Infrared evolution and phase structure of a gauge theory containing different fermion representations

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We study the evolution of an asymptotically free vectorial SU(N) gauge theory from the ultraviolet to the infrared and the resultant phase structure in the general case in which the theory contains fermions transforming according to several different representations of the gauge group. We discuss the sequential fermion condensation and dynamical mass generation that occur, and comment on the effect of bare fermion mass terms.

I. INTRODUCTION

The phase structure of a non-Abelian gauge theory depends on its fermion content. Here we consider an asymptotically free vectorial gauge theory [in (3 + 1) dimensions, at zero temperature and chemical potential] with an SU(N) gauge group and fermions corresponding to several different representations of the gauge group. We denote the running gauge coupling of the theory as $g(\mu)$, with $\alpha(\mu) = g(\mu)^2/(4\pi)$, where μ is the Euclidean energy/momentum scale (which will often be suppressed in the notation). Since the gauge interaction is asymptotically free, at a sufficiently high energy scale μ, $\alpha(\mu)$ is small and the theory is perturbatively calculable. We will study a theory which contains several Dirac fermions transforming according to different representations of SU(N). We denote a representation as R, the set of fermion representations in the theory as $\{R\} \equiv \{R_1, \ldots, R_k\}$, the number of Dirac fermions in each representation R_i as N_{R_i}, and the set of these numbers as $\{N_R\} \equiv \{N_{R_1}, \ldots, N_{R_k}\}$ [1]. We will first consider the case in which all of these fermions are massless or have bare masses in the high-scale Lagrangian that are small compared with the scale where α grows to a size of order unity and the theory becomes strongly coupled. One interesting property of this type of theory is that it can exhibit fermion condensations at different energy scales, with fermions with larger quadratic Casimir invariants condensing and gaining dynamical masses at higher scales. This theory could arise from a larger one which is a chiral gauge theory, in which fermion masses would generically be forbidden. However, if we consider the theory by itself, then, since it is vectorial, and hence fermion mass terms do not violate the SU(N) gauge symmetry, it is natural to consider a more complicated situation in which some fermions have masses that are comparable to or greater than the scale where the coupling α grows to O(1). We shall also briefly comment on this latter possibility.

Although our work is an abstract field-theoretic study, not an effort to construct a phenomenological model, we note that there has been considerable interest recently in the analysis of vectorial non-Abelian gauge theories with fermions in higher-dimensional representations, partly motivated by technicolor model building [2,3]. We note in passing that in the early development of the standard model, the possibility was considered that the color SU(3)$_c$ sector might contain not just quarks but also other fermions transforming as higher-dimensional representations of the color group [4]. Fermions in higher-dimensional representations have also been used in constructions of chiral gauge theories, but here we restrict our consideration to vectorial gauge theories.

II. GENERAL THEORETICAL FRAMEWORK

A. Beta function

In this section we review the general theoretical framework that we will use in our calculations. The beta function of the theory is denoted $\beta = dg/dt$, where $dt = d\ln\mu$. In terms of α, this can be written as

$$\frac{d\alpha}{dt} = -\frac{\alpha^2}{2\pi}\left[b_1 + \frac{b_2\alpha}{4\pi} + O(\alpha^2)\right], \qquad (2.1)$$

where the coefficient b_ℓ arises at ℓ-loop order in perturbation theory, and the first two coefficients, b_1 and b_2, are scheme independent. These are [5]

$$b_1 = \frac{1}{3}\left[11C_2(G) - 4\sum_R N_R T(R)\right] \qquad (2.2)$$

and [6]

$$b_2 = \frac{1}{3}\left[34C_2(G)^2 - 4\sum_R (5C_2(G) + 3C_2(R))N_R T(R)\right]. \qquad (2.3)$$

Here $C_2(R)$ is the quadratic Casimir invariant and $T(R)$ is the trace invariant for the representation R [7], with $C_2(G) \equiv C_2(\mathrm{adj})$ and $C_2(\mathrm{SU}(N)) = N$. The condition that the theory be asymptotically free, i.e., that $b_1 > 0$, yields the upper bound

$$\sum_R N_R T(R) < \frac{11N}{4}. \qquad (2.4)$$

Since all of the terms on the left-hand side contribute positively, this implies the upper bound on the number of fermions in each representation $N_R < N_{R,\max}$, where

$$N_{R,\max} = \frac{11N}{4T(R)}. \tag{2.5}$$

Here and below, we implicitly carry out an analytic continuation of N_R from non-negative integers to non-negative real numbers; however, it is understood that physically they are, of course, non-negative integers. If there are few fermions, then also $b_2 > 0$, so that the two-loop beta function has a zero only at the origin, $\alpha = 0$.

A sufficient increase in the numbers of fermions in various representations leads to a reversal in the sign of b_2 from positive to negative, while still satisfying the condition of asymptotic freedom, (2.4). For a set of fermion representations $\{N_R\}$ with this property, the two-loop beta function has a zero away from the origin at

$$\alpha_{\mathrm{IR}} = -\frac{4\pi b_1}{b_2} = \frac{4\pi b_1}{|b_2|}. \tag{2.6}$$

For the theory with a single type of fermion representation, we denote the value of N_R where $b_2 = 0$ as $N_{R,\mathrm{IR}}$. This is

$$N_{R,\mathrm{IR}} = \frac{17C_2(G)^2}{2[5C_2(G) + 3C_2(R)]T(R)}. \tag{2.7}$$

The fact that $N_{R,\mathrm{IR}} < N_{R,\max}$ is evident because for $N = N_{R,\mathrm{IR}}$, b_1 has the positive (i.e., asymptotically free) value

$$b_1 = \frac{C_2(G)[6C_2(G) + 11C_2(R)]}{5C_2(G) + 3C_2(R)} > 0 \quad \text{for } N_R = N_{R,\mathrm{IR}}. \tag{2.8}$$

If $b_2 < 0$, so that there is an infrared zero of the beta function, then as the scale μ decreases from large values, $\alpha(\mu)$ increases toward this value. The infrared behavior then depends on whether or not the value of the coupling α_{IR} is sufficiently large as to cause spontaneous chiral-symmetry breaking [8]. If the properties of the theory are such that no fermion condensates form, then this is an exact infrared fixed point (IRFP) of the (perturbatively calculated) renormalization group equation for α. If, on the other hand, some fermions do condense, so that they get dynamically generated masses and are integrated out of the low-energy effective theory applicable below the scale(s) of condensation, then, since the beta function changes, the original value of α_{IR} is only an approximate IRFP. Since the coefficients b_1 and b_2 are the maximal set of coefficients in the beta function that are scheme independent, it follows that conclusions obtained from the two-loop beta function should be at least qualitatively reliable physically. However, since we will deal with values of α_{IR} of order unity, i.e., strongly coupled gauge interactions, it is understood that there are inevitably significant theoretical uncertainties in the results. In this context, we recall that the two-loop perturbative beta function is an asymptotic ex-

pansion in α and does not include a number of important effects, including confinement and instantons. Indeed, instanton effects involve factors like $\exp(-c\pi/\alpha)$ (where c is a constant), which cannot be seen to any order of perturbation theory. Moreover, it should be noted that even if there is no zero of the two-loop beta function away from the origin, i.e., a perturbative IRFP, the beta function may exhibit a nonperturbative slowing of the running associated with the fact that at energy scales below the confinement scale, the physics is not accurately described in terms of the Lagrangian degrees of freedom (fermions and gluons) [9–11]. We observe that one can calculate α_{IR} more accurately using the higher-order coefficients of the beta function. Finally, although an asymptotically free vectorial SU(N) gauge theory of the type that we consider here does not require an ultraviolet completion, it could, as remarked above, arise as the low-energy effective field theory resulting from the breaking of a larger, chiral, gauge symmetry. In this case, one would also want to assess the effects of residual higher-dimensional operators from this larger gauge theory (e.g., [12]).

B. Results from approximate solution of Dyson-Schwinger equation for fermion propagator

A solution of the Dyson-Schwinger (DS) equation for the propagator of a fermion ψ in the representation R of the gauge group, with zero bare mass, in the approximation of one-gluon (also called ladder) exchange, yields a nonzero, dynamically generated mass if the coupling $\alpha(\mu)$ exceeds a critical value $\alpha_{R,\mathrm{cr}}$ given by [13–16]

$$\frac{3C_2(R)\alpha_{R,\mathrm{cr}}}{\pi} = 1. \tag{2.9}$$

In the same ladder approximation, the anomalous dimension for the fermion (bilinear) mass operator is $\gamma = 1$ at $\alpha = \alpha_{\mathrm{cr},R}$. Some lattice studies have reported initial results on measurements of γ [17,18]. Corrections to the one-gluon exchange approximation have been analyzed and found not to be too large [19]. To assess the implications of these corrections for the boundary of the chirally symmetric phase, one also calculates α_{IR} to the corresponding higher order. Since the dynamically generated mass for this fermion is the coefficient of the bilinear fermion operator in an effective Lagrangian, this indicates the formation of a condensate of the fermions in the representation R, and associated spontaneous chiral-symmetry breaking (SχSB) by the gauge interaction, as α increases through the critical value $\alpha_{R,\mathrm{cr}}$ Some early studies with lattice simulations of chiral-symmetry breaking were carried out for SU(2) and SU(3) for various fermion representations in [20–22]. There has been considerable recent lattice work, mainly on the group SU(3) with fermions in the fundamental representation or rank-2 symmetric (sextet) representation and on SU(2) with fermions in the adjoint (equivalent to

rank-2 symmetric) representation. Some of the rapidly increasing number of papers reporting results from numerical lattice simulations include Refs. [17,18]. To our knowledge, there have not been lattice studies of chiral-symmetry breaking in a theory containing dynamical fermions in two or more different representations (simultaneously present).

The analysis of the gauge coupling evolution and chiral-symmetry realization in vectorial asymptotically free gauge theories has been of particular interest in the context of technicolor (TC) theories [23], especially in the context of the most promising such theories, which exhibit a slowly running ("walking") gauge coupling associated with an approximate infrared fixed point of the renormalization group [15] (see also [24]). In the actual application to theories of dynamical electroweak symmetry breaking, one must embed the technicolor sector in a larger theory, extended technicolor (ETC) in order to give masses to quarks and leptons and to account for their generational structure [25]. A necessary property of TC/ETC theories is that the ETC symmetry must break in a series of stages to the TC symmetry, which is an asymptotically free, vectorial theory that becomes strongly coupled on the TeV scale, producing bilinear technifermion condensates that break the electroweak gauge symmetry. ETC is constructed as an asymptotically free chiral gauge symmetry, which becomes strongly coupled and hence forms condensates that self-break the ETC symmetry. In reasonably ultraviolet-complete ETC models [26] it is also necessary to include another auxiliary, strongly coupled gauge interaction. Accounting for the large mass splitting between the t and b quarks may require additional mechanisms [27] (recent reviews of TC/ETC include [3,28,29]). In this paper we do not try to construct quasirealistic models of dynamical electroweak symmetry breaking but instead focus on the SU(N) vectorial gauge theory with fermions in different representations as an interesting problem in abstract nonperturbative field theory.

It should be mentioned that, in principle, an asymptotically free, vectorial gauge theory with a certain set of massless fermions might confine without producing any spontaneous chiral-symmetry breaking. The spectrum would thus include a set of massless gauge-singlet composite fermions. A necessary (but not sufficient) condition for this to occur is that there should be a matching of the global chiral anomalies between the fermion fields in the Lagrangian and the gauge-singlet massless composite fermions [30]. In our present study we will focus on the situation in which, as suggested by the analysis of the Dyson-Schwinger equation for the fermion propagator(s), there is spontaneous chiral-symmetry breaking. In this context, we recall a simple heuristic physical argument that confinement produces SχSB, namely, that as a massless fermion heading outward from the interior of a gauge-singlet state is "reflected" back at the boundary, its chi-

rality flips, and this is equivalent to the presence of a mass term in the effective Lagrangian [31]. However, although our analysis is restricted to nonsupersymmetric gauge theories, we note for completeness that supersymmetric SU(N) gauge theories can, for a certain range in the number of chiral superfields, exhibit confinement without SχSB [32].

C. βDS Method for determining chiral phase boundary

Here we recall a method to estimate the critical value, $N_{R,\mathrm{cr}}$ of the number of fermions in a single representation R beyond which the theory goes from a phase with spontaneous chiral-symmetry breaking to a phase without such breaking [16]. The method combines an analysis of the beta function and coupling constant evolution into the infrared with an expression for the critical coupling from an approximation solution of the DS equation, and hence we call it the βDS method.

Let us first consider the theory with N_R fermions transforming according to a single representation R. If N_R is sufficiently small that $b_2 > 0$, then as the reference scale μ decreases from large values, $\alpha(\mu)$ increases until it exceeds the critical value $\alpha_{R,\mathrm{cr}}$ at which there is the formation of a bilinear condensate of the fermions

$$\langle \bar{\psi}\psi \rangle \equiv \sum_{j=1}^{\dim(R_j)} \langle \bar{\psi}_j \psi_j \rangle = \sum_{j=1}^{\dim(R_j)} \langle \bar{\psi}_{j,L} \psi_{j,R} \rangle + \mathrm{H.c.} \quad (2.10)$$

[For the gauge group SU(2), the condensate can be written in terms of a product of same-chirality fermions, as discussed below.] If N_R is sufficiently large that $b_2 < 0$, then the two-loop beta function has an infrared zero at α_{IR}. The value of α_{IR} is a monotonically decreasing function of N_R, with partial derivative

$$\frac{\partial \alpha_{\mathrm{IR}}}{\partial N_R} = -\frac{12\pi T(R)C_2(G)[7C_2(G) + 11C_2(R)]}{[17C_2(G)^2 - 2N_R\{5C_2(G) + 3C_2(R)\}]^2}. \quad (2.11)$$

If the theory has only one type of fermion representation R, then as N_R increases through a critical value $N_{R,\mathrm{cr}}$, and the value of α_{IR} decreases through the critical value $\alpha_{R,\mathrm{cr}}$, the condensate vanishes and the theory goes over to one without any spontaneous breaking of chiral symmetry. Setting

$$\alpha_{\mathrm{IR}} = \alpha_{R,\mathrm{cr}} \quad (2.12)$$

yields a solution for the critical number $N_{R,\mathrm{cr}}$ for this case where the theory has fermions in only one representation, R. Stated in other terms, if $N_R < N_{R,\mathrm{cr}}$, then as the theory evolves into the infrared, $\alpha(\mu)$ eventually increases above the critical value $\alpha_{R,\mathrm{cr}}$, the fermions condense and gain dynamical masses of order the condensation scale, and the evolution further into the infrared of the low-energy effective theory applicable below this scale is governed by a different beta function. Thus, as noted above, in this case, α_{IR} is only an approximate infrared fixed point. Here, with

fermions in a single representation, below the condensation scale, the beta function would be that of the pure gauge theory with no fermions, and hence would not have a perturbative infrared fixed point. The only light degrees of freedom in this theory would be the Nambu-Goldstone bosons (NGB's) resulting from the breaking of the global chiral symmetry by the fermion condensates, and these, being derivatively coupled, become noninteracting as the energy scale goes to zero. If, on the other hand, $N_R > N_{R,cr}$, then $\alpha_{IR} < \alpha_{R,cr}$, so that no condensates form, there is thus no spontaneous chiral-symmetry breaking, and α_{IR} is an exact infrared fixed point. As N_R increases to $N_{R,max}$ so that b_1 decreases to zero, the value of b_2 approaches a nonzero value, so that $\alpha_{IR} \to 0$. The value of b_2 at $N_R = N_{R,max}$ is $N(7N + 11C_2(R))$.

We next consider the general case of massless fermions transforming according to several different types of representations, denoted, as above, by the set of numbers $\{N_R\}$. As the reference scale μ decreases from large values where the coupling $\alpha(\mu)$ is small, this coupling increases. There are then two possibilities: (i) $b_2 > 0$, so that the two-loop beta function does not have an infrared zero, and the coupling $\alpha(\mu)$ increases until it exceeds the critical value for fermion condensation; (ii) $b_2 < 0$, so that the two-loop beta function does have an infrared zero, and $\alpha(\mu)$ increases toward this value. Under category (ii) there are two subcategories, just as there were for the case of a single type of fermion representation: (iia) the numbers $\{N_R\}$ are sufficiently small that α_{IR} is greater than the critical value for some condensate to form, and (iib) the numbers $\{N_R\}$ are sufficiently large so that α_{IR} is less than the critical value for any condensate to form. In cases (iia) and (iib), α_{IR} is an approximate and exact infrared fixed point, respectively.

Let us assume that the set $\{N_R\}$ is such that either case (i) or case (iia) holds. Then as the scale μ decreases from large values, the coupling $\alpha(\mu)$ increases sufficiently so that there is condensation in the most attractive channel. For a channel in which fermions of representations R_1 and R_2 form a condensate transforming as R_{cond},

$$R_1 \times R_2 \to R_{cond}, \qquad (2.13)$$

a measure of the attractiveness is

$$\Delta C_2 = C_2(R_1) + C_2(R_2) - C_2(R_{cond}). \qquad (2.14)$$

The maximization of ΔC_2 implies that in a vectorial gauge theory, the most attractive channels are always of the form

$$R \times \bar{R} \to 1 \qquad (2.15)$$

for various R, which preserve the gauge invariance. Furthermore, for channels of the form (2.15), $\Delta C_2 = 2C_2(R)$, so that the criterion for the critical coupling is, in the one-gluon exchange approximation to the DS equation,

$$\frac{3\alpha\Delta C_2}{2\pi} = \frac{3\alpha C_2}{\pi} = 1, \qquad (2.16)$$

as in Eq. (2.9). It follows that as the theory evolves from high scales μ to lower scales, as $\alpha(\mu)$ increases, if it exceeds a critical value for condensation, the one-gluon exchange approximation predicts that this will occur first in the channel (2.15) with the largest value of $C_2(R)$. Let us denote the scale where this occurs as Λ_1. That is, with this one-gluon approximation to the Dyson-Schwinger equation, the fermion with the largest value of $C_2(R)$ has the smallest value of $\alpha_{R,cr}$ and hence forms a condensate at this highest condensation scale. Associated with this condensation, the fermions transforming according to this representation gain a dynamical mass of order Λ_1. In the low-energy effective field theory that is applicable at scales below Λ_1, these fermions are then integrated out, and the theory evolves in a manner determined by a new beta function, calculated without these fermions.

For sets of numbers $\{N_R\}$ for which case (iia) holds, we again find that the partial derivative of α_{IR} with respect to one of the numbers N_R, denoted N_{R_i}, with the others, N_{R_j} with $j \neq i$, held fixed, is negative:

$$\frac{\partial \alpha_{IR}}{\partial N_{R_i}} < 0. \qquad (2.17)$$

Hence, the same logic applies as before. We can start with a set of fermions $\{N_R\}$ which is such that $b_2 < 0$, so that there is an infrared zero of the two-loop beta function, and the numbers N_R are sufficiently small that α_{IR} is large, and the theory forms chiral-symmetry breaking condensates. We can then increase one of the numbers, N_{R_i}, with the others held fixed. As we do this, α_{IR} decreases, and eventually decreases through the critical value given in Eq. (2.9) for condensation in the channel $R_i \times \bar{R}_i \to 1$, at which point this condensate vanishes. The condition in Eq. (2.12) then defines a critical value $N_{R_i,crit}$. However, in contrast to the simpler case of the theory with fermions in only a single representation, now the critical value $N_{R_i,cr}$ for a given R_i depends on the values of the numbers, N_{R_j}, $j \neq i$, of fermions transforming according to other representations of the gauge group. Another tool that has been applied to analyze chiral-symmetry breaking is a conjectured inequality concerning thermal degrees of freedom [33,34].

D. Global chiral symmetry

For a vectorial SU(N) theory with $N \neq 2$ with massless fermions in a set of representations $\{R\} \equiv \{R_1, R_2, \ldots R_k\}$, such that the numbers of (Dirac) fermions are $\{N_R\} \equiv \{N_{R_1}, N_{R_2}, \ldots, N_{R_k}\}$, the formal (classical) global chiral symmetry is $\prod_{i=1}^{k} U(N_{R_i})_L \times U(N_{R_i})_R$. For each R_i, the group $U(N_{R_i})_L \times U(N_{R_i})_R$ can be rewritten as

$$SU(N_{R_i})_L \times SU(N_{R_i})_R \times U(1)_{R_i,V} \times U(1)_{R_i,A}. \qquad (2.18)$$

The vectorial global symmetry $U(1)_{R_i,V}$ represents the conservation of fermion number for the fermions in the representation R_i. Each of the k axial global symmetries $U(1)_{R_i,A}$ is broken by $SU(N)$ instantons [35], with divergences of the corresponding axial-vector currents $\partial_\lambda J_{R_i}^{A,\lambda} \propto [\alpha/(4\pi)]T(R_i)F_{\mu\nu}^a \tilde{F}^{a,\mu\nu}$. From these k broken symmetries $U(1)_{R_i,A}$, $i = 1, \ldots, k$, one can construct $k-1$ linear combinations that are conserved in the presence of instantons, which we denote $\mathcal{U}(1)_{s,A}$, $s = 1, \ldots, k-1$ with currents $\mathcal{J}_s^{A\lambda}$. Let us define

$$\hat{J}_{R_i}^{A,\lambda} \equiv \frac{J_{R_i}^{A,\lambda}}{T(R_i)}. \tag{2.19}$$

One of the conserved currents is (up to a normalization factor)

$$\mathcal{J}_1^{A\lambda} \propto \hat{J}_{R_1}^{A,\lambda} - \hat{J}_{R_2}^{A,\lambda}. \tag{2.20}$$

The others are constructed by Gram-Schmidt orthonormalization. For example, for $k = 3$, the other one is

$$\mathcal{J}_2^{A\lambda} \propto \tfrac{1}{2}[\hat{J}_{R_1}^{A,\lambda} + \hat{J}_{R_2}^{A,\lambda} - 2\hat{J}_{R_3}^{A,\lambda}]. \tag{2.21}$$

Thus the actual (continuous) nonanomalous global symmetry of the theory, before any fermion condensates form, is

$$G_{\text{global}} = \left[\prod_{i=1}^{k} SU(N_{R_i})_L \times SU(N_{R_i})_R \times U(1)_{R_i,V}\right]$$
$$\times \left[\prod_{s=1}^{k-1} \mathcal{U}(1)_{A,s}\right]. \tag{2.22}$$

The resultant realization of this global symmetry depends on the gauge coupling evolution and whether the coupling α increases above the critical value for condensation of the fermions in the R_i representation. As an example, let us assume that all of the fermions condense, at the respective different scales $\Lambda_{N_{R_i}}$, $i = 1, \ldots, k$. For our discussion here we shall label the representation with the largest value of $C_2(R_i)$ as R_1; from the one-gluon exchange approximation, it then follows that the R_1 fermions condense at the highest scale, Λ_1. In accordance with the most attractive channel arguments recalled above, the fermion condensate of the form $\langle \bar{\psi}_{R_1} \psi_{R_1} \rangle$ preserves the global $U(1)_{R_1,V}$ and breaks the non-Abelian global symmetry from $SU(N_{R_1})_L \times SU(N_{R_1})_R$ to its diagonal, vectorial subgroup, $SU(N_{R_1})_V$. This condensate also breaks each of the $k-1$ $\mathcal{U}(1)_{s,A}$ axial symmetries. In the low-energy effective field theory applicable at scales $\mu < \Lambda_1$, with the fermions in the R_1 representation having gained dynamical masses of order Λ_1 and having been integrated out, one can construct $k-2$ appropriate linear combinations of the former $k-1$ $\mathcal{U}(1)_{s,A}$ axial symmetries that exclude the R_1 fermions and are preserved in the presence of instantons. We denote these as $\mathcal{U}(1)'_{s,A}$. The continuous global

symmetry group of this low-energy effective theory below Λ_{R_1} is then

$$G'_{\text{global}} = \left[\prod_{i=2}^{k} SU(N_{R_i})_L \times SU(N_{R_i})_R\right] \times \left[\prod_{i=1}^{k} U(1)_{R_i,V}\right]$$
$$\times \left[\prod_{s=1}^{k-2} \mathcal{U}(1)'_{A,s}\right]. \tag{2.23}$$

The number of broken generators of continuous global Lie algebras at the first scale is $N_{R_1}^2 - 1$ from the breaking of the non-Abelian group, plus one for the breaking of one linear combination of the $k-1$ nonanomalous axial $U(1)$ symmetries, for a total of $N_{\text{NGB},\Lambda_{R_1}} = N_{R_1}^2$ Nambu-Goldstone bosons resulting from this first level of fermion condensation. One repeats this process at each of the various condensation scales. The NGB's produced at each level couple derivatively, and hence become progressively more weakly interacting as powers of μ/f_{R_i}, where f_{R_i} is the generalization of the pion decay constant applicable to the condensation of the R_i fermions.

In the case $N = 2$, because $SU(2)$ has only (pseudo)real representations, the analysis of the global symmetry is different than in the case of $SU(N)$ with $N \neq 2$. If, for example, one has an $SU(2)$ theory with N_f (Dirac) fermions in the fundamental representation, then one can reexpress these fermions as a set of $2N_f$ chiral (say, left-handed) fermions, and the covariant derivative term has the form $\bar{\psi}_L \gamma \cdot D\psi_L$, where ψ is a $2N_f$-dimensional vector of left-handed fermions. It follows that the formal (classical) global symmetry in this case is $U(2N_f)_L$, or equivalently, $SU(2N_f)_L \times U(1)_L$. The $U(1)_L$ is broken by the $SU(2)$ instantons [35], so that the nonanomalous global symmetry is $SU(2N_f)_L$. The condensates are of the form $\langle \epsilon_{ab} \psi_{p,L}^{aT} C \psi_{p',L}^b \rangle$, where ϵ_{ab} is the antisymmetric tensor density for $SU(2)$ and $1 \leq p, p' \leq 2N_f$. If the fermions are in the rank-2 symmetric (equivalently, the adjoint) representation, of the form $\psi_{p,L}^{ab}$ with $1 \leq p \leq 2N_f$, then the condensates are of the form $\langle \epsilon_{ar}\epsilon_{bs} \psi_{p,L}^{abT} C \psi_{p',L}^{rs} \rangle$, and so forth for higher-dimensional representations. These condensates break the $SU(2N_f)_L$ down to its symplectic subgroup, $Sp(2N_f)_L$. In this case there are thus $N_{\text{NGB}} = 2N_f^2 - N_f - 1$ Nambu-Goldstone bosons.

III. $SU(N)$ GAUGE THEORY WITH FERMIONS IN A SINGLE REPRESENTATION

In this section we review some results on an $SU(N)$ gauge theory with fermions in a single representation, which will serve as a useful background for our analysis of the theory with fermions in multiple different representations.

A. Fundamental representation

For the SU(N) theory with N_F Dirac fermions in the fundamental representation F ($= \square$ in Young tableau notation), the condition for asymptotic freedom yields the upper bound $N_F < N_{F,\max}$, where

$$N_{F,\max} = \frac{11N}{2}. \qquad (3.1)$$

The coefficient b_2 changes sign from positive to negative as N_F increases through the value

$$N_{F,\mathrm{IR}} = \frac{34N^3}{13N^2 - 3}, \qquad (3.2)$$

which is always less than $N_{F,\max}$. For $N_{F,\mathrm{IR}} < N_F < N_{F,\max}$, the beta function has a zero away from the origin at

$$\alpha_{\mathrm{IR}} = \frac{4\pi(11N - 2N_F)}{-34N^2 + N_F(13N - 3N^{-1})}. \qquad (3.3)$$

The estimate for the critical value for condensation [from Eq. (2.9)] is

$$\alpha_{F,\mathrm{cr}} = \frac{2\pi N}{3(N^2 - 1)}. \qquad (3.4)$$

Setting $\alpha_{\mathrm{IR}} = \alpha_{F,\mathrm{cr}}$ and solving for N_F, one obtains the critical value of N_F [16]

$$N_{F,\mathrm{cr}} = \frac{2N(50N^2 - 33)}{5(5N^2 - 3)}. \qquad (3.5)$$

As $N \to \infty$, this has the series expansion

$$N_{F,\mathrm{cr}} = N\left[1 - \frac{3}{50N^2} - \frac{9}{250N^4} - O\left(\frac{1}{N^6}\right)\right]. \qquad (3.6)$$

For $N = 2$, $N_{F,\mathrm{cr}} \simeq 8$ and for $N = 3$, $N_{F,\mathrm{cr}} \simeq 12$. Recent lattice measurements for the $N = 3$ case are in broad agreement, to within the uncertainties, with this prediction [17].

The DS equation analysis is semiperturbative in the sense that it contains polynomial dependence on α, and it neglects nonperturbative effects associated with confinement and instantons. The DS equation is an integral equation, and the standard analysis of this equation involves an integration over Euclidean loop momentum k from $k = 0$ to $k = \infty$. If the theory confines, then the lower bound for the Euclidean loop momentum should actually not be $k = 0$, but instead $k = k_{\mathrm{min.}} \sim r_c^{-1}$ where r_c is the spatial confinement scale [9]. The use of $k = 0$ thus overestimates the tendency toward SχSB. Instantons enhance SχSB, and the neglect of instanton effects amounts to an underestimate of the tendency toward SχSB; since these two neglected aspects of the physics—confinement and instantons—produce errors that are of opposite sign as regards the tendency for SχSB, it is plausible that these errors tend to cancel out, so this may help to explain why the usual DS analysis may be reasonably accurate [9], at

least in the case $N = 3$ where recent lattice results are broadly consistent with it.

B. Rank-2 symmetric and antisymmetric representations

In this section we consider the two separate cases of the SU(N) theory with (i) N_S fermions in the symmetric rank-2 representation, $S \equiv \square\square$ and (ii) N_A fermions in the antisymmetric rank-2 representation, $A \equiv \begin{array}{c}\square\\\square\end{array}$. Since a number of formulas are similar for these two cases, we include them together in this section. In the case of $R = \square\square$, our analysis applies for any $N \geq 2$, while for $R = \begin{array}{c}\square\\\square\end{array}$, we take $N \geq 4$, since for $N = 2$, $\begin{array}{c}\square\\\square\end{array}$ is the singlet and for $N = 3$,

$$\begin{array}{c}\square\\\square\end{array}$$

is not a distinct representation, but is instead equivalent to $\bar{\square}$. For the SU(N) theory with N_s Dirac fermions in the symmetric rank-2 representation $\square\square$, the condition for asymptotic freedom yields the upper bounds $N_S < N_{S,\max}$, where

$$N_{S,\max} = \frac{11N}{2(N + 2)}, \qquad (3.7)$$

and $N_A < N_{A,\max}$, where

$$N_{A,\max} = \frac{11N}{2(N - 2)}. \qquad (3.8)$$

The coefficient b_2 changes sign from positive to negative as N_S and N_A increase through the respective values

$$N_{S,\mathrm{IR}} = \frac{17N^3}{(N + 2)(8N^2 + 3N - 6)} \qquad (3.9)$$

and

$$N_{A,\mathrm{IR}} = \frac{17N^3}{(N - 2)(8N^2 - 3N - 6)}, \qquad (3.10)$$

which are always less than the respective values $N_{S,\max}$ and $N_{A,\max}$.

For the theory with just N_S fermions in the S representation, and $N_{S,\mathrm{IR}} < N_S < N_{S,\max}$, the beta function (2.1) has a zero away from the origin at

$$\alpha_{\mathrm{IR},S} = \frac{2\pi(11N - 2N_S(N + 2))}{-17N^2 + N_S(8N^2 + 19N - 12N^{-1})}. \qquad (3.11)$$

The estimate for the critical value for condensation [from Eq. (2.9)] is

$$\alpha_{S,\mathrm{cr}} = \frac{\pi N}{3(N + 2)(N - 1)}. \qquad (3.12)$$

Setting $\alpha_{\mathrm{IR},S} = \alpha_{S,\mathrm{cr}}$ and solving for N_S, we obtain the critical value of N_S,

$$N_{S,\mathrm{cr}} = \frac{N(83N^2 + 66N - 132)}{5(N + 2)(4N^2 + 3N - 6)}. \quad (3.13)$$

For $N \to \infty$, this has the series expansion

$$N_{S,\mathrm{cr}} = \frac{83}{20} - \frac{649}{80N} + \frac{5027}{320N^2} + O\left(\frac{1}{N^3}\right). \quad (3.14)$$

For $N = 2$, $N_{S,\mathrm{cr}} \simeq 2.1$, while for $N = 3$, $N_{S,\mathrm{cr}} \simeq 2.5$. Some Lattice measurements for the $N = 3$ case are reported in [18]. As N increases from 2 to ∞, $N_{S,\mathrm{cr}}$ increases monotonically from $83/40 \simeq 2.08$ to $83/20 = 4.15$. [As before, although we quote the exact fractions and give the floating-point numbers to three significant figures, we emphasize that because of the strong-coupling nature of the physics and the approximations involved, these numbers have estimated theoretical uncertainties of O(1). This applies to all such estimates of $N_{R,\mathrm{cr}}$ values in this paper.]

For the theory with just N_A fermions in the A representation, and $N_{A,\mathrm{IR}} < N_A < N_{A,\max}$, the beta function (2.1) has a zero away from the origin at

$$\alpha_{\mathrm{IR},A} = \frac{2\pi(11N - 2N_A(N - 2))}{-17N^2 + N_A(8N^2 - 19N + 12N^{-1})}. \quad (3.15)$$

The estimate for the critical value for condensation [from Eq. (2.9)] is

$$\alpha_{A,\mathrm{cr}} = \frac{\pi N}{3(N - 2)(N + 1)}. \quad (3.16)$$

Setting $\alpha_{\mathrm{IR},A} = \alpha_{A,\mathrm{cr}}$ and solving for N_A, we obtain the critical value

$$N_{A,\mathrm{cr}} = \frac{N(83N^2 - 66N - 132)}{5(N - 2)(4N^2 - 3N - 6)}. \quad (3.17)$$

For $N \to \infty$, this has the series expansion

$$N_{A,\mathrm{cr}} = \frac{83}{20} + \frac{649}{80N} + \frac{5027}{320N^2} + O\left(\frac{1}{N^3}\right). \quad (3.18)$$

As N increases from 3 to ∞, $N_{A,\mathrm{cr}}$ decreases monotonically from $417/35 \simeq 11.9$ to $83/20 \simeq 4.15$.

C. Adjoint representation

For the case of N_{Adj} Dirac fermions, or equivalently, $2N_{\mathrm{Adj,Maj}}$ Majorana fermions, in the adjoint representation Adj, the condition for asymptotic freedom is $N_{\mathrm{Adj}} < N_{\mathrm{Adj,max}}$, where

$$N_{\mathrm{Adj,max}} = \frac{11}{4}, \quad (3.19)$$

i.e., $N_{\mathrm{Adj}} \leq 2$. Majorana fermions in the adjoint representation of the gauge group appear naturally in supersymmetric theories. In the present nonsupersymmetric context, we shall restrict ourselves to adjoint fermions of Dirac type. The coefficient b_2 changes sign from positive to negative as N_{Adj} increases through the value

$$N_{\mathrm{Adj,IR}} = \frac{17}{16}. \quad (3.20)$$

For $N_{\mathrm{Adj,IR}} < N_{\mathrm{Adj}} < N_{\mathrm{Adj,max}}$, the beta function has a zero away from the origin at

$$\alpha_{\mathrm{IR}} = \frac{2\pi(11 - 4N_{\mathrm{Adj}})}{N(-17N + 16N_{\mathrm{Adj}})}. \quad (3.21)$$

Setting

$$\alpha_{\mathrm{Adj,cr}} = \frac{\pi}{3N} \quad (3.22)$$

equal to $\alpha_{\mathrm{Adj,cr}}$, one solves for

$$N_{\mathrm{Adj,cr}} = \frac{83}{40} = 2.075. \quad (3.23)$$

IV. SU(2) GAUGE GROUP

For the simplest non-Abelian Yang-Mills gauge group, SU(2), we can give a rather compact general treatment that includes all possible representations. We recall that this group has only (pseudo-)real representations R, which are labeled by a single Dynkin index, the non-negative integer $p_1 = 2I$, where I will be labeled as the "isospin" (not to be confused with the actual gauged weak isospin). $I = 1/2$ is the fundamental representation, $\square$; $I = 1$ is the adjoint or equivalently, rank-2 symmetric representation, $\square\square$; $I = 3/2$ is the rank-3 symmetric representation, and so forth. The following SU(2) relations will be useful:

$$C_2(I) = I(I + 1) \quad (4.1)$$

and

$$T(I) = \frac{(2I + 1)I(I + 1)}{3}. \quad (4.2)$$

The asymptotic freedom condition (2.4) reads

$$\sum_I N_I T(I) < \frac{11}{2}, \quad (4.3)$$

where the sum over I is formally over all positive integral and half-integral values, but actually truncates, because of the fact that $C_2(I) > 11/2$ for $I \geq 2$. Hence, (4.3) reduces to the Diophantine inequality

$$\frac{1}{2}N_{1/2} + 2N_1 + 5N_{3/2} < \frac{11}{2}. \quad (4.4)$$

The nontrivial solutions to this include cases with only one type of fermion representation present. In these cases, the allowed numbers of fermions of each type are

$$N_{1/2} \leq 10, \quad (4.5)$$

$$N_1 \leq \left[\frac{11}{4}\right]_\ell = 2, \quad (4.6)$$

and

$$N_{3/2} \le \left[\frac{11}{10}\right]_\ell = 1, \qquad (4.7)$$

where here $[\nu]_\ell$ denotes the greatest integer less than or equal to ν and it is understood in each case that the N_I's for other I's are zero. We also find the following solutions of the asymptotic freedom condition with two different fermion representations present (and $N_{3/2} = 0$):

$$1 \le N_{1/2} \le 6, \qquad N_1 = 1, \qquad (4.8)$$

and

$$1 \le N_{1/2} \le 2, \qquad N_1 = 2. \qquad (4.9)$$

Substituting the general result for $C_2(I)$ in Eq. (4.1) in Eq. (2.9), we have, in this approximation,

$$\alpha_{I,\mathrm{cr}} = \frac{\pi}{3I(I+1)}. \qquad (4.10)$$

The predictions for the case of $N_{1/2}$ massless Dirac fermions in the fundamental representation are well known [16]. The two-loop coefficient b_2 reverses sign from positive to negative as $N_{1/2}$ increases through the value $272/49 \simeq 5.55$ and decreases through negative values as $N_{1/2}$ increases. The zero of the two-loop beta function occurs at

$$\alpha_{\mathrm{IR}} = \frac{4\pi(11 - N_{1/2})}{49N_{1/2} - 272} SU(2), \qquad I = 1/2. \qquad (4.11)$$

Equating $\alpha_{\mathrm{IR}} = \alpha_{1/2,\mathrm{cr}} = 4\pi/9$ or substituting $N_c = 2$ into Eq. (3.5), one obtains the critical value for the case with only fermions in the $I = 1/2$ representation, $N_{1/2,\mathrm{cr}} = 668/85 \simeq 7.9$.

There are two other cases where the theory involves only fermions of a single type of representation, namely, those with $I = 1$ and $I = 3/2$. For the symmetric rank-2 tensor, or equivalently adjoint, representation, $I = 1$, substituting $N_c = 2$ into Eq. (3.9) or using (3.20) shows that b_2 reverses sign from positive to negative as N_1 increases through the value $17/16$. Similarly, substituting $N_c = 2$ into Eq. (3.11) or using (3.21), one derives that

$$\alpha_{\mathrm{IR},I=1} = \frac{\pi(11 - 4N_1)}{16N_1 - 17}. \qquad (4.12)$$

Setting this equal to $\alpha_{\mathrm{cr},I=1} = \frac{\pi}{6} \simeq 0.52$ or using Eq. (3.23) directly, one has $N_{\mathrm{cr},I=1} = 83/40 = 2.075$.

Finally, among the cases with a single fermion representation present, there is the case of fermions with $I = 3/2$. For this case, b_2 reverses sign from positive to negative as $N_{3/2}$ increases through the value $N_{3/2} = 8/25 = 0.32$. The two-loop beta function has a zero away from the origin at

$$\alpha_{\mathrm{IR},I=3/2} = \frac{8\pi(11 - 10N_{3/2})}{17(25N_{3/2} - 8)}. \qquad (4.13)$$

Setting this equal to $\alpha_{\mathrm{cr},I=3/2} = 4\pi/45 \simeq 0.28$, we get the critical value

$$N_{\mathrm{cr},I=3/2} = \frac{1126}{1325} \simeq 0.85. \qquad (4.14)$$

Since α_{IR} decreases with increasing $N_{3/2}$ and since the minimal nonzero value $N_{f,\mathrm{cr},I=3/2}$ is 1, this predicts that with one such Dirac fermion with $I = 3/2$, the infrared fixed point is below the value for condensation and hence is an exact IR fixed point. That is, the gauge coupling will evolve to this point without any condensate involving the $I = 3/2$ forming, so that it does not gain any dynamical mass and remains massless. Thus, in the infrared limit of this theory the fermion is massless. We summarize our results for SU(2) in Table I.

TABLE I. Some numerical results for the SU(2) theory. IRFP denotes an (exact or approximate) infrared fixed point of the renormalization group equation for α. "nIRFP" means that the two-loop beta function does not have such an IRFP, i.e., a zero away from the origin. In the columns marked c_I for $I = 1/2, 1$ we indicate with a y (yes) or n (no) whether the βDS method with the one-gluon (ladder) approximation to the DS equation predicts that there is condensation of the isospin I fermions. The notation m means "maybe," reflecting the substantial theoretical uncertainties in the βDS predictions due to the strong-coupling nature of the physics. If the theory with all of its massless fermions has a IRFP, this is marked as $\alpha_{\mathrm{IR},h}$, where h stands for "highest scale." If the low-energy effective field theory applicable for energies below the highest condensation scale has an IRFP, this is denoted $\alpha_{\mathrm{RI},\ell}$, where ℓ stands for "lower scale." In cases where no condensation occurs for any of the isospin I fermions, $\alpha_{\mathrm{IR},h} = \alpha_{\mathrm{IR},\ell}$.

$N_{1/2}$	N_1	$\alpha_{\mathrm{IR},h}$	$\alpha_{\mathrm{IR},\ell}$	$c_{1/2}$	c_1
1	0	nIRFP	$\cdots$	y	$\cdots$
2	0	nIRFP	$\cdots$	y	$\cdots$
3	0	nIRFP	$\cdots$	y	$\cdots$
4	0	nIRFP	$\cdots$	y	$\cdots$
5	0	mIRFP	$\cdots$	y	$\cdots$
6	0	11.4	$\cdots$	y	$\cdots$
7	0	2.83	$\cdots$	y	$\cdots$
8	0	1.26	$\cdots$	m	$\cdots$
9	0	0.59	$\cdots$	n	$\cdots$
10	0	0.23	$\cdots$	n	$\cdots$
0	1	nIRFP	$\cdots$	$\cdots$	$\cdots$
1	1	nIRFP	nIRFP	y	y
2	1	11.42	nIRFP	y	y
3	1	2.83	nIRFP	y	y
4	1	1.26	nIRFP	y	y
5	1	0.59	mIRFP	y	y
6	1	0.23	0.23	n	n
0	2	1.26	$\cdots$	$\cdots$	y
1	2	0.59	nIRFP	y	m
2	2	0.23	0.23	n	n

SU(2) theory with fermions in several representations

We next consider the SU(2) theory with fermions in several different representations. As discussed above, the requirement of asymptotic freedom limits the possible numbers of fermions, delineated by the numbers $N_{1/2}$, N_1, and $N_{3/2}$. For the case of $N_{3/2} = 0$, we have

$$b_2 = \frac{1}{3}\left[136 - \frac{N_{1/2}}{2} - 128N_1\right]. \qquad (4.15)$$

For $N_1 = 1$ and $N_{1/2} = 0, 1$, it follows that $b_2 > 0$, so that the beta function has no infrared zero away from the origin. This means that as the scale μ decreases, α increases until it exceeds the value $\alpha_{\mathrm{cr},I=1}$, and the $I = 1$ fermions condense. They are then integrated out, and the theory evolves further into the infrared as governed by the beta function with only the $I = 1/2$ fermions present. The coupling thus increases further until it exceeds the value $\alpha_{\mathrm{cr},I=1/2}$, at which point these $I = 1/2$ fermions condense.

For $N_1 = 1$ and $2 \leq N_{1/2} \leq 6$, $b_2 < 0$ and so the beta function has an infrared zero away from the origin, at

$$\alpha_{\mathrm{IR}} = \frac{\pi(1328 - 145N_{1/2} - 640N_1)}{6(49N_{1/2} + 256N_1)}. \qquad (4.16)$$

If this is less than the critical value (2.9), then no fermion condensates form. Setting this α_{IR} equal to the smaller of the two critical values, $\alpha_{\mathrm{cr},I=1}$, one derives the condition for condensation of the $I = 1$ fermions. This is

$$N_{1/2} + \frac{128}{29}N_1 < \frac{1328}{145} \simeq 9.16. \qquad (4.17)$$

In addition to the cases with $N_1 = 0$ dealt with above, this condition is satisfied for $N_1 = 1$ and $2 \leq N_{1/2} \leq 5$ (check). For $N_1 = 1$ and $N_{1/2} = 6$, the two-loop beta function has $\alpha_{\mathrm{IR}} = 0.23$, below the value for condensation of both the $I = 1$ and $I = 1/2$ fermions. Hence, in this case, this is an exact infrared fixed point, and the theory evolves into the infrared without any spontaneous chiral-symmetry breaking.

For $N_1 = 2$, the condition of asymptotic freedom, $2N_{1/2} + 8N_1 < 22$, is $N_{1/2} < 3$. Aside from the case $N_{1/2} = 0$ dealt with above, for the cases $N_{1/2} = 1$ and $N_{1/2} = 2$, the two-loop beta function has an infrared zero at the respective values $\alpha = 32\pi/289 \simeq 0.35$ and $\alpha = 8\pi/169 \simeq 0.15$, both of which are smaller than the estimate $\alpha_{\mathrm{cr},I=1} = \pi/6$, so that the βDS analysis predicts that no condensate occurs and the theory evolves into the infrared in a phase without any spontaneous chiral-symmetry breaking.

V. SU(3) GAUGE GROUP

It is also of interest to investigate properties of a (vectorial, asymptotically free) SU(3) gauge theory with multiple fermion representations. We recall that the representations of SU(3) are labeled by a set of two Dynkin indices

(p_1, p_2), where p_i are non-negative integers. We use the following results from group theory. The dimension of the representation is

$$\dim(p_1, p_2) \equiv d(p_1, p_2)$$
$$= (1 + p_1)(1 + p_2)\left(1 + \frac{p_1 + p_2}{2}\right). \qquad (5.1)$$

The quadratic Casimir invariant is

$$C_2(p_1, p_2) = \tfrac{1}{3}[p_1^2 + p_2^2 + p_1 p_2 + 3(p_1 + p_2)] \qquad (5.2)$$

and the trace invariant is

$$T(p_1, p_2) = \frac{\dim(p_1, p_2)C_2(p_1, p_2)}{8}. \qquad (5.3)$$

The asymptotic freedom condition (2.4) reads

$$\sum_R N_R T(R) < \frac{33}{4}, \qquad (5.4)$$

where, again, the sum over representations truncates because for sufficiently large values of p_1 and/or p_2, $T(p_1, p_2) > 33/4$. We find that it is satisfied by the following nonsinglet representations labeled by their dimension and values of (p_1, p_2):

$$R_{(p_1, p_2)} = 3_{(1,0)}6_{(2,0)}, 8_{(1,1)}, 10_{(3,0)}. \qquad (5.5)$$

The asymptotic freedom condition (5.4) thus can be written explicitly as the Diophantine inequality

$$\frac{1}{2}N_3 + \frac{5}{2}N_6 + 3N_8 + \frac{15}{2}N_{10} < \frac{33}{4}. \qquad (5.6)$$

In the case where the theory has fermions in only one of these representations $R = (p_1, p_2)$, the upper bounds on the corresponding number N_R are, in addition to $N_3 \leq [33/2]_\ell = 16$,

$$N_6 \leq \left[\frac{33}{10}\right]_\ell = 3, \qquad (5.7)$$

$$N_8 \leq \left[\frac{11}{4}\right]_\ell = 2, \qquad (5.8)$$

and

$$N_{10} \leq \left[\frac{11}{10}\right]_\ell = 1. \qquad (5.9)$$

In each of these inequalities, it is understood that the N_R's for other representations are zero.

For the case of multiple fermion representations, we find that the asymptotic freedom condition is satisfied for the following combinations of two fermion representations (where N_R's that do not appear are zero):

$$1 \leq N_3 \leq 11, \qquad N_6 = 1, \qquad (5.10)$$

$$1 \leq N_3 \leq 6, \qquad N_6 = 2, \qquad (5.11)$$

$$N_3 = 1, \qquad N_6 = 3, \qquad (5.12)$$

$$1 \le N_3 \le 10, \qquad N_8 = 1, \qquad (5.13)$$

$$1 \le N_3 \le 4, \qquad N_8 = 2, \qquad (5.14)$$

$$1 \le N_6 \le 2, \qquad N_8 = 1, \qquad (5.15)$$

and

$$N_3 = 1, \qquad N_{10} = 1. \qquad (5.16)$$

We also find the asymptotic freedom condition allows the following combination of three fermion representations:

$$1 \le N_3 \le 5, \qquad N_6 = 1, \qquad N_8 = 1. \qquad (5.17)$$

It is straightforward to calculate the values of b_2 for each of the various sets $\{N_R\}$ involving one or several different fermion representations. As before, if $b_2 > 0$, then α definitely increases past $\alpha_{R,\text{cr}}$ for at least one of the fermion representations R, and one analyzes the sequential condensations accordingly. If $b_2 < 0$, then one determines whether the behavior is of type (iia) or (iib) in the classification discussed above, i.e. whether α_{IR} is greater than $\alpha_{R,\text{cr}}$ for some R or α_{IR} is less than the minimum $\alpha_{R,\text{cr}}$. All of these types of behavior are exhibited by various sets $\{R\}$ among those allowed by asymptotic freedom.

VI. RELATIVE SCALES OF CONDENSATION

In an SU(N) gauge theory with N_R fermions in a single representation R and with a small, perturbatively calculable value of $\alpha(\mu_{\text{UV}})$ at some high scale, μ_{UV}, provided that N_R is sufficiently small that there exists a scale $\mu = \Lambda_R$ at which $\alpha(\mu)$ increases beyond the critical value $\alpha_{R,\text{cr}}$ for condensation, then one can estimate this scale by integrating the renormalization group equation, with the leading-order result

$$\Lambda_R \simeq \mu_{\text{UV}} \exp\left[-\frac{2\pi}{b_1(R)} (\alpha(\mu_{\text{UV}})^{-1} - \alpha_{R,\text{cr}}^{-1}) \right], \quad (6.1)$$

where we have indicated explicitly the dependence of the beta function coefficient $b_1 = (1/3)(11N_c - 4N_R T(R))$ on R. One can, of course, calculate Λ_R to greater accuracy by including higher-order terms in the beta function, as well as estimates of important physics effects not included in the perturbative beta function, such as instantons, but this leading-order result will be sufficient for our discussion here. From Eq. (6.1), it follows that if one compares an SU(N) theory with fermions in the single representation R_i with a different SU(N) theory with fermions in the single representation R_j, the ratio of the condensation scales, $\Lambda_{R_i}/\Lambda_{R_j}$, depends on all of the parameters μ_{UV}, N_{R_i}, and N_{R_j}, as well as the ladder estimates for the respective critical couplings, $\alpha_{R_i,\text{cr}}$ and $\alpha_{R_j,\text{cr}}$. For fixed values of μ_{UV}, $\alpha_{\text{UV}}(\mu)$, and N_{R_i}, one may ask how $\Lambda_{R_i}/\mu_{\text{UV}}$ depends on R_i. There are two countervailing effects that are relevant here: (i) as the dimension dim(R_i) of a representation R_i increases, the value of $C_2(R_i)$ also tends to increase (although the dependence is necessarily monotonic [36]), and hence the critical value of the coupling, $\alpha_{R_i,\text{cr}}$, decreases; if this increase were the only effect, then $\Lambda_{R_i}/\mu_{\text{UV}}$ would increase with increasing size of R_i. However, there is an effect that goes in the opposite direction, namely, (ii) as the dimension dim(R_i) of the representation R_i increases, the value of $T(R_i)$ also increases, thereby reducing $b_1(R_i)$, and slowing down the increase of α as μ descends from μ_{UV}. Indeed, a sufficient increase in the size of the representation R_i, for a fixed N_{R_i}, can even change the infrared behavior of the theory to preclude any spontaneous chiral-symmetry breaking and condensate formation. Thus, for a general R_i, one cannot draw a very robust conclusion about how, for fixed values of μ_{UV}, $\alpha(\mu_{\text{UV}})$, and N_{R_i}, the condensation scale Λ_{R_i} depends on the size of R_i.

In situations in which the theory has fermions in two or more different representations and these form condensates at different mass scales, it is of interest to calculate the ratio(s) of these scales. In carrying out this analysis, one acknowledges that, owing to the fact that the theory is strongly coupled at these scales, it is only possible to obtain rough estimates of such a ratio of condensation scales. Let us consider the SU(N) theory with the specific set of Dirac fermions $\{N_R\} = \{N_{R_1}, N_{R_2}\}$, say, where the R_i, $i = 1, 2$, are two different (nonsinglet) representations of SU(N). Without loss of generality, we label the representations such that $C_2(R_1) > C_2(R_2)$. As always, we require that this set $\{N_R\}$ have the property that the theory is asymptotically free, and here we also require that the set is such that condensates of both types of fermions occur, since otherwise there is no ratio to estimate. Again, we assume that at the high reference scale μ_{UV} the coupling $\alpha(\mu_{\text{UV}})$ is small and the theory is perturbatively calculable. As μ decreases from μ_{UV}, the first condensation occurs when $\alpha(\mu) = \alpha_{R_1,\text{cr}}$, where $\alpha_{R_1,\text{cr}}$ was given in Eq. (2.9), from the solution of the Dyson-Schwinger equation in the approximation of one-gauge-boson exchange. Solving the renormalization group equation to leading order, we have, for the scale at which this condensation occurs, the result

$$\Lambda_1 \simeq \mu_{\text{UV}} \exp\left[-\frac{2\pi}{b_1} (\alpha(\mu_{\text{UV}})^{-1} - \alpha_{R_1,\text{cr}}^{-1}) \right]$$

$$\simeq \mu_{\text{UV}} \exp\left[-\frac{2\pi}{b_1} \left(\alpha(\mu_{\text{UV}})^{-1} - \frac{3C_2(R_1)}{\pi} \right) \right], \quad (6.2)$$

where b_1 is given by the appropriate special case of Eq. (2.2) with the full set $\{N_{R_1}, N_{R_2}\}$ of fermions. The N_{R_1} fermions in the condensates gain dynamical masses of order Λ_1 and are integrated out of the low-energy effective field theory applicable for $\mu < \Lambda_1$. The coupling

$\alpha(\mu)$ continues to grow, as governed by the beta function of this low-energy effective theory, which differs from that of the high-scale theory by the removal of the N_{R_1} fermions in the representation R_1. Insofar as the coupling α is not too large to prevent one from using the perturbative beta function to track its evolution reliably for $\mu < \Lambda_1$, one has

$$\alpha(\mu)^{-1} = \alpha^{-1}(\Lambda_1) + \frac{b_1(R_2)}{2\pi} \ln\left(\frac{\Lambda_1}{\mu}\right), \qquad (6.3)$$

where $b_1(R_1)$ is the value of b_1 from Eq. (2.2) for the low-energy effective field theory with only N_{R_2} fermions in R_2 present. Then, given our assumptions about the set $\{N_{R_1}, N_{R_2}\}$, at a lower scale Λ_2, condensation occurs for the fermions in the representation R_2, when $\alpha(\mu) = \alpha_{R_2,\mathrm{cr}}$. Solving for the ratio of these two condensations scales in this rough approximation, we obtain

$$\frac{\Lambda_2}{\Lambda_1} \simeq \exp\left[-\frac{6}{b_1(R_2)}(C_2(R_1) - C_2(R_1))\right], \qquad (6.4)$$

where $b_1(R_2) = (1/3)(11N - 4N_{R_2}T(R_2))$. As an example, consider the SU(2) theory with R_1 and R_2 being the $I = 1$ and $I = 1/2$ representations, respectively, and numbers $N_{R_1} \equiv N_1$ and $N_{R_2} \equiv N_{1/2}1$ for which there are two condensations, as indicated in Table I. Then

$$\frac{\Lambda_{1/2}}{\Lambda_1} \simeq \exp\left[-\frac{45}{4(11 - N_{1/2})}\right]. \qquad (6.5)$$

As $N_{1/2}$ increases from 2 to 4, this ratio $\Lambda_{1/2}/\Lambda_1$ decreases from about 0.3 to 0.2. These are comparable to the sort of ratios of condensation scales that would characterize the sequential breaking of reasonably ultraviolet-complete extended-technicolor theories (e.g., [37,38]).

VII. EFFECTS OF NONZERO INTRINSIC MASSES FOR FERMIONS

In the discussion up to this point we have assumed that the fermions have zero intrinsic masses in the Lagrangian describing the high-scale physics, and the only masses that they acquire arise dynamically if they are involved in condensates that form as the gauge interaction becomes sufficiently strongly coupled in the infrared. This is a well-motivated assumption if the vectorial gauge theory arises as a low-energy effective field theory from an ultraviolet completion which is a chiral gauge theory. This is natural if the latter theory becomes strongly coupled, since it can then form fermion condensates that self-break it down to the vectorial subgroup symmetry. However, one may also choose to focus on the vectorial gauge theory as an ultraviolet-complete theory in itself. In a vectorial gauge theory, an intrinsic (bare) mass term for a fermion ψ, $\mathcal{L}_m = -m\bar{\psi}\psi$, is allowed by the gauge invariance. [For an SU(2) theory, with fermions written as left-handed chiral fields, the gauge-invariant mass term can be expressed in a Majorana form, e.g., for the fundamental

representation, $m'\epsilon_{ij}\psi_L^{iT}C\psi_L^j$.] Hence, one may consider a more general situation in which the fermions may have such intrinsic masses in the high-scale Lagrangian. Quantum chromodynamics (QCD) provides an example of this, in which the quarks have hard (also called current-quark) masses [39] that span a large range, from m_u of a few MeV to $m_t \simeq 172$ GeV. In particular, this range extends both far below and far above, the scale $\Lambda_{\mathrm{QCD}} \simeq 300$ MeV where the QCD coupling $\alpha_s(\mu)$ becomes O(1) and the theory confines and spontaneously breaks chiral symmetry.

The main effect of intrinsic fermion masses here is the same as in QCD; as the reference scale μ decreases below the value of such a mass of some fermion m_f, the beta function changes from one that includes this to one that excludes this in the set of light, active fermions. For a theory with a set $\{N_R\}$ such that $b_2 < 0$ at a high scale, and hence evolution toward an approximate or exact infrared fixed point, the reduction of one or more numbers N_R can reverse the sign of b_2, making it positive and removing this infrared fixed point. Indeed, in principle, a theory could have sufficiently large numbers of fermions in various representations $\{N_R\}$ that it is not asymptotically free at a high energy scale above the fermion masses, but as this scale decreases below some of these masses, the modified beta function describing the gauge coupling evolution in the resulting low-energy effective field theory is asymptotically free.

VIII. CONCLUSIONS

In this paper we have studied the evolution of an asymptotically free vectorial SU(N) gauge theory from high scales to the infrared and the resultant phase structure in the general case in which the theory contains fermions transforming according to several different representations of the gauge group. Using information from the beta function and results from an approximate analysis of the Dyson-Schwinger equation for the fermion(s), we have investigated examples that illustrate a wide range of possible behavior. In one type of model, the theory contains sufficiently few fermions that the coupling α increases as the reference scale decreases, but the 2-loop beta function does not have an infrared zero away from the origin. In this case, as α increases and exceeds a critical value for the formation of a condensate of fermions with the largest $C_2(R)$, this forms, the fermions gain dynamical masses, and these fermions are then integrated out of the low-energy effective field theory applicable below this highest condensation scale. In the low-energy theory, the coupling α continues to evolve, but according to a different beta function, and there is then condensation of the fermions with the next largest value of $C_2(R)$, and so forth. In another type of model, the theory contains enough fermions in various representations that the beta function does have an infrared zero. In this case, there are two main

categories of behavior. In one type, the value of α_{IR} is larger than the critical value for condensation of the fermions with the largest $C_2(R)$, so this condensation occurs and is followed by sequential condensation(s) at lower scales. In a second type, the value of α_{IR} is sufficiently small that there are no condensates formed, there is no spontaneous chiral-symmetry breaking, and α_{IR} is an exact infrared fixed point of the renormalization group. We have given explicit examples of each of these types of behavior in the case of an SU(2) gauge theory. We have also briefly discussed the effects of nonzero intrinsic fermion masses.

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