Does stability of relativistic dissipative fluid dynamics imply causality?

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We investigate the causality and stability of relativistic dissipative fluid dynamics in the absence of conserved charges. We perform a linear stability analysis in the rest frame of the fluid and find that the equations of relativistic dissipative fluid dynamics are always stable. We then perform a linear stability analysis in a Lorentz-boosted frame. Provided that the ratio of the relaxation time for the shear stress tensor τ_π to the sound attenuation length $\Gamma_s = 4\eta/3(\varepsilon + P)$ fulfills a certain asymptotic causality condition, the equations of motion give rise to stable solutions. Although the group velocity associated with perturbations may exceed the velocity of light in a certain finite range of wave numbers, we demonstrate that this does not violate causality, as long as the asymptotic causality condition is fulfilled. Finally, we compute the characteristic velocities and show that they remain below the velocity of light if the ratio τ_π/Γ_s fulfills the asymptotic causality condition.

I. INTRODUCTION

Data from the Relativistic Heavy-Ion Collider (RHIC) on the collective flow of matter in nucleus-nucleus collisions have delivered a surprising result: the elliptic flow coefficient v_2 is sufficiently large [1–4] to be compatible with calculations performed in the framework of *ideal* fluid dynamics [5]. This has given rise to the notion that "RHIC physicists serve up the perfect liquid" [6–8].

Of course, no real liquid can have zero viscosity: for all weakly coupled theories, i.e., theories with well-defined quasiparticles, in the dilute limit there is a lower bound which one can derive from the uncertainty principle [9]: the ratio of shear viscosity-to-entropy density $\eta/s \gtrsim 1/12$. For certain strongly coupled theories without quasiparticles, there is also a lower bound which can be obtained from the AdS/CFT conjecture [10], $\eta/s \geq 1/(4\pi)$, i.e., surprisingly close to the bound for dilute, weakly coupled systems.

In order to see whether the shear viscosity of the hot and dense matter created in nuclear collisions at RHIC is close to the lower bound, one has to perform calculations in the framework of relativistic *dissipative* fluid dynamics. This program has only been recently initiated, but has already led to an enormous activity in the literature [11–30].

Fluid dynamics is an effective theory for the long-wavelength, small-frequency modes of a given theory. In order to see this, let us introduce three length scales: (a) a microscopic length scale $\ell_{\rm micro}$. In all theories, at sufficiently large temperatures this length scale can be defined as the thermal wavelength $\lambda_{\rm th} \sim 1/T$. In weakly coupled theories with well-defined quasiparticles, this can be interpreted as the interparticle distance. (b) A mesoscopic length scale $\ell_{\rm meso}$. In weakly coupled theories and in the dilute limit, this can be identified with the mean-free path of particles between collisions. In strongly coupled theories, such a scale is not known and should be identified with $\ell_{\rm micro}$. (c) A macroscopic length scale $\ell_{\rm macro}$. This is the scale over which the conserved densities (e.g., the charge density n or the energy density ε) of the theory vary. Thus, $\ell_{\rm macro}^{-1} \sim |\partial\varepsilon|/\varepsilon$; i.e., $\ell_{\rm macro}^{-1}$ is proportional to the gradients of the conserved quantities.

We now define the quantity $K \equiv \ell_{\rm meso}/\ell_{\rm macro}$. For dilute systems, this quantity is identical to the so-called Knudsen number. If K is sufficiently small, fluid dynamics as an effective theory can be derived in a controlled way as a power series in terms K. Since $K \sim \ell_{\rm macro}^{-1}$, this series expansion is equivalent to a gradient expansion.

To zeroth order in K, one obtains the equations of ideal fluid dynamics. To first order in K, one obtains the Navier-Stokes (NS) equations. So-called second-order theories contain terms of second order in K. Examples for the latter are the Burnett equations [31], the Israel-Stewart equations for relativistic dissipative fluid dynamics [32], the memory function theory [25,28], extended thermodynamics [28,33], and others [34]. The main difference between first- and second-order theories is the velocity of signal propagation. The relativistic NS equations allow for infinite signal propagation speeds and are therefore acausal. On the other hand, all second-order theories are considered to be causal in the sense that all signal velocities are smaller than the speed of light, provided that the parameters of the theory are suitably chosen.

The stability and causality of fluid-dynamical theories are usually studied around a hydrostatic state (i.e., for vanishing macroscopic flow velocity) which is in thermo-dynamical equilibrium. However, if a theory is stable around a hydrostatic state, it does not necessarily imply that it is stable in a state of nonzero flow velocity.

Following this idea, the stability and causality of first- and second-order fluid dynamics for a state with nonzero background flow velocity (mathematically realized by a Lorentz boost) were studied for the case of nonzero bulk viscosity, but for vanishing shear stress and heat flow in Ref. [27]. There it was found that causality and stability are intimately related: for all parameters considered, the theory becomes unstable if and only if there is a mode which propagates faster than the speed of light.

In this paper, we extend this analysis to the case of nonvanishing shear viscosity in second-order theories of relativistic dissipative fluid dynamics. A similar analysis for a hydrostatic background has already been done by Hiscock, Lindblom, and Olson [35,36], but they discussed exclusively the low- and high-wave-number limits [36]. As we shall show in this paper, their analysis missed a divergence of the group velocity of a shear mode at intermediate wave numbers. This anomalous behavior is generic; i.e., it cannot be removed by tuning the parameters of the theory, e.g., the relaxation time for the shear stress tensor τ_π and the shear viscosity η. However, if the ratio τ_π/Γ_s, where $\Gamma_s = 2(D-2)\eta/[(D-1)(\varepsilon+P)]$ is the sound attenuation length in D space-time dimensions, is chosen such that the large-momentum limit of the group velocity associated with the perturbation remains below the velocity of light (the so-called asymptotic causality condition), one can ensure that the divergence is restricted to a finite range of momenta. It will be demonstrated that in this case, the causality of the theory is not compromised. On the other hand, second-order fluid dynamics is always stable in the rest frame of the fluid, even if we use a parameter set which violates the asymptotic causality condition.

We also study the causality and stability for a state with nonzero background flow velocity, i.e., in a Lorentz-boosted frame. We find that the divergence of the group velocity is removed. However, depending on the boost velocity the group velocity of either the shear or the sound mode may still exceed the speed of light in a certain range of wave numbers. Nevertheless, provided that the ratio τ_π/Γ_s fulfills the asymptotic causality condition, we can show that the equations are stable. In contrast to the analysis in the rest frame, however, they become unstable if the asymptotic causality condition is violated. We shall demonstrate that if the asymptotic causality condition is fulfilled, the causality of the theory as a whole is not compromised. In this sense, causality and stability are intimately related.

So far, the discussion was limited to the fluid-dynamical equations in the linear approximation. Therefore, we expect the results to be valid for all versions of second-order theories presently discussed in the literature, since they differ only by nonlinear terms. We also compute the characteristic velocities for the so-called simplified Israel-Stewart equations [15] without linearizing these equations. Our analysis strongly indicates that the characteristic ve-

locities remain below the velocity of light if the ratio τ_π/Γ_s is chosen such that the asymptotic causality condition is fulfilled.

The asymptotic causality condition implies that, for a given $\Gamma_s \sim \eta$, τ_π must not be arbitrarily small. This explains why relativistic NS theory is acausal, because there $\tau_\pi \to 0$, while η is nonzero. It also implies that second-order theories are not *per se* causal; they can violate causality (and become unstable) if a too small value for τ_π is chosen. The statement that second-order theories automatically cure the shortcomings of NS theory is therefore not true.

This paper is organized as follows. In Sec. II, we discuss the causality and stability of the linearized second-order fluid-dynamical equations in the local rest frame. We also extend this analysis to nonzero bulk viscosity and show that the divergence of the group velocity still exists in this case. In Sec. III, this discussion is generalized to a Lorentz-boosted frame. We discuss Lorentz boosts both in and orthogonal to the direction of propagation of the perturbation. It will be demonstrated that superluminal group velocities will not compromise the causality of the theory as long as the asymptotic causality condition is fulfilled. In Sec. IV, we compute the characteristic velocities in the nonlinear case. A summary of our results concludes this work in Sec. V. An appendix contains details of our calculations in Sec. IV. The metric tensor is $g^{\mu\nu} = \text{diag}(+, -, -, -)$; our units are $\hbar = c = k_B = 1$.

II. STABILITY IN THE REST FRAME

As mentioned in the introduction, there are several approaches to formulate a second-order theory of relativistic dissipative fluids [25,27,28,32–34]. These approaches differ only by nonlinear (second-order) terms. However, since we shall apply a linear stability analysis in the following, these differences vanish and all approaches lead to the same set of linearized fluid-dynamical equations. In this work, we do not consider any conserved charges and thus are left with energy-momentum conservation:

$$\partial_\mu T^{\mu\nu} = 0, \tag{1}$$

where

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \tag{2}$$

is the energy-momentum tensor. Here, ε and P are the energy density and the pressure, respectively, while u^μ, Π, and $\pi^{\mu\nu}$ are the fluid velocity, the bulk viscous pressure, and the shear stress tensor, respectively. We also introduced the projection operator

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \tag{3}$$

which projects onto the $(D-1)$-dimensional subspace orthogonal to the fluid velocity. We compute in the

Landau frame [37], where there is no energy flow in the local rest frame.

In second-order theories of relativistic dissipative fluid dynamics, the bulk viscous pressure and the shear stress tensor are determined from evolution equations. In D space-time dimensions ($D \geq 3$), these equations are given by

$$\tau_\Pi \frac{d}{d\tau} \Pi + \Pi = -\zeta \partial_\mu u^\mu, \tag{4a}$$

$$\tau_\pi P^{\mu\nu\alpha\beta} \frac{d}{d\tau} \pi_{\alpha\beta} + \pi^{\mu\nu} = 2\eta P^{\mu\nu\alpha\beta} \partial_\alpha u_\beta; \tag{4b}$$

possible other second-order terms [23] can be neglected for the purpose of a linear stability analysis. In Eqs. (4), the comoving derivative is denoted by $u^\mu \partial_\mu \equiv d/d\tau$. The relaxation times for the bulk viscous pressure and the shear stress tensor are denoted by τ_Π and τ_π, respectively. The coefficients ζ and η are the bulk and shear viscosities, respectively. We also introduced the symmetric rank-four projection operator

$$P^{\mu\nu\alpha\beta} = \frac{1}{2}(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\nu\alpha}\Delta^{\mu\beta}) - \frac{1}{D-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}. \tag{5}$$

The shear stress tensor is traceless $\pi^\mu_\mu = 0$ and orthogonal to the fluid velocity $u_\mu \pi^{\mu\nu} = 0$.

The stability and causality of a relativistic dissipative fluid with bulk viscous pressure only have been investigated in Ref. [27]. Thus, for the sake of simplicity, we shall first ignore the effects from bulk viscous pressure and discuss the properties of the fluid-dynamical equations of motion including only shear viscosity. The interplay between shear and bulk viscosity will be discussed afterwards.

A. Shear viscosity only

For convenience, we introduce the following parameterization:

$$\eta = as, \tag{6a}$$

$$\tau_\pi = \frac{\eta}{\varepsilon + P} b = \frac{ab}{T}, \tag{6b}$$

where s and T are the entropy density and the temperature, respectively. From the second equation we obtain $\tau_\pi(\varepsilon + P)/\eta = b$. The parametrization (6) is motivated by the leading-order results for the causal shear viscosity coefficient and the relaxation time obtained in Ref. [30], where the relation $\tau_\pi = \eta/P$ was found. For a massless ideal gas equation of state $\varepsilon = (D-1)P$, this result is reproduced by choosing $b = D$.

In this section, we discuss the stability of second-order relativistic fluid dynamics in the local rest frame. Following Refs. [27,35], let us introduce a perturbation $\sim e^{i\omega t - ikx}$ around the hydrostatic equilibrium state:

$$\varepsilon = \varepsilon_0 + \delta\varepsilon e^{i\omega t - ikx}, \tag{7a}$$

$$\pi^{\mu\nu} = \pi_0^{\mu\nu} + \delta\pi^{\mu\nu} e^{i\omega t - ikx}, \tag{7b}$$

$$u^\mu = u_0^\mu + \delta u^\mu e^{i\omega t - ikx}, \tag{7c}$$

where $\varepsilon_0 = \text{const}$, $\pi_0^{\mu\nu} = 0$, and $u_0^\mu = (1, 0, 0, \ldots)$, respectively. In the linear approximation, the velocity perturbation has no zeroth component,

$$\delta u^\mu = (0, \delta u^1, \delta u^2, \ldots, \delta u^{D-1}), \tag{8}$$

because $u^\mu u_\mu = 1$. Moreover, in the local rest frame, $\delta\pi^{0\nu} \equiv 0$ on account of the orthogonality condition $u_\mu \pi^{\mu\nu} = 0$. Since $\pi^{\mu\nu}$ is traceless, $\delta\pi^{(D-1)(D-1)}$ is not an independent variable. Taking all of this into account, the linearized fluid-dynamical equations can be written as

$$AX = 0, \tag{9}$$

where

$$X = (\delta\varepsilon, \delta u^1, \delta\pi^{11}, \delta u^2, \delta\pi^{12}, \ldots, \delta u^{D-1}, \delta\pi^{1(D-1)},$$

$$\delta\pi^{22}, \delta\pi^{33}, \ldots, \delta\pi^{(D-2)(D-2)}, \delta\pi^{23}, \delta\pi^{24}, \ldots,$$

$$\delta\pi^{2(D-1)}, \delta\pi^{34}, \ldots, \delta\pi^{(D-2)(D-1)})^T.$$

The matrix A is expressed as

$$A = \begin{pmatrix} T & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ G & 0 & C & 0 \\ 0 & 0 & 0 & E \end{pmatrix}, \tag{10}$$

with

$$T = \begin{pmatrix} i\omega & f_1 & 0 \\ -ikc_s^2 & f_2 & -ik \\ 0 & \Gamma & f \end{pmatrix}, \tag{11a}$$

$$B = \text{diag}(B_0, \ldots, B_0)_{(D-2)\times(D-2)},$$

$$B_0 = \begin{pmatrix} f_2 & -ik \\ \Gamma_1 & f \end{pmatrix}, \tag{11b}$$

$$G = \begin{pmatrix} 0 & \Gamma_2 & 0 \\ & \ldots & \\ 0 & \Gamma_2 & 0 \end{pmatrix}_{(D-3)\times 3}, \tag{11c}$$

$$C = \text{diag}(f, \ldots, f)_{(D-3)\times(D-3)}, \tag{11d}$$

$$E = \text{diag}(f, \ldots, f)_{(1/2)(D-2)(D-3)\times(1/2)(D-2)(D-3)}, \tag{11e}$$

where $c_s = \sqrt{\partial P/\partial\varepsilon}$ is the velocity of sound. Here, we introduced the abbreviations

$$f = i\omega\tau_\pi + 1, \qquad f_1 = -ik(\varepsilon + P),$$

$$f_2 = i\omega(\varepsilon + P), \qquad \Gamma = -ik\frac{2(D-2)}{D-1}\eta,$$

$$\Gamma_1 = -ik\eta, \qquad \Gamma_2 = ik\frac{2}{D-1}\eta.$$

For nontrivial solutions of Eq. (9), the determinant of the matrix A should vanish. This leads to the following conditions for the dispersion relations $\omega(k)$:

$$f = 0, \tag{12a}$$

$$\det B = (\det B_0)^{D-2} = 0, \tag{12b}$$

$$\det T = \det \begin{pmatrix} i\omega & f_1 & 0 \\ -ikc_s^2 & f_2 & -ik \\ 0 & \Gamma & f \end{pmatrix} = 0. \tag{12c}$$

Equation (12a) gives a purely imaginary frequency

$$\omega = \frac{i}{\tau_\pi}, \tag{13}$$

which corresponds to a nonpropagating mode. The degeneracy of this mode is $(D - 3)[1 + (D - 2)/2]$.

Equation (12b) leads to a complex frequency

$$\omega = \frac{1}{2\tau_\pi}\left(i \pm \sqrt{\frac{4\eta\tau_\pi}{\varepsilon + P}k^2 - 1}\right), \tag{14}$$

corresponding to two propagating modes, if k is larger than the critical wave number

$$k_c = \sqrt{\frac{\varepsilon + P}{4\eta\tau_\pi}} \equiv \frac{\sqrt{b}}{2\tau_\pi}. \tag{15}$$

Following Ref. [38], we shall call these modes shear modes. There are in total $2(D - 2)$ shear modes.

Equation (12c) gives the same dispersion relation as Eq. (16) of Ref. [27], after replacing $2(D - 2)\eta/(D - 1)$ with ζ_0. Introducing the sound attenuation length in D space-time dimensions

$$\Gamma_s \equiv \frac{2(D - 2)}{D - 1}\frac{\eta}{\varepsilon + P} \equiv \frac{2(D - 2)}{D - 1}\frac{\tau_\pi}{b}, \tag{16}$$
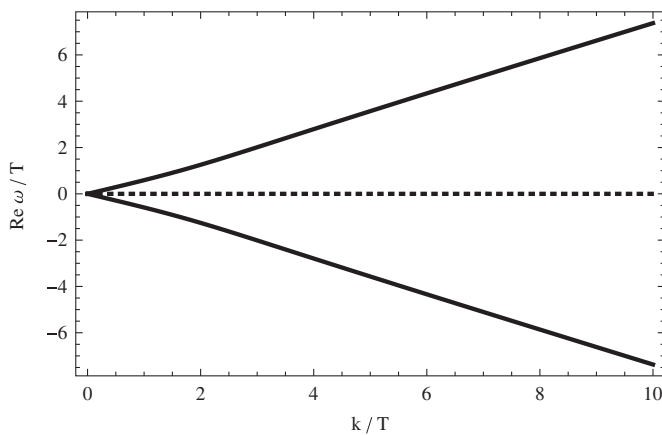
the analytic solution in the limit of small wave number k is

$$\omega = \begin{cases} \frac{i}{\tau_\pi}, \\ \pm kc_s + i\frac{\Gamma_s}{2}k^2, \end{cases} \tag{17}$$

while for a large wave number we obtain

$$\omega = \begin{cases} \frac{i}{\tau_\pi}\left[1 + \frac{\Gamma_s}{\tau_\pi c_s^2}\right]^{-1}, \\ \pm kc_s\sqrt{1 + \frac{\Gamma_s}{\tau_\pi c_s^2}} + \frac{i}{2\tau_\pi}\left[1 + \frac{\tau_\pi c_s^2}{\Gamma_s}\right]^{-1}. \end{cases} \tag{18}$$

This corresponds to another nonpropagating mode and two propagating modes which we call sound modes in accordance with Ref. [38]. All imaginary parts are positive and therefore the nonpropagating as well as the shear and sound modes are stable around the hydrostatic equilibrium state. This fact is already known from the study of Hiscock and Lindblom [35].

In order to discuss the issue of causality, we follow Refs. [27,35] and study the group velocity defined as

$$v_g = \frac{\partial \operatorname{Re}\omega}{\partial k}. \tag{19}$$

For the two nonpropagating modes, $\operatorname{Re}\omega = 0$. Consequently, in order to discuss causality, we have to consider the behavior of the imaginary part [27]. Let us digress for the moment and consider the diffusion equation with diffusion constant D_0. There is a nonpropagating mode with dispersion relation $\omega = iD_0k^2$. Moreover, it is known that the diffusion equation is acausal. Therefore, we conjecture that a k^2 dependence of any nonpropagating mode can be considered a sign of acausality. In our case, the nonpropagating modes either are independent of k or have a weak k dependence (cf. Fig. 1). According to our conjecture, we conclude that the nonpropagating modes do not violate causality.

The dispersion relations resulting from Eq. (12c) are shown in Fig. 1, and the corresponding group velocity
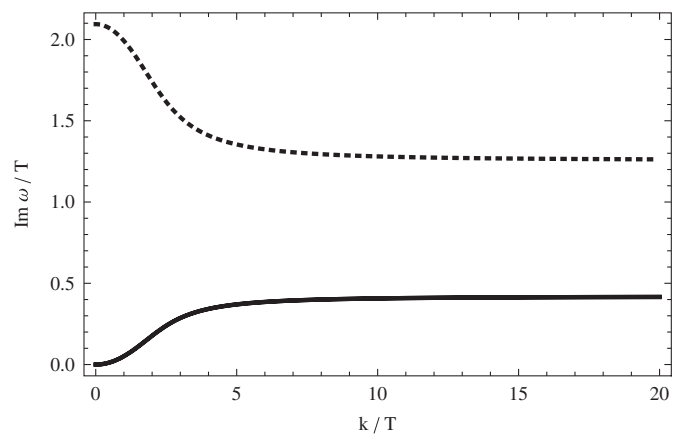

FIG. 1. The real parts (left panel) and the imaginary parts (right panel) of the dispersion relations for the sound modes (full lines) and the nonpropagating mode (dashed line) obtained from Eq. (12c). The parameters are $a = \frac{1}{4\pi}$, $b = 6$, and $c_s^2 = \frac{1}{3}$ for the 3 + 1-dimensional case $D = 4$.
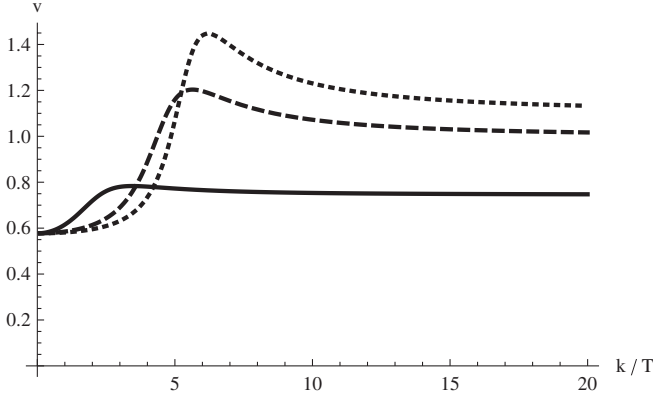
FIG. 2.   The group velocity (22) for $a = 1/(4\pi)$, $D = 4$, $c_s^2 = \frac{1}{3}$, and $b = 6$ (full line), $b = 2$ (dashed line), as well as $b = 1.5$ (dotted line).

resulting from Eq. (19) in Fig. 2. The group velocity has a maximum for a finite value of k/T and approaches its asymptotic value ($k \rightarrow \infty$) from above. For small values of b, it may thus happen that the group velocity becomes superluminal. Nevertheless, in Sec. III C we shall show that only the asymptotic value determines whether the theory as a whole is causal or not. The asymptotic value of the group velocity is

$$v_{g,\text{sound}}^{\text{as}} = \lim_{k \to \infty} \frac{\partial \operatorname{Re}\omega}{\partial k} = c_s \sqrt{1 + \frac{\Gamma_s}{\tau_\pi c_s^2}}. \qquad (20)$$

Consequently, for the asymptotic group velocity of sound waves to be less than the speed of light, τ_π and Γ_s should satisfy the following, so-called asymptotic causality condition:

$$\frac{\Gamma_s}{\tau_\pi} \leq 1 - c_s^2 \Leftrightarrow \frac{1}{b} \equiv \frac{\eta}{\tau_\pi(\varepsilon + P)} \leq \frac{D-1}{2(D-2)}(1 - c_s^2). \qquad (21)$$

This is similar to the causality condition for the group

velocity in the case of bulk viscosity, Eq. (21) of Ref. [27]. For conformal fluids, where $c_s^2 = 1/(D-1)$, the condition (21) simplifies to $\Gamma_s \leq (D-2)\tau_\pi/(D-1)$ or, equivalently, $b \geq 2$. For example, for the values of η and τ_π deduced from the AdS/CFT correspondence [38–40], $\eta = s/(4\pi)$ and $\tau_\pi = (2 - \ln 2)/(2\pi T)$, the condition (21) is always satisfied because $b = 2(2 - \ln 2) \simeq 2.614 > 2$.

The dispersion relations for the shear modes resulting from Eq. (12b) change their behavior from nonpropagating to propagating at the critical wave number (15), as shown in Fig. 3. It should be noted that a similar behavior is observed in the case of bulk viscosity; cf. Fig. 1 in Ref. [27]. For wave numbers larger than k_c, the (modulus of the) group velocity of the propagating mode is

$$v_g = v_{g,\text{shear}}^{\text{as}} \frac{k/k_c}{\sqrt{(k/k_c)^2 - 1}}, \qquad (22)$$

where

$$v_{g,\text{shear}}^{\text{as}} \equiv \frac{1}{\sqrt{2\tau_\pi k_c}} \equiv \sqrt{\frac{\eta}{\tau_\pi(\varepsilon + P)}} \equiv \frac{1}{\sqrt{b}} \qquad (23)$$

is the asymptotic value of v_g in the large-wave-number limit. If the asymptotic causality condition (21) is satisfied, $v_{g,\text{shear}}^{\text{as}} \leq \sqrt{(D-1)(1 - c_s^2)/2(D-2)}$. This is smaller than 1 for any value of c_s and $D \geq 3$. However, near the critical wave number k_c the group velocity diverges, as shown in Fig. 4. From the definitions of k_c [Eq. (15)] and the parameters a and b [Eqs. (6)], we observe that $k_c/T = (2a\sqrt{b})^{-1}$. The $1/a$ scaling of k_c/T for fixed b can be nicely observed in Fig. 4.

In Sec. III C we shall show that the apparent violation of causality of the group velocity does not cause the theory as a whole to become acausal. The important issue is whether the asymptotic causality condition is fulfilled. If yes, the theory is causal.
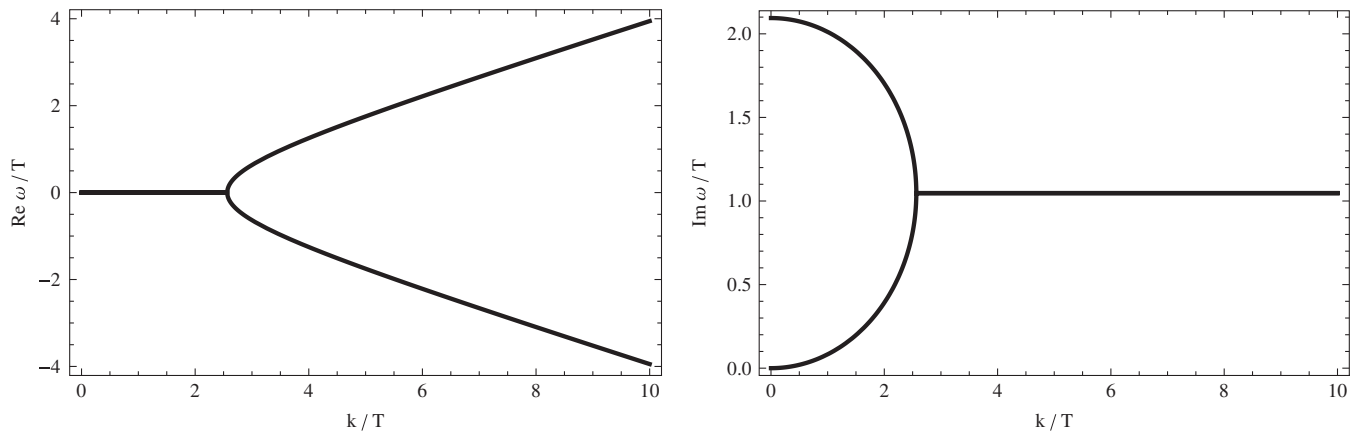

FIG. 3.   The real parts (left panel) and the imaginary parts (right panel) of the dispersion relations for the shear modes obtained from Eq. (12b). The parameters are $a = \frac{1}{4\pi}$, $b = 6$, and $c_s^2 = \frac{1}{3}$ for the 3 + 1-dimensional case $D = 4$.
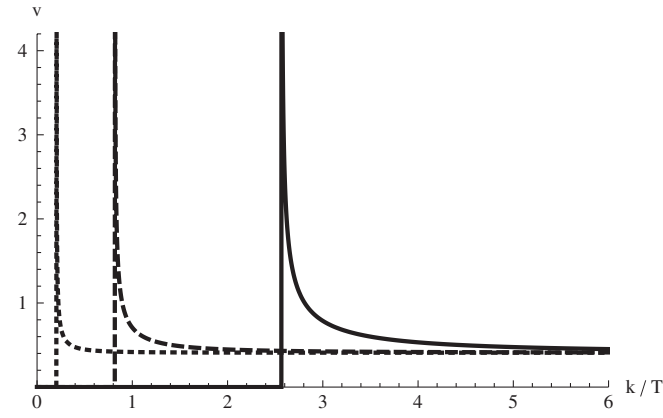
FIG. 4. The group velocity (22) for $D = 4$, $b = 6$, $c_s^2 = \frac{1}{3}$, and $a = 1/(4\pi)$ (full line), $a = 1/4$ (dashed line), as well as $a = 1$ (dotted line).

We remark that, *in the local rest frame*, the *stability* of the system of fluid-dynamical equations is *not* affected if we choose a parameter set which violates the asymptotic

causality condition (21), for instance, a conformal fluid in $D = 4$ dimensions and $b = 1$. This is demonstrated for the sound modes in Fig. 5 and for the shear modes in Fig. 6.

B. Competition of bulk and shear

The question we would like to answer in this section is whether the problem of the divergent group velocity can be removed by adding bulk viscosity to the discussion. For the sake of simplicity, we consider only the $2 + 1$-dimensional case, i.e., $D = 3$. Similarly to Eqs. (6), we introduce the parametrization

$$\zeta = a_1 s, \qquad \tau_\Pi = \frac{\zeta}{\varepsilon + P} b_1. \qquad (24)$$

As before, the equations of motion (4) have to be linearized, yielding Eq. (9), where now

$$X = (\delta\varepsilon, \delta u^x, \delta\pi^{xx}, \delta u^y, \delta\pi^{xy}, \delta\Pi)^T, \qquad (25)$$

and

$$A = \begin{pmatrix} i\omega & -ik(\varepsilon + P) & 0 & 0 & 0 & 0 \\ -ikc_s^2 & i\omega(\varepsilon + P) & -ik & 0 & 0 & -ik \\ 0 & -ik\eta & i\omega\tau_\pi + 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\omega(\varepsilon + P) & -ik & 0 \\ 0 & 0 & 0 & -ik\eta & i\omega\tau_\pi + 1 & 0 \\ 0 & -ik\zeta & 0 & 0 & 0 & i\omega\tau_\Pi + 1 \end{pmatrix}. \qquad (26)$$

Then, the dispersion relations are given by solving the following equations:

$$k^2\eta + i\omega(1 + i\omega\tau_\pi)(\varepsilon + P) = 0, \qquad (27a)$$

$$i\omega k^2(1 + i\omega\tau_\Pi)\eta + (1 + i\omega\tau_\pi)[i\omega k^2\zeta + (1 + i\omega\tau_\Pi)(\varepsilon + P)(c_s^2 k^2 - \omega^2)] = 0. \qquad (27b)$$

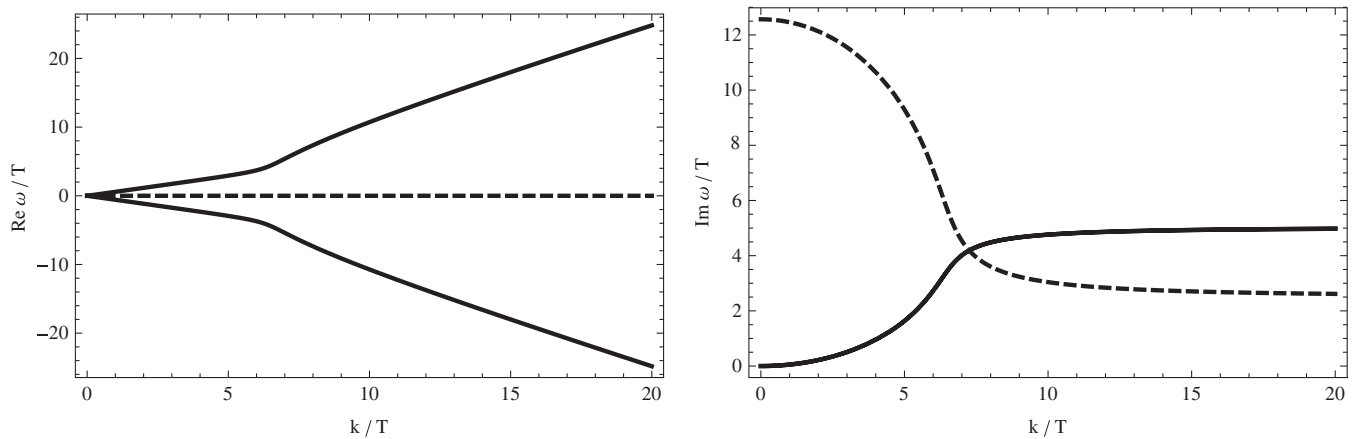The dispersion relations resulting from sound and bulk viscous modes [Eq. (27b)] are

FIG. 5. The real parts (left panel) and the imaginary parts (right panel) of the dispersion relations for the sound modes obtained from Eq. (12c). The parameters are $a = \frac{1}{4\pi}$, $b = 1$, and $c_s^2 = \frac{1}{3}$ for the $3 + 1$-dimensional case $D = 4$.
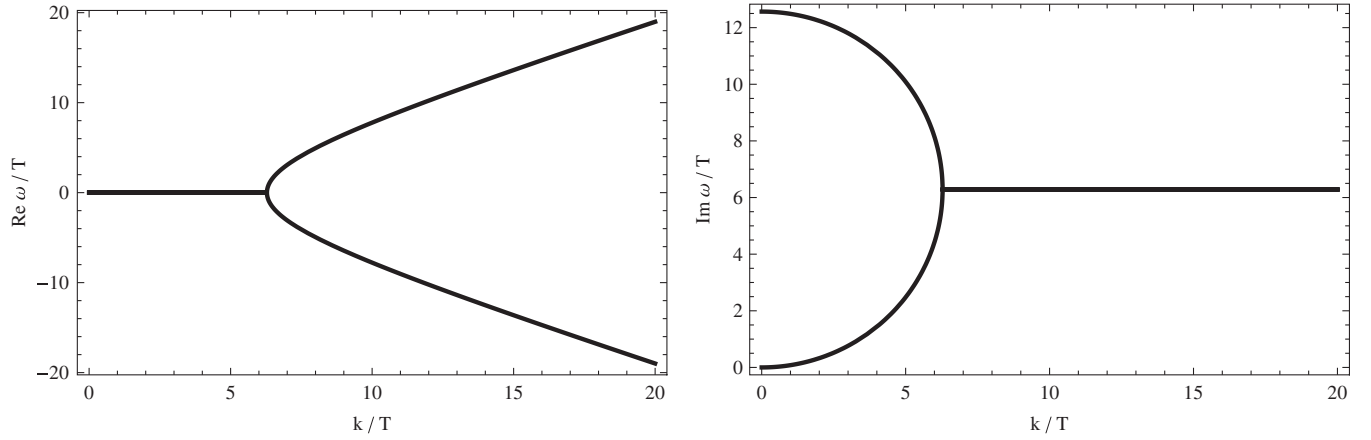
FIG. 6.　The real parts (left panel) and the imaginary parts (right panel) of the dispersion relations for the shear modes obtained from Eq. (12b). The parameters are $a = \frac{1}{4\pi}$, $b = 1$, and $c_s^2 = \frac{1}{3}$ for the $3 + 1$-dimensional case $D = 4$.

$$
\omega = \begin{cases} \frac{T}{2aa_1(b+b_1+bb_1c_s^2)}\{ia(1+bc_s^2) + ia_1(1+b_1c_s^2) \pm [4aa_1c_s^2(b+b_1+bb_1c_s^2) - (a+a_1+abc_s^2+a_1b_1c_s^2)^2]^{1/2}\}, \\[2ex] \pm k\sqrt{\frac{1}{b}+\frac{1}{b_1}+c_s^2} + \frac{iT}{2(b+b_1+bb_1c_s^2)}\left(\frac{b}{a_1b_1}+\frac{b_1}{ab}\right), \end{cases} \tag{28}
$$

for large k, and

$$
\omega = \begin{cases} \frac{i}{\tau_\pi}, \\ \frac{i}{\tau_\Pi}, \\ \pm c_s^2 k, \end{cases} \tag{29}
$$

for small k.

Thus the asymptotic causality condition reads

$$
\frac{1}{b_1} + \frac{1}{b} \equiv \frac{\zeta}{\tau_\Pi(\varepsilon+P)} + \frac{\eta}{\tau_\pi(\varepsilon+P)} \leq 1 - c_s^2. \tag{30}
$$

On the other hand, the equation for the shear modes [Eq. (27a)] is the same as Eq. (12b) and hence the corresponding group velocity again shows a divergence. Thus, the inclusion of bulk viscosity does not solve the problem of the divergent group velocity.

III. STABILITY IN LORENTZ-BOOSTED FRAME

The discussion of causality and stability in the case of nonzero bulk viscosity in a Lorentz-boosted frame in Ref. [27] has shown that causality and stability are intimately related. Relativistic dissipative fluid dynamics becomes unstable if the group velocity exceeds the speed of light. If this is still true in the case of nonzero shear viscosity, the divergence of the group velocity found in the rest frame may induce an instability in a moving frame. In order to investigate this question, we consider the stability of the hydrostatic state observed from a Lorentz-boosted frame, following Ref. [27]. In this section, we restrict our investigations to the case $D = 4$.

We consider a frame moving with a velocity $\vec{V}$ with respect to the hydrostatic state. Then, the total fluid velocity u'^μ is given by

$$
u'^\mu = \begin{pmatrix} \gamma_V & V\gamma_V\vec{n}^T \\ V\gamma_V\vec{n} & \gamma_V P_\parallel + Q_\perp \end{pmatrix} u^\mu, \tag{31}
$$

where $\gamma_V = 1/\sqrt{1-V^2}$, $P_\parallel = \vec{n}\vec{n}^T$, and $Q_\perp = 1 - P_\parallel$, with $\vec{n} = \vec{V}/|\vec{V}|$. We consider the two cases where the direction of the Lorentz boost is parallel and where it is perpendicular to the direction of propagation of the perturbation; the latter we take to be the x direction.

A. Boost along the x direction

The perturbation of the fluid velocity is given by

$$
u'^\mu = u_0'^\mu + \delta u'^\mu e^{i\omega t - ikx}, \tag{32}
$$

where

$$
u_0'^\mu = \gamma_V(1, V, 0, 0), \tag{33a}
$$
$$
\delta u'^\mu = (V\gamma_V\delta u^x, \gamma_V\delta u^x, \delta u^y, \delta u^z), \tag{33b}
$$

where δu^μ is the velocity perturbation in the local rest frame. The linearized fluid-dynamical equations are again given by Eq. (9), with

$$
X = (\delta\varepsilon, \delta u^x, \delta\pi^{xx}, \delta u^y, \delta\pi^{xy}, \delta u^z, \delta\pi^{xz}, \delta\pi^{yy}, \delta\pi^{yz})^T, \tag{34}
$$

and

$$A = \begin{pmatrix} T_1 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ G_1 & 0 & C_1 & 0 \\ 0 & 0 & 0 & E_1 \end{pmatrix}. \tag{35}$$

The submatrices are given by

$$T_1 = \gamma_V^2 \begin{pmatrix} i\omega(1 + V^2 c_s^2) - ikV(1 + c_s^2) & i[2\omega V - k(1 + V^2)](\varepsilon + P) & i\gamma_V^{-2} V(\omega V - k) \\ i\omega V(1 + c_s^2) - ik(V^2 + c_s^2) & i[\omega(1 + V^2) - 2kV](\varepsilon + P) & i\gamma_V^{-2}(\omega V - k) \\ 0 & \frac{4}{3} i\eta\gamma_V(\omega V - k) & \gamma_V^{-2} F \end{pmatrix}, \tag{36a}$$

$$B_1 = \mathrm{diag}(B_{01}, B_{01}), \qquad B_{01} = \begin{pmatrix} i\gamma_V(\omega - kV)(\varepsilon + P) & i(\omega V - k) \\ i\eta\gamma_V^2(\omega V - k) & F \end{pmatrix}, \tag{36b}$$

$$G_1 = \begin{pmatrix} 0 & -\frac{2}{3} i\eta\gamma_V(\omega V - k) & 0 \end{pmatrix}, \tag{36c}$$

$$C_1 = E_1 = F. \tag{36d}$$

Here we abbreviated

$$F = i\gamma_V(\omega - kV)\tau_\pi + 1. \tag{36e}$$

Obviously,

$$\det A = \det T_1 \times \det B_1 \times F^2. \tag{37}$$

From $F^2 = 0$, we only obtain two trivial propagating modes

$$\omega = \frac{i}{\gamma_V \tau_\pi} + kV. \tag{38}$$

The group velocity is $v_g = V$, which implies that these modes correspond to the nonpropagating modes in the local rest frame.

From $\det B_1 = 0$, we obtain

$$[iT + ab\gamma_V(kV - \omega)](kV - \omega) + a\gamma_V(kV - \omega)^2 T = 0, \tag{39}$$

corresponding to the shear modes. There are in total four modes satisfying this relation. The solutions are given by

$$\omega_\pm = \frac{1}{2a(b - V^2)\gamma_V}[iT - 2a(1 - b)kV\gamma_V \\ \pm \sqrt{-T^2 + 4iakTV\gamma_V^{-1} + 4a^2bk^2\gamma_V^{-2}}]. \tag{40}$$
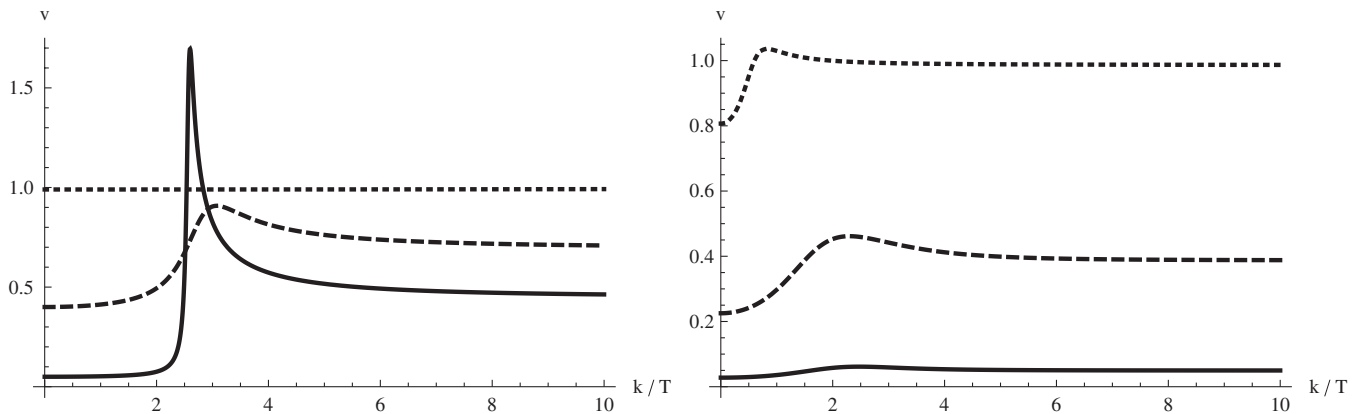

FIG. 7. The group velocity calculated for one of the shear modes (left panel) and one of the sound modes (right panel). We set $a = 1/(4\pi)$, $b = 6$, and $c_s^2 = 1/3$. The solid line is for a boost velocity $V = 0.05$, the dashed line for $V = 0.4$, and the dotted line for $V = 0.99$.
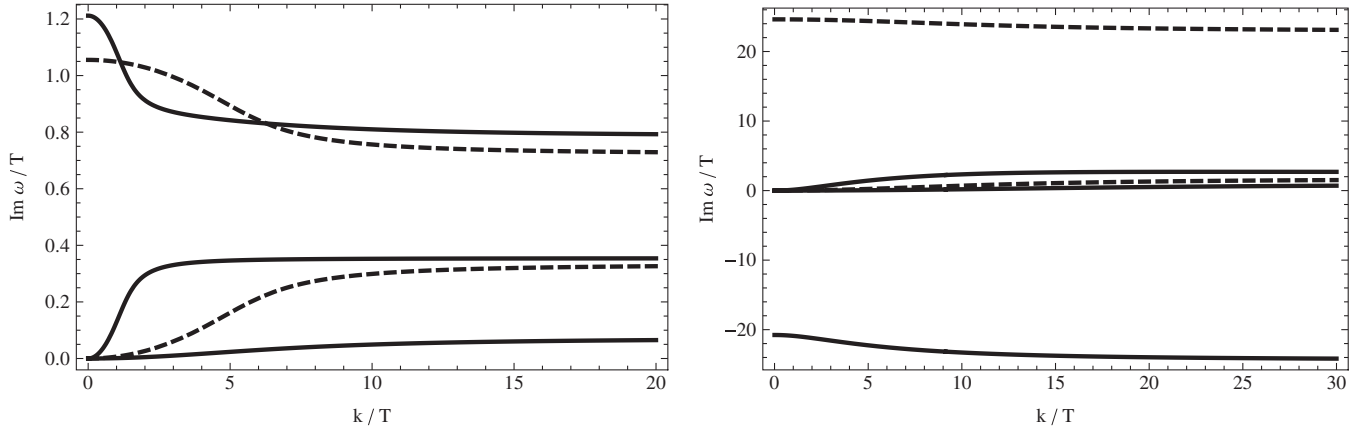
FIG. 8.   The imaginary parts of the dispersion relations for a boost in the x direction with velocity $V = 0.9$. The left panel shows the results for the parameter set $a = \frac{1}{4\pi}$, $b = 6$, $c_s^2 = \frac{1}{3}$, which fulfills the asymptotic causality condition, while the right panel is for $a = \frac{1}{4\pi}$, $b = 1$, $c_s^2 = \frac{1}{3}$, which violates this condition. The dashed lines are for the shear modes, while the solid lines are for the sound modes.

On the other hand, the sound modes result from

$$c_s^2(\varepsilon + P)[1 - i\gamma_V\tau_\pi(kV - \omega)]\{k^2[V^2 + (V - 1)^2V\gamma_V^2 + 1]$$
$$+ 2kV\omega[(V - 1)V\gamma_V^2 - 1] + V^2\omega^2 - c_s^{-2}(\omega - kV)^2\}$$
$$+ \tfrac{4}{3}i\gamma_V\eta(k - V\omega)^2\{kV[c_s^2\gamma_V^2V(1 - V) - 1] + \omega\} = 0.$$

$$(41)$$

In Fig. 7, the dependence of the group velocity on the wave number is shown for various values of the boost velocity V. The left panel shows the behavior of one of the shear modes and the right panel one of the sound modes. The parameter set used here is $a = \frac{1}{4\pi}$, $b = 6$, $c_s^2 = \frac{1}{3}$, which satisfies the asymptotic causality condition. We observe that the divergence of the group velocity of the shear mode in the rest frame is tempered by the Lorentz boost to result in a peak of finite height. However, the group velocity may still exceed the speed of light in a certain range of wave numbers. As we increase the boost velocity, the peak height diminishes, until the group velocity remains below the speed of light for all wave numbers. However, further increasing the boost velocity leads to an acausal group velocity in the *sound* mode.

Although the group velocity of the shear or the sound mode may exceed the speed of light, as long as the asymptotic causality condition is fulfilled, the theory is still stable. This is demonstrated in the left panel of Fig. 8, where the imaginary parts of the modes are shown for the parameter set $a = \frac{1}{4\pi}$, $b = 6$, $c_s^2 = \frac{1}{3}$. We observe that all imaginary parts are positive, indicating the stability of the theory.

In contrast to the rest frame, where the theory is stable even for parameters which violate the asymptotic causality condition (21), this is no longer the case in a Lorentz-boosted frame. In the right panel of Fig. 8, the imaginary parts of the modes are calculated with the parameter set $a = \frac{1}{4\pi}$, $b = 1$, $c_s^2 = \frac{1}{3}$. Now one observes the appearance of negative imaginary parts, indicating that the theory becomes unstable.

B. Boost along the y direction

Now we consider a Lorentz boost along the y direction. The perturbation of the fluid velocity is given by

$$u'^\mu = u_0'^\mu + \delta u'^\mu e^{i\omega t - ikx}, \qquad (42)$$

where

$$u_0'^\mu = \gamma_V(1, 0, V, 0), \qquad (43a)$$
$$\delta u'^\mu = (V\gamma_V\delta u^y, \delta u^x, \gamma_V\delta u^y, \delta u^z). \qquad (43b)$$

Similarly to the preceding discussion, the linearized fluid-dynamical equations take the form (9), where the matrix A is

$$A = \begin{pmatrix} T_2 & H_1 & H_2 & 0 \\ H_3 & B_2 & H_4 & H_5 \\ G_2 & H_6 & C_2 & 0 \\ 0 & H_7 & 0 & E_2 \end{pmatrix}, \qquad (44)$$

with

$$T_2 = \begin{pmatrix} i\omega\gamma_V^2(1 + c_s^2 V^2) & -ik\gamma_V(\varepsilon + P) & 0 \\ -ikc_s^2 & i\omega\gamma_V(\varepsilon + P) & -ik \\ 0 & -\frac{4}{3}ik\eta & F_1 \end{pmatrix}, \tag{45a}$$

$$H_1 = \begin{pmatrix} 2i\omega V(\varepsilon + P)\gamma_V^2 & -ikV & 0 & 0 \\ 0 & i\omega V & 0 & 0 \\ -\frac{2}{3}i\omega V\eta\gamma_V & 0 & 0 & 0 \end{pmatrix}. \tag{45b}$$

$$H_2 = (i\omega V^2 \quad 0 \quad 0)^T, \tag{45c}$$

$$H_3 = \begin{pmatrix} i\omega V\gamma_V^2(1 + c_s^2) & -ikV\gamma_V(\varepsilon + P) & 0 \\ 0 & i\omega V\gamma_V^2\eta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{45d}$$

$$B_2 = \begin{pmatrix} i\omega\gamma_V^2(1 + V^2)(\varepsilon + P) & -ik & 0 & 0 \\ -ik\gamma_V\eta & F_1 & 0 & 0 \\ 0 & 0 & i\omega\gamma_V(\varepsilon + P) & -ik \\ 0 & 0 & -ik\eta & F_1 \end{pmatrix}, \tag{45e}$$

$$H_4 = (i\omega V \quad 0 \quad 0 \quad 0)^T, \qquad H_5 = (0 \quad 0 \quad i\omega V \quad 0)^T, \tag{45f}$$

$$G_2 = (0 \quad \tfrac{2}{3}ik\gamma_V^2\eta \quad 0), \qquad H_6 = (\tfrac{4}{3}i\omega V\gamma_V^3\eta \quad 0 \quad 0 \quad 0), \tag{45g}$$

$$H_7 = (0 \quad 0 \quad i\omega V\gamma_V^2\eta \quad 0), \qquad C_2 = E_2 = F_1. \tag{45h}$$

Here we abbreviated

$$F_1 = i\omega\gamma_V\tau_\pi + 1.$$

The condition det$A = 0$ leads again to the following nine modes: three nonpropagating modes, four shear modes, and two sound modes.

The nonpropagating mode has almost the same form as that in the local rest frame,

$$\omega = \frac{i}{\gamma_V\tau_\pi}. \tag{46}$$

The shear modes are given by the solution of the following equation:

$$k^2\eta + \gamma_V\omega[V^2\gamma_V\eta\omega + (\varepsilon + P)(i - \gamma_V\tau_\pi\omega)] = 0, \tag{47}$$

and the solutions are given by

$$\omega_\pm = \frac{1}{2a(b - V^2)\gamma_V}[iT \pm \sqrt{-T^2 + 4a^2bk^2 - 4a^2k^2V^2}]. \tag{48}$$

We find that the critical wave number is now given by $\tilde{k}_c = T/(2a\sqrt{b - V^2})$, below which the shear modes become nonpropagating modes.

On the other hand, the sound modes and another non-propagating mode result from

$$3c_s^2(\varepsilon + P)(-i + \gamma_V\tau_\pi\omega)(k^2 + V^2\gamma_V^2\omega^2)$$
$$+ \gamma_V\omega\{4k^2\eta + \gamma_V\omega[3i(\varepsilon + P) + 4V^2\gamma_V\eta\omega$$
$$- 3(\varepsilon + P)\gamma_V\tau_\pi\omega]\} = 0. \tag{49}$$

The real and imaginary parts of this dispersion relation are calculated with a parameter set satisfying the asymptotic causality condition. The results are shown in Fig. 9. One observes that the real parts are symmetric around $\omega = 0$. This symmetry is due to the fact that the direction of the Lorentz boost is orthogonal to the direction of the perturbation. The critical wave number $\tilde{k}_c$ where the shear mode changes from nonpropagating to propagating mode can be clearly seen. The imaginary parts are seen to be positive. We confirmed that the imaginary parts become negative if we use a parameter set which violates the asymptotic causality condition.

C. Causality of wave propagation

In the preceding discussion we have seen that the theory is stable if the asymptotic causality condition is fulfilled. The reverse is in general not true, as the discussion in the local rest frame has shown, since a stable theory may also violate the asymptotic causality condition. However, the discussion in the Lorentz-boosted frame has revealed that the stability of a theory is contingent upon whether the *asymptotic* causality condition is fulfilled.
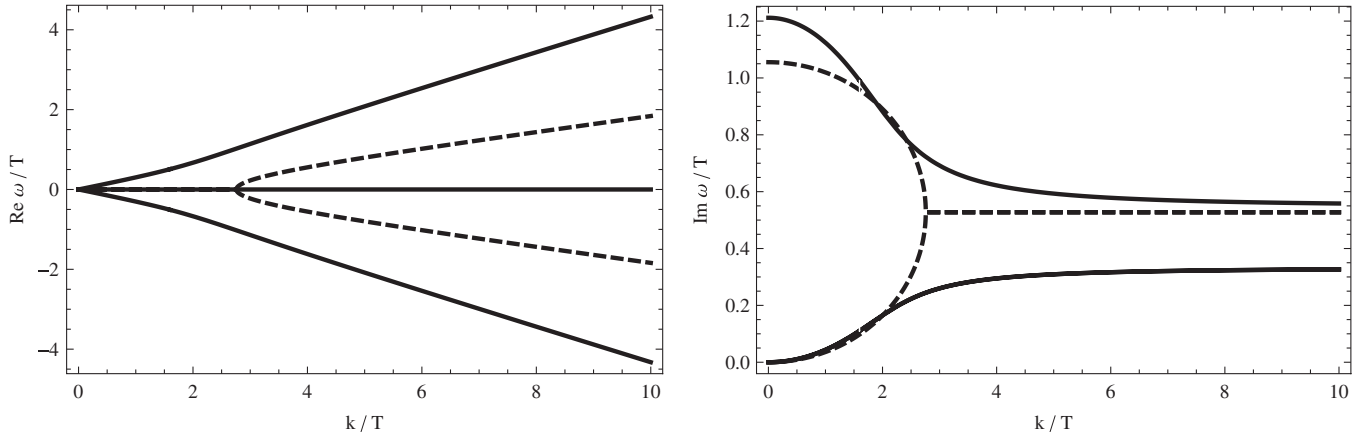
FIG. 9.   The real and imaginary parts for the dispersion relations of the shear modes (dashed lines) and sound modes (solid lines), for a Lorentz boost in the y direction. We use $a = \frac{1}{4\pi}$, $b = 6$, $c_s^2 = \frac{1}{3}$, and $V = 0.9$ in the $3 + 1$-dimensional case.

In this section, we shall show that the causality of the theory as a whole is guaranteed if the asymptotic stability condition is fulfilled. The group velocity may become superluminal, or even diverge, as long as this apparent violation of causality is restricted to a finite range of momenta. The argument leading to this conclusion is analogous to that of Sommerfeld and Brillouin in classical electrodynamics [41,42]. For instance, in the case of anomalous dispersion the group velocity may become superluminal, but the causality of the theory as a whole is not affected.

The change in a fluid-dynamical variable induced by a general perturbation is given by

$$\delta X(x, t) = \sum_j \int d\omega \, \widetilde{\delta X}_j(\omega) e^{i\omega t - i k_j(\omega) x}, \qquad (50)$$

where $\delta X(x, t)$ stands for $\delta \varepsilon$, δu^μ, and $\delta \pi^{\mu\nu}$. The index j denotes the different modes, i.e., the shear modes, the sound modes, etc. The function $k_j(\omega)$ is the inverted dispersion relation $\omega_j(k)$ of the respective mode. The Fourier components are given by

$$\sum_j \widetilde{\delta X}_j(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, \delta X(0, t) e^{-i\omega t}. \qquad (51)$$

We assume that the incident wave has a well-defined front that reaches $x = 0$ not before $t = 0$. Thus $\delta X(0, t) = 0$ for $t < 0$. This condition on $\delta X(0, t)$ ensures that $\sum_j \widetilde{\delta X}_j(\omega)$ is analytic in the lower half of the complex ω plane [41]. On the other hand, in Sec. II A we have found that the group velocity of the shear modes diverges for certain values of k. These divergences correspond to singularities in the complex ω plane. However, if the asymptotic causality condition is fulfilled, the imaginary part of the dispersion relation is always positive; i.e., the singularities only appear in the upper half of the complex ω plane. In this case, the system is also stable. On the other

hand, if the asymptotic causality condition is violated, the singularities may appear also in the lower half-plane, i.e., for the negative imaginary part of the dispersion relation, and the system is unstable.

We shall now demonstrate that the divergences in the group velocity do not violate causality as long as the asymptotic causality condition is satisfied, i.e., as long as the asymptotic group velocity remains subluminal. To this end, we compute Eq. (50) by contour integration in the complex ω plane. To close the contour, we have to know the asymptotic behavior of the dispersion relations. In our calculation, we found that the real part of the dispersion relation at large k is proportional to k [see Eq. (18)], with a coefficient which is the large-k limit of the group velocity, i.e., v_{gj}^{as},

$$\lim_{k \to \infty} \mathrm{Re}\,\omega_j(k) = v_{gj}^{as} k. \qquad (52)$$

Then, in the large-k limit, the exponential becomes

$$\exp[i\omega t - i k_j(\omega) x] \to \exp\left[-i \frac{\omega}{v_{gj}^{as}} (x - v_{gj}^{as} t) \right]. \qquad (53)$$

In the case $x > v_{gj}^{as} t$, we have to close the integral contour in the lower half-plane. If the asymptotic causality condition is fulfilled, there are no singularities in the lower half-plane, and Eq. (50) vanishes. On the other hand, the contour should be closed in the upper half-plane if $x \leq v_{gj}^{as} t$. Then, because of the singularities, Eq. (50) may have a nonzero value. However, as long as we choose a parameter set for which the asymptotic group velocity v_{gj}^{as} is smaller than the speed of light, i.e., for which the asymptotic causality condition is fulfilled, the signal propagation does not violate causality, since the locations x where the disturbance has travelled lie within the cone given by v_{gj}^{as} which, in turn, lies within the light cone, Q.E.D.

To conclude this section, we have shown that the asymptotic causality condition not only implies stability in a

general (Lorentz-boosted) frame, but also causality of the theory as a whole.

IV. CHARACTERISTIC VELOCITIES

So far, we have analyzed the causality and stability of relativistic dissipative fluid dynamics with shear viscosity using a linear stability analysis. However, there is another possibility to analyze causality, namely, by studying the characteristic velocities. For the sake of simplicity, we consider the $2 + 1$-dimensional case with shear viscosity only. The fluid-dynamical equations can be written in the following form:

$$(A^t_{ab}\partial_t + A^x_{ab}\partial_x + A^y_{ab}\partial_y)Y_b = B_a, \qquad (54)$$

where $Y^T_b = (\varepsilon, u^x, u^y, \pi^{xx}, \pi^{xy})$ and $B^T_a = (0, 0, 0, \pi^{xx}, \pi^{xy})$. The expressions for the components of A are given in the appendix. Then, as discussed in Ref. [35], the characteristic velocities are defined as the roots of the following equations:

$$\det(v_x A^t - A^x) = 0, \qquad (55a)$$
$$\det(v_y A^t - A^y) = 0. \qquad (55b)$$

For the case of bulk viscosity, see Ref. [27].

For the sake of simplicity, we consider $u^\mu = (1, 0, 0)$ and $\pi^{xx} = \pi^{xy} = 0$. Then, the characteristic velocities are given by

$$v_x = v_y = \begin{cases} 0, \\ \pm\sqrt{\dfrac{1}{b}}, \\ \pm\sqrt{\dfrac{1}{b} + c_s^2}. \end{cases} \qquad (56)$$

Interestingly, the second velocity is identical to the asymptotic group velocity (23) for the shear modes and the third velocity is the same as the asymptotic group velocity (20) for the sound modes (since $D = 3$). As a matter of fact, if

the asymptotic causality condition (21) is satisfied, the velocity (56) is smaller than the speed of light.

In Fig. 10, we show the b dependence of one of the five characteristic velocities. We set $u^\mu = (\sqrt{5}/2, 1/2, 0)$, $\pi^{xx} = \pi^{xy} = 0$, and $c_s^2 = 1/2$. The velocity exhibits a divergence at small values of b and thus exceeds the speed of light. This divergence occurs also for at least one other characteristic velocity. As far as we have checked numerically, in order to satisfy causality, one should use a value of b which is larger than about 2. This condition is consistent with the asymptotic causality condition (21).

V. CONCLUDING REMARKS

In this work, we have discussed the stability and causality of relativistic dissipative fluid dynamics, based on a linear stability analysis around a hydrostatic state. Following the usual argument, we calculated the group velocity from the dispersion relation of the perturbation. We found that the group velocity diverges at a critical wave number k_c. The appearance of the divergence is independent of the dimensionality of space-time and can be removed neither by tuning the parameters of the theory nor by adding bulk viscosity to the discussion.

Nevertheless, in the rest frame of the background this acausal group velocity does not cause the fluid to become unstable. Moreover, investigating causality and stability in a Lorentz-boosted frame, we found that the fluid-dynamical equations of motion are stable, if we choose parameters which satisfy a so-called asymptotic causality condition. They become unstable if this condition is violated. In this sense, the problems of acausality and instability are still correlated even in the case of shear viscosity, as was already found for the case of bulk viscosity [27].

We have then demonstrated that the causality of the theory as a whole is guaranteed if the asymptotic causality condition is fulfilled. Therefore, a superluminal group velocity in a finite range of momenta can cause the theory
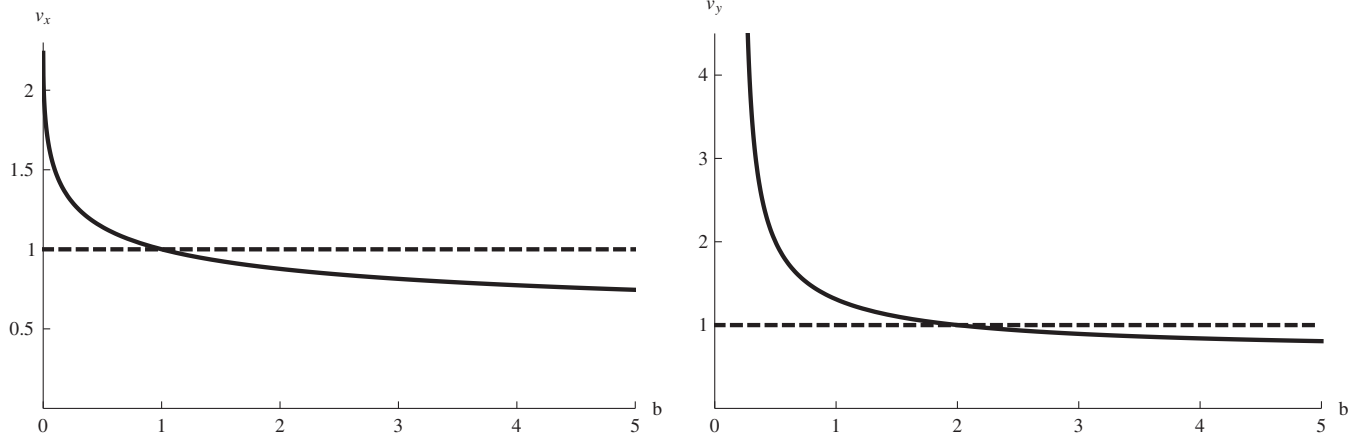

FIG. 10. One of the five characteristic velocities determined from the roots of Eqs. (55). The left panel is for v_x and the right panel is for v_y. We set $u^\mu = (\sqrt{5}/2, 1/2, 0)$, $\pi^{xx} = \pi^{xy} = 0$, and $c_s^2 = 1/2$.

to become neither acausal nor unstable. Finally, we studied the characteristic velocities and found a violation of causality for small values of $\tau_\pi(\varepsilon + P)/\eta$, but not for values which satisfy the asymptotic causality condition.

The asymptotic causality condition requires that the ratio τ_π/Γ_s is sufficiently large, i.e., that the time scale τ_π over which the shear viscous pressure relaxes towards its NS value is not too small compared to the sound attenuation length $\Gamma_s \sim \eta/(\varepsilon + P) \equiv \eta/(Ts)$. This is an important finding for practitioners of fluid dynamics, who frequently consider τ_π and the shear viscosity-to-entropy density ratio η/s to be independent from each other. We have demonstrated that this is not the case if one wants the theory to remain causal. Therefore, second-order theories of relativistic dissipative fluid dynamics are not automatically causal by construction. Our findings also illuminate why NS theory violates causality from a different perspective, because there $\tau_\pi \to 0$ while η remains nonzero.

APPENDIX: MATRIX ELEMENTS IN EQ. (54)

The fluid-dynamical equations can be expressed in the form (54). Let us parameterize the velocity of the fluid as $u^\mu = (\cosh\theta, \sinh\theta\cos\phi, \sinh\theta\sin\phi)$. The matrix elements of A_{ab}^x are

$$A_{11}^x = (c_s^2 + 1)\sinh\theta\cosh\theta\cos\phi,$$

$$A_{12}^x = \tfrac{1}{2}\text{sech}^3\theta\{2\sinh^2\theta[(2w + \pi^{xx})\sin^2\phi + 3w\cos^2\phi - \pi^{xy}\sin\phi\cos\phi] + w\sinh^4\theta(\cos(2\phi) + 3) + w + \pi^{xx}\},$$

$$A_{13}^x = \text{sech}^3\theta\{\sinh^2\theta\cos\phi[(w - \pi^{xx})\sin\phi + \pi^{xy}\cos\phi] + w\sinh^4\theta\sin\phi\cos\phi + \pi^{xy}\},$$

$$A_{14}^x = \tanh\theta\cos\phi,$$

$$A_{15}^x = \tanh\theta\sin\phi,$$

$$A_{21}^x = (c_s^2 + 1)\sinh^2\theta\cos^2\phi + c_s^2,$$

$$A_{22}^x = 2w\sinh\theta\cos\phi,$$

$$A_{24}^x = A_{35}^x = 1,$$

$$A_{31}^x = (c_s^2 + 1)\sinh^2\theta\sin\phi\cos\phi,$$

$$A_{32}^x = w\sinh\theta\sin\phi,$$

$$A_{33}^x = w\sinh\theta\cos\phi,$$

$$A_{42}^x = \text{sech}^2\theta\{\sinh^4\theta\cos^2\phi[\eta + \tau_\pi\pi^{xx}\cos(2\phi) - \tau_\pi\pi^{xx} + \tau_\pi\pi^{xy}\sin(2\phi)] + \sinh^2\theta[2(\eta - \tau_\pi\pi^{xx})\cos^2\phi + \eta\sin^2\phi] + \eta\},$$

$$A_{43}^x = -2\tau_\pi\tanh^2\theta\cos^2\phi[\sinh^2\theta\cos\phi(\pi^{xy}\cos\phi - \pi^{xx}\sin\phi) + \pi^{xy}],$$

$$A_{44}^x = A_{55}^x = \tau_\pi\sinh\theta\cos\phi,$$

$$A_{52}^x = \frac{\tanh^2\theta\cos\phi}{2(\sinh^2\theta\cos^2\phi + 1)}\{-2\sinh^2\theta(\pi^{xx}\sin^3\phi + 2\pi^{xx}\sin\phi\cos^2\phi + \pi^{xy}\cos^3\phi) + \sinh^4\theta\sin^2(2\phi) \times (\pi^{xy}\cos\phi - 2\pi^{xx}\sin\phi) - 2\pi^{xx}\sin\phi - 2\pi^{xy}\cos\phi\},$$

$$A_{53}^x = \tfrac{1}{2}\text{sech}^2\theta\{2\sinh^4\theta\cos^2\phi[\eta - \tau_\pi\pi^{xx}\cos(2\phi) + \tau_\pi\pi^{xx} - \tau_\pi\pi^{xy}\sin(2\phi)] + \sinh^2\theta[(\eta + \tau_\pi\pi^{xx}) \times \cos(2\phi) + 3\eta + \tau_\pi\pi^{xx} - \tau_\pi\pi^{xy}\sin(2\phi)] + 2\eta\}.$$

The matrix elements of A_{ab}^t are given by

$$A_{11}^t = \tfrac{1}{2}[(c_s^2 + 1)\cosh(2\theta) - c_s^2 + 1],$$

$$A_{12}^t = \frac{2\sinh\theta}{(\sinh^2\theta\cos^2\phi + 1)^2}\{\sinh^2\theta\cos\phi(2w\cos^2\phi + \pi^{xx}\sin^2\phi - \pi^{xy}\sin\phi\cos\phi) + w\sinh^4\theta\cos^5\phi + (w + \pi^{xx})\cos\phi + \pi^{xy}\sin\phi\},$$

$$A_{13}^t = 2\sinh\theta\left(w\sin\phi + \frac{\pi^{xy}\cos\phi - \pi^{xx}\sin\phi}{\sinh^2\theta\cos^2\phi + 1}\right),$$

$$A_{14}^t = \frac{\cos(2\phi)}{\text{csch}^2\theta + \cos^2\phi},$$

$$A_{15}^t = \frac{\sin(2\phi)}{\text{csch}^2\theta + \cos^2\phi},$$

$$A_{21}^t = (c_s^2 + 1)\sinh\theta\cosh\theta\cos\phi,$$

$$A_{31}^t = (c_s^2 + 1) \sinh\theta \cosh\theta \sin\phi,$$

$$A_{22}^t = \frac{\mathrm{sech}^3\theta}{2}\{2\sinh^2\theta[(2w + \pi^{xx})\sin^2\phi + 3w\cos^2\phi \\ - \pi^{xy} \sin\phi \cos\phi] + w\sinh^4\theta[\cos(2\phi) + 3] + 2w \\ + 2\pi^{xx}\},$$

$$A_{23}^t = \mathrm{sech}^3\theta\{\sinh^2\theta \cos\phi[w\sinh^2\theta \sin\phi + (w - \pi^{xx}) \\ \times \sin\phi + \pi^{xy} \cos\phi] + \pi^{xy}\},$$

$$A_{24}^t = \tanh\theta \cos\phi,$$

$$A_{25}^t = \tanh\theta \sin\phi,$$

$$A_{32}^t = \frac{\mathrm{sech}^3\theta}{(\sinh^2\theta\cos^2\phi + 1)^2}\{\sinh^2\theta[(w + 3\pi^{xx}) \sin\phi \cos\phi \\ + 3\pi^{xy}\sin^2\phi + 2\pi^{xy}\cos^2\phi] + \sinh^4\theta[3(w + \pi^{xx}) \\ \times \sin\phi\cos^3\phi + (w + 5\pi^{xx})\sin^3\phi \cos\phi \\ + 2\pi^{xy}\sin^4\phi + \pi^{xy}\cos^4\phi] + \frac{1}{16}\sinh^6\theta[10\sin(2\phi) \\ + \sin(4\phi)][(w - \pi^{xx})\cos(2\phi) + w + \pi^{xx} \\ - \pi^{xy}\sin(2\phi)] + w\sinh^8\theta \sin\phi\cos^5\phi + \pi^{xy}\},$$

$$A_{33}^t = \frac{\mathrm{sech}^3\theta}{8(\sinh^2\theta\cos^2\phi + 1)}\{\sinh^4\theta[4(w + 2\pi^{xx})\cos(2\phi) \\ + (\pi^{xx} - w)\cos(4\phi) + 21w - 9\pi^{xx} \\ + 10\pi^{xy}\sin(2\phi) + \pi^{xy}\sin(4\phi)] + 4\sinh^2\theta[6w \\ + 2\pi^{xx}\cos(2\phi) - 4\pi^{xx} + 3\pi^{xy}\sin(2\phi)] \\ - 4w\sinh^6\theta\cos^2\phi[\cos(2\phi) - 3] + 8w - 8\pi^{xx}\},$$

$$A_{34}^t = -\frac{\tanh\theta \sin\phi(\sinh^2\theta\sin^2\phi + 1)}{\sinh^2\theta\cos^2\phi + 1},$$

$$A_{35}^t = \frac{\tanh\theta \cos\phi}{2\sinh^2\theta\cos^2\phi + 2}\{2 - \sinh^2\theta[\cos(2\phi) - 3]\},$$

$$A_{42}^t = \tanh\theta \cos\phi\{\sinh^2\theta\{2\sin\phi[(\eta - \tau_\pi\pi^{xx})\sin\phi \\ + \tau_\pi\pi^{xy}\cos\phi] + \eta\cos^2\phi\} + \eta - 2\tau_\pi\pi^{xx}\},$$

$$A_{43}^t = -\tanh\theta\{\sinh^2\theta\cos^2\phi[(\eta - 2\tau_\pi\pi^{xx})\sin\phi \\ + 2\tau_\pi\pi^{xy}\cos\phi] + \eta\sin\phi + 2\tau_\pi\pi^{xy}\cos\phi\},$$

$$A_{44}^t = A_{55}^t = \tau_\pi \cosh\theta,$$

$$A_{52}^t = \frac{\tanh\theta}{4\sinh^2\theta\cos^2\phi + 4}\{-2\sinh^2\theta\{\sin\phi[-2\eta \\ + \tau_\pi\pi^{xx}\cos(2\phi) + 3\tau_\pi\pi^{xx}] + 2\tau_\pi\pi^{xy}\cos^3\phi\} \\ + \sinh^4\theta\sin^2(2\phi)[(\eta - 2\tau_\pi\pi^{xx})\sin\phi \\ + 2\tau_\pi\pi^{xy}\cos\phi] + 4(\eta - \tau_\pi\pi^{xx})\sin\phi \\ - 4\tau_\pi\pi^{xy}\cos\phi\},$$

$$A_{53}^t = \tanh\theta\{\sinh^2\theta[\eta\cos^3\phi + \tau_\pi\pi^{xx}\sin\phi\sin(2\phi) \\ - 2\tau_\pi\pi^{xy}\sin\phi\cos^2\phi] + (\eta + \tau_\pi\pi^{xx})\cos\phi \\ - \tau_\pi\pi^{xy}\sin\phi\}.$$

The matrix elements of A_{ab}^y are

$$A_{11}^y = (c_s^2 + 1)\sinh\theta\cosh\theta\sin\phi,$$

$$A_{21}^y = (c_s^2 + 1)\sinh^2\theta\sin\phi\cos\phi,$$

$$A_{12}^y = \frac{\mathrm{sech}^3\theta}{(\sinh^2\theta\cos^2\phi + 1)^2}\{\sinh^2\theta[(w + 3\pi^{xx})\sin\phi\cos\phi \\ + 3\pi^{xy}\sin^2\phi + 2\pi^{xy}\cos^2\phi] + \sinh^4\theta[3(w + \pi^{xx}) \\ \times \sin\phi\cos^3\phi + (w + 5\pi^{xx})\sin^3\phi\cos\phi \\ + 2\pi^{xy}\sin^4\phi + \pi^{xy}\cos^4\phi] + \frac{1}{16}\sinh^6\theta[10\sin(2\phi) \\ + \sin(4\phi)][(w - \pi^{xx})\cos(2\phi) + w + \pi^{xx} \\ - \pi^{xy}\sin(2\phi)] + w\sinh^8\theta\sin\phi\cos^5\phi + \pi^{xy}\},$$

$$A_{13}^y = \frac{\mathrm{sech}^3\theta}{8(\sinh^2\theta\cos^2\phi + 1)}\{\sinh^4\theta[4(w + 2\pi^{xx})\cos(2\phi) \\ + (\pi^{xx} - w)\cos(4\phi) + 21w - 9\pi^{xx} \\ + 10\pi^{xy}\sin(2\phi) + \pi^{xy}\sin(4\phi)] + 4\sinh^2\theta[6w \\ + 2\pi^{xx}\cos(2\phi) - 4\pi^{xx} + 3\pi^{xy}\sin(2\phi)] \\ - 4w\sinh^6\theta\cos^2\phi[\cos(2\phi) - 3] + 8w - 8\pi^{xx}\},$$

$$A_{14}^y = -\frac{\tanh\theta\sin\phi(\sinh^2\theta\sin^2\phi + 1)}{\sinh^2\theta\cos^2\phi + 1},$$

$$A_{15}^y = \frac{\tanh\theta\cos\phi}{2\sinh^2\theta\cos^2\phi + 2}\{2 - \sinh^2\theta[\cos(2\phi) - 3]\},$$

$$A_{22}^y = w\sinh\theta\sin\phi,$$

$$A_{23}^y = w\sinh\theta\cos\phi,$$

$$A_{25}^y = 1,$$

$$A_{31}^y = (c_s^2 + 1)\sinh^2\theta\sin^2\phi + c_s^2,$$

$$A^y_{32} = \frac{2\sinh\theta[\sinh^2\theta\sin\phi\cos\phi(\pi^{xx}\sin\phi - \pi^{xy}\cos\phi) + \pi^{xx}\cos\phi + \pi^{xy}\sin\phi]}{(\sinh^2\theta\cos^2\phi + 1)^2},$$

$$A^y_{33} = 2\sinh\theta\left(w\sin\phi + \frac{\pi^{xy}\cos\phi - \pi^{xx}\sin\phi}{\sinh^2\theta\cos^2\phi + 1}\right),$$

$$A^y_{34} = -\frac{\sinh^2\theta\sin^2\phi + 1}{\sinh^2\theta\cos^2\phi + 1},$$

$$A^y_{35} = \frac{\sin(2\phi)}{\mathrm{csch}^2\theta + \cos^2\phi},$$

$$A^y_{42} = \tanh^2\theta\sin\phi\cos\phi\{\sinh^2\theta[2\eta + \tau_\pi\pi^{xx}\cos(2\phi)$$
$$- \tau_\pi\pi^{xx} + \tau_\pi\pi^{xy}\sin(2\phi)] + 2\eta - 2\tau_\pi\pi^{xx}\},$$

$$A^y_{43} = -\frac{\mathrm{sech}^2\theta}{2}\{2\sinh^4\theta\cos^2\phi[\eta + \tau_\pi\pi^{xx}\cos(2\phi)$$
$$- \tau_\pi\pi^{xx} + \tau_\pi\pi^{xy}\sin(2\phi)] + \sinh^2\theta\{\eta[\cos(2\phi)$$
$$+ 3] + 2\tau_\pi\pi^{xy}\sin(2\phi)\} + 2\eta\},$$

$$A^y_{44} = A^y_{55} = \tau_\pi\sinh\theta\sin\phi,$$

$$A^y_{52} = \frac{\tanh^2\theta}{8(\sinh^2\theta\cos^2\phi + 1)}\{\sinh^2\theta[(\tau_\pi\pi^{xx} - \eta)\cos(4\phi)$$
$$+ 9\eta + 4\tau_\pi\pi^{xx}\cos(2\phi) - 5\tau_\pi\pi^{xx}$$
$$- 8\tau_\pi\pi^{xy}\sin\phi\cos^3\phi] + 2\sinh^4\theta\sin^2(2\phi)[\eta$$
$$+ \tau_\pi\pi^{xx}\cos(2\phi) - \tau_\pi\pi^{xx} + \tau_\pi\pi^{xy}\sin(2\phi)]$$
$$+ 4[4\eta + \tau_\pi\pi^{xx}\cos(2\phi) - \tau_\pi\pi^{xx}$$
$$- \tau_\pi\pi^{xy}\sin(2\phi)] + 8\eta\,\mathrm{csch}^2\theta\},$$

$$A^y_{53} = \tau_\pi\tanh^2\theta\sin\phi[\sinh^2\theta\sin(2\phi)(\pi^{xx}\sin\phi$$
$$- \pi^{xy}\cos\phi) + \pi^{xx}\cos\phi - \pi^{xy}\sin\phi],$$

where we defined $w = \varepsilon + P$. All other elements vanish.

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