(5)}\Lambda}{\phi_0} = {}^{(5)}\Lambda_{\mathrm{eff}}. \tag{3.11}
$$

The Eq. (3.11) shows that in the case of a constant scalaron, the term ${}^{(5)}\Lambda_{\mathrm{eff}}$ plays the role of a five-dimensional effective cosmological constant in the bulk. We can independently fix the value of the ϕ_0 by a choice of the shape of the function $W(\phi)$ near the brane using (2.23) as

$$
\phi_0 = W^{-1}(-\tfrac{10}{3}{}^{(5)}\Lambda_{\mathrm{eff}}). \tag{3.12}
$$

Now for a fixed value of $V(\phi_0) \equiv V_0$, Eq. (3.11) gives

$$
{}^{(5)}\Lambda_{\mathrm{eff}} = \frac{V_0 - 2\kappa_5^{2(5)}\Lambda}{2\phi_0}. \tag{3.13}
$$

Combining (3.10) with (3.11), and assuming that $w = -1$, we obtain

$$
3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) = {}^{(5)}\Lambda_{\mathrm{eff}}. \tag{3.14}
$$

The nonstatic solution of (3.14) takes the form

$$a = \tilde{a}_0 \exp(H_0 t), \tag{3.15}$$

and it is consistent with the solution of (3.5) for $\mathcal{F} = 0$. For $w \neq -1$, the continuity Eq. (3.7) implies a constant scale factor $a(t) = a_0$, and the generalized Friedmann Eqs. (3.5) and (3.6) become inconsistent. The solution (3.15) describes an embedding of a de Sitter (dS_4) brane in an anti-de Sitter (AdS_5) bulk provided that

$$H_0 = \sqrt{\frac{^{(5)}\Lambda_{\text{eff}}}{6}} = \sqrt{\frac{V_0 - 2\kappa_5^{2\,(5)}\Lambda}{12\phi_0}} \tag{3.16}$$

(note that because of the assumption that $^{(5)}\Lambda_{\text{eff}} > 0$, we need $V_0 > 2\kappa_5^{2\,(5)}\Lambda$). The Eq. (3.16) is a fine-tuning condition for the value of the bulk cosmological constant $^{(5)}\Lambda$ and the potential $V(\phi)$, which is responsible for the value of ϕ_0 and V_0. The special case with

$$V_0 = 2\kappa_5^{2\,(5)}\Lambda \tag{3.17}$$

gives $H_0 = 0$, and the solution (3.15) describes a static Minkowski brane which is a flat analogue of the Einstein static universe. In fact, the condition (3.17) is a special case of the fine-tuning relation (3.16), and can be interpreted as a necessary and a sufficient condition for the existence of a static brane in the model with $w = -1$ and a vanishing Weyl tensor contribution $\mathcal{F} = 0$.

If the Weyl tensor contribution is nonvanishing, i.e., if $\mathcal{F} \neq 0$, it is then possible to embed a static Friedmann-Robertson-Walker brane (3.2) in a bulk with the cosmological constant $^{(5)}\Lambda$ only [24]. In such a case, the solution of (3.1) for the metric (3.2) has the form [24]

$$a^2(n) = f(n), \quad \text{where } f(n) = \gamma e^{2H_0|n|} + \delta e^{-2H_0|n|}, \tag{3.18}$$

$$b^2(n) = \frac{e^2(n)}{f(n)}, \quad \text{where } e(n) = \gamma e^{2H_0|n|} - \delta e^{-2H_0|n|}, \tag{3.19}$$

with the brane at $n = 0$. Using the transformation of the metric components

$$f(n) \to \frac{f(n)}{\gamma + \delta} \quad \text{and} \quad \frac{e^2(n)}{f(n)} \to \frac{e^2(n)}{f(n)} \frac{\gamma + \delta}{(\gamma - \delta)^2}, \tag{3.20}$$

which is equivalent to a rescaling of the coordinates, we obtain a Minkowski brane (for $n = 0$). We then compute the nonvanishing components of the electric part of the Weyl tensor $E^a{}_b$ and the corresponding term $\mathcal{F}$ at the brane ($n = 0$) as

$$E^1{}_1 = E^2{}_2 = E^3{}_3 = -E^0{}_0 = \mathcal{F} = \frac{2\gamma\delta}{(\gamma + \delta)^2}\,{}^{(5)}\Lambda_{\text{eff}}. \tag{3.21}$$

For $w \neq -1$, the solution of Eqs. (3.7), (3.5), and (3.6) gives

$$w = \frac{1 - 3M \pm 2\sqrt{M}}{9M - 1}, \qquad \text{where } M = \frac{\phi_0(V_0 - 2\kappa_5^2\Lambda)}{3(\kappa_5^2\lambda)^2} > 0, \tag{3.22}$$

$$\mathcal{F} = \frac{3}{2}\left(\frac{\kappa_5^2\lambda}{\phi_0}\right)^2 \sqrt{M}. \tag{3.23}$$

Note that (3.22) and (3.23) requires that $\phi_0 > 0$. The solution above describes a flat static Minkowski brane which is a flat analogue of the Einstein static universe. In fact, from (3.21) and (3.23), we have

$$\frac{2\gamma\delta}{(\gamma + \delta)^2}\,{}^{(5)}\Lambda_{\text{eff}} = \frac{3}{2}\left(\frac{\kappa_5^2\lambda}{\phi_0}\right)^2 \sqrt{M}. \tag{3.24}$$

The condition (3.24) means that to support a flat static Minkowski brane one needs to fine-tuned the values of the parameters ϕ_0, V_0, $^{(5)}\Lambda$, λ, γ, and δ.

IV. CONCLUSIONS

In this paper we have studied brane universes within the framework of the fourth-order $f(R)$ gravity theory. We applied the junction conditions obtained in our earlier papers [9,10] in order to get the set of the field equations which were applicable to cosmology. We conclude that for the matter with a barotropic equation of state $p = w\rho$ on the nonstatic Friedmann-Robertson-Walker brane (3.2) embedded in a five-dimensional anti-de Sitter (AdS_5) bulk (with vanishing Weyl tensor contribution $\mathcal{F} = 0$), and the matter in the bulk having the form of a cosmological constant $^{(5)}\Lambda$, there is only one case with $w = -1$ that possesses the solution in the form of the exponential evolution (3.15) which is a four-dimensional de Sitter (dS_4) brane embedded in a five-dimensional anti-de Sitter (AdS_5) bulk. The case with the Friedmann-Robertson-Walker brane (3.2) embedded in a bulk with the cosmological constant $^{(5)}\Lambda$ and nonvanishing Weyl tensor contribution $\mathcal{F} \neq 0$ allows the solution in the form of the flat static Minkowski universe (3.22) only. The cosmological constant $^{(5)}\Lambda$ in the bulk implies the constant curvature and whence the vanishing of the trace of the brane energy-momentum tensor S. This is an extremely strong condition that makes the energy density constant. In conclusion, we claim that more nonstatic configurations on the brane are possible if we assume a dynamical matter energy-momentum tensor in the bulk.

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APPENDIX A: DERIVATION OF USEFUL GEOMETRIC FORMULAS FROM SEC. II

The formula (2.20) can be obtained as follows:

$$\begin{aligned}{}^{(5)}\Box\phi &= g^{ab}\nabla_a\nabla_b\phi = h^{ab}\nabla_a\nabla_b\phi + n^a n^b\nabla_a\nabla_b\phi\\ &= h^{ab}D_aD_b\phi + h^{ab}K_{ab}(n^c\nabla_c)\phi + (n^c\nabla_c)^2\phi\\ &= {}^{(4)}\Box\phi + K(n^c\nabla_c)\phi + (n^c\nabla_c)^2\phi,\end{aligned} \tag{A1}$$

where we have used

$$\begin{aligned}n^c n^d\nabla_c\nabla_d\phi &= n^c\nabla_c(n^d\nabla_d H) - n^c(\nabla_c n^d)(\nabla_d\phi)\\ &= n^c\nabla_c(n^d\nabla_d\phi) = (n^c\nabla_c)^2\phi.\end{aligned} \tag{A2}$$

The formula (2.21) can be obtained as follows:

$$\begin{aligned}h^c{}_a h^d{}_b\nabla_c\nabla_d\phi &= h^c{}_a h^d{}_b\nabla_c(g^e{}_d\nabla_e\phi) = h^c{}_a h^d{}_b\nabla_c[(h^e{}_d + n^e n_d)\nabla_e\phi]\\ &= h^c{}_a h^d{}_b\nabla_c(h^e{}_d\nabla_e\phi) + h^c{}_a h^d{}_b(\nabla_c n^e)n_d(\nabla_e\phi) + h^c{}_a h^d{}_b(\nabla_c n_d)(n^e\nabla_e\phi) + h^c{}_a h^d{}_b n^e n_d\nabla_c\nabla_e\phi\\ &= D_a(h^e{}_b\nabla_e\phi) + h^c{}_a h^d{}_b\nabla_c n_d(n^e\nabla_e\phi) = D_aD_b\phi + K_{ad}h^d{}_b(n^c\nabla_c)\phi = D_aD_b\phi + K_{ab}(n^c\nabla_c)\phi.\end{aligned} \tag{A3}$$

It is also useful to prove that

$$\begin{aligned}h^d{}_a n^c\nabla_d\nabla_c\phi &= h^d{}_a\nabla_d(n^c\nabla_c\phi) - h^d{}_a(\nabla_d n^c)(\nabla_c\phi) = h^d{}_a\nabla_d(n^c\nabla_c)\phi - K_a{}^c\nabla_c\phi = D_a(n^c\nabla_c)\phi - K_a{}^e g_e{}^c\nabla_c\phi\\ &= D_a(n^c\nabla_c)\phi - K_a{}^e(h_e{}^c + n_e n^c)\nabla_c\phi = D_a(n^c\nabla_c)\phi - K_a{}^e h_e{}^c\nabla_c\phi - K_a{}^e n_e n^c\nabla_c\phi\\ &= D_a(n^c\nabla_c\phi) - K_a{}^e D_e\phi.\end{aligned} \tag{A4}$$

APPENDIX B: DERIVATION OF THE BRANE ENERGY-MOMENTUM TENSOR CONSERVATION

The conservation law for the brane energy-momentum tensor can be obtained as follows. We take the covariant derivative of the left hand side of the Eq. (2.13) multiplied by ϕ and get

$$D_a(\phi K^a{}_b) = K^a{}_b D_a\phi + \phi D_a K^a{}_b. \tag{B1}$$

Next, we use the Gauss-Coddazzi Eq. (2.18) together with the condition (2.7) gives

$$D_a(\phi K^a{}_b) = K^a{}_b D_a\phi - \phi\,{}^{(5)}R_{cd}h^d{}_b n^c. \tag{B2}$$

Contracting (2.22) with the induced metric $h^a{}_b$ and the normal vector n^a one obtains

$${}^{(5)}R_{cd}h^d{}_a n^c = -\frac{1}{\phi}h^d{}_a n^c\phi_{;cd}. \tag{B3}$$

After substitution of (B3) to (B2) one gets

$$D_a(\phi K^a{}_b) = K^a{}_b D_a\phi + h^d{}_b n^c\phi_{;cd}. \tag{B4}$$

Applying (A4) to (B4) we have

$$D_a(\phi K^a{}_b) = K^a{}_b D_a\phi + D_b(n^c\nabla_c\phi) - K^a{}_b D_a\phi. \tag{B5}$$

Substituting (2.12) into (B5), we obtain the relation

$$D_a(\phi K^a{}_b) = -\frac{\chi}{16}D_b S. \tag{B6}$$

Now taking the covariant derivative of the right-hand side of the Eq. (2.13) multiplied by ϕ, we have

$$D_a\left\{\frac{\chi}{4}\left(S^a{}_b - \frac{1}{4}h^a{}_b S\right)\right\} = \frac{\chi}{4}D_a S^a{}_b - \frac{\chi}{16}D_a S. \tag{B7}$$

Comparison of (B6) with (B7) gives the conservation law of the brane energy-momentum tensor

$$D_a S^a{}_b = 0, \tag{B8}$$

as required.

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).

[2] M. Visser, Phys. Lett. **159B**, 22 (1985); N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Nucl. Phys. **B516**, 70 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436**, 257 (1998); N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Rev. D **59**, 086004 (1999).

[3] P. Binétruy, C. Deffayet, and D. Langlois, Nucl. Phys. **B565**, 269 (2000); Phys. Lett. B **477**, 285 (2000); M. Sasaki, T. Shiromizu, and K. I. Maeda, Phys. Rev. D **62**, 024008 (2000); T. Shiromizu, K. I. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000); S. Mukohyama, T. Shiromizu, and K. I. Maeda, Phys. Rev. D **62**, 024028 (2000); A. N. Aliev and A. E. Gümrükçüglu, Classical Quantum Gravity **21**, 5081 (2004).

[4] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980); G. Magnano and L. M. Sokołowski, Phys. Rev. D **50**, 5039 (1994); S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003); G. J. Olmo, Phys. Rev. Lett. **98**, 061101 (2007); S. Capozziello, V. F. Cardone, and A. Troisi, Phys. Rev. D**71**, 043503 (2005); S. Capozziello, S. Nojiri, S. D. Odintsov, and A. Troisi, Phys. Lett. B **639**, 135 (2006); L. Amendola, D. Polarski, and S. Tsujikawa, Phys. Rev. Lett. **98**, 131302 (2007); S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006); *Problems of Modern Theoretical Physics, A Volume in Honour of Prof. I. L. Buchbinder in the Occasion of His 60th Birthday*, (Tomsk State Pedagogical University Publishing, Tomsk, Russia, 2008), p. 266.

[5] T. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010).

[6] S. Nojiri and S. D. Odintsov, Int. J. Geom. Methods Mod. Phys. **4**, 115 (2007).

[7] N. Deruelle and T. Doležel, Phys. Rev. D **62**, 103502 (2000); C. Charmousis and J. F. Dufaux, Classical Quantum Gravity **19**, 4671 (2002); S. C. Davis, Phys. Rev. D **67**, 024030 (2003); J. F. Dufaux, J. E. Lidsey, R. Maartens, and M. Sami, Phys. Rev. D **70**, 083525 (2004); K. I. Maeda and T. Torii, Phys. Rev. D **69**, 024002 (2004); A. N. Aliev, H. Cebeci, and T. Dereli, Classical Quantum Gravity **23**, 591 (2006); H. Maeda, V. Sahni, and Y. Shtanov, Phys. Rev. D **76**, 104028 (2007); P. S. Apostopoulos *et al.*, Phys. Rev. D **76**, 084029 (2007).

[8] K. A. Meissner and M. Olechowski, Phys. Rev. Lett. **86**, 3708 (2001).

[9] A. Balcerzak and M. P. Dąbrowski, Phys. Rev. D **77**, 023524 (2008).

[10] A. Balcerzak and M. P. Dąbrowski, J. Cosmol. Astropart. Phys. 01 (2009) 018.

[11] H. H. V. Borzeszkowski and V. P. Frolov, Ann. Phys. (Leipzig) **492**, 285 (1980).

[12] E. Dyer and K. Hinterbichler, Phys. Rev. D **79**, 024028 (2009).

[13] S. Nojiri and S. D. Odintsov, J. High Energy Phys. 07 (2000) 049; S. Nojiri, S. D. Odintsov, and S. Ogushi, Phys. Rev. D **65**, 023521 (2001); S. Nojiri and S. D. Odintsov, Gen. Relativ. Gravit. **37**, 1419 (2005).

[14] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[15] W. Israel, Nuovo Cimento B **44**, 1 (1966).

[16] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, Eur. Phys. J. C **55**, 337 (2008).

[17] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, arXiv:0811.4609.

[18] M. Parry, S. Pichler, and D. Deeg, J. Cosmol. Astropart. Phys. 04 (2005) 014.

[19] N. Deruelle, M. Sasaki, and Y. Sendouda, Prog. Theor. Phys. **119**, 237 (2008).

[20] V. I. Afonso, D. Bazeia, R. Menezes, and A. Yu. Petrov, Phys. Lett. B **658**, 71 (2007).

[21] S. W. Hawking and J. C. Lutrell, Nucl. Phys. **B247**, 250 (1984).

[22] R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[23] Ø. Grøn and S. Hervik, *Einstein's General Theory of Relativity with Modern Application in Cosmology* (Springer, New York, 2007).

[24] P. D. Mannheim, *Brane Localized Gravity* (World Scientific, Singapore, 2005).