

Derivation of the Planck spectrum for relativistic classical scalar radiation from thermal equilibrium in an accelerating frame

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The Planck spectrum of thermal scalar radiation is derived suggestively within classical physics by the use of an accelerating coordinate frame. The derivation has an analogue in Boltzmann's derivation of the Maxwell velocity distribution for thermal particle velocities by considering the thermal equilibrium of noninteracting particles in a uniform gravitational field. For the case of radiation, the gravitational field is provided by the acceleration of a Rindler frame through Minkowski spacetime. Classical zero-point radiation and relativistic physics enter in an essential way in the derivation, which is based upon the behavior of free radiation fields and the assumption that the field correlation functions contain but a single correlation time in thermal equilibrium. The work has connections with the thermal effects of acceleration found in relativistic quantum field theory.

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I. INTRODUCTION

Many textbooks present Boltzmann's derivation [1] of the Maxwell velocity distribution for free particles in thermal equilibrium in a box. In his analysis, Boltzmann introduced a uniform gravitational field, followed the implications of thermal equilibrium under gravity, and finally took the zero-gravity limit. The derivation is striking because it uses only the physics of free nonrelativistic particles moving in a gravitational field. By the principle of equivalence, the gravitational field can be replaced by an accelerating coordinate frame. But then thermodynamic consistency requires that the interactions of particles that lead to equilibrium in an inertial frame must be consistent with the equilibrium determined by the physics of free particles in an accelerating frame. The natural question arises as to whether the analogue of this procedure can be applied to the much more complicated problem of thermal equilibrium for relativistic radiation with its infinite number of normal modes. In this paper we show that an analogous derivation is indeed possible for relativistic classical scalar radiation. We introduce a relativistic accelerating coordinate frame (a Rindler frame, which is the closest relativistic equivalent to a uniform gravitational field), consider the implications for thermal radiation equilibrium, make the assumption that thermal equilibrium involves but a single correlation time, and finally take the limit of zero acceleration to obtain the thermal radiation spectrum in an inertial frame. The use of an accelerating coordinate frame to obtain the thermal equilibrium spectrum seems striking because only noninteracting free radiation fields are needed for the derivation. However, we expect that any other interaction that produces equilibrium must be consistent with the equilibrium determined by the accelerating frame. In particular, the use of nonrelativistic nonlinear scattering systems [2], which lead to the Rayleigh-Jeans spectrum for radiation equilibrium, are

inconsistent with special relativity and relativistic accelerating coordinate frames.

The derivation here for the Planck spectrum is provided within the context of relativistic classical scalar field theory which includes classical zero-point radiation. This classical *scalar* field theory is analogous to the classical *electromagnetic* theory with classical electromagnetic zero-point radiation. The classical electromagnetic theory has been shown to provide classical explanations for a number of phenomena which are usually regarded as belonging to the exclusive domain of quantum theory, including Casimir forces, van der Waals forces, diamagnetism, specific heats of solids [3,4], and the ground state of hydrogen [5]. The description of thermal radiation in terms of classical radiation with random phases, which is used in the classical theories, is a standard procedure dating from 19th century physics. The choice of zero-point radiation as the "vacuum" situation in classical physics is required in order to describe the experimentally observed Casimir forces, but this choice then gives natural classical explanations for other phenomena. The Lorentz invariance of classical zero-point radiation determines the spectrum up to one unknown multiplicative constant giving the scale of the zero-point radiation. The scale of zero-point radiation is chosen to give numerical agreement with experimental measurements of Casimir forces. It turns out that the unknown multiplicative constant takes a numerical value, which is immediately recognizable as Planck's constant \hbar . Thus, Planck's constant \hbar enters the classical theory as the scale factor of classical zero-point radiation and not as the quantum of action familiar in current quantum theory.

There have been many indignant objections to work involving "classical" zero-point radiation; the claim is made that zero-point radiation is exclusively a "quantum" concept. The classical electromagnetic theory treated earlier and the classical scalar theory discussed here are both

classical in the sense that they contain no intrinsically discrete aspects of energy or action. The zero-point radiation is classical random radiation chosen as a perfectly valid homogeneous boundary condition on the classical field equations. The concept of zero-point radiation (random radiation fluctuations at the zero of temperature) can appear in both classical and quantum theories. Zero-point radiation cannot be regarded as belonging exclusively to quantum theory any more than the concepts of mass, energy, and gravity can be claimed as exclusively classical concepts because they appeared first in the context of classical mechanics.

The outline of our presentation is as follows. In Sec. II, we review the basics of classical relativistic scalar field theory. We introduce the random phases between normal modes for stationary distributions of random classical radiation, and then we calculate the two-point field correlation function associated with a general stationary spectrum of random radiation. In Sec. III, we discuss thermal radiation in an inertial frame. We start with the two fundamental ideas required for understanding thermal radiation equilibrium within classical physics. These include the presence of a divergent spectrum of classical zero-point radiation and the presence of a finite density of thermal radiation above the zero-point spectrum. We note that thermodynamic ideas give us Wien's displacement law and the Stefan-Boltzmann relation. Although Wien's law gives a restriction on the form of the radiation spectrum and also on the form of the two-point field correlation function, thermodynamics in an inertial frame does not determine the spectrum of thermal radiation. In Sec. IV, we introduce a relativistic coordinate frame (a Rindler frame) accelerating through Minkowski spacetime. We note the role played by acceleration in breaking the homogeneity and isotropy of Minkowski spacetime. Then we review some preliminary information regarding the Rindler accelerated coordinate frame, and use the thermodynamics of pressure equilibrium to show that temperature and acceleration have the same spatial dependence throughout a Rindler frame. Next we recalculate the two-point correlation function for classical zero-point radiation in terms of Rindler coordinates. But then one sees a natural behavior for the thermal correlation function which follows from the known correlation function involving zero-point radiation. Finally, we take the limit of vanishing acceleration and so recover the Planck spectrum as the classical radiation spectrum of thermal equilibrium. The article closes with allusions to related but vastly different work in relativistic quantum field theory.

II. SCALAR FIELD THEORY FOR RANDOM FIELDS

A. The relativistic scalar field

In an inertial frame, the relativistic free scalar field $\phi(ct, x, y, z)$ is specified by the Lagrangian density [6]

$$\mathcal{L} = \frac{1}{8\pi} \left[\frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial x} \right)^2 - \left(\frac{\partial \phi}{\partial y} \right)^2 - \left(\frac{\partial \phi}{\partial z} \right)^2 \right], \quad (1)$$

which leads to the wave equation as the equation of motion

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (2)$$

The energy U in the field follows from the Lagrangian density in (1) as [6]

$$U = \int d^3x \frac{1}{8\pi} \left[\frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right]. \quad (3)$$

The radiation in a box can be described by a complete set of either standing waves or running waves with appropriate wave vectors \mathbf{k} . In the present case, we are not interested in any special conditions holding at the walls of the rectangular box of dimensions $L_x \times L_y \times L_z$, and so we will choose periodic running waves where

$$\mathbf{k} = \hat{\mathbf{x}}(n_x 2\pi/L_x) + \hat{\mathbf{y}}(n_y 2\pi/L_y) + \hat{\mathbf{z}}(n_z 2\pi/L_z) \quad (4)$$

and the integers n_x, n_y, n_z run over all positive and negative values. Then the radiation field in the box can be written as

$$\begin{aligned} \phi(ct, x, y, z) = & \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} \frac{f(c\mathbf{k})}{(L_x L_y L_z)^{1/2}} \\ & \times \cos[\mathbf{k} \cdot \mathbf{r} - kct - \theta(\mathbf{k})], \end{aligned} \quad (5)$$

where $k = |\mathbf{k}|$, and $\theta(\mathbf{k})$ is an appropriate phase. Each mode $\phi_{\mathbf{k}}(ct, x, y, z) = [f(c\mathbf{k})/(L_x L_y L_z)^{1/2}] \times \cos[\mathbf{k} \cdot \mathbf{r} - kct - \theta(\mathbf{k})]$ labeled by n_x, n_y, n_z has the energy $U_{\mathbf{k}}$ found by substituting into Eq. (3),

$$U_{\mathbf{k}} = \frac{1}{8\pi} k^2 f^2(c\mathbf{k}). \quad (6)$$

B. Two-point correlation function for random radiation

Coherent radiation involves fixed phase relations $\theta(\mathbf{k}) - \theta(\mathbf{k}')$ between the various modes $\phi_{\mathbf{k}}$ and $\phi_{\mathbf{k}'}$, which are used to decompose a radiation pattern. Random radiation involves the opposite situation. Random radiation can be written in the form of Eq. (5) where the phases $\theta(\mathbf{k})$ are randomly distributed on the interval $[0, 2\pi)$ and are independently distributed for each \mathbf{k} . It is convenient to characterize random radiation by taking the two-point correlation function of the fields $\langle \phi(ct, x, y, z) \phi(ct', x', y', z') \rangle$ obtained by averaging over the random phases as

$$\langle \cos\theta(\mathbf{k}) \cos\theta(\mathbf{k}') \rangle = \langle \sin\theta(\mathbf{k}) \sin\theta(\mathbf{k}') \rangle = (1/2) \delta_{\mathbf{k}, \mathbf{k}'}, \quad (7)$$

$$\langle \cos\theta(\mathbf{k}) \sin\theta(\mathbf{k}') \rangle = 0. \quad (8)$$

Then the two-point correlation function for a general isotropic distribution of classical scalar waves is [7]

$$\begin{aligned}
\langle \phi(ct, x, y, z)\phi(ct', x', y'z') \rangle &= \left\langle \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} \frac{f(c\mathbf{k})}{(L_x L_y L_z)^{1/2}} \cos[\mathbf{k} \cdot \mathbf{r} - kct - \theta(\mathbf{k})] \right. \\
&\quad \times \left. \sum_{n'_x=-\infty}^{\infty} \sum_{n'_y=-\infty}^{\infty} \sum_{n'_z=-\infty}^{\infty} \frac{f(c\mathbf{k}')}{(L_x L_y L_z)^{1/2}} \cos[\mathbf{k}' \cdot \mathbf{r}' - k'ct' - \theta(\mathbf{k}')] \right\rangle \\
&= \frac{1}{2} \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} \frac{f^2(c\mathbf{k})}{L_x L_y L_z} \cos[\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}') - kc(t - t')]. \tag{9}
\end{aligned}$$

For a very large box, the normal modes are closely spaced and the sums over the integers n_x, n_y, n_z can be replaced by integrals of the form $dk_x = (2\pi/L_x)dn_x, dk_y = (2\pi/L_y)dn_y, dk_z = (2\pi/L_z)dn_z$ so that the correlation function of Eq. (9) becomes

$$\langle \phi(ct, x, y, z)\phi(ct', x', y'z') \rangle = \frac{1}{16\pi^3} \int d^3k f^2(c\mathbf{k}) \cos[\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}') - \omega(t - t')]. \tag{10}$$

In the work to follow, we will be interested in isotropic distributions of random radiation so that the function $f(c\mathbf{k})$ involves only the magnitude $ck = \omega$. In this case, we can carry out the angular integrations for \mathbf{k} in Eq. (10) [7]

$$\begin{aligned}
\langle \phi(ct, x, y, z)\phi(ct', x', y'z') \rangle &= \frac{1}{16\pi^3} \int d^3k f^2(ck) \cos[\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}') - \omega(t - t')] \\
&= \frac{1}{8\pi^2 c^2 |\mathbf{r} - \mathbf{r}'|} \int_0^\infty d\omega \omega f^2(\omega) \{ \sin[(\omega/c)(|\mathbf{r} - \mathbf{r}'| - c(t - t'))] \\
&\quad + \sin[(\omega/c)(|\mathbf{r} - \mathbf{r}'| + c(t - t'))] \}. \tag{11}
\end{aligned}$$

This integral expression (11) is as far as we can carry the evaluation of the two-field correlation function for random radiation without knowing something (beyond isotropy) about the spectral function $f^2(\omega)$.

III. THERMAL RADIATION IN AN INERTIAL FRAME

A. Two fundamental ideas of classical thermal radiation

In order to understand thermal radiation within classical physics, two fundamental ideas are needed. The first needed idea is the presence of classical zero-point radiation as the universal homogeneous boundary condition on radiation equations. This random zero-point radiation corresponds to the “vacuum state” of classical physics [3]. Zero-point radiation exists only for massless waves within the context of relativistic theory. This wave situation is in contrast with nonrelativistic physics where all momentum is carried by mass. Within relativistic classical physics, zero-point radiation is random classical radiation with a Lorentz-invariant spectrum. Thus, the same zero-point spectrum appears in every inertial frame. The requirement of Lorentz invariance fixes the spectrum of random zero-point radiation up to one overall multiplicative constant [7–9]. The second needed idea is that thermal radiation with $T > 0$ represents a finite density of random radiation above the divergent spectrum of zero-point radiation. It is the divergent spectrum of zero-point radiation that prevents the finite energy density of thermal radiation from leaking out to the infinite number of high-frequency modes. For each

normal mode, the thermal energy is added on top of the zero-point energy, which is always present. Now the idea of a finite energy density on top of a divergent energy density may give one pause. However, any particle system with mass interacts with only low-frequency modes; the very-high-frequency modes act too rapidly to influence a massive system. Thus, for example, a box with real conducting walls becomes transparent to very high-frequency electromagnetic waves; radio waves are reflected by a copper sheet, while gamma rays easily pass through. Thermal radiation can be confined in a box whose walls reflect the low-frequency waves, while the high-frequency modes carrying only zero-point energy penetrate through the walls. Indeed, since the zero-point spectrum is invariant under an adiabatic compression, the thermal radiation can be compressed while the zero-point spectrum remains unchanged. Thermodynamic equilibrium involves the distribution of energy among a very large number of weakly interacting systems. In the case of radiation, each normal mode of oscillation in the confining box can be regarded as a separate system, and the problem of classical thermal radiation is to determine the equilibrium distribution of energy among the modes of the box.

B. Thermodynamics of a normal mode

Each normal mode of oscillation for the radiation in the box acts as a harmonic oscillator system [10]. In a discussion of the thermodynamics of harmonic oscillator systems in an inertial frame [11], it was shown that ther-

modynamics alone requires that the energy $U(\omega, T)$ per normal mode at frequency ω must take the form

$$U(\omega, T) = \omega F(T/\omega), \quad (12)$$

where T is the temperature of the system, and F is some unknown function. This result corresponds to Wien's displacement theorem. In the limit as the temperature goes to zero $T \rightarrow 0$, the energy of a normal mode becomes independent of T and reduces to the zero-point value

$$U(\omega, 0) = (1/2)\hbar\omega \quad (13)$$

(corresponding to $F(T/\omega) \rightarrow \hbar/2$), where the functional dependence $U(\omega, 0) = \text{const} \times \omega$ agrees with Lorentz invariance, and the scale $\hbar/2$ is determined so as to give agreement with Casimir forces. The thermal energy U_T of the mode is the energy above the zero-point energy and is just the difference

$$U_T(\omega, T) = U(\omega, T) - U(\omega, 0). \quad (14)$$

In the limit as the temperature T becomes large, the mode energy becomes independent of the frequency ω

$$U(\omega, T) \rightarrow k_B T \quad (15)$$

giving the Rayleigh-Jeans spectrum of radiation (corresponding to $F(T/\omega) \rightarrow T/\omega$). The general thermal radiation spectrum $U(\omega, T) = \omega F(T/\omega)$ must interpolate between these two limits where the ratio T/ω goes to zero or to infinity. In the earlier discussion [11] of the thermodynamics of a harmonic oscillator, it was noted that the smoothest interpolation mathematically between the entropy of these limits was that corresponding to the Planck radiation spectrum including zero-point energy. Here, rather than using mathematical considerations, we wish to use physical ideas involving accelerating frames to obtain the spectral function connecting the high- and low-temperature limits.

C. Thermal radiation in a box

Thermal radiation involves a finite energy density above the zero-point radiation spectrum. Thus, in a container of volume V with radiation modes, the total thermal energy $\mathcal{U}(T)$ is a sum over all normal modes of the thermal energy $U_T(\omega, T)$ in each mode, thermal energy being energy above the zero-point energy as in (14)

$$\mathcal{U}(T) = \sum_{\omega} U_T(\omega, T) = \sum_{\omega} [U(\omega, T) - U(\omega, 0)]. \quad (16)$$

The thermal radiation will be isotropic in the inertial frame where the box is at rest. For finite temperature $T > 0$, radiation can be "thermal" in only one coordinate frame in which its spectrum is isotropic; any observer moving with finite velocity relative to this frame detects a spectrum that is not isotropic. On the other hand, the (divergent) Lorentz-invariant zero-point radiation is isotropic in every inertial frame.

The number of normal modes per unit volume per unit frequency interval is [12]

$$d\mathcal{N} = \omega^2 d\omega / (2\pi^2 c^3) \quad (17)$$

which, except for a factor of 2, is the same as the familiar electromagnetic case. The thermal energy density $u = \mathcal{U}(T)/V$ is then

$$\begin{aligned} u(T) &= \int d\mathcal{N} U_T(\omega, T) \\ &= \int_0^{\infty} d\omega \frac{\omega^2}{2\pi^2 c^3} \omega [F(T/\omega) - F(0)] \\ &= T^4 \int_0^{\infty} dz \frac{z^2}{2\pi^2 c^3} z [F(1/z) - F(0)] = a_{SS} T^4, \end{aligned} \quad (18)$$

where $U_T(\omega, T)$ is the thermal energy (above the zero-point energy) in a mode of frequency ω , the function F is the unknown function appearing in the thermal radiation spectrum of Eq. (12), and a_{SS} is a constant playing the same role as Stefan's constant [13], but now for the scalar radiation field. Also, we expect the pressure p to be a function of temperature alone and to satisfy

$$p(T) = (1/3)u(T), \quad (19)$$

where the factor of 1/3 arises from the isotropic angular dependence of the radiation.

D. Field correlation for thermal radiation

In the case of thermal radiation, we know from Eqs. (6) and (12) that the spectral function $f_T(\omega)$ for the radiation field takes the form $f_T^2(\omega) = 8\pi c^2 U_{\omega} / \omega^2 = 8\pi c^2 F(\omega/T) / \omega$ so that the two-point correlation function in Eq. (11) becomes

$$\begin{aligned} &\langle \phi_T(ct, x, y, z) \phi_T(ct', x', y', z') \rangle \\ &= \frac{1}{\pi |\mathbf{r} - \mathbf{r}'|} \int_0^{\infty} d\omega F(\omega/T) \\ &\quad \times \{ \sin[(\omega/c)(|\mathbf{r} - \mathbf{r}'| - c(t - t'))] \\ &\quad + \sin[(\omega/c)(|\mathbf{r} - \mathbf{r}'| + c(t - t'))] \} \\ &= T^2 \frac{1}{\pi(T|\mathbf{r} - \mathbf{r}'|)} \int_0^{\infty} dv F(v) \\ &\quad \times \{ \sin[(v/c)(T|\mathbf{r} - \mathbf{r}'| - Tc(t - t'))] \\ &\quad + \sin[(v/c)(T|\mathbf{r} - \mathbf{r}'| + Tc(t - t'))] \}. \end{aligned} \quad (20)$$

We can also specialize the situation to the case where the spatial separation becomes small $|\mathbf{r} - \mathbf{r}'| \rightarrow 0$. In this limit, the field correlation function at a single spatial coordinate point $(x, y, z) = (x', y', z')$ but at two different times t and t' becomes

$$\begin{aligned}
& \langle \phi_T(ct, x, y, z) \phi_T(ct', x, y, z) \rangle \\
&= T^2 \frac{2}{\pi c} \int_0^\infty dv v F(v) \cos[vT(t-t')] \\
&= T^2 \tilde{\mathfrak{F}}[T(t-t')], \tag{21}
\end{aligned}$$

where $\tilde{\mathfrak{F}}[T(t-t')]$ is some unknown function of temperature multiplied by time.

E. Zero-point radiation correlation function in an inertial frame

In our discussion so far, we do not know the spectral function $f_T^2(\omega) = 8\pi c^2 U(\omega, T)/\omega^2 = 8\pi c^2 F(\omega/T)/\omega$ for thermal radiation at nonzero temperature. However, we have stated the spectral form for zero-point radiation in Eq. (13) based upon the Lorentz invariance of the spectrum. Substituting the expression of Eq. (13) into Eq. (11), we find that the two-point field correlation function of the zero-point radiation field $\phi_0(ct, x, y, z)$ can be calculated in closed form as [14]

$$\begin{aligned}
& \langle \phi_0(ct, x, y, z) \phi_0(ct', x', y', z') \rangle \\
&= \frac{-\hbar c}{\pi [c^2(t-t')^2 - (x-x')^2 - (y-y')^2 - (z-z')^2]}. \tag{22}
\end{aligned}$$

The subscript 0 on the fields on the left-hand side indicates that zero-point radiation is involved. The denominator on the right-hand side involves exactly the Lorentz-invariant square of the spacetime interval between the two coordinate points and shows clearly the Lorentz-invariant character of the random zero-point radiation. The denominator corresponds to the square of the proper time interval between the two points as measured in any inertial frame.

If the spatial coordinates x, y, z are the same for the two points, then the correlation function in (22) becomes

$$\langle \phi_0(ct, x, y, z) \phi_0(ct', x, y, z) \rangle = \frac{-\hbar c}{\pi c^2(t-t')^2}. \tag{23}$$

We notice that this form is consistent with the thermal expression in Eq. (21) provided $\tilde{\mathfrak{F}}(z)$ goes as the inverse of its argument squared at small arguments $\tilde{\mathfrak{F}}(z) \sim -\hbar/(\pi c z^2)$

$$T^2 \tilde{\mathfrak{F}}[T(t-t')] \approx T^2 \frac{-\hbar}{\pi c [T(t-t')]^2} = -\frac{\hbar c}{\pi c^2(t-t')^2}. \tag{24}$$

Small arguments for the function $\tilde{\mathfrak{F}}[T(t-t')]$ in Eq. (21) can refer to either small temperatures at finite time differences or small time differences at finite temperatures. Agreement of the correlation functions in Eqs. (23) and (24) shows that we expect that at large frequencies (corresponding to short correlation times) the spectral function goes over to the zero-point spectrum for any temperature. Indeed, this is what we expect when we think of the

thermal radiation as being distributed among only the lower frequency modes.

The correlation function (23) for the zero-point fields involves simply the time difference between the two space-time points in the inertial frame without any characteristic correlation time appearing in the expression. In contrast, we expect that thermal radiation will indeed involve a correlation time associated with the finite density of thermal radiation. Unfortunately, the correlation for zero-point radiation given in Eq. (23) gives us no hint about the low-frequency (long-time-correlation) behavior of the thermal radiation spectrum.

IV. USE OF ACCELERATION TO DERIVE THE THERMAL DISTRIBUTION

A. Review of Boltzmann's derivation for the Maxwell velocity distribution

The use of mechanical and thermodynamic ideas in an inertial frame allows one to obtain significant information about the thermal equilibrium distributions of particles or of waves. Thus, momentum transfer to the walls of a container relates the pressure p of a gas of free particles or of radiation to the energy density u at the walls; $p = (2/3)u$ for free particles and $p = (1/3)u$ for radiation. The equations of state, $pV = Nk_B T$ for free particles and the assumption that the energy density u is a function of temperature T alone for radiation, when combined with thermodynamic ideas, allow determinations of the entropy of free particles and the energy density and entropy for radiation. Indeed Wein's displacement theorem [given here in Eq. (12) from the thermodynamics of each normal mode] is consistent with the Stefan-Boltzmann law $u = a_{SS} T^4$ appearing here in Eq. (18).

Although these mechanical and thermodynamic arguments give us considerable information about the thermodynamics of free particles and of radiation, these arguments do not tell us the thermal distribution of particle velocities or the spectrum of thermal radiation in an inertial frame. We expect the thermal distributions to be homogeneous in space and isotropic in direction; however, the distribution in energy does not follow from symmetries under space translation and rotation. The presence of gravity or acceleration breaks the symmetry of the space and so allows one to distinguish thermal systems. For free non-relativistic particles, this situation is familiar to most physicists. Boltzmann [1] assumed that thermal equilibrium exists for noninteracting nonrelativistic particles in a uniform gravitational field, or equivalently, in a uniformly accelerating box. Thermodynamic arguments about cyclic lifting of a harmonic oscillator between the bottom and top of the box indicate that the temperature must be uniform throughout the box. Indeed, such arguments show that in equilibrium, the temperature is constant throughout any nonrelativistic system. The temperature alone determines the velocity distribution at any height. But then the velocity

distribution required to maintain the equilibrium spatial pressure gradient against gravity or against acceleration is unique. If one now allows the gravitational field or acceleration to go to zero, then one recovers the Maxwell distribution for the equilibrium velocity distribution of particles in thermal equilibrium in an inertial frame [1].

In this article we wish to carry through an analogous argument for the derivation of the equilibrium spectrum of classical relativistic scalar radiation in an inertial frame. Unfortunately, the derivation is not as simple as that for nonrelativistic particles because the radiation derivation must use relativistic ideas, and these are not as familiar as those of nonrelativistic mechanics.

B. The Rindler frame

Following the analogy with Boltzmann's work, we would like to discuss radiation in a box undergoing uniform acceleration. Since we are dealing with relativistic classical radiation, we would like to consider a box undergoing uniform acceleration through Minkowski spacetime. In the frame of the box, the acceleration should be constant in time, and the dimensions of the box should not change so that the radiation pattern can be assumed steady state. However, relativity introduces some complications which are quite different from nonrelativistic kinematics. When viewed from an inertial frame where the box is momentarily at rest at some $t = 0$, the acceleration a of a point of the box will appear to change according to the Lorentz transformation for accelerations, $a = a'/\gamma^3 = a'(1 - v^2/c^2)^{3/2}$, with the acceleration a (seen in the inertial frame) becoming smaller as the velocity v of the box becomes larger even though the acceleration a' in the frame of the box is constant in time. Furthermore, in order for the box to maintain a constant length in its own rest frame, the box must be found to undergo a length contraction in the inertial frame. But this requires that different points of the box must undergo different accelerations as seen in any inertial frame, and indeed, in any inertial frame momentarily at rest with respect to the box. Thus, the proper acceleration of each point of the box must vary with height. This relativistic situation has been explored in the literature [15] and the coordinate frame associated with the box is termed a Rindler frame. If the coordinates of an inertial frame are given by ct, x, y, z , the coordinates of the associated Rindler frame which is at rest with respect to the inertial frame at $t = 0$ are specified as [16]

$$ct = \xi \sinh(\eta), \quad (25)$$

$$x = \xi \cosh(\eta), \quad (26)$$

with y and z remaining unchanged between the frames and $\xi > 0$. Using these transformations, we see that the space-time interval changes from the Minkowski form in (ct, x, y, z) over to a new form in the Rindler coordinates (η, ξ, y, z)

$$\begin{aligned} ds^2 &= c^2 dt^2 - dx^2 - dy^2 - dz^2 \\ &= \xi^2 d\eta^2 - d\xi^2 - dy^2 - dz^2. \end{aligned} \quad (27)$$

If we consider a point with fixed spatial coordinates ξ, y, z in a Rindler frame, then (by introducing Eqs. (25) and (26) into the relation $\cosh^2 \eta - \sinh^2 \eta = 1$) we find that in the inertial frame this point follows the trajectory

$$x = (\xi^2 + c^2 t^2)^{1/2} \quad (28)$$

and has a constant proper acceleration given by

$$a = c^2/\xi. \quad (29)$$

We notice from Eq. (29) that no single acceleration can be assigned to a Rindler frame. Rather the acceleration varies with the coordinate ξ , becoming very small for large ξ and diverging as ξ goes to zero. The surface at $\xi = 0$ [where the acceleration in Eq. (29) diverges] is termed the ‘‘event horizon’’ of the Rindler frame. At any instant of Rindler time η , the spatial coordinates of the Rindler frame are in agreement with those of a Minkowski frame, which is instantaneously at rest with respect to the Rindler frame.

C. Variation in temperature for thermal radiation in a Rindler frame

Although the temperature of thermal radiation is constant throughout nonrelativistic systems in equilibrium, this constancy is not true in relativistic gravitational physics, and, in particular, it is not true in a Rindler frame. There are clearly profound differences between the thermodynamics of nonrelativistic and relativistic physics. These profound differences can be seen immediately in the contrasting determinations of the forces \mathbf{F}_1 and \mathbf{F}_2 needed to accelerate from rest, respectively, 1) a box of interacting particles and 2) a box of radiation. In nonrelativistic physics, the force \mathbf{F}_1 accelerating the box of particles is $\mathbf{F}_1 = M\mathbf{a}$, where M is the total mechanical mass of the particles (independent of the potential energies in the box), and the force \mathbf{F}_2 accelerating the box of radiation is zero $\mathbf{F}_2 = 0$ since no mechanical mass is present. In relativistic physics, both forces are given by $\mathbf{F} = (U/c^2)\mathbf{a}$, where U is the total energy in the box. In a Rindler frame, all unsupported systems will tend to fall relative to the Rindler coordinates because the coordinates of the frame are accelerating. Therefore, for thermal equilibrium in this relativistic system, the pressure (and hence the temperature) must increase at lower depths in order to support the energy above it. (We can imagine introducing massless horizontal reflecting surfaces into the box of radiation and determining the pressure needed to support the thermal radiation above the surface.) In relativity, the change of pressure p with height due to acceleration depends upon the sum $p + u$ of pressure plus thermal energy density [17]. Since the thermal energy density u depends on the temperature as T^4 , as shown in Eq. (18), we expect

$$\frac{dp}{d\xi} = -\frac{p(T) + u(T)}{c^2} a = -\left(\frac{1/3 + 1}{c^2} a_{Ss} T^4\right) \left(\frac{c^2}{\xi}\right), \quad (30)$$

where we have use the connection of Eq. (19) $p = (1/3)u(T) = (1/3)a_{Ss}T^4$. Thus, the pressure at any point depends upon the temperature at that point, which depends in some unknown fashion upon the distance ξ to the event horizon. Therefore, Eq. (30) becomes

$$\frac{d}{d\xi} \left(\frac{1}{3} a_{Ss} T^4 \right) = \frac{4}{3} a_{Ss} T^3 \frac{dT}{d\xi} = -\frac{4}{3} a_{Ss} T^4 \frac{1}{\xi} \quad (31)$$

or $dT/d\xi = -T/\xi$. This equation has the solution $\ln(T) = -\ln(\xi) + \text{const}$, giving $T\xi = \text{const}$ or

$$T = \text{const}/\xi. \quad (32)$$

This result is consistent with the Tolman-Ehrenfest relation of general relativity [18] $T(g_{00})^{1/2} = \text{const}$ for the change of temperature, when we note from Eq. (27) that in the Rindler frame $(g_{00})^{1/2} = \xi$. Thus, we have found that there is no single temperature which can be assigned to the thermal radiation in equilibrium in a Rindler frame.

$$\begin{aligned} \langle \phi_{R0}(\eta, \xi, y, z) \phi_{R0}(\eta', \xi', y, z) \rangle &= \frac{-\hbar c}{\pi} [(\xi \sinh \eta - \xi' \sinh \eta')^2 - (\xi \cosh \eta - \xi' \cosh \eta')^2 - (y - y')^2 - (z - z')^2]^{-1} \\ &= \frac{-\hbar c}{\pi [2\xi\xi' \cosh(\eta - \eta') - \xi^2 - \xi'^2 - (y - y')^2 - (z - z')^2]}. \end{aligned} \quad (34)$$

Although the correlation functions given in Eqs. (22) and (34) look quite different, they actually involve the same zero-point radiation but described in different coordinates.

If we evaluate the zero-point correlation function at a single Rindler time $\eta = \eta'$, this corresponds to a single time $t = t'$ in the inertial frame which is momentarily at rest with respect to the Rindler frame. In this case, the correlation functions in the inertial frame and in the Rindler frame agree exactly, both involving the inverse square of the same spatial distance between the field points. There is no spatial correlation length appearing in the Rindler frame, just as there was none in the inertial frame. The system is still a zero-point radiation system with no possibility of defining a finite local energy density or local entropy density. However, if we consider a single spatial point $(x, y, z) = (x', y', z')$ in an inertial frame at two different times t and t' , or a single spatial point $(\xi, y, z) = (\xi', y', z')$ in the Rindler frame at two different times η and η' , then the field correlation functions for zero-point radiation take quite different forms. The correlation function for zero-point radiation in the inertial frame is given in Eq. (23), while the expression in the Rindler frame follows from Eq. (34) as

Rather the temperature varies with the distance ξ from the event horizon, going to zero at infinite distance and diverging on approach to the event horizon.

D. Two-point correlation function for zero-point radiation in a Rindler frame

Within classical physics, the zero-point radiation of the “vacuum state” is present throughout spacetime and takes the same spectrum in any inertial frame. This zero-point radiation will also be found in the Rindler frame, which is accelerating through Minkowski spacetime. Thus, next we wish to obtain the expression for the field correlation function for zero-point radiation as evaluated in the Rindler frame. The correlation function can be expressed in terms of the fields $\phi_R(\eta, \xi, y, z)$ seen in the Rindler frame. The value of a scalar field at any spacetime point is independent of the coordinate frame in which it is evaluated,

$$\begin{aligned} \phi_R(\eta, \xi, y, z) &= \phi(ct, x, y, z) \\ &= \phi(\xi \sinh(\eta), \xi \cosh(\eta), y, z). \end{aligned} \quad (33)$$

Therefore, merely substituting Eqs. (25) and (26) into Eq. (22), gives

$$\begin{aligned} &\langle \phi_{R0}(\eta, \xi, y, z) \phi_{R0}(\eta', \xi, y, z) \rangle_0 \\ &= \frac{-\hbar c}{\pi [2\xi^2 \cosh(\eta - \eta') - 2\xi^2]} \\ &= \frac{-\hbar c}{\pi [2\xi \sinh\{(\eta - \eta')/2\}]^2} \\ &= \frac{-\hbar c}{\pi [2\xi \sinh\{(\tau_R - \tau'_R)/(2\xi/c)\}]^2} \\ &\quad \times \frac{-\hbar c a^2}{\pi [2c^2 \sinh\{(\tau_R - \tau'_R)a/(2c)\}]^2}, \end{aligned} \quad (35)$$

where in the third line we have introduced the proper time interval $(\tau_R - \tau'_R) = \xi(\eta - \eta')$ measured by a clock at rest in the Rindler frame at the spatial coordinates ξ, y, z , and in the fourth line we have introduced the proper acceleration $a = c^2/\xi$. In an inertial frame, there are no characteristic lengths or times connected to any coordinate point, and the field correlation function for zero-point radiation given in Eq. (22) involves simply the Minkowski proper time interval between any two spacetime points. However, each spatial coordinate point ξ, y, z of the Rindler frame has associated a characteristic

time ξ/c (corresponding to the time for light to travel to the event horizon at speed c), and this is the characteristic time appearing in the third line of Eq. (35) for the correlation function for zero-point radiation in Rindler coordinates. The zero-point radiation in the Rindler frame is measured in units of time, which already contain a characteristic correlation time, and this correlation time is imposed on the zero-point correlation function given in Eq. (35).

E. Example of a horizontal light clock in a Rindler frame

We wish to emphasize strongly that at every point of a Rindler frame, there is a correlation time $\tau_R = \xi/c = c/a$ and an associated correlation frequency $\Omega_R = 1/\tau_R = a/c$ that is unrelated to the temperature of any thermal radiation which may be present. Thus, for example, consider a horizontal light clock of horizontal length l at rest at height ξ in a Rindler frame. The Rindler coordinate time interval η_l read by the light clock corresponds to the time required for light to travel the horizontal length l . Now in the inertial frame, which was momentarily at rest with respect to the length l when the light pulse started, the light is seen to follow a diagonal path; this diagonal path has length l in the direction perpendicular to the Rindler-frame acceleration and a distance $x(t) - x(0) = \xi \cosh \eta_l - \xi \cosh 0$ in the direction parallel to the Rindler-frame acceleration and occurs during an inertial-frame time interval $t - 0 = (\xi/c) \sinh \eta_l - (\xi/c) \sinh 0$. Since in the inertial frame the light travels with speed c , we have $(ct)^2 = [x(t) - x(0)]^2 + l^2$. This corresponds to

$$(\xi \sinh \eta_l)^2 = (\xi \cosh \eta_l - \xi)^2 + l^2 \quad (36)$$

or

$$\begin{aligned} l &= 2\xi \sinh(\eta_l/2) = 2\xi \sinh[\tau_{IR}/(2\xi/c)] \\ &= 2\xi \sinh[\tau_{IR}(a/(2c))], \end{aligned} \quad (37)$$

where $\tau_{IR} = \xi \eta_l$ is the proper time read by a clock at height ξ in the Rindler frame. Thus, the connection between the proper time interval τ_{IR} and the length l of the horizontal light clock is governed by the hyperbolic sine function as in Eq. (37). Accordingly, we find that for a small horizontal light clock in the Rindler frame, the time interval τ_{IR} read by this horizontal light clock is given by the linear relation $\tau_{IR} = \xi \eta_l = l/c$, whereas for a large horizontal light clock, there is an exponential connection between l and τ_{IR} . The transition length between two these two regimes is given by the length $l \approx \xi$, the time $\tau_{IR} \approx \xi/c$, and the associated frequency $\Omega_R \approx c/\xi = a/c$.

F. Assumption of a single correlation time for thermal radiation in a Rindler frame

At this point we want to consider the two-point field correlation function in time for the radiation field ϕ_{RT} in the Rindler frame when thermal radiation at temperature $T > 0$ is present. We notice from Eq. (35) that in a Rindler frame the two-point field correlation function for zero-point radiation at a single spatial point already includes the finite correlation time $\xi/c = c/a$, which is characteristic of the acceleration a of a point ξ above the event horizon. Thus for small times τ_R (where the high-frequency zero-point radiation contributes) the correlation function in Eq. (35) behaves as $-\hbar c/(\pi c^2 \tau_R^2)$, whereas for long times the behavior is as $-\hbar c a^2/\{\pi c^4 \exp[\tau_R a/c]\}$. Accordingly in a Rindler frame, both the acceleration a and the finite nonzero temperature T will contribute finite correlation times to the two-field correlation function at fixed height ξ . Thus, one might expect three different time regions for the two-point field correlation function: i) the short-time region dominated by high-frequency zero-point radiation, ii) the region dominated by the acceleration-related correlation time, and iii) the region dominated by the temperature-related correlation time. Depending upon the relative magnitude of the acceleration a and the temperature T , the two-point field correlation function and the associated frequency spectrum would take on varying forms. This variation in form would allow us to distinguish the relative temperature in the Rindler frame compared to the acceleration, or the acceleration relative to the temperature. Since we have seen in Eq. (32) that in a Rindler frame the temperature of the radiation must behave with height ξ as $T = \text{const}/\xi$ while the acceleration given in Eq. (29) behaves with height ξ as $a = c^2/\xi$, the ratio of temperature T to acceleration a remains the same throughout the Rindler frame in thermal equilibrium. We notice in Eq. (35) that in zero-point radiation the correlation function in the Rindler frame is a monotonic function of the acceleration a . Thermodynamics requires that the correlation function for thermal radiation must also be a monotonic function of temperature T . Clearly we want the field correlation function at two different times η and η' when thermal radiation is present to fit with the correlation function (35) when only zero-point radiation is present. The simplest possibility is that the two-point correlation function of the fields at a single height involves not two distinct correlation times but rather only a single correlation time. This situation corresponds to substituting $(a + \text{const} \times T)$ in place of the acceleration a in the correlation function of Eq. (35). This increase in the argument of the correlation function corresponds to increasing the energy density in the normal modes above the zero-point value, exactly as is appropriate for thermal radiation. The only combination of fundamental units with the correct dimensions requires the combination $a + \zeta 2\pi c k_B T/\hbar$ where ζ is a dimensionless number. The correlation function for ther-

mal radiation in the Rindler frame then takes the form

$$\begin{aligned} & \langle \phi_{RT}(\eta, \xi, y, z) \phi_{RT}(\eta', \xi, y, z) \rangle \\ &= \frac{-\hbar c}{\pi} \left(\frac{(a + \zeta 2\pi c k_B T / \hbar)^2}{[2c^2 \sinh\{(\tau_R - \tau'_R)(a + \zeta 2\pi c k_B T / \hbar) / (2c)\}]^2} \right). \end{aligned} \quad (38)$$

Furthermore, if we make this substitution, then we find that the asymptotic limits are appropriate. We recall that at large values of ξ , the acceleration of the Rindler frame becomes small so that the Rindler frame has behavior similar to that of an inertial frame. But then in this small-acceleration limit $a \rightarrow 0$, we notice that the field correlation function in time Eq. (38) for the Rindler frame takes just the same form as the field correlation function in time Eq. (21) for an inertial frame; both involve $T^2 \tilde{\gamma} [T(\tau - \tau')]$. In the small-temperature limit $T \rightarrow 0$, the correlation function (38) returns to the zero-point correlation function (35). At short time differences $(\tau_R - \tau'_R)$, the correlation function (38) still goes over to the zero-point radiation result of Eq. (23), which is independent of both a and of T . At long time differences $(\tau_R - \tau'_R)$, the correlation function decreases exponentially, but now with the combination $a + \zeta 2\pi c k_B T / \hbar$.

The correlation function in Eq. (38) corresponds to amplitudes for the normal modes which have monotonically larger amplitudes with increasing temperature (corresponding to increased energy due to the assignment of thermal energy) than the amplitudes of the normal modes involving zero-point radiation alone. At high temperature and fixed frequency, the thermal radiation dominates the spectrum. On the other hand, in the limit as the temperature T goes to zero, the correlation function becomes the zero-point correlation function of the Rindler frame given in Eq. (35). All of these considerations suggest that the field correlation function given in Eq. (38) is indeed the thermal correlation function for scalar radiation in a Rindler frame. In a Rindler frame, thermal radiation is constrained to fit with zero-point radiation which appeared from Lorentz invariance in a Minkowski frame.

The appearance of only a single correlation time in the two-point field correlation function at fixed height, as in Eq. (38), serves to hide the acceleration of the system from any spatially-local measurement which considers only time correlations. An observer who has access only to the time correlation at a fixed spatial point would not be able to determine the acceleration of the system since the correlation might represent any combination of finite-temperature thermal radiation and acceleration through zero-point radiation. However, measurements of spatial correlations in the fields at fixed time will indeed allow separation of the acceleration and finite-temperature aspects. In a sense, this behavior is analogous to the suppression of acceleration information in a local measurement of particle velocities in a nonrelativistic thermal distribution

in an accelerating frame. The velocity distribution at a fixed height in an accelerating frame is the Maxwell distribution. The information about the acceleration of the frame is contained in the spatial change in particle density with height.

G. Planck radiation spectrum

In the limit as the acceleration a goes to zero while the temperature T is held fixed, the proper time τ_R in the Rindler frame becomes equal to the proper time τ_M in the Minkowski frame, and the field correlation of the Rindler frame becomes that of thermal radiation in a Minkowski frame. Indeed, if we consider the Minkowski frame limit $a \rightarrow 0$ in Eq. (38), then we find

$$\begin{aligned} & \lim_{a \rightarrow 0} \langle \phi_{RT}(\eta, \xi, y, z) \phi_{RT}(\eta', \xi, y, z) \rangle \\ &= \langle \phi_T(ct, x, y, z) \phi_T(ct', x, y, z) \rangle \\ &= \frac{-\hbar c}{\pi} \left(\frac{(\zeta 2\pi c k_B T / \hbar)^2}{[2c^2 \sinh\{(t - t')(\zeta 2\pi c k_B T / \hbar) / (2c)\}]^2} \right). \end{aligned} \quad (39)$$

By taking the (singular) Fourier cosine transform of this correlation function [19], we obtain the thermal radiation spectrum in a Minkowski frame

$$\begin{aligned} \frac{\omega^2 f^2(\omega)}{8\pi c^2} &= \int_0^\infty d\tau \frac{-\hbar c}{\pi} \left(\frac{(\zeta 2\pi c k_B T / \hbar)^2}{[2c^2 \sinh\{\tau(\zeta 2\pi c k_B T / \hbar) / (2c)\}]^2} \right) \\ &\quad \times \cos(\omega\tau) \\ &= \frac{1}{2} \hbar \omega \coth\left(\frac{\hbar\omega}{2\zeta k_B T}\right) \end{aligned} \quad (40)$$

corresponding to an energy per normal mode from Eq. (6)

$$\begin{aligned} U(\omega, T) &= \frac{1}{2} \hbar \omega \coth\left(\frac{\hbar\omega}{2\zeta k_B T}\right) \\ &= \frac{\hbar\omega}{\exp[\hbar\omega / (\zeta k_B T)] - 1} + \frac{1}{2} \hbar \omega, \end{aligned} \quad (41)$$

which is exactly the usual Planck scalar radiation result including zero-point radiation when we set the unknown constant $\zeta = 1$. At high frequencies ω , the energy $U(\omega)$ becomes $U(\omega) = (1/2)\hbar\omega$. At low frequencies the energy $U(\omega)$ becomes $U(\omega) = k_B T$.

V. DISCUSSION

The analysis given here has ties to work appearing in quantum field theory [20,21]. In connection with Hawking's ideas regarding the quantum evaporation of black holes [22] and Fulling's nonuniqueness of the field quantization [23], Davies [24] and Unruh [25] noted the appearance of the Planck correlation function when a point was accelerated through the quantum vacuum of Minkowski spacetime. Within the quantum literature, a mechanical system accelerating through the vacuum is

often said to experience a thermal bath at temperature $T = \hbar a / (2\pi c k_B)$ and to take on a thermal distribution. There have been controversies as to whether or not the acceleration turns the “virtual photons” of the vacuum into “real photons.” In this paper, the analysis has been entirely within classical physics.

The analysis of thermal radiation given here is totally different from the discussions that appear in the textbooks of modern physics [12]. The present analysis is crucially dependent upon relativistic physics, whereas the historical treatments combine nonrelativistic and relativistic aspects

so that the combination satisfies neither Galilean nor Lorentz invariance. The emphasis upon a relativistic treatment in the present article is consistent with recent analysis showing that scattering by relativistic (as opposed to non-relativistic) mechanical systems leaves the zero-point spectrum invariant [26].

The classical analysis of thermodynamics in a Rindler frame has important implications for the connections between classical and quantum theories, which will be pursued elsewhere.

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