PHYSICAL REVIEW D **81,** 107701 (2010)

Nonperturbative results for the mass dependence of the QED fermion determinant

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(Received 26 February 2010; published 19 May 2010)

The fermion determinant in four-dimensional quantum electrodynamics in the presence of $O(2) \times O(3)$ symmetric background gauge fields with a nonvanishing global chiral anomaly is considered. It is shown that the leading mass singularity of the determinant's nonperturbative part is fixed by the anomaly. It is also shown that for a large class of such fields there is at least one value of the fermion mass at which the determinant's nonperturbative part reduces to its noninteracting value.

DOI: 10.1103/PhysRevD.81.107701 PACS numbers: 11.15.Tk, 11.10.Kk, 12.20.Ds

Every physical process calculable within the standard model ultimately depends on the model's fermion determinants. These are part of the effective functional measure for the gauge fields when the fermion fields are integrated. Without them, charge and color screening, quark fragmentation into hadrons and unitarity would be lost. Accordingly, they are fundamental, and the nonperturbative structure of the standard model requires corresponding information about its determinants.

Taking Schwinger's 1951 paper [1] on vacuum polarization as the beginning of the modern era of fermion determinants, progress on elucidating their nonperturbative properties has been extremely slow. In this paper we focus on the fermion mass dependence of the determinant in four-dimensional QED. We note here the recent advance in determining the quark mass dependence of the QCD_4 instanton determinant [2].

We begin by summarizing the main analytic results for QED_4's fermion determinant on noncompact, Euclidean space-time. Formally, a fermion field integration produces the ratio of determinants $\det(\not{p} - e\not{A} + m)/\det(\not{p} + m) = \det(1 - eS\not{A})$, where S is the free fermion propagator. Since the operator $S\not{A}$ is not trace class, $\det(1 - eS\not{A})$ is undefined no matter how well behaved the gauge field A_μ is. Nevertheless, sense is made of it based on the following results:

(a) The operator $S\not{A}$ is a non-Hermitian compact operator in the trace ideal $\mathcal{I}_p$ for $p > 4$ and fermion mass $m \neq 0$ provided $A_\mu \in L^p(\mathbb{R}^4)$ [3,4]. This includes the instanton-like case of A_μ having a $1/r$ falloff. The theorem means that the traces $\mathrm{Tr}(S\not{A})^n$, $n \geq 5$, are absolutely convergent and really do correspond to sums of eigenvalues of $S\not{A}$.

(b) A renormalized determinant can be defined:

$$\det{}_{\mathrm{ren}} = \exp(\Pi_2 + \Pi_3 + \Pi_4)\det{}_5(1 - eS\not{A}), \qquad (1)$$

where

$$\mathrm{ln}\det{}_5 = \mathrm{Tr}\left[\ln(1 - eS\not{A}) + \sum_{n=1}^{4} \frac{(eS\not{A})^n}{n}\right], \qquad (2)$$

and $\Pi_{2,3,4}$ are the second-, third-, and fourth-order contributions to the one-loop effective action, $\mathrm{ln}\det_{\mathrm{ren}}$, defined by some consistent regularization procedure [5]. The graph Π_2 contains a charge renormalization subtraction. The regularization should result in $\Pi_3 = 0$ by C invariance and give the unique gauge invariant result for Π_4.

(c) As corollaries of (a) and (b), $\det_5$ is an entire function of the coupling e and can be represented in terms of the discrete complex eigenvalues $1/e_n$ of $S\not{A}$:

$$\det{}_5 = \prod_n \left[\left(1 - \frac{e}{e_n}\right)\exp\left(\sum_{k=1}^{4} \frac{(e/e_n)^k}{k}\right)\right]. \qquad (3)$$

By C invariance and the reality of $\det_5$ for real e the eigenvalues can appear in quartets $\pm e_n$, $\pm \bar{e}_n$ or as imaginary pairs [6].

(d) $\det_{\mathrm{ren}}$ has no zeros for real e when $m \neq 0$ [7], and since $\det_{\mathrm{ren}}(e = 0) = 1$, $\det_{\mathrm{ren}} > 0$ for real e.

(e) $\det_{\mathrm{ren}}$ is an entire function of e of order 4 since $S\not{A} \in \mathcal{I}_{4+\epsilon}$ when $A_\mu \in L^{4+\epsilon}(\mathbb{R}^4)$, $\epsilon > 0$ [8]. This conclusion was first reached for a restricted class of gauge fields by Adler [9] and later by Balian *et al.* [10], for another restricted class of fields. The growth of $\det_{\mathrm{ren}}$ for real values of e is unknown.

(f) $\det_5$ is an analytic function of m throughout the complex m plane cut along the negative real axis [11].

This comprises the general analytic knowledge of $\det_{\mathrm{ren}}$ obtained since 1951. The present sparse knowledge of such a central part of the standard model is noteworthy. There is a large body of results for background gauge fields which do not fall off sufficiently rapidly in all directions in $\mathbb{R}^4$ to satisfy the theorem in (a) [12]. These require the introduction of an *ad hoc* volume cutoff, and none of the results (b)–(f) necessarily hold for such fields.

We report here on an extension of results (a)–(f) for a large class of $O(2) \times O(3)$ symmetric background gauge fields of the form $A_\mu(x) = M_{\mu\nu}x_\nu a(r^2)$, where the profile function $a(r^2)$ is at least 3 times differentiable, regular at the origin, and $a(r^2) = \nu/r^2$ for $r > R$, R being a range parameter, $r^2 = x_\mu x_\mu$. The constant ν is assumed positive without loss of generality. For $r < R$, $a(r^2)$ may have multiple zeros. The constant antisymmetric matrix M has nonvanishing entries $M_{12} = M_{30} = 1$. Letting ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$ and noting that ${}^*FF = \partial_\alpha(\epsilon_{\alpha\beta\mu\nu}A_\beta F_{\mu\nu})$, it is evident that A_μ must have a $1/r$ falloff for the Dirac operator $\not{D} = \not{P} - e\not{A}$ to have a nonvanishing global chiral anomaly $\mathcal{A} = -\int d^4x^* F_{\mu\nu}F_{\mu\nu}/16\pi^2$. In our case $\mathcal{A} = \nu^2/2$. From here on we set $e = 1$ to reduce notation.

In [11] we proved a vanishing theorem for this choice of A_μ. Consequently all the square-integrable zero modes of $\not{D}$ have positive chirality, and such modes first appear when $\nu > 2$. The zero modes can be shifted to the negative chirality sector by replacing M with the antisymmetric matrix N with entries $N_{03} = N_{12} = 1$. Since $F_{\mu\nu}$ is not self-dual it extends the vanishing theorem of [13] to such U(1) fields. Because the calculation here is in noncompact Euclidean space-time the index theorem has to be modified to account for the continuum part of the spectrum of $\not{D}^2$ extending down to zero energy, thereby contributing an additional part to the index of $\not{D}$ [14,15]. These low-energy states play an essential role in the analysis discussed below.

We can now add the following results to the list (a)–(f) above:

(g) For the class of gauge fields defined above the leading mass singularity of $\ln\det_5$ is governed by the chiral anomaly, that is,

$$\ln\det_5 \underset{m\to 0}{\sim} \frac{\nu^2}{4}\ln m^2 + \text{ less singular}, \tag{4}$$

so that $\ln\det_5$ becomes negative as $m \to 0$. Inconclusive evidence for (4) was first reported in [11].

The presence of a zero mode in the spectrum of the Dirac operator and its control of the leading quark mass dependence of QCD_4 instanton determinant has been known for many years [16]. Establishing (4) relies on the above vanishing theorem and showing that the zero-mode-free negative chirality sector contributes terms to (4) less singular than $\ln m^2$. With more effort it should be possible to prove (4) under the more general assumption that $a(r^2) \sim \nu/r^2$, for $r \gg R$.

(h) If

$$\int_0^{R^2} dr^2[2r^{14}a'^6 + 12r^{12}aa'^5 + 23r^{10}a^2a'^4 + 12r^8a^3a'^3 - 19r^6a^4a'^2] < \frac{9\nu^6}{2R^8}, \tag{5}$$

then $\ln\det_5$ becomes positive before dropping off to zero for $m \to \infty$ [11]. This and (4) imply that there is at least one value of m for which $\ln\det_5 = 0$. That is, $\ln\det_5$ has a mass zero, or possibly an odd number of such zeros, at which $\det_5 = 1$.

Subject to the conditions on a stated above, (5) can be satisfied in general for an $a(r^2)$ that can oscillate between positive and negative values before curving downward to join smoothly with its long-range form ν/r^2 at $r = R$.

As ν is varied the eigenvalues e_n in (3) will shift, presumably shifting the position of the mass zero as well. That a particular value of m can cause $\det_5$ to assume its noninteracting value indicates that mass has a profound effect on the distribution of the eigenvalues e_n in the complex plane.

To establish (4) we begin by defining $\det_{\text{ren}}$ in (1) by

$$\ln\det_{\text{ren}} = \frac{1}{2}\lim_{\delta\to 0}\int_\delta^\infty \frac{dt}{t} \times \Big[\mathrm{Tr}(e^{-tP^2} - e^{-t[(P-A)^2+(1/2)\sigma F]})e^{-tm^2} + \frac{1}{24\pi^2}\int d^4x F^2(x)e^{-t\mu^2}\Big], \tag{6}$$

where μ is the renormalization scale. Although $\det_{\text{ren}}$ is finite with on-shell renormalization of Π_2 when $F_{\mu\nu} \sim 1/r^2$, this complicates the small-mass analysis of $\det_5$, and so we prefer to deal with the off-shell case. One can always go back on shell once $\det_5$ is understood.

In the representation where γ_5 is diagonal with entries $\pm\mathbb{1}_2$, $(P-A)^2 + \frac{1}{2}\sigma F$ is diagonal with corresponding positive and negative chirality entries $H_\pm = (P-A)^2 - \boldsymbol{\sigma}\cdot(\mathbf{B} \pm \mathbf{E})$. Differentiating (6) with respect to m^2 yields the renormalization independent result

$$m^2\frac{\partial}{\partial m^2}\ln\det_5 = \frac{1}{2}m^2\,\mathrm{Tr}[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}] + m^2\int_0^\infty dt e^{-tm^2} \times \int d^4x\,\mathrm{tr}\langle x|e^{-tH_-} - e^{-tP^2}|x\rangle - m^2\partial\Pi_2/\partial m^2 - m^2\partial\Pi_4/\partial m^2, \tag{7}$$

where we have taken δ to zero and set $\Pi_3 = 0$. The first and last terms in (7) are well defined for the background fields considered here, but the second and third terms are not. Specifically, the perturbative expansion of (6) gives $\partial\Pi_2/\partial m^2 \sim \int_0 dk/k$, which must be cancelled.

The strategy is this: There must be a corresponding infrared divergence in the second term in (7) that cancels that in $\partial\Pi_2/\partial m^2$, as the left-hand side of (7) is well defined for $m^2 > 0$. The second term in (7) will be calculated by summing over the exact eigenstates of H_-. As already noted, these are scattering states only. An infrared regulator is introduced by cutting off the low-energy spectra of H_- and P^2 at λ^2. Then the infrared divergent part is isolated; it must be second order. Identify it as the divergent part of $\partial\Pi_2/\partial m^2$ to effect the cancellation of infrared divergences. That is, the second term in (7) defines the divergent part of $\partial\Pi_2/\partial m^2$ according to (9) and (13) below, consistent with our way of calculating $\det_{\text{ren}}$. Then set $\lambda = 0$ and finally study the $m \to 0$ limit of (7) to find the small-mass dependence of the well-defined quantity $\ln\det_5$. It is essential to proceed in this way. Pulling out the contribution to $\partial\Pi_2/\partial m^2$ from the second term in (7) by a straightforward perturbation expansion results in a gauge invariant remainder that is a sum of separately nongauge invariant terms, leading to a computational impasse.

For the fields under consideration we find Π_4 is well defined for $m \neq 0$ and is less singular than $\ln m^2$ as $m \to 0$ and so gives a vanishing contribution to the right-hand side of (7) as $m \to 0$. This result relies in part on the finiteness at $m = 0$ of the photon-photon scattering subgraph in Π_4 [17].

We now turn to the calculation of the rest of the right-hand side of (7). Here, we use the definition of the chiral anomaly on noncompact manifolds due to Musto *et al.* [15], $\mathcal{A} = \lim_{m\to 0} m^2 \mathrm{Tr}[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}]$. This combined with $\mathcal{A} = \nu^2/2$, Eq. (7), and an integration with respect to m^2 gives result (4), provided that the remainder in (7) contributes terms to (4) less singular than $\ln m^2$ for $m \to 0$.

Denote the second term on the right-hand side of (7) by I and obtain for $m \to 0$

$$I(m^2) = m^2 \lim_{\lambda\to 0} \lim_{L\to\infty} \int_{\lambda^2}^{\Lambda^2} \frac{dk^2}{k^2 + m^2} \int_0^L dr r^3 \int d\Omega_4 \times \sum_{jMmm'} (|\Psi^-_{EjMmm'}(x)|^2 - |\Psi^0_{EjMmm'}(x)|^2), \tag{8}$$

where $\Psi^-_{EjMmm'}$ are the eigenstates H_- derived in [11], and $\Psi^0_{EjMmm'}$ are the associated free-particle states. Here, $E = k^2$, and $j = 0, \frac{1}{2}, 1, \ldots$; $M = -j - \frac{1}{2}, \ldots, j + \frac{1}{2}$; $m = -j, \ldots, j$; $m' = \pm\frac{1}{2}$ are the quantum numbers associated with the $O(2) \times O(3)$ symmetry of the background fields; λ is the infrared cutoff introduced above, and Λ, with $\Lambda R < 1$, limits the range of k needed to study the small-mass dependence of I.

Divide I into $I_>(I_<)$, the exterior (interior) parts of I from $r > R$ $(r < R)$, and consider first the most singular part in m^2, $I_>$. The radial wave functions associated with Ψ^- for $r > R$ are calculated from the outgoing wave combination of Bessel functions $\sqrt{r} J_\sigma(kr) \cos\Delta_\alpha(k) - \sqrt{r} Y_\sigma(kr) \sin\Delta_\alpha(k)$, where α denotes jMm', $\sigma = \sqrt{(2j+1)^2 + 4\nu M + \nu^2}$, and Δ_α is the energy-dependent part of the low-energy phase shifts δ_α, $\Delta_\alpha(k) = \pi(\sigma - 2j - 1)/2 + \delta_\alpha(k)$, mod π. Since $\Delta_\alpha(0) = 0$, we can expand in powers of Δ_α. For $|M| \neq j + \frac{1}{2}$, $\tan\Delta_\alpha = C_\alpha(\sigma)(kR/2)^{2\sigma}(\sigma\Gamma^2(\sigma))^{-1}[1 + O((kR)^2, (kR)^{2\sigma})]$, where $C_\alpha(\sigma)$ is a bounded function of σ [11]. The rapid falloff of Δ_α with j and energy allows one to terminate the expansion after Δ_α^2. Terms in $I_>$ containing Δ_α and Δ_α^2 are uniformly convergent and can be integrated term by term and the limit $L = \infty$ taken. There are some oscillating k integrals containing $\cos(2kL)$ and $\sin(2kL)$. These are set equal to zero by the Riemann-Lebesque lemma following the sequence of limits in (8). The result is contributions to $I_>$ less singular than $m^2 \ln m^2$ and $O(1)$ contributions to $\ln\det_5$. Terms from $M = j + \frac{1}{2}$ contribute $O(m^2)$ terms to $I_>$. Terms of $O(\Delta_\alpha^0)$ will be considered below.

The zero modes of H_+ appear in the sector $M = -j - \frac{1}{2}$ for values of j satisfying $\nu > 2j + 2$, $j = 0, \frac{1}{2}, \ldots$. The most singular contribution to $I_>$ occurs at the zero mode thresholds $M = -j - \frac{1}{2}$, $\nu = 2j + 2$ at which Δ_α's energy dependence drops to $\tan\Delta_\alpha = \frac{\pi}{2}(1 + O(kR)^2) \times [\ln(kR) + C + O(kR)^2 \ln(kR)]^{-1}$ where C is a negative k-independent constant [11]. This results in a contribution to $I_>$ of $O(1/\ln(mR))$ and a $\ln|\ln(mR)|$ contribution to $\ln\det_5$ in (4). This covers all terms in (8) from Δ_α and Δ_α^2.

The zero mode thresholds also dominate the region $r < R$. Specifically, they are responsible for the radial wave function contributing to (8) with the slowest k falloff, whose form is $(\ln kR + C)^{-1}\psi(k^2, r)$, $\psi(0, r) \neq 0$, and C as above. Here, ψ is analytic in k^2 and is a smooth function of r behaving near $r = 0$ as $r^{2j+3/2}$. This results in contributions of $O(m^2)$ to $I_<$ and $O(1)$ to $\ln\det_5$. Other cases in the $M = -j - \frac{1}{2}$ sector have a faster small k falloff. The study of the $M \neq -j - \frac{1}{2}$ sectors is facilitated by the $1/(2j+1)!$ falloff of the radial wave functions (also true for $M = -j - \frac{1}{2}$), their small k falloff of at least $(kR)^\sigma$, and their $r^{2j+1/2}$ behavior near $r = 0$ [11]. These results allow the $m \to 0$ limit of $I_<$ to be taken term by term, giving a final $O(1)$ contribution to $\ln\det_5$.

Now consider the terms in $I_>$ of $O(\Delta_\alpha^0)$, here denoted by $I_>^0$. For fixed L the integral and sum over j in (8) can be interchanged since $|J_\sigma(z)| \leq |z/2|^\sigma/\Gamma(\sigma + 1)$, z real, and because only J_σ is present in $I_>^0$. The result for the L-dependent terms is

$$I_>^0 = \frac{m^2}{4} \lim_{\lambda\to 0} \lim_{L\to\infty} \int_{\lambda^2}^{\Lambda^2} \frac{dk^2}{k^2 + m^2} L^2(S_1(kL) + S_2(kL)), \tag{9}$$

where

$$S_1 = \sum_{j=0,\frac{1}{2},\ldots} (2j+1)[J^2_{|2j+2-\nu|} - J_{|2j+1-\nu|}J_{|2j+3-\nu|} + (\nu \to -\nu) - (\nu = 0)], \tag{10}$$

$$S_2 = \sum_{j=\frac{1}{2},1,\ldots} (2j+1) \sum_{M=-j+(1/2)}^{j-(1/2)} [J^2_{\sigma+1} - J_{\sigma+2}J_\sigma + J^2_{\sigma-1} - J_{\sigma-2}J_\sigma - (\nu = 0)]. \tag{11}$$

The Bessel functions are evaluated at kL. These series are not uniformly convergent and must be summed before taking $L \to \infty$. S_1 can be summed to give for $kL \gg 1$

$$S_1 = \frac{2\nu^2}{\pi kL} + \frac{1}{\pi(kL)^2}\cos(2kL)\sin^2\left(\frac{\pi\nu}{2}\right) + O(kL)^{-3}. \tag{12}$$

The leading term in (12) must be cancelled by S_2 to make $I_>^0$ finite.

Up to this point all calculations have been nonperturbative. We have not been able to sum S_2 without resorting to its perturbative expansion in ν. This is a well-behaved expansion as it occurs in the Bessel function's order, and J_σ is an entire function of σ. To $O(\nu^2)$ we find for $kL \gg 1$

$$S_2 = -\frac{2\nu^2}{\pi kL} + \left(\frac{\nu}{kL}\right)^2 \times \left[C + \frac{\pi}{12}\cos(2kL) + \left(\frac{7}{30} + \frac{2}{3}\ln 2\right)\sin(2kL)\right] + O(kL)^{-3}. \tag{13}$$

As expected, the leading term in (13) cancels that in (12). Referring to (9), the second term in (13) results in the expected infrared divergent term discussed above. The constant C is given by a complicated, but absolutely convergent, series of Bessel functions. Its value is irrelevant to our analysis as it will be cancelled by the counterterm $\partial\Pi_2/\partial m^2$. The remaining oscillating terms in (12) and (13) give vanishing contribution to $I^0_>$ by the Riemann-Lebesque lemma.

The second-order calculation may be extended to all orders in ν. Structures generated in second order appear again in higher orders differentiated with respect to Bessel function order. No further infrared divergences appear, only $\cos(2kL)$ and $\sin(2kL)$ terms as in (12) and (13). Because S_1 and S_2 are closely related, and S_1's summed series has an infinite radius of convergence when expanded in ν, we are confident no information has been lost in the expansion of S_2. The R-dependent terms from the lower bound of integration of $I^0_>$ are uniformly convergent, and no expansion is necessary. They result in $O(1)$ contributions to $\ln\det_5$. This establishes Eq. (4).

The conclusion that $\det_5$ can be reduced to its noninteracting value by varying its mass for a class of background gauge fields points to an unexpected nonperturbative role of mass in QED_4's effective action. It would be surprising if result (4)—the chiral anomaly's control of $\ln\det_5$'s leading mass singularity—is limited to our background fields. Presumably it is generally true and, if so, mass zero(s) in $\ln\det_5$ are also present more generally.

The communication of C. Schubert that Π_4 in scalar QED_4 with a r^{-2} falloff profile function has no $\ln m^2$ singularity is gratefully acknowledged.

Note added in proof.—The form of the result (13) for the second-order term in S_2 as well as all higher order terms in S_2's expansion hold for arbitrarily large values of ν; no information is lost in these expansions. The proof of this result will appear in the arXiv version of this paper.

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