

Liouville-Lifshitz theory in 3 + 1 dimensionsJ. Alexandre,^{1,*} K. Farakos,^{2,†} and A. Tsapalis^{2,3,‡}¹*King's College London, Department of Physics, WC2R 2LS, UK*²*Department of Physics, National Technical University of Athens, Zografou Campus, 157 80 Athens, Greece*³*Hellenic Naval Academy, Hatzikyriakou Avenue, Pireaus 185 39, Greece*

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We consider a four-dimensional theory in the $z = 3$ Lifshitz context, with an exponential (Liouville) potential. We determine the exact renormalized potential of the theory and derive the nonperturbative relation between the renormalized and bare couplings. In addition, we show that Lorentz symmetry is naturally generated by quantum fluctuations in the infrared regime, and conclude that the model can be relevant to high energy physics.

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I. INTRODUCTION

Recently, quantum field theories in the Lifshitz context have attracted attention, exhibiting interesting renormalization properties [1]. Lifshitz-type models are based on an anisotropy between space and time directions, which is characterized by the dynamical critical exponent z , determining the properties of space-time coordinates under scale transformations: $t \rightarrow b^z t$ and $\mathbf{x} \rightarrow b\mathbf{x}$. For $z > 1$ the higher powers of momentum in the propagators lower the superficial degree of divergence of graphs, yielding the renormalizability of new interactions, such as the four-fermion interaction [2]. Also, divergences of renormalizable interactions in the standard model become softer [3] as, for example, in the Yukawa model [4], where only logarithmic divergences appear.

While absent from the classical action, Lorentz symmetry is naturally generated in Lifshitz-type models through quantum corrections, since the corresponding kinetic term is a relevant operator and dominates the dispersion relation of the modes in the infrared regime (IR). Note, however, that recovering the speed of light, in theories with more than one species of interacting fields, requires fine-tuning of bare parameters [5].

In the context of Lifshitz-type models, the scalar field has dimensionality $[\phi] = (D - z)/2$, such that for $D = z$ it is dimensionless. As a consequence, any power ϕ^n represents a classically marginal operator. This is also the case of Liouville theory, in 1 + 1 dimensions, where the potential is $\mu^2 \exp(g\phi)$. It is known that, after quantization, the potential maintains its exponential form, with renormalized parameters μ_r and g_r . In this theory, the renormalized coupling g_r receives finite corrections and its exact relation to the bare coupling g is known [6,7]. We show in this paper that these results hold for the 3 + 1 dimensional Liouville potential in the $z = 3$ Lifshitz the-

ory. Our proof is based on both exact functional properties and the complete resummation of diverging graphs. Note that this approach offers an independent derivation for the known results of the 1 + 1 dimensional Liouville theory.

The outline of the paper is the following. In Sec. II we present the classical model and its quantization via path integral. The exact functional form of the renormalized potential is derived in Sec. III. As an illustration, we calculate the one-loop renormalized mass in Sec. IV, where we also show that the Lorentz-restoring kinetic term is generated in the quantum theory.

II. LIOUVILLE-LIFSHITZ MODEL**A. Classical action**

The Liouville-Lifshitz model for $D = z = 3$ is defined by the classical action,

$$S_\mu = \int dt d^3\mathbf{x} \left(\frac{1}{2} (\dot{\phi})^2 - \frac{1}{2} \partial^k \phi \Delta^2 \partial_k \phi - \frac{\mu^6}{g^2} e^{g\phi} \right), \quad (1)$$

where $[\phi] = [g] = 0$ and $[\mu] = 1$. From naive power counting, the theory is expected to be renormalizable. An essential property of the model (1) is the following: a constant shift in the field $\phi(x) \rightarrow \phi(x) + \eta$ is equivalent to a redefinition of the only dimensionful parameter as

$$\mu^6 \rightarrow \tilde{\mu}^6 = \mu^6 e^{g\eta}, \quad (2)$$

and we will show how this property enables us to determine exactly the functional form of the renormalized potential of the theory. Because of the higher order spatial derivatives in the classical action, the propagator to be used in the diagrammatic analysis is

$$G(\omega, \mathbf{p}) = \frac{i}{\omega^2 - (\mathbf{p}^2)^3 - \mu^6 + i\epsilon}, \quad (3)$$

as determined by the quadratic part of the action (1).

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B. Path integral quantization

The path integral for the model (1) is

$$Z_\mu[j] = \int \mathcal{D}[\phi] \exp\left(iS_\mu[\phi] + i \int j\phi\right) = \exp(iW_\mu[j]), \quad (4)$$

where j is the source and $W_\mu[j]$ is the connected graphs generating functional. The classical field is then defined as

$$\phi_{\text{cl}}(x) = \frac{\delta W_\mu[j]}{\delta j(x)}. \quad (5)$$

Shifting the field by a constant $\phi(x) \rightarrow \phi(x) + \eta$ in the path integral (4) leads to

$$W_\mu[j] = W_{\tilde{\mu}}[j] + \int j(x)\eta \quad (6)$$

from which a functional derivative with respect to $j(x)$ gives

$$\tilde{\phi}_{\text{cl}}(x) = \phi_{\text{cl}}(x) - \eta, \quad (7)$$

where $\tilde{\phi}_{\text{cl}}(x)$ is the classical field calculated with the parameter $\tilde{\mu}$ defined by Eq. (2). If we are interested in the effective potential only, it is enough to consider a constant source j_0 from which the corresponding constant classical fields are denoted ϕ_0 and $\tilde{\phi}_0$ for the parameters μ and $\tilde{\mu}$, respectively, such that $\tilde{\phi}_0 = \phi_0 - \eta$. The partition function, regularized by the cut off Λ , is then

$$Z_\mu[j_0] = \exp\left(i \int U(\phi_0) + i \int j_0\phi_0\right), \quad (8)$$

where $U(\phi_0)$, depending on Λ , is the effective potential, defined as the derivative independent part of the proper graphs generating functional. The latter is the Legendre transform of $W[j_0]$, where j_0 has to be understood as a function of the classical field ϕ_0 . The cut off dependence will be taken into account through the dimensionless parameter $t = \ln(2^{1/3}\Lambda/\mu)$, where, for convenience, the factor $2^{1/3}$ has been absorbed in the logarithm [see the origin of this factor in Eq. (18)]. It is then easy to see that the properties (6) and (7) lead to the exact identity

$$U(\tilde{\mu}, \tilde{t}, \tilde{\phi}_0) = U(\mu, t, \phi_0), \quad (9)$$

where

$$\tilde{t} = \ln\left(\frac{2^{1/3}\Lambda}{\tilde{\mu}}\right) = t - \frac{g\eta}{6}, \quad (10)$$

and the dependence on g is understood on both sides of Eq. (9).

As a consequence, the effective potential U must be a function of invariant combinations of μ , t , ϕ_0 as these parameters change to $\tilde{\mu}$, \tilde{t} , $\tilde{\phi}_0$, and the only possibility is

$$U = \frac{\mu^6}{g^2} e^{g\phi_0} F(z), \quad (11)$$

where F is a function of $z = g\phi_0 - 6t$ (F can also depend on g , independently of z). Also, the invariance expressed in Eq. (9) leads to

$$\begin{aligned} 0 &= \left(\frac{dU(\tilde{\mu}, \tilde{t}, \tilde{\phi}_0)}{d\eta}\right)_{\eta=0} \\ &= \left(\frac{\partial U}{\partial \tilde{\mu}^6} \frac{\partial \tilde{\mu}^6}{\partial \eta} + \frac{\partial U}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial \eta} + \frac{\partial U}{\partial \tilde{\phi}_0} \frac{\partial \tilde{\phi}_0}{\partial \eta}\right)_{\eta=0} \\ &= g\mu^6 \frac{\partial U}{\partial \mu^6} - \frac{g}{6} \dot{U} - U', \end{aligned} \quad (12)$$

where a prime denotes a derivative with respect to ϕ_0 and a dot denotes a derivative with respect to t . Together with Eq. (11) we obtain

$$U' = gU - \frac{g}{6} \dot{U}. \quad (13)$$

As will be seen in the next section, the partial differential equation (13) will lead us to the exact field dependence of the effective potential U , in the limit where $t \rightarrow \infty$.

C. Loop expansion

Quantum corrections to the potential are calculable in the loop expansion by standard methods [8] by summing all the vacuum diagrams of the theory, leading to an expansion of $F(z)$ in \hbar . We show here that the loop expansion is simultaneously an expansion in g^2 . The perturbative treatment of the model requires the expansion of the potential in power series of the coupling g and therefore generates an infinite series of vertices. Besides the tadpole $\mu^6\phi/g$ and the mass term $\mu^2\phi^2/2$, n -point vertices γ_n are generated by the expansion of the exponential

$$\gamma_n = \mu^6 g^{n-2}, \quad n \geq 3. \quad (14)$$

The number L of loops of a given vacuum graph is related to the number P of propagators and the number V of vertices by

$$L = P - V + 1. \quad (15)$$

If V_n denotes the number of vertices with $n \geq 3$ legs, we have

$$V = \sum_{n=3}^N V_n, \quad \text{and} \quad \sum_{n=3}^N nV_n = 2P, \quad (16)$$

where N is the highest number of legs joining at the same vertex, in the specific graph which is considered. Since a vertex with n legs is proportional to g^{n-2} , the vacuum graph is proportional to a power of g equal to

$$\sum_{n=3}^N (n-2)V_n = 2P - 2V = 2L - 2. \quad (17)$$

Hence, if we consider the factor g^{-2} in Eq. (11), one can see that a L -loop graph in the expansion of the potential U is proportional to $(g^2)^L$: the expansion in \hbar is equivalent to an expansion in g^2 .

III. EXACT RENORMALIZED POTENTIAL

A. Diagrammatic analysis and counterterms

By power counting, one can see that the *only* source of divergence in this model is a loop made of one propagator only, that we denote 1PL for ‘‘one propagator loop,’’ and which is equal to

$$\begin{aligned} C_{1\text{PL}} &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d^3 p}{(2\pi)^3} \frac{i}{\omega^2 - p^6 - \mu^6} \\ &= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{p^6 + \mu^6}} = \frac{1}{12\pi^2} \sinh^{-1}\left(\frac{\Lambda^3}{\mu^3}\right) \\ &\simeq \frac{1}{12\pi^2} \ln\left(2\frac{\Lambda^3}{\mu^3}\right) \equiv \frac{t}{4\pi^2}. \end{aligned} \quad (18)$$

The ‘‘1PL’’ loop may appear a multitude of times adjacent to a n -point vertex (see Fig. 1), in fact up to k times as long as $2k \leq n$. A n -point vertex will be responsible therefore for divergences coming from the appearance of k 1PL loops, for $k \leq \lfloor \frac{n}{2} \rfloor$.

We now derive the structure of the counterterms of the theory, using a proof by induction.

(i) One-loop.

Counting the number of ways to stick one 1PL to a n -point vertex, the one-loop divergence corresponding to a n -point function is

$$C_n^{(1)} = \frac{\mu^6 g^{n-2}}{n!} \frac{n(n-1)}{2} C_{1\text{PL}}, \quad (19)$$

and the summation over the vertices in the corresponding effective potential gives

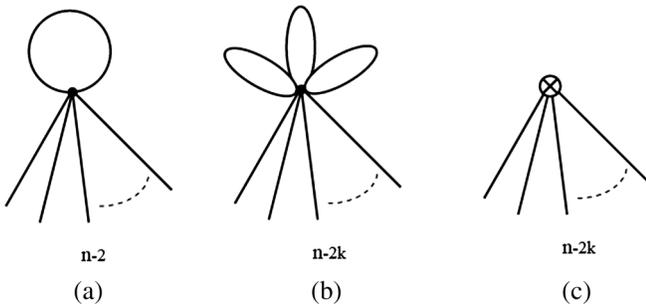


FIG. 1. Diagrams contributing to infinities in the Liouville theory from the n -point vertex: (a) one-loop infinity renormalized by an $(n-2)$ -point counterterm, (b) k -loop infinity renormalized by the $(n-2k)$ -point counterterm, (c)

$$\begin{aligned} \sum_{n=2}^{\infty} C_n^{(1)} \phi_0^{n-2} &= \sum_{n=2}^{\infty} \frac{\mu^6 g^{n-2}}{(n-2)!} \frac{C_{1\text{PL}}}{2} \phi_0^{n-2} \\ &= \frac{\mu^6 t}{8\pi^2} e^{g\phi_0}. \end{aligned} \quad (20)$$

In order to cure this divergence, in the minimal subtraction scheme, we add the one-loop counterterm to the bare mass parameter $\mu^6 \rightarrow \mu^6 + \delta\mu_{(1)}^6$, with

$$\delta\mu_{(1)}^6 = -\mu^6 \frac{g^2 t}{8\pi^2}. \quad (21)$$

(ii) k -loops.

A k -loop diagram contains at most k 1PLs, and those containing less than k 1PLs are cured by the $(k-1)$ -loop counterterms. As a consequence, the new divergence appearing at k loops is carried by graphs where all the 1PLs are stuck at *the same vertex*. Such a n -point vertex [see Fig. 1(b)] contributes with a divergence

$$\begin{aligned} C_n^{(k)} &= \frac{\mu^6 g^{n-2}}{n!} \frac{n(n-1)(n-2)\dots(n-2k+1)}{2^k k!} \\ &\quad \times (C_{1\text{PL}})^k. \end{aligned} \quad (22)$$

The summation of the dominant divergences over the vertices n contributes to the corresponding effective potential

$$\begin{aligned} \sum_{n=2k}^{\infty} C_n^{(k)} \phi_0^{n-2k} &= \sum_{n=2k}^{\infty} \frac{\mu^6 g^{n-2}}{(n-2k)!} \frac{1}{k!} \left(\frac{C_{1\text{PL}}}{2}\right)^k \phi_0^{n-2k} \\ &= \frac{\mu^6}{g^2 k!} \left(\frac{g^2 t}{8\pi^2}\right)^k e^{g\phi_0}. \end{aligned} \quad (23)$$

In order to cure this divergence, in a minimal subtraction scheme, we add the k -loop counterterm to the $(k-1)$ -loop mass parameter $\mu_{(k-1)}^6 \rightarrow \mu_{(k-1)}^6 + \delta\mu_{(k)}^6$, with

$$\delta\mu_{(k)}^6 = -\frac{\mu^6}{k!} \left(\frac{g^2 t}{8\pi^2}\right)^k. \quad (24)$$

(iii) Complete resummation

The complete cancellation of infinities requires therefore the introduction of the *unique* counterterm (up to finite terms)

$$U_{ct} = \sum_{k=1}^{\infty} \frac{\delta\mu_{(k)}^6}{g^2} e^{g\phi_0} = \frac{\mu^6}{g^2} \left[1 - \exp\left(\frac{\hbar g^2 t}{8\pi^2}\right) \right] e^{g\phi_0}, \quad (25)$$

where the factor \hbar is explicitly written to emphasize that the loop expansion is also an expansion in g^2 . We conclude that all the divergences are contained in

the mass parameter of the model, and that the coupling g receives finite quantum corrections, which we now calculate.

B. Renormalized coupling and mass parameter

We write the effective potential as the sum of the contributions $U^{(k)}$ from each loop order k :

$$U = U_{\text{bare}} + \sum_{k=1}^{\infty} U^{(k)}, \quad (26)$$

where the bare potential is $U_{\text{bare}} \equiv U^{(0)}$, and $U^{(k)}$ is a polynomial of t :

$$U^{(k)} = \sum_{l=0}^k a_l^{(k)}(\phi_0) t^l. \quad (27)$$

Although it is not possible to know exactly all the coefficients $a_l^{(k)}$, we know the dominant divergence $a_k^{(k)} t^k$ from the previous discussion, and one can write for $k \geq 1$,

$$U^{(k)} = \frac{\mu^6}{g^2 k!} \left(\frac{g^2 t}{8\pi^2} \right)^k e^{g\phi_0} [1 + \mathcal{O}(t^{-1})], \quad (28)$$

where the orders $\mathcal{O}(t^{-1})$ also depend on ϕ_0 . As a consequence, we have for $k \geq 1$,

$$\dot{U}^{(k)} = \frac{g}{8\pi^2} \frac{\partial}{\partial \phi_0} U^{(k-1)} [1 + \mathcal{O}(t^{-1})]. \quad (29)$$

Taking into account $\dot{U}^{(0)} = \dot{U}_{\text{bare}} = 0$, the summation over the loops gives

$$\dot{U} = \frac{g}{8\pi^2} U' [1 + \mathcal{O}(t^{-1})]. \quad (30)$$

Substituting this result in Eq. (13) leads to

$$\left(1 + \frac{g^2}{48\pi^2} \right) U' = gU [1 + \mathcal{O}(t^{-1})], \quad (31)$$

which integrates as

$$U = C(t) \exp(g_r \phi_0) [1 + \mathcal{O}(t^{-1})], \quad (32)$$

where $C(t)$ does not depend on ϕ_0 and

$$g_r \equiv g \left(1 + \frac{g^2}{48\pi^2} \right)^{-1}. \quad (33)$$

This relation between g_r and g is exact, and gives the field dependence of the renormalized potential. In order to completely determine the latter, one still needs to specify the renormalized mass parameter μ_r^6 . This is done by following the usual procedure, which consists in adding the counterterm (25) to the bare potential:

$$U_{\text{bare}} + U_{ct} = \frac{\mu^6}{g^2} \left(2 - \exp\left(\frac{\hbar g^2 t}{8\pi^2}\right) \right) e^{g\phi_0}, \quad (34)$$

and, loop after loop, removing divergences perturbatively in order to obtain the (finite) renormalized potential $U_r =$

$A \exp(g_r \phi_0)$ in the limit $t \rightarrow \infty$. The constant A can be determined in perturbation theory only, since it contains all the finite graphs of the theory. Identifying A with μ_r^6/g_r^2 defines the renormalized mass parameter μ_r^6 , such that the renormalized potential is finally

$$U_r = \frac{\mu_r^6}{g_r^2} \exp(g_r \phi_0). \quad (35)$$

Note that, in general, μ_r^6 depends on the subtraction scheme, and is fixed here by the minimal subtraction scheme in which the counterterm is given by Eq. (25). In the next section, we will explicitly calculate the one-loop renormalized mass parameter in this scheme.

IV. ONE-LOOP THEORY

In this section, we first illustrate the result (35) with the explicit calculation of the one-loop effective potential, which determines the one-loop renormalized mass parameter μ_r^6 . We then calculate the one-loop kinetic term quadratic in derivatives, showing the restoration of Lorentz symmetry in the IR of the quantum theory.

A. One-loop effective potential

The one-loop effective potential is

$$U^{(1)} = \frac{\mu^6}{g^2} e^{g\phi_0} + \frac{1}{2} \int \frac{d\omega}{2\pi} \frac{d^3\mathbf{p}}{(2\pi)^3} \times \ln \left(\frac{\omega^2 + (\mathbf{p}^2)^3 + \mu^6 e^{g\phi_0}}{\omega^2 + (\mathbf{p}^2)^3 + \mu^6} \right), \quad (36)$$

and it is easy to calculate its field derivative

$$\begin{aligned} \frac{\partial U^{(1)}}{\partial \phi_0} &= \frac{\mu^6}{g} e^{g\phi_0} + \frac{\mu^6 g e^{g\phi_0}}{8\pi^3} \\ &\times \int p^2 dp \int \frac{d\omega}{\omega^2 + p^6 + \mu^6 e^{g\phi_0}} \\ &= \frac{\mu^6}{g} e^{g\phi_0} + \frac{\mu^6 g e^{g\phi_0}}{24\pi^2} \sinh^{-1} \left(\frac{\Lambda^3}{\mu^3 e^{g\phi_0/2}} \right) \\ &= \frac{\mu^6}{g} e^{g\phi_0} + \frac{\mu^6 g e^{g\phi_0}}{8\pi^2} \left(t - \frac{g\phi_0}{6} \right) + \mathcal{O}(\mu/\Lambda)^2, \end{aligned} \quad (37)$$

such that, ignoring terms vanishing in the limit $\Lambda \rightarrow \infty$,

$$U^{(1)} = \frac{\mu^6}{g^2} e^{g\phi_0} \left\{ 1 + \frac{g^2}{8\pi^2} \left(t - \frac{g\phi_0}{6} + \frac{1}{6} \right) \right\}. \quad (38)$$

The field dependence in the expression (38) can be written

$$e^{g\phi_0} \left(1 - \frac{g^3}{48\pi^2} \phi_0 \right) = \exp \left\{ \left(g - \frac{g^3}{48\pi^2} \right) \phi_0 \right\} + \mathcal{O}(g^4), \quad (39)$$

which corresponds to the one-loop approximation of the renormalized coupling in Eq. (33), since

$$g_r = g \left(1 + \frac{g^2}{48\pi^2} \right)^{-1} = g - \frac{g^3}{48\pi^2} + \mathcal{O}(g^5). \quad (40)$$

As far as the one-loop renormalized mass is concerned, from Eq. (38) one can see that the addition of the one-loop counterterm $U_{ct}^{(1)} = g^{-2} \delta\mu_{(1)}^6 \exp(g\phi_0)$, where $\delta\mu_{(1)}^6$ is given in Eq. (21), leads to the definition

$$\frac{\mu_r^6}{g_r^2} \equiv \frac{\mu^6}{g^2} \left(1 + \frac{g^2}{48\pi^2} \right) + \mathcal{O}(g^2), \quad (41)$$

such that

$$\mu_r^6 = \mu^6 \left(1 - \frac{g^2}{48\pi^2} \right) + \mathcal{O}(g^4), \quad (42)$$

where the relation (40) was used.

B. Lorentz symmetry restoration

Finally, we calculate the one-loop kinetic term $\partial^k \phi \partial_k \phi$ generated by quantum fluctuations, such that the IR effective theory exhibits Lorentz symmetry:

$$S_{\text{eff}} = \int dt d^3x \left(\frac{1 + \zeta_0}{2} (\dot{\phi})^2 - \frac{\zeta_1}{2} \mu^4 \partial^k \phi \partial_k \phi - \frac{\mu_r^6}{g_r^2} e^{g_r \phi} + \mathcal{O}(\partial^4) \right), \quad (43)$$

where $\zeta_0 = \mathcal{O}(\hbar)$ and $\zeta_1 = \mathcal{O}(\hbar)$ are generated dynamically. The one-loop Feynman graph responsible for the generation of this kinetic term arises from the insertion of two three-point vertices in the propagator (the interaction $\mu^2 g \phi^3$ in the Liouville potential). Indeed, the tadpole $\mu^6 \phi/g$ and the interaction $\mu^6 g^2 \phi^4$ give corrections independent of the external momentum \mathbf{k} , and renormalize the mass only. Also, the higher orders $\mu^6 g^{n-2} \phi^n$ do not contribute at one-loop. This graph is

$$\frac{(-ig\mu^6)^2}{2} \int \frac{d\omega}{2\pi} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{i}{\omega^2 - (\mathbf{p}^2)^3 - \mu^6 + i\varepsilon} \times \frac{i}{(\omega + \nu)^2 - ((\mathbf{p} + \mathbf{k})^2)^3 - \mu^6 + i\varepsilon}. \quad (44)$$

The contribution proportional to ν^2 is (after a Wick rotation)

$$\begin{aligned} i\zeta_0 \nu^2 &= \frac{ig^2 \mu^{12}}{2(2\pi)^4} \int d\omega d^3\mathbf{p} \left(\frac{-\nu^2}{(\omega^2 + (\mathbf{p}^2)^3 + \mu^6)^3} \right. \\ &\quad \left. + \frac{4\omega^2 \nu^2}{(\omega^2 + (\mathbf{p}^2)^3 + \mu^6)^4} \right) \\ &= -\frac{ig^2 \mu^{12}}{64\pi^2} \nu^2 \int \frac{p^2 dp}{(p^6 + \mu^6)^{5/2}} = -i \frac{g^2}{288\pi^2} \nu^2, \end{aligned} \quad (45)$$

and the contribution proportional to k^2 is

$$\begin{aligned} -i\zeta_1 \mu^4 k^2 &= \frac{ig^2 \mu^{12}}{2(2\pi)^4} \int d\omega d^3\mathbf{p} \left(\frac{-7(\mathbf{p}^2)^2 \mathbf{k}^2}{(\omega^2 + (\mathbf{p}^2)^3 + \mu^6)^3} \right. \\ &\quad \left. + \frac{36(\mathbf{p}^2)^4 (\mathbf{p} \cdot \mathbf{k})^2}{(\omega^2 + (\mathbf{p}^2)^3 + \mu^6)^4} \right) \\ &= -\frac{21ig^2 \mu^{12}}{64\pi^2} k^2 \int \frac{p^6 dp}{(p^6 + \mu^6)^{5/2}} \\ &\quad + \frac{15ig^2 \mu^{12}}{32\pi^2} k^2 \int \frac{p^{12} dp}{(p^6 + \mu^6)^{7/2}} \\ &= -i \frac{3g^2 I}{64\pi^2} \mu^4 k^2, \end{aligned} \quad (46)$$

where

$$I = \int_0^\infty du \frac{7u^6 - 3u^{12}}{(1 + u^6)^{7/2}} = \frac{4}{9\sqrt{\pi}} \Gamma(4/3) \Gamma(13/6) \simeq 0.242. \quad (47)$$

We therefore see that the Lorentz symmetric effective action (43) is indeed generated dynamically by quantum fluctuations, with $\zeta_0 = -g^2/(288\pi^2)$ and $\zeta_1 = 3g^2 I/(64\pi^2)$. The dispersion relation for the low momentum modes reads

$$\tilde{\omega}^2 = m^2 + \tilde{k}^2 + \mathcal{O}(\tilde{k}^4/\mu^2), \quad (48)$$

where frequency and momentum are appropriately rescaled as

$$\tilde{\omega} = \frac{\omega}{\mu^2} \sqrt{1 + \zeta_0}, \quad \tilde{k} = k\sqrt{\zeta_1}, \quad \text{and} \quad m^2 = \frac{\mu_r^6}{\mu^4}. \quad (49)$$

Hence, Lorentz violating effects are suppressed in the IR by powers of $(\tilde{k}/\mu)^2$.

V. CONCLUSIONS

In this work, we studied a renormalizable Lifshitz-type model with the Liouville potential, in 3 + 1 dimensions, and we demonstrated that the exponential form of the potential does not change after quantization. An important result is the exact relation (33) between the bare and renormalized couplings, which can also be written in the form

$$\frac{1}{g_r} = \frac{1}{g} + \frac{g}{48\pi^2}. \quad (50)$$

This result is similar to the one obtained in the 1 + 1 dimensional Liouville theory [6,7], where the factor $48\pi^2$ is replaced by 8π . Furthermore, the steps in our derivation of the renormalized potential can be repeated exactly for the 1 + 1 dimensional Liouville theory. The possibility to obtain the above features arises from the use of exact functional properties and the complete resummation of graphs, which is specific to the exponential form of the Liouville potential.

In addition to the knowledge of the exact renormalized coupling, the IR regime of the quantum theory exhibits Lorentz symmetry, with a relativistic dispersion relation induced dynamically. An open question concerns the possibility to have exact results for the different kinetic terms of the renormalized theory, beyond one-loop. Such a study has been done in [9] for the 1 + 1 dimensional Liouville theory, where it is shown, in the framework of a gradient expansion valid to all loop orders, that the wave function renormalization vanishes. A similar study in the Lifshitz context might also lead us to interesting properties, and this is left for a future work.

An interesting possibility is to study the Liouville-Lifshitz model in curved space-time, in the context of Horava-Lifshitz gravity [10]. This can address a variety

of cosmological questions [11], in particular, in the framework of quintessence models, where the exponential potential was first studied in [12].

Finally, the lack of a translationally invariant ground state for the quantum Liouville field theory [6] motivates the extension of our study to potentials which involve more than one exponential, where a ground state can exist.

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