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Superqubits

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We provide a supersymmetric generalization of n quantum bits by extending the local operations and classical communication entanglement equivalence group $[SU(2)]^n$ to the supergroup $[uOSp(1|2)]^n$ and the stochastic local operations and classical communication equivalence group $[SL(2,\mathbb{C})]^n$ to the supergroup $[OSp(1|2)]^n$. We introduce the appropriate supersymmetric generalizations of the conventional entanglement measures for the cases of $n=2$ and $n=3$. In particular, super-Greenberger-Horne-Zeilinger states are characterized by a nonvanishing superhyperdeterminant.

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I. INTRODUCTION

The question of computable entanglement measures for arbitrary quantum systems is, to a large extent, an open one. However, substantial progress has been made utilizing the paradigms of *local operations and classical communication* (LOCC) and *stochastic local operations and classical communication* (SLOCC). For example, 2-qubit and 3-qubit systems both admit concise, but nontrivial, SLOCC classifications, which reveal a number of important qualitative features of multipartite entanglement [1–6]. In particular, 2-qubit Bell states and 3-qubit Greenberger-Horne-Zeilinger (GHZ) states are characterized, respectively, by nonvanishing determinant and hyperdeterminant.

Here we propose a supersymmetric generalization of the qubit, the *superqubit*. We proceed by extending the n-qubit SLOCC equivalence group $[SL(2\mathbb{C},)]^n$ and the LOCC equivalence group $[SU(2)]^n$ to the supergroups $[OSp(1|2)]^n$ and $[uOSp(1|2)]^n$, respectively. A single superqubit forms a three-dimensional representation of $OSp(1|2)$ consisting of two commuting "bosonic" components and one anticommuting "fermionic" component. For $n=2$ and $n=3$ we introduce the appropriate supersymmetric generalizations of the conventional entanglement measures. In particular, super-Bell and super-GHZ states are characterized, respectively, by nonvanishing superdeterminant (distinct from the Berezinian) and superhyperdeterminant.[1]

This mathematical construction seems a very natural one. Moreover, from a physical point of view, it makes contact with various condensed-matter systems. For example, the three-dimensional representation of $OSp(1|2)$ is encountered in the supersymmetric t-J model where it describes spinons and holons on a one-dimensional lattice [8–12]. It also shows up in the quantum Hall effect [13] and Affleck-Kennedy-Lieb-Tasaki models of superconductivity [14].

In order to facilitate the introduction of a super Hilbert space, super LOCC and superqubits in IV, we first recall some familiar properties of ordinary Hilbert space, LOCC, and qubits in II. Similarly, in order to discuss the super-entanglement of two and three superqubits in V, we first review the ordinary entanglement of two and three qubits in III.

II. QUBITS

A. Hilbert space

A complex Hilbert space $\mathcal{H}$ is equipped with a one-to-one map into its dual space $\mathcal{H}^\dagger$,

$$\dagger\colon \mathcal{H} \to \mathcal{H}^\dagger, \qquad |\psi\rangle \mapsto (|\psi\rangle)^\dagger := \langle\psi|, \tag{1}$$

which defines an inner product $\langle\psi|\phi\rangle$ and satisfies the following properties:

(1) For all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, and any complex number α we have,

$$(\alpha|\psi\rangle)^\dagger = \langle\psi|\alpha^*, \qquad (|\psi\rangle + |\phi\rangle)^\dagger = \langle\psi| + \langle\phi|. \tag{2}$$

(2) For all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$,

$$\langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle. \tag{3}$$

(3) For all $|\psi\rangle \in \mathcal{H}$,

$$\langle\psi|\psi\rangle \geq 0 \tag{4}$$

with equality holding if and only if $|\psi\rangle$ is the null vector.

In particular, a qubit lives in the two-dimensional complex Hilbert space $\mathbb{C}^2$. An arbitrary n-qubit system is then simply a vector in the n-fold tensor product Hilbert space $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = [\mathbb{C}^2]^n$.

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[1] The present work was in part inspired by the construction of the superhyperdeterminant in [7].

B. LOCC and SLOCC

Two states are said to be LOCC equivalent if and only if they may be transformed into one another with certainty using LOCC protocols. Reviews of the LOCC paradigm and entanglement measures may be found in [15,16]. It is well known that two states of a composite system are LOCC equivalent if and only if they are related by the group of local unitaries (which we will refer to as the *LOCC equivalence group*), unitary transformations that factorize into separate transformations on the component parts [17]. In the case of n qubits the group of local unitaries is given (up to a global phase) by $[SU(2)]^n$.

Similarly, two quantum states are said to be SLOCC equivalent if and only if they may be transformed into one another with some *nonvanishing probability* using LOCC operations [2,17]. The set of SLOCC transformations relating equivalent states forms a group (which we will refer to as the *SLOCC equivalence group*). For n qubits the SLOCC equivalence group is given (up to a global complex factor) by the n-fold tensor product, $[SL(2,\mathbb{C})]^n$, one factor for each qubit [2]. Note, the LOCC equivalence group forms a compact subgroup of the larger SLOCC equivalence group.

The Lie algebra $\mathfrak{sl}(2)$ may be conveniently summarized as

$$[P_{A_1A_2}, P_{A_3A_4}] = 2\varepsilon_{(A_1(A_3}P_{A_4)A_2)}, \tag{5}$$

where $A = 0, 1$, and throughout this paper we use "strength one" (anti)symmetrization, so that

$$X_{(A_1A_2)} \equiv \tfrac{1}{2}(X_{A_1A_2} + X_{A_2A_1}). \tag{6}$$

We permit the indices to be raised/lowered by the $SL(2,\mathbb{C})$-invariant epsilon tensors according to the rules:

$$V_{A_1} = \varepsilon_{A_1A_2}V^{A_2}, \qquad V^{A_1} = \varepsilon^{A_1A_2}V_{A_2}, \tag{7}$$

where we adopt the following conventions:

$$\varepsilon_{A_1A_2} = -\varepsilon^{A_1A_2}, \qquad \varepsilon_{A_1A_2}\varepsilon^{A_2A_3} = \delta^{A_3}_{A_1}. \tag{8}$$

Consequently,

$$U^AV_A = -U_AV^A. \tag{9}$$

The compact subalgebra $\mathfrak{su}(2)$ is given by

$$\mathfrak{su}(2) := \{X \in \mathfrak{sl}(2) | X^\dagger = -X\}. \tag{10}$$

An arbitrary element $X \in \mathfrak{su}(2)$ may be written as

$$X = \xi_i A_i, \tag{11}$$

where $\xi_i \in \mathbb{R}$ and

$$\begin{aligned} A_1 &= \frac{i}{2}(P_{00} - P_{11}),\\ A_2 &= \frac{1}{2}(P_{00} + P_{11}),\\ A_3 &= iP_{01},\\ A_i^\dagger &= -A_i. \end{aligned} \tag{12}$$

C. One qubit

The one-qubit system (Alice) is described by the state

$$|\Psi\rangle = a_A|A\rangle, \tag{13}$$

and the Hilbert space has dimension 2. The SLOCC equivalence group is $SL(2,\mathbb{C})_A$, under which a_A transforms as a **2**.

The norm squared $\langle\Psi|\Psi\rangle$ is given by

$$\langle\Psi|\Psi\rangle = \delta^{A_1A_2}a^*_{A_1}a_{A_2} \tag{14}$$

and is invariant under $SU(2)_A$. The one-qubit density matrix is given by

$$\rho := |\Psi\rangle\langle\Psi| = a_{A_1}a^*_{A_2}|A_1\rangle\langle A_2|. \tag{15}$$

The norm squared is then given by

$$\langle\Psi|\Psi\rangle = \mathrm{tr}(\rho). \tag{16}$$

Unnormalized pure state density matrices satisfy

$$\rho^2 = \mathrm{tr}(\rho)\rho. \tag{17}$$

D. Two qubits

The two-qubit system (Alice and Bob) is described by the state

$$|\Psi\rangle = a_{AB}|AB\rangle, \tag{18}$$

and the Hilbert space has dimension $2^2 = 4$. The SLOCC equivalence group is $SL(2,\mathbb{C})_A \times SL(2,\mathbb{C})_B$ under which a_{AB} transforms as a $(\mathbf{2},\mathbf{2})$.

The norm squared $\langle\Psi|\Psi\rangle$ is given by

$$\langle\Psi|\Psi\rangle = \delta^{A_1A_2}\delta^{B_1B_2}a^*_{A_1B_1}a_{A_2B_2} \tag{19}$$

and is invariant under $SU(2)_A \times SU(2)_B$. The two-qubit density matrix is given by

$$\rho := |\Psi\rangle\langle\Psi| = a_{A_1B_1}a^*_{A_2B_2}|A_1B_1\rangle\langle A_2B_2|. \tag{20}$$

The reduced density matrices are defined using the partial trace

$$\rho_A = \mathrm{tr}_B|\Psi\rangle\langle\Psi|, \qquad \rho_B = \mathrm{tr}_A|\Psi\rangle\langle\Psi|, \tag{21}$$

or

$$\begin{aligned} (\rho_A)_{A_1A_2} &= \delta^{B_1B_2}a_{A_1B_1}a^*_{A_2B_2},\\ (\rho_B)_{B_1B_2} &= \delta^{A_1A_2}a_{A_1B_1}a^*_{A_2B_2}. \end{aligned} \tag{22}$$

E. Three qubits

The three-qubit system (Alice, Bob, Charlie) is described by the state

$$|\Psi\rangle = a_{ABC}|ABC\rangle, \tag{23}$$

and the Hilbert space has dimension $2^3 = 8$. The SLOCC equivalence group is $SL(2,\mathbb{C})_A \times SL(2,\mathbb{C})_B \times SL(2,\mathbb{C})_C$ under which a_{ABC} transforms as a $(\mathbf{2},\mathbf{2},\mathbf{2})$.

The norm squared $\langle\Psi|\Psi\rangle$ is given by

$$\langle\Psi|\Psi\rangle = \delta^{A_1A_2}\delta^{B_1B_2}\delta^{C_1C_2}a^*_{A_1B_1C_1}a_{A_2B_2C_2} \tag{24}$$

and is invariant under $SU(2)_A \times SU(2)_B \times SU(2)_C$. The three-qubit density matrix is given by

$$\rho := |\Psi\rangle\langle\Psi| = a_{A_1B_1C_1}a^*_{A_2B_2C_2}|A_1B_1C_1\rangle\langle A_2B_2C_2|. \tag{25}$$

The singly reduced density matrices are defined using the partial trace

$$\begin{aligned}\rho_{AB} &= \mathrm{tr}_C|\Psi\rangle\langle\Psi|,\\ \rho_{BC} &= \mathrm{tr}_A|\Psi\rangle\langle\Psi|,\\ \rho_{CA} &= \mathrm{tr}_B|\Psi\rangle\langle\Psi|,\end{aligned} \tag{26}$$

or

$$\begin{aligned}(\rho_{AB})_{A_1A_2B_1B_2} &= \delta^{C_1C_2}a_{A_1B_1C_1}a^*_{A_2B_2C_2},\\ (\rho_{BC})_{B_1B_2C_1C_2} &= \delta^{A_1A_2}a_{A_1B_1C_1}a^*_{A_2B_2C_2},\\ (\rho_{CA})_{C_1C_2A_1A_2} &= \delta^{B_1B_2}a_{A_1B_1C_1}a^*_{A_2B_2C_2}.\end{aligned} \tag{27}$$

The doubly reduced density matrices are defined using the partial traces

$$\begin{aligned}\rho_{A} &= \mathrm{tr}_{BC}|\Psi\rangle\langle\Psi|,\\ \rho_{B} &= \mathrm{tr}_{CA}|\Psi\rangle\langle\Psi|,\\ \rho_{C} &= \mathrm{tr}_{AB}|\Psi\rangle\langle\Psi|,\end{aligned} \tag{28}$$

or

$$\begin{aligned}(\rho_{A})_{A_1A_2} &= \delta^{B_1B_2}\delta^{C_1C_2}a_{A_1B_1C_1}a^*_{A_2B_2C_2},\\ (\rho_{B})_{B_1B_2} &= \delta^{C_1C_2}\delta^{A_1A_2}a_{A_1B_1C_1}a^*_{A_2B_2C_2},\\ (\rho_{C})_{C_1C_2} &= \delta^{A_1A_2}\delta^{B_1B_2}a_{A_1B_1C_1}a^*_{A_2B_2C_2}.\end{aligned} \tag{29}$$

III. ENTANGLEMENT

A. Two qubits

For two qubits there are only two distinct SLOCC entanglement classes—two qubits are either entangled or not. The two classes are distinguished by the SLOCC invariant, $\det a_{AB}$. For separable states $\det a_{AB} = 0$, while it is nonzero for any entangled state.

There are two independent $[SU(2)]^2$ invariants, the norm $\langle\Psi|\Psi\rangle^{1/2}$ and the 2-tangle τ_{AB} [1,18],

$$\tau_{AB} = 4\det\rho_A = 4\det\rho_B = 4|\det a_{AB}|^2. \tag{30}$$

The 2-tangle is maximized, $\tau_{AB} = 1$, by the Bell state:

$$|\Psi\rangle_{\mathrm{Bell}} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{31}$$

B. Three qubits

For three qubits there are six distinct SLOCC entanglement classes [2,4–6]. These classes and their representative states are summarized as follows:

Separable: Zero entanglement orbit for completely factorizable product states,

$$A\text{-}B\text{-}C\text{: } |000\rangle. \tag{32}$$

Biseparable: Three classes of bipartite entanglement

$$\begin{aligned}&A\text{-}BC\text{: } |010\rangle + |001\rangle,\\ &B\text{-}CA\text{: } |100\rangle + |001\rangle,\\ &C\text{-}AB\text{: } |010\rangle + |100\rangle.\end{aligned} \tag{33}$$

W: Three-way entangled states that do not maximally violate Bell-type inequalities in the same way as the GHZ class discussed below. However, they are robust in the sense that tracing out a subsystem generically results in a bipartite mixed state that is maximally entangled under a number of criteria [2],

$$W\text{: } |100\rangle + |010\rangle + |001\rangle. \tag{34}$$

GHZ: Genuinely tripartite entangled Greenberger-Horne-Zeilinger [19] states. These maximally violate Bell's inequalities but, in contrast to class W, are fragile under the tracing out of a subsystem since the resultant state is completely unentangled,

$$\mathrm{GHZ}\text{: } |000\rangle + |111\rangle. \tag{35}$$

The six classes may be distinguished either by appealing to simple arguments concerning the conservation of reduced density matrix ranks as in [2] or by considering the vanishing or not of five algebraically independent covariants/invariants as in [6]. For our purposes it is more convenient to follow the latter approach as it better facilitates our supersymmetric extension. The five covariants/invariants are given as follows:

(1) Three covariants

$$\begin{aligned}(\gamma^A)_{A_1A_2} &= a_{A_1}{}^{BC}a_{A_2BC},\\ (\gamma^B)_{B_1B_2} &= a^A{}_{B_1}{}^{C}a_{AB_2C},\\ (\gamma^C)_{C_1C_2} &= a^{AB}{}_{C_1}a_{ABC_2},\end{aligned} \tag{36}$$

transforming, respectively, as a $(\mathbf{3},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{3},\mathbf{1})$,

TABLE I. The entanglement classification of three qubits.

Class	Vanishing	Nonvanishing
A-B-C	γ^A, γ^B, γ^C	a_{ABC}
A-BC	γ^B, γ^C	γ^A
B-CA	γ^A, γ^C	γ^B
C-AB	γ^A, γ^B	γ^C
W	$\mathrm{Det}\, a_{ABC}$	T_{ABC}
GHZ	$\cdots$	$\mathrm{Det}\, a_{ABC}$

and $(\mathbf{1}, \mathbf{1}, \mathbf{3})$ under $SL_A(2, \mathbb{C}) \times SL_B(2, \mathbb{C}) \times SL_C(2, \mathbb{C})$.

(2) One covariant T_{ABC} transforming as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ under $[SL(2, \mathbb{C})]^3$, which may be written in one of three equivalent forms

$$\begin{aligned} T_{ABC} &= (\gamma^A)_{AA'} a^{A'}{}_{BC}, \\ T_{ABC} &= (\gamma^B)_{BB'} a_A{}^{B'}{}_C, \\ T_{ABC} &= (\gamma^C)_{CC'} a_{AB}{}^{C'}. \end{aligned} \tag{37}$$

(3) Cayley's hyperdeterminant $\mathrm{Det} a_{ABC}$ [4,5,20], the unique quartic $[SL(2, \mathbb{C})]^3$ invariant, where

$$\mathrm{Det}\, a_{ABC} = -\det\gamma^A = -\det\gamma^B = -\det\gamma^C. \tag{38}$$

The entanglement classification as determined by these covariants/invariants is summarized in Table I.

There are six independent $[SU(2)]^3$ pure state invariants [21]: the norm, the three local entropies $4\det\rho_A$, $4\det\rho_B$, $4\det\rho_C$, the Kempe invariant [22], and finally the all important 3-tangle τ_{ABC} [1],

$$\tau_{ABC} = 4|\mathrm{Det} a_{ABC}|. \tag{39}$$

The 3-tangle is maximized, $\tau_{ABC} = 1$, by the GHZ state:

$$|\Psi\rangle_{\mathrm{GHZ}} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{40}$$

IV. SUPERQUBITS

A. Super Hilbert space and $uOSp(1|2)$

1. The dual space

With one important difference, explained below, our definition of a super Hilbert space follows that of DeWitt [23]. We define a super Hilbert space to be a supervector space $\mathcal{H}$ equipped with an injection to its dual space $\mathcal{H}^\ddagger$,

$$\ddagger\colon \mathcal{H} \to \mathcal{H}^\ddagger, \qquad |\psi\rangle \mapsto (|\psi\rangle)^\ddagger := \langle\psi|. \tag{41}$$

Details of even and odd Grassmann numbers and supervectors may be found in Appendix A. A basis in which all basis vectors are pure even or odd is said to be pure. Such a basis may always be found [23].

The map $\ddagger\colon \mathcal{H} \to \mathcal{H}^\ddagger$ defines an inner product $\langle\psi|\phi\rangle$ and satisfies the following axioms:

(1) $\ddagger$ sends pure bosonic (fermionic) supervectors in $\mathcal{H}$ into bosonic (fermionic) supervectors in $\mathcal{H}^\ddagger$.

(2) $\ddagger$ is linear

$$(|\psi\rangle + |\phi\rangle)^\ddagger = \langle\psi| + \langle\phi|. \tag{42}$$

(3) For pure even/odd α and $|\psi\rangle$

$$(|\psi\rangle\alpha)^\ddagger = (-)^{\alpha\psi}\alpha^\#\langle\psi| \tag{43}$$

and

$$(\alpha\langle\psi|)^\ddagger = (-)^{\psi+\alpha\psi}|\psi\rangle\alpha^\#, \tag{44}$$

where $^\#$ is the superstar introduced in Appendix A. In particular,

$$|\psi\rangle^{\ddagger\ddagger} = (-)^\psi|\psi\rangle. \tag{45}$$

Note, an α (or ψ and the like) appearing in the exponent of $(-)$ is shorthand for its grade, $\deg(\alpha)$, which takes the value 0 or 1 according to whether α is even or odd. The impure case follows from the linearity of $\ddagger$.

In a pure even/odd orthonormal basis $\{|i\rangle\}$ we adopt the following convention:

$$|\psi\rangle = |i\rangle\psi_i \tag{46}$$

so that for pure even/odd ψ (43) and (44) imply

$$\begin{aligned} (|i\rangle\psi_i)^\ddagger &= (-)^{\psi_i i}\psi_i^\#\langle i| = (-)^{i+i\psi}\psi_i^\#\langle i|((-)^{i+i\psi}\psi_i^\#\langle i|)^\ddagger \\ &= (-)^\psi|i\rangle\psi_i, \end{aligned} \tag{47}$$

where we have used $\deg(\psi_i) = \deg(i) + \deg(\psi)$. This is consistent with (A20).

2. Inner product

For all pure even/odd $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, the inner product $\langle\psi|\phi\rangle$ satisfies

$$\langle\psi|\phi\rangle^\# = (-)^{\psi+\psi\phi}\langle\phi|\psi\rangle. \tag{48}$$

Consequently,

$$\langle\psi|\phi\rangle^{\#\#} = (-)^{\psi+\phi}\langle\phi|\psi\rangle, \tag{49}$$

as would be expected of a pure even/odd Grassmann number since $\deg(\langle\phi|\psi\rangle) = \deg(\psi) + \deg(\phi)$. In a pure even/odd orthonormal basis we find

$$\langle\phi|\psi\rangle = (-)^{i+i\phi}\phi_i^\#\psi_i. \tag{50}$$

In using the superstar we depart from the formalism presented in [23], which uses the ordinary star. A comparison of the star and superstar may be found in Appendix A. The use of the superstar anticipates the implementation of

$uOSp(1|2)$ as the compact subgroup of $OSp(1|2)$ as will be explained in IV B.

3. Linear superoperators and the superadjoint

A linear superoperator A: $\mathcal{H} \to \mathcal{H}$ is required to satisfy the following properties:

(1) $A(|\psi\rangle + |\phi\rangle) = A|\psi\rangle + A|\phi\rangle$,

(2) $A(|\psi\rangle\alpha) = (A|\psi\rangle)\alpha$.

Linear superoperators may be combined using

(1) $(A + B)|\psi\rangle = A|\psi\rangle + B|\psi\rangle$,

(2) $(AB)|\psi\rangle = A(B|\psi\rangle)$.

A linear superoperator is said to be pure even (odd) if it takes pure even supervectors into pure even (odd) supervectors and pure odd supervectors into pure odd (even) supervectors.

The superadjoint of a pure even/odd linear superoperator is defined through

$$(A|\phi\rangle)^{\ddagger} = (-)^{\phi A}\langle\phi|A^{\ddagger}. \tag{51}$$

This is, in fact, equivalent to

$$\langle\phi|A^{\ddagger}|\psi\rangle = (-)^{\psi+\phi\psi+(\phi+\psi)A}\langle\psi|A|\phi\rangle^{\#}, \tag{52}$$

which is the natural supersymmetric generalization of the conventional definition of the adjoint. This equivalence may be established by simply inserting the identity operator, $\mathbb{1} = |i\rangle\langle i|$, in (51),

$$\begin{aligned}
&(|i\rangle\langle i|A|\phi\rangle)^{\ddagger} = (-)^{\phi A}\langle\phi|A^{\ddagger}|i\rangle\langle i| \\
\Rightarrow\quad &(-)^{i(i+A+\phi)}\langle i|A|\phi\rangle^{\#}\langle i| = (-)^{\phi A}\langle\phi|A^{\ddagger}|i\rangle\langle i| \\
\Rightarrow\quad &(-)^{i+i\phi+(i+\phi)A}\langle i|A|\phi\rangle^{\#} = \langle\phi|A^{\ddagger}|i\rangle \\
\Rightarrow\quad &\sum_i(-)^{i+i\phi+(i+\phi)A}\langle i|A|\phi\rangle^{\#}\psi_i = \sum_i\langle\phi|A^{\ddagger}|i\rangle\psi_i \\
\Rightarrow\quad &\sum_i(-)^{i+i\phi+(i+\phi)A+\psi_i(i+A+\phi)+\psi_i}(\psi_i^{\#}\langle i|A|\phi\rangle)^{\#} = \sum_i\langle\phi|A^{\ddagger}|i\rangle\psi_i \\
\Rightarrow\quad &(-)^{\psi+\phi\psi+(\psi+\phi)A}\langle\psi|A|\phi\rangle^{\#} = \langle\phi|A^{\ddagger}|\psi\rangle,
\end{aligned} \tag{53}$$

where we have defined $|\psi\rangle = |i\rangle\psi_i$ and used $\deg(\psi) = \deg(\psi_i) + \deg(i)$. The converse implication follows from a similar treatment, which we omit. From (52) we also have

$$(\langle\phi|A)^{\ddagger} = (-)^{\phi+\phi A}A^{\ddagger}|\phi\rangle. \tag{54}$$

Moreover,

$$A^{\ddagger\ddagger} = (-)^{A}A, \tag{55}$$

which is consistent with the properties of supermatrices and the supermatrix superadjoint given in Appendix A.

In a pure even/odd orthonormal basis the supermatrix representation of a linear operator A is given by

$$A_{ij} := \langle i|A|j\rangle. \tag{56}$$

In particular, (52) implies that the component form of the adjoint is given by

$$(A^{\ddagger})_{ij} = (-)^{j+ij+(i+j)A}A^{\#}_{ji}, \tag{57}$$

where an index in the exponent of $(-)$ is understood to take the value 0 or 1 according to whether it corresponds to an even or odd basis vector. This is just the conventional supermatrix superadjoint used to define $uOSp(1|2)$ in IV B.

For any linear operator of the form $|\psi\rangle\langle\phi|$ one obtains

$$(|\psi\rangle\langle\phi|)^{\ddagger} = (-)^{\phi+\phi\psi}|\phi\rangle\langle\psi|. \tag{58}$$

For pure even/odd $|\psi\rangle$ the butterfly operator $|\psi\rangle\langle\psi|$ is manifestly self-adjoint.

The inner product is invariant under the action of all even operators satisfying the superunitary condition

$$A^{\ddagger}A = \mathbb{1}, \qquad A^{\ddagger}_{ij}A_{jk} = \delta_{ik}. \tag{59}$$

Let $|\psi\rangle$ be a pure even/odd supervector and

$$|\tilde{\psi}\rangle = A|\psi\rangle. \tag{60}$$

Then, in a pure orthonormal basis $\{|i\rangle\}$

$$\tilde{\psi}_i = \langle i|\tilde{\psi}\rangle = \langle i|A|j\rangle\psi_j = A_{ij}\psi_j. \tag{61}$$

Hence, for pure even/odd supervectors $|\phi\rangle$ and $|\psi\rangle$ and even A the transformed inner product is given by

$$\begin{aligned}
\langle\tilde{\phi}|\tilde{\psi}\rangle &= (-)^{i+i\tilde{\phi}}\tilde{\phi}^{\#}_i\tilde{\psi}_i = (-)^{i+i\phi}(A_{ij}\phi_j)^{\#}A_{ik}\psi_k \\
&= (-)^{i+i\phi+(j+\phi)(i+j)}\phi^{\#}_jA^{\#}_{ij}A_{ik}\psi_k \\
&= (-)^{i+i\phi+(j+\phi)(i+j)}\phi^{\#}_j(-)^{i+ij}A^{st\#}_{ji}A_{ik}\psi_k \\
&= (-)^{(j+j\phi)}\phi^{\#}_jA^{\ddagger}_{ji}A_{ik}\psi_k = (-)^{(j+j\phi)}\phi^{\#}_j\psi_j \\
&= \langle\phi|\psi\rangle,
\end{aligned} \tag{62}$$

where we have used $\deg(A_{ij}) = \deg(i) + \deg(j)$.

4. Physical states

For all $|\psi\rangle \in \mathcal{H}$

$$\langle\psi|\psi\rangle_{\mathcal{B}} \geq 0. \tag{63}$$

Here $z_{\mathcal{B}} \in \mathbb{C}$ denotes the purely complex number component of the Grassmann number z and is referred to as the *body*, a terminology introduced in [23]. The *soul* of z, denoted $z_{\mathcal{S}}$, is the purely Grassmannian component. Any Grassmann number may be decomposed into body and soul, $z = z_{\mathcal{B}} + z_{\mathcal{S}}$.

A Grassmann number has an inverse iff it has a nonvanishing body. Consequently, a state $|\psi\rangle$ is normalizable iff $\langle\psi|\psi\rangle_{\mathcal{B}} > 0$. The state may then be normalized,

$$|\hat{\psi}\rangle = N_{\psi}|\psi\rangle, \qquad N_{\psi} = \langle\psi|\psi\rangle^{-1/2}, \tag{64}$$

where N_{ψ} is given by the general definition of an analytic function f on the space of Grassmann numbers (A3). Explicitly,

$$\langle\psi|\psi\rangle^{-1/2} = \sum_{k=0}^{\infty}\frac{1}{k!2^k}\prod_{j=0}^{k}(1-2j)\langle\psi|\psi\rangle_{\mathcal{B}}^{-((2k+1)/2)}\langle\psi|\psi\rangle_{\mathcal{S}}^{k}. \tag{65}$$

Motivated by the above considerations a state $|\psi\rangle$ is said to be *physical* iff $\langle\psi|\psi\rangle_{\mathcal{B}} > 0$. We restrict our attention to physical states throughout.

B. Super LOCC and SLOCC

We promote the conventional SLOCC equivalence group $SL(2, \mathbb{C})$ to its minimal supersymmetric extension $OSp(1|2)$ [24,25]. The orthosymplectic superalgebras and $OSp(1|2)$, in particular, are described in Appendix B.

The three even elements $P_{A_1A_2}$ form an $\mathfrak{sl}(2)$ subalgebra generating the bosonic SLOCC equivalence group, under which Q_A transforms as a spinor.

The supersymmetric generalization of the conventional group of local unitaries is given by $uOSp(1|2)$, a compact subgroup of $OSp(1|2)$ [25,26]. It has a supermatrix representation as the subset of $OSp(1|2)$ supermatrices satisfying the additional superunitary condition

$$M^{\ddagger}M = \mathbb{1}, \tag{66}$$

where ‡ is the superadjoint given by

$$M^{\ddagger} = (M^{st})^{\#}. \tag{67}$$

The $uOSp(1|2)$ algebra is given by

$$\mathfrak{u}\,\mathfrak{osp}(1|2) := \{X \in \mathfrak{osp}(1|2) | X^{\ddagger} = -X\}. \tag{68}$$

An arbitrary element $X \in \mathfrak{uosp}(1|2)$ may be written as

$$X = \xi_i A_i + \eta^{\#}Q_0 + \eta Q_1, \tag{69}$$

where ξ_i and η are pure even/odd Grassmann numbers, respectively, and

$$A_1 = \frac{i}{2}(P_{00} - P_{11}), \qquad A_2 = \frac{1}{2}(P_{00} + P_{11}),$$
$$A_3 = iP_{01}, \qquad Q_A^{\ddagger} = \varepsilon_{AA'}Q_{A'}, \qquad A_i^{\ddagger} = -A_i. \tag{70}$$

C. One superqubit

The one-superqubit system (Alice) is described by the state

$$\Psi = |A\rangle a_A + |\bullet\rangle a_{\bullet}, \tag{71}$$

where a_A is commuting with $A = 0, 1$ and $a_{\bullet}$ is anticommuting. That is to say, the state vector is promoted to a supervector. The super Hilbert space has dimension 3, two "bosons," and one "fermion." In more compact notation we may write

$$|\Psi\rangle = |X\rangle a_X, \tag{72}$$

where $X = (A, \bullet)$.

The super SLOCC equivalence group for a single qubit is $OSp(1|2)_A$. Under the $SL(2)_A$ subgroup a_A transforms as a **2** while $a_{\bullet}$ is a singlet as shown in Table II. The super LOCC entanglement equivalence group, i.e. the group of local unitaries, is given by $uOSp(1|2)_A$, the unitary subgroup of $OSp(1|2)_A$.

The norm squared $\langle\Psi|\Psi\rangle$ is given by

$$\langle\Psi|\Psi\rangle = \delta^{A_1A_2}a^{\#}_{A_1}a_{A_2} - a^{\#}_{\bullet}a_{\bullet}, \tag{73}$$

where $\langle\Psi| = (|\Psi\rangle)^{\ddagger}$ and $\langle\Psi|\Psi\rangle$ is the conventional inner product that is manifestly $uOSp(1|2)$ invariant. The one-superqubit state may then be normalized.

As explained in Appendix A the n-superqubit Hilbert space is defined over a 2^{n+1}-dimensional Grassmann algebra for which $z_{\mathcal{S}}^{2n+1} = 0$ for all z. So (65) terminates after a finite number of terms:

$$\langle\Psi|\Psi\rangle^{-1/2} = \sum_{k=0}^{n}\frac{1}{k!2^k}\prod_{j=0}^{k}(1-2j)\langle\Psi|\Psi\rangle_{\mathcal{B}}^{-((2k+1)/2)}\langle\Psi|\Psi\rangle_{\mathcal{S}}^{k}, \tag{74}$$

where the sum only runs to n since the bracket $\langle\Psi|\Psi\rangle_{\mathcal{S}}$ is at least quadratic in Grassmann variables. For one superqubit, with a_A pure body, this gives

$$\langle\Psi|\Psi\rangle^{-1/2} = (\delta^{A_1A_2}a^{*}_{A_1}a_{A_2})^{-1/2} + \tfrac{1}{2}(\delta^{A_1A_2}a^{*}_{A_1}a_{A_2})^{-3/2}a^{\#}_{\bullet}a_{\bullet} \tag{75}$$

TABLE II. The action of the $\mathfrak{osp}(1|2)$ generators on the superqubit fields.

Generator	Field acted upon a_{A_3}	$a_{\bullet}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\|A_3}a_{\|A_2)}$	0
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_{\bullet}$	a_{A_1}

so the normalized wave function $|\hat{\Psi}\rangle$, for which $\langle\hat{\Psi}|\hat{\Psi}\rangle = 1$, is

$$|\hat{\Psi}\rangle = |A\rangle\hat{a}_A + |\bullet\rangle\hat{a}_\bullet, \tag{76}$$

where

$$\hat{a}_A = a_A[(\delta^{A_1A_2}a^*_{A_1}a_{A_2})^{-1/2} + \tfrac{1}{2}(\delta^{A_1A_2}a^*_{A_1}a_{A_2})^{-3/2}a^\#_\bullet a_\bullet],$$
$$\hat{a}_\bullet = a_\bullet(\delta^{A_1A_2}a^*_{A_1}a_{A_2})^{-1/2}. \tag{77}$$

The one-superqubit density matrix is given by

$$\begin{aligned}\rho &:= |\Psi\rangle\langle\Psi| \\ &= (-)^{X_2}|X_1\rangle a_{X_1}a^\#_{X_2}\langle X_2| = |A_1\rangle a_{A_1}a^\#_{A_2}\langle A_2| - |A_1\rangle \\ &\quad\times a_{A_1}a^\#_\bullet\langle\bullet| + |\bullet\rangle a_\bullet a^\#_{A_2}\langle A_2| - |\bullet\rangle a_\bullet a^\#_\bullet\langle\bullet|.\end{aligned} \tag{78}$$

Alternatively, in components, we may write

$$\rho_{X_1X_2} = \langle X_1|\rho|X_2\rangle = (-)^{X_2}a_{X_1}a^\#_{X_2}. \tag{79}$$

The density matrix is self-superadjoint,

$$\begin{aligned}\rho^\ddagger_{X_1X_2} &= (\rho^{st}_{X_1X_2})^\# = (-)^{X_2+X_1X_2}\rho^\#_{X_2X_1} \\ &= (-)^{X_2+X_1X_2}(-)^{X_1}a^\#_{X_2}a^{\#\#}_{X_1} = (-)^{X_2}a_{X_1}a^\#_{X_2} = \rho_{X_1X_2}.\end{aligned} \tag{80}$$

The norm squared is then given by the supertrace

$$\begin{aligned}\mathrm{str}\,(\rho) &= (-)^{X_1}\delta^{X_1X_2}\langle X_1|\rho|X_2\rangle = \sum_X a_X a^\#_X \\ &= \sum_X(-)^X a^\#_X a_X = \langle\Psi|\Psi\rangle\end{aligned} \tag{81}$$

as one would expect.

Unnormalized pure state super density matrices satisfy $\rho^2 = \mathrm{str}(\rho)\rho$,

$$\begin{aligned}\rho^2 &= (-)^{X_2}a_{X_1}a^\#_{X_2}\delta^{X_2X_3}(-)^{X_4}a_{X_3}a^\#_{X_4} \\ &= \delta^{X_2X_3}a_{X_2}a^\#_{X_3}(-)^{X_4}a_{X_1}a^\#_{X_4} = \mathrm{str}(\rho)\rho,\end{aligned} \tag{82}$$

the appropriate supersymmetric version of the conventional pure state density matrix condition (17).

D. Two superqubits

The two-superqubit system (Alice and Bob) is described by the state

$$\Psi = |AB\rangle a_{AB} + |A\bullet\rangle a_{A\bullet} + |\bullet B\rangle a_{\bullet B} + |\bullet\bullet\rangle a_{\bullet\bullet}, \tag{83}$$

where a_{AB} is commuting, $a_{A\bullet}$ and $a_{\bullet B}$ are anticommuting, and $a_{\bullet\bullet}$ is commuting. The super Hilbert space has dimension 9: 5 bosons and 4 fermions. The super SLOCC group for two superqubits is $OSp(1|2)_A \times OSp(1|2)_B$. Under the $SL(2)_A \times SL(2)_B$ subgroup a_{AB} transforms as a $(\mathbf{2}, \mathbf{2})$, $a_{A\bullet}$ as a $(\mathbf{2}, \mathbf{1})$, $a_{\bullet B}$ as a $(\mathbf{1}, \mathbf{2})$, and $a_{\bullet\bullet}$ as a $(\mathbf{1}, \mathbf{1})$ as summarized in Table III. The coefficients may also be assembled into a $(2|1) \times (2|1)$ supermatrix

TABLE III. The action of the $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ generators on the 2-superqubit fields.

Generator	Field acted upon			
	Bosons		Fermions	
	$a_{A_3B_3}$	$a_{\bullet\bullet}$	$a_{A_3\bullet}$	$a_{\bullet B_3}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\|A_3}a_{\|A_2)B_3}$	0	$\varepsilon_{(A_1\|A_3}a_{\|A_2)\bullet}$	0
$P_{B_1B_2}$	$\varepsilon_{(B_1\|B_3}a_{A_3\|B_2)}$	0	0	$\varepsilon_{(B_1\|B_3}a_{\bullet\|B_2)}$
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_{\bullet B_3}$	$a_{A_1\bullet}$	$\varepsilon_{A_1A_3}a_{\bullet\bullet}$	$a_{A_1B_3}$
$2Q_{B_1}$	$\varepsilon_{B_1B_3}a_{A_3\bullet}$	$-a_{\bullet B_1}$	$a_{A_3B_1}$	$-\varepsilon_{B_1B_3}a_{\bullet\bullet}$

$$\langle XY|\Psi\rangle = a_{XY} = \begin{pmatrix} a_{AB} & a_{A\bullet} \\ a_{\bullet B} & a_{\bullet\bullet}\end{pmatrix}. \tag{84}$$

See Fig. 1.

The norm squared $\langle\Psi|\Psi\rangle$ is given by

$$\begin{aligned}\langle\Psi|\Psi\rangle &= (-)^{X_1+Y_1}\delta^{X_1X_2}\delta^{Y_1Y_2}a^\#_{X_1Y_1}a_{X_2Y_2} \\ &= \delta^{A_1A_2}\delta^{B_1B_2}a^\#_{A_1B_1}a_{A_2B_2} - \delta^{A_1A_2}a^\#_{A_1\bullet}a_{A_1\bullet} \\ &\quad - \delta^{B_1B_2}a^\#_{\bullet B_1}a_{\bullet B_1} + a^\#_{\bullet\bullet}a_{\bullet\bullet},\end{aligned} \tag{85}$$

where $\langle\Psi| = (|\Psi\rangle)^\ddagger$ and $\langle\Psi|\Psi\rangle$ is the conventional inner product that is manifestly $uOSp(1|2)_A \times uOSp(1|2)_B$ invariant.

The two-superqubit density matrix is given by

$$\rho = |\Psi\rangle\langle\Psi| = (-)^{X_2+Y_2}|X_1Y_1\rangle a_{X_1Y_1}a^\#_{X_2Y_2}\langle X_2Y_2|. \tag{86}$$

The reduced density matrices for Alice and Bob are given by the partial supertraces:

$$\rho_A = \sum_Y(-)^Y\langle Y|\rho|Y\rangle = \sum_Y(-)^{X_2}|X_1\rangle a_{X_1Y}a^\#_{X_2Y}\langle X_2|, \tag{87a}$$

$$\rho_B = \sum_X(-)^X\langle X|\rho|X\rangle = \sum_X(-)^{Y_2}|Y_1\rangle a_{XY_1}a^\#_{XY_2}\langle Y_2|. \tag{87b}$$

In component form the reduced density matrices are given by

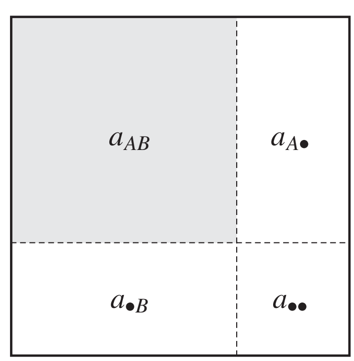

FIG. 1. The 3×3 square supermatrix.

TABLE IV. The action of the $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ generators on the 3-superqubit fields.

Generator	Bosons acted upon			
	$a_{A_3B_3C_3}$	$a_{A_3\bullet\bullet}$	$a_{\bullet B_3\bullet}$	$a_{\bullet\bullet C_3}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)B_3C_3}$	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)\bullet\bullet}$	0	0
$P_{B_1B_2}$	$\varepsilon_{(B_1\vert B_3}a_{A_3\vert B_3)C_2}$	0	$\varepsilon_{(B_1\vert B_3}a_{\bullet\vert A_2)\bullet}$	0
$P_{C_1C_2}$	$\varepsilon_{(C_1\vert C_3}a_{A_3B_3\vert C_2)}$	0	0	$\varepsilon_{(C_1\vert C_3}a_{\bullet\bullet\vert C_2)}$
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_{\bullet B_3C_3}$	$\varepsilon_{A_1A_3}a_{\bullet\bullet\bullet}$	$a_{A_1B_3\bullet}$	$a_{A_1\bullet C_3}$
$2Q_{B_1}$	$\varepsilon_{B_1B_3}a_{A_3\bullet C_3}$	$a_{A_3B_1\bullet}$	$-\varepsilon_{B_1B_3}a_{\bullet\bullet\bullet}$	$-a_{\bullet B_1C_3}$
$2Q_{C_1}$	$\varepsilon_{C_1C_3}a_{A_3B_3\bullet}$	$-a_{A_3\bullet C_1}$	$-a_{\bullet B_3C_1}$	$\varepsilon_{C_1C_3}a_{\bullet\bullet\bullet}$
	Fermions acted upon			
	$a_{A_3B_3\bullet}$	$a_{A_3\bullet C_3}$	$a_{\bullet B_3C_3}$	$a_{\bullet\bullet\bullet}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)B_3\bullet}$	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)\bullet C_3}$	0	0
$P_{B_1B_2}$	$\varepsilon_{(B_1\vert B_3}a_{A_3\vert B_3)\bullet}$	0	$\varepsilon_{(B_1\vert B_3}a_{\bullet\vert B_3)C_2}$	0
$P_{C_1C_2}$	0	$\varepsilon_{(C_1\vert C_3}a_{A_3\bullet\vert C_2)}$	$\varepsilon_{(C_1\vert C_3}a_{\bullet B_3\vert C_2)}$	0
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_{\bullet B_3\bullet}$	$\varepsilon_{A_1A_3}a_{\bullet\bullet C_3}$	$a_{A_1B_3C_3}$	$a_{A_1\bullet\bullet}$
$2Q_{B_1}$	$\varepsilon_{B_1B_3}a_{A_3\bullet\bullet}$	$a_{A_3B_1C_3}$	$-\varepsilon_{B_1B_3}a_{\bullet\bullet C_3}$	$-a_{\bullet B_1\bullet}$
$2Q_{C_1}$	$a_{A_3B_3C_1}$	$-\varepsilon_{C_1C_3}a_{A_3\bullet\bullet}$	$-\varepsilon_{C_1C_3}a_{\bullet B_3\bullet}$	$a_{\bullet\bullet C_1}$

$$(\rho_A)_{X_1X_2} = \sum_Y (-)^{X_2} a_{X_1Y} a^{\#}_{X_2Y},$$
$$(\rho_B)_{Y_1Y_2} = \sum_X (-)^{Y_2} a_{XY_1} a^{\#}_{XY_2}, \tag{88}$$

and

$$\rho_A = \mathrm{str}\rho_B = \langle\Psi|\Psi\rangle. \tag{89}$$

E. Three superqubits

The three-superqubit system (Alice, Bob, and Charlie) is described by the state

$$\begin{aligned}|\Psi\rangle = {} & |ABC\rangle a_{ABC} + |AB\bullet\rangle a_{AB\bullet} + |A\bullet C\rangle a_{A\bullet C} \\ & + |\bullet BC\rangle a_{\bullet BC} + |A\bullet\bullet\rangle a_{A\bullet\bullet} + |\bullet B\bullet\rangle a_{\bullet B\bullet} \\ & + |\bullet\bullet C\rangle a_{\bullet\bullet C} + |\bullet\bullet\bullet\rangle a_{\bullet\bullet\bullet},\end{aligned} \tag{90}$$

where a_{AB} is commuting, $a_{AB\bullet}$ $a_{A\bullet C}$ $a_{\bullet BC}$ are anticommuting, $a_{A\bullet\bullet}$ $a_{\bullet B\bullet}$ $a_{\bullet\bullet C}$ are commuting, and $a_{\bullet\bullet\bullet}$ is anticommuting. The super Hilbert space has dimension 27: 14 bosons and 13 fermions. The super SLOCC group for three superqubits is $OSp(1|2)_A \times OSp(1|2)_B \times OSp(1|2)_C$. Under the $SL(2)_A \times SL(2)_B \times SL(2)_C$ subgroup a_{ABC} transforms as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$, $a_{AB\bullet}$ as a $(\mathbf{2}, \mathbf{1}, \mathbf{1})$, $a_{A\bullet C}$ as a $(\mathbf{2}, \mathbf{1}, \mathbf{2})$, $a_{\bullet BC}$ as a $(\mathbf{1}, \mathbf{2}, \mathbf{2})$, $a_{A\bullet\bullet}$ as a $(\mathbf{2}, \mathbf{1}, \mathbf{1})$, $a_{\bullet B\bullet}$ as a $(\mathbf{1}, \mathbf{2}, \mathbf{1})$, $a_{\bullet\bullet C}$ as a $(\mathbf{1}, \mathbf{1}, \mathbf{2})$, and $a_{\bullet\bullet\bullet}$ as a $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ as summarized in Table IV. The coefficients may also be assembled into a $(2|1) \times (2|1) \times (2|1)$ superhypermatrix

$$\langle XYZ|\Psi\rangle = a_{XYZ}. \tag{91}$$

See Fig. 2.

The norm squared $\langle\Psi|\Psi\rangle$ is given by

$$\begin{aligned}\langle\Psi|\Psi\rangle = {} & (-)^{X_1+Y_1+Z_1}\delta^{X_1X_2}\delta^{Y_1Y_2}\delta^{Z_1Z_2}a^{\#}_{X_1Y_1Z_1}a_{X_2Y_2Z_2} \\ = {} & \delta^{A_1A_2}\delta^{B_1B_2}\delta^{C_1C_2}a^{\#}_{A_1B_1C_1}a_{A_2B_2C_2} - \delta^{A_1A_2}\delta^{B_1B_2}a^{\#}_{A_1B_1\bullet}a_{A_2B_2\bullet} - \delta^{A_1A_2}\delta^{C_1C_2}a^{\#}_{A_1\bullet C_1}a_{A_2\bullet C_2} - \delta^{B_1B_2}\delta^{C_1C_2}a^{\#}_{\bullet B_1C_1}a_{\bullet B_2C_2} \\ & + \delta^{A_1A_2}a^{\#}_{A_1\bullet\bullet}a_{A_2\bullet\bullet} + \delta^{B_1B_2}a^{\#}_{\bullet B_1\bullet}a_{\bullet B_2\bullet} + \delta^{C_1C_2}a^{\#}_{\bullet\bullet C_1}a_{\bullet\bullet C_2} - a^{\#}_{\bullet\bullet\bullet}a_{\bullet\bullet\bullet},\end{aligned} \tag{92}$$

where $\langle\Psi| = (|\Psi\rangle)^{\ddagger}$ and $\langle\Psi|\Psi\rangle$ is the conventional inner product that is manifestly $uOSp(1|2)_A \times uOSp(1|2)_B \times uOSp(1|2)_C$ invariant.

The three-superqubit density matrix is given by

$$\begin{aligned}\rho &= |\Psi\rangle\langle\Psi| \\ &= (-)^{X_2+Y_2+Z_2}|X_1Y_1Z_1\rangle a_{X_1Y_1Z_1}a^{\#}_{X_2Y_2Z_2}\langle X_2Y_2Z_2|.\end{aligned} \tag{93}$$

The singly reduced density matrices are defined using the partial supertraces

$$\begin{aligned}\rho_{AB} &= \sum_Z (-)^Z\langle Z|\rho|Z\rangle, \\ \rho_{BC} &= \sum_X (-)^X\langle X|\rho|X\rangle, \\ \rho_{CA} &= \sum_Y (-)^Y\langle Y|\rho|Y\rangle,\end{aligned} \tag{94}$$

or

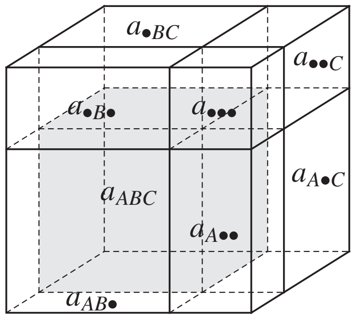

FIG. 2. The $3 \times 3 \times 3$ cubic superhypermatrix.

$$\begin{aligned} \rho_{AB} &= \sum_{Z} (-)^{X_2+Y_2} |X_1 Y_1\rangle a_{X_1 Y_1 Z} a^{\#}_{X_2 Y_2 Z} \langle X_2 Y_2|, \\ \rho_{BC} &= \sum_{X} (-)^{Y_2+Z_2} |Y_1 Z_1\rangle a_{X Y_1 Z_1} a^{\#}_{X Y_2 Z_2} \langle Y_2 Z_2|, \\ \rho_{CA} &= \sum_{Y} (-)^{X_2+Z_2} |X_1 Z_1\rangle a_{X_1 Y Z_1} a^{\#}_{X_2 Y Z_2} \langle X_2 Z_2|. \end{aligned} \tag{95}$$

The doubly reduced density matrices for Alice, Bob, and Charlie are given by the partial supertraces

$$\begin{aligned} \rho_A &= \sum_{Y,Z} (-)^{Y+Z} \langle YZ|\rho|YZ\rangle, \\ \rho_B &= \sum_{X,Z} (-)^{X+Z} \langle XZ|\rho|XZ\rangle, \\ \rho_C &= \sum_{X,Y} (-)^{X+Y} \langle XY|\rho|XY\rangle, \end{aligned} \tag{96}$$

or

$$\begin{aligned} \rho_A &= \sum_{Y,Z} (-)^{X_2} |X_1\rangle a_{X_1 Y Z} a^{\#}_{X_2 Y Z} \langle X_2|, \\ \rho_B &= \sum_{X,Z} (-)^{Y_2} |Y_1\rangle a_{X Y_1 Z} a^{\#}_{X Y_2 Z} \langle Y_2|, \\ \rho_C &= \sum_{X,Y} (-)^{Z_2} |Z_1\rangle a_{X Y Z_1} a^{\#}_{X Y Z_2} \langle Z_2|. \end{aligned} \tag{97}$$

V. SUPER ENTANGLEMENT

A. Two superqubits

In seeking a supersymmetric generalization of the 2-tangle (30) one might be tempted to replace the determinant of a_{AB} by the Berezinian of a_{XY}

$$\mathrm{Ber}\, a_{XY} = \det(a_{AB} - a_{A\bullet} a_{\bullet\bullet}^{-1} a_{\bullet B}) a_{\bullet\bullet}^{-1}. \tag{98}$$

See Appendix A. However, although the Berezinian is the natural supersymmetric extension of the determinant, it is not defined for vanishing $a_{\bullet\bullet}$, making it unsuitable as an entanglement measure.

A better candidate follows from writing

$$\det a_{AB} = \tfrac{1}{2} a^{AB} a_{AB} = \tfrac{1}{2} \mathrm{tr}(a^t \varepsilon a \varepsilon^t) = \tfrac{1}{2} \mathrm{tr}[(a\varepsilon)^t \varepsilon a]. \tag{99}$$

This expression may be generalized by a straightforward promotion of the trace and transpose to the supertrace and supertranspose and replacing the $SL(2)$ invariant tensor ε with the $OSp(1|2)$ invariant tensor E. See Appendix A. This yields a quadratic polynomial, which we refer to as the superdeterminant, denoted sdet:

$$\begin{aligned} \mathrm{sdet} a_{XY} &= \tfrac{1}{2} \mathrm{str}[(aE)^{\mathrm{st}} E a] \\ &= \tfrac{1}{2}(a^{AB} a_{AB} - a^{A\bullet} a_{A\bullet} - a^{\bullet B} a_{\bullet B} - a^{\bullet\bullet} a_{\bullet\bullet}) \\ &= (a_{00} a_{11} - a_{01} a_{10} + a_{0\bullet} a_{1\bullet} + a_{\bullet 0} a_{\bullet 1}) - \tfrac{1}{2} a_{\bullet\bullet}{}^2, \end{aligned} \tag{100}$$

which is clearly not equal to the Berezinian, but is nevertheless supersymmetric since Q_A annihilates $a^{AB} a_{AB} - a^{\bullet B} a_{\bullet B}$ and $a^{A\bullet} a_{A\bullet} + a^{\bullet\bullet} a_{\bullet\bullet}$, while Q_B annihilates $a^{AB} a_{AB} - a^{A\bullet} a_{A\bullet}$ and $a^{\bullet B} a_{\bullet B} + a^{\bullet\bullet} a_{\bullet\bullet}$. Satisfyingly, (100) reduces to $\det a_{AB}$ when $a_{A\bullet}$, $a_{\bullet B}$, and $a_{\bullet\bullet}$ are set to zero. We then define the super 2-tangle as

$$\tau_{XY} = 4 \mathrm{sdet} a_{XY} (\mathrm{sdet} a_{XY})^{\#}. \tag{101}$$

In summary, 2-superqubit entanglement seems to have the same two entanglement classes as 2-qubits with the invariant $\det a_{AB}$ replaced by its supersymmetric counterpart $\mathrm{sdet} a_{XY}$.

Nonsuperentangled states are given by product states for which $a_{AB} = a_A b_B$, $a_{A\bullet} = a_A b_\bullet$, $a_{\bullet B} = a_\bullet b_B$, $a_{\bullet\bullet} = a_\bullet b_\bullet$, and $\mathrm{sdet} a_{XY}$ vanishes. This provides a nontrivial consistency check.

An example of a normalized physical superentangled state is given by

$$|\Psi\rangle = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + i|\bullet\bullet\rangle) \tag{102}$$

for which

$$\mathrm{sdet}\, a_{XY} = \tfrac{1}{3} + \tfrac{1}{2} \cdot \tfrac{1}{3} = \tfrac{1}{2} \tag{103}$$

and

$$\tau_{XY} = 4\, \mathrm{sdet} a_{XY} (\mathrm{sdet} a_{XY})^{\#} = 1. \tag{104}$$

So this state is not only entangled but maximally entangled, just like the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \tag{105}$$

for which $\mathrm{sdet} a_{XY} = 1/2$ and $\tau_{XY} = 1$. Another more curious example is

$$|\Psi\rangle = i|\bullet\bullet\rangle, \tag{106}$$

which is not a product state since $a_{\bullet\bullet}$ is pure body and hence could never be formed by the product of two odd Grassmann numbers. In fact, $\mathrm{sdet} a_{XY} = 1/2$ and $\tau_{XY} = 1$, so this state is also maximally entangled.

We may interpolate between these two examples with the normalized state

$$(|\alpha|^2 + |\beta|^2)^{-1/2}[\alpha|\Psi\rangle_{\rm Bell} + \beta|\bullet\bullet\rangle], \qquad (107)$$

where $\alpha, \beta \in \mathbb{C}$, for which we have

$$\text{sdet}\, a_{XY} = \frac{1}{2}\frac{\alpha^2 - \beta^2}{|\alpha|^2 + |\beta|^2}, \qquad \tau_{XY} = \frac{|\alpha^2 - \beta^2|^2}{(|\alpha|^2 + |\beta|^2)^2}. \qquad (108)$$

The entanglement for this state is displayed as a function of the complex parameter β in Fig. 3 for the case $\alpha = 1$. Note, in particular, that while the entanglement is maximized for arbitrary pure imaginary β, it has its minimum value on the real axis at $\beta = \pm 1$ as shown in Fig. 4.

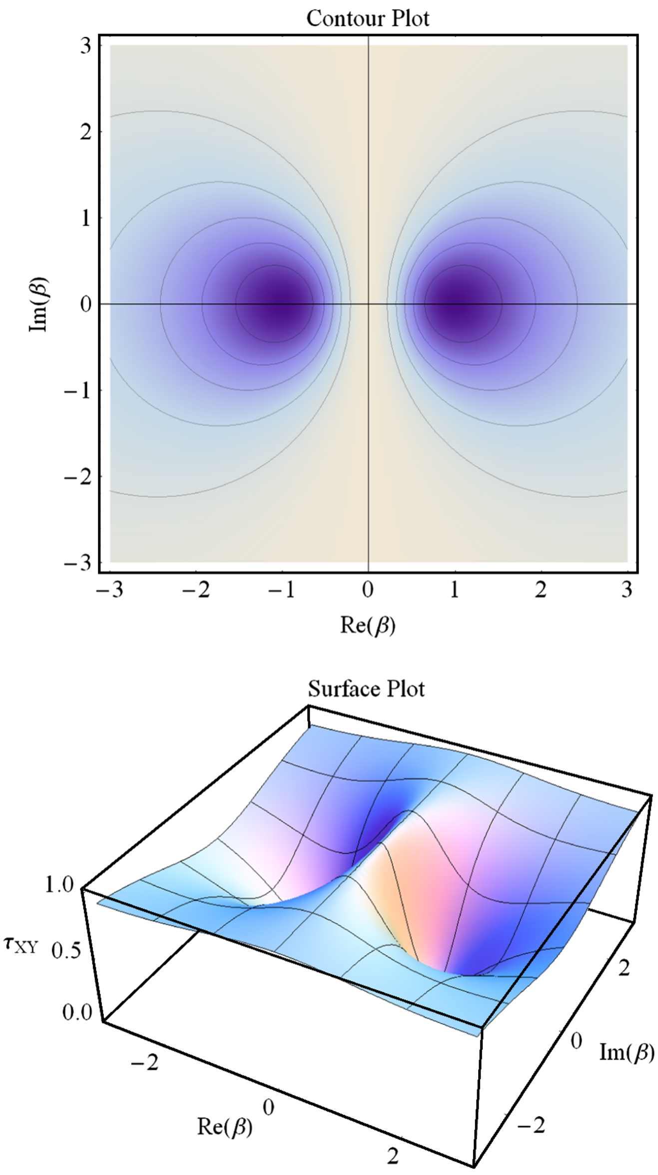

FIG. 3 (color online). The 2-tangle τ_{XY} for the state (107) for a complex parameter β.

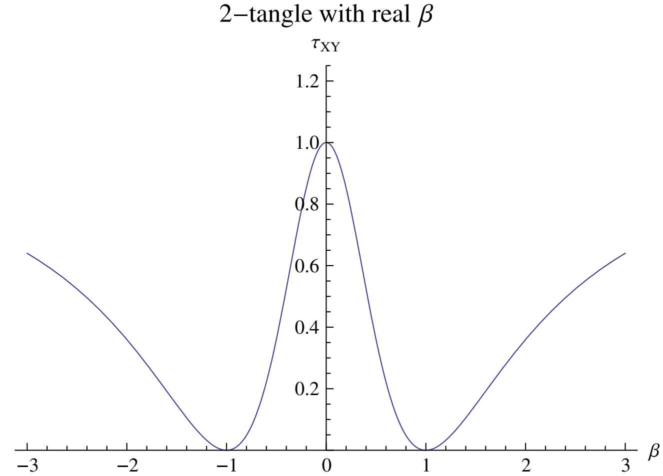

FIG. 4 (color online). The 2-tangle τ_{XY} for the state (107) for a real parameter β.

B. Three superqubits

In seeking to generalize the 3-tangle (39), invariant under $[SL(2)]^3$, to a supersymmetric object, invariant under $[OSp(1|2)]^3$, we need to find a quartic polynomial that reduces to Cayley's hyperdeterminant when $a_{AB\bullet}$, $a_{A\bullet C}$, $a_{\bullet BC}$, $a_{A\bullet\bullet}$, $a_{\bullet B\bullet}$, $a_{\bullet\bullet C}$, and $a_{\bullet\bullet\bullet}$ are set to zero. We do this by generalizing the γ matrices:

$$\gamma A_1A_2 := a_{A_1}{}^{BC}a_{A_2BC} - a_{A_1}{}^{B\bullet}a_{A_2B\bullet} - a_{A_1}{}^{\bullet C}a_{A_2\bullet C} - a_{A_1}{}^{\bullet\bullet}a_{A_2\bullet\bullet}, \qquad (109a)$$

$$\gamma_{A_1\bullet} := a_{A_1}{}^{BC}a_{\bullet BC} + a_{A_1}{}^{B\bullet}a_{\bullet B\bullet} + a_{A_1}{}^{\bullet C}a_{\bullet\bullet C} - a_{A_1}{}^{\bullet\bullet}a_{\bullet\bullet\bullet}, \qquad (109b)$$

$$\gamma_{\bullet A_2} := a_{\bullet}{}^{BC}a_{A_2BC} - a_{\bullet}{}^{B\bullet}a_{A_2B\bullet} - a_{\bullet}{}^{\bullet C}a_{A_2\bullet C} - a_{\bullet}{}^{\bullet\bullet}a_{A_2\bullet\bullet}, \qquad (109c)$$

together with their B and C counterparts; notice that the building blocks with two indices are bosonic and those with one index are fermionic. The final bosonic possibility, $\gamma_{(\bullet\bullet)}$, vanishes identically. The simple supersymmetry relations are given by

$$Q_{A_1}\gamma_{A_2A_3} = \varepsilon_{A_1(A_2}\gamma_{A_3)\bullet}, \qquad Q_{A_1}\gamma_{A_2\bullet} = \tfrac{1}{2}\gamma_{A_1A_2},$$
$$Q_B\gamma_{A_1A_2} = 0 = Q_C\gamma_{A_1A_2}, \qquad Q_B\gamma_{A\bullet} = 0 = Q_C\gamma_{A\bullet}. \qquad (110)$$

Using these expressions we define the superhyperdeterminant, denoted sDet$a$:

$$\text{sDet}\, a_{XYZ} = \tfrac{1}{2}(\gamma^{A_1A_2}\gamma_{A_1A_2} - \gamma^{A\bullet}\gamma_{A\bullet} - \gamma^{\bullet A}\gamma_{\bullet A}), \quad (111)$$

which is invariant under the action of the superalgebra. The corresponding expressions singling out superqubits B and C are also invariant and equal to (111). sDeta_{XYZ} can be seen as the definition of the super-Cayley determinant of the cubic superhypermatrix given in Fig. 2.

Writing

$$\Gamma^A := \begin{pmatrix} \gamma_{A_1A_2} & \gamma_{A_1\bullet} \\ \gamma_{\bullet A_2} & \gamma_{\bullet\bullet} \end{pmatrix} = \begin{pmatrix} \gamma_{A_1A_2} & \gamma_{A_1\bullet} \\ \gamma_{A_2\bullet} & 0 \end{pmatrix}, \tag{112}$$

we obtain an invariant analogous to (100)

$$\mathrm{sDet}\, a_{XYZ} = \tfrac{1}{2}\mathrm{str}[(\Gamma^A E)^{\mathrm{st}} E \Gamma^A] \tag{113}$$

so that

$$\mathrm{sDet}\, a_{XYZ} = -\mathrm{sdet}\Gamma^A \tag{114}$$

in analogy to the conventional three-qubit identity (38). This result for sDet agrees with that of [7].

Finally, using Γ^A we are able to define the supersymmetric generalization T_{XYZ} of the 3-qubit tensor T_{ABC} as defined in (37),

$$T_{XYZ} = \Gamma^A_{XX'} a^{X'}{}_{YZ}. \tag{115}$$

It is not difficult to verify that T_{XYZ} transforms in precisely the same way as a_{XYZ} (as given in Table IV) under $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$. The superhyperdeterminant may then also be written as

$$\begin{aligned} \mathrm{sDet} a_{XYZ} = {} & T_{ABC}a^{ABC} + T_{\bullet BC}a^{\bullet BC} - T_{A\bullet C}a^{A\bullet C} \\ & - T_{AB\bullet}a^{AB\bullet} - T_{A\bullet\bullet}a^{A\bullet\bullet} + T_{\bullet B\bullet}a^{\bullet B\bullet} \\ & + T_{\bullet\bullet C}a^{\bullet\bullet C} - T_{\bullet\bullet\bullet}a^{\bullet\bullet\bullet}. \end{aligned} \tag{116}$$

In this sense $\mathrm{sDet} a_{XYZ}$, $(\Gamma^A)_{X_1X_2}$, and T_{XYZ} are the natural supersymmetric generalizations of the hyperdeterminant, $\mathrm{Det} a_{ABC}$, and the covariant tensors, $(\gamma^A)_{A_1A_2}$ and T_{ABC}, of the conventional 3-qubit treatment summarized in III B. Finally we are in a position to define the super 3-tangle:

$$\tau_{XYZ} = 4\sqrt{\mathrm{sDet} a_{XYZ}(\mathrm{sDet} a_{XYZ})^{\#}}. \tag{117}$$

In summary, 3-superqubit entanglement seems to have the same five entanglement classes as that of 3-qubits shown in Table I, with the covariants a_{ABC}, γ^A, γ^B, γ^C, T_{ABC}, and $\mathrm{Det} a_{ABC}$ replaced by their supersymmetric counterparts a_{XYZ}, Γ^A, Γ^B, Γ^C, T_{XYZ}, and $\mathrm{sDet} a_{ABC}$.

Completely separable nonsuperentangled states are given by product states for which $a_{ABC} = a_A b_B c_C$, $a_{AB\bullet} = a_A b_B c_\bullet$, $a_{A\bullet C} = a_A b_\bullet c_C$, $a_{\bullet BC} = a_\bullet b_B c_C$, $a_{A\bullet\bullet} = a_A b_\bullet c_\bullet$, $a_{\bullet B\bullet} = a_\bullet b_B c_\bullet$, $a_{\bullet\bullet C} = a_\bullet b_\bullet c_C$, $a_{\bullet\bullet\bullet} = a_\bullet b_\bullet c_\bullet$, and $\mathrm{sDet} a_{XYZ}$ vanishes. This provides a nontrivial consistency check.

An example of a normalized physical biseparable state is provided by

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |011\rangle + |0\bullet\bullet\rangle) \tag{118}$$

for which

$$(\Gamma^A)_{00} = \tfrac{1}{3} \tag{119}$$

and Γ^B, Γ^C, T_{XYZ}, and $\mathrm{sDet} a_{XYZ}$ vanish. More generally, one can consider the combination

$$\begin{aligned} |\Psi\rangle = {} & (|\alpha|^2 + |\beta|^2)^{-1/2} \\ & \times \left[\frac{1}{\sqrt{2}}\alpha(|000\rangle + |011\rangle) + \beta|0\bullet\bullet\rangle\right] \end{aligned} \tag{120}$$

for which

$$(\Gamma^A)_{00} = \frac{\alpha^2 - \beta^2}{|\alpha|^2 + |\beta|^2} \tag{121}$$

and the other covariants vanish.

An example of a normalized physical W state is provided by

$$\begin{aligned} |\Psi\rangle = {} & \frac{1}{\sqrt{6}}(|110\rangle + |101\rangle + |011\rangle + |\bullet\bullet 1\rangle + |\bullet 1\bullet\rangle \\ & + |1\bullet\bullet\rangle) \end{aligned} \tag{122}$$

for which

$$(\Gamma^A)_{11} = (\Gamma^B)_{11} = (\Gamma^C)_{11} = -\tfrac{1}{2} \tag{123}$$

and

$$T_{111} = \frac{1}{2\sqrt{6}} \tag{124}$$

while $\mathrm{sDet} a_{XYZ}$ vanishes. One could also consider

$$\begin{aligned} |\Psi\rangle = {} & \frac{1}{\sqrt{3}}(|\alpha|^2 + |\beta|^2)^{-1/2}[\alpha(|110\rangle + |101\rangle + |011\rangle) \\ & + \beta(|\bullet\bullet 1\rangle + |\bullet 1\bullet\rangle + |1\bullet\bullet\rangle)] \end{aligned} \tag{125}$$

for which

$$(\Gamma^A)_{11} = (\Gamma^B)_{11} = (\Gamma^C)_{11} = -\frac{2\alpha^2 + \beta^2}{3(|\alpha|^2 + |\beta|^2)} \tag{126}$$

and

$$T_{111} = \frac{\alpha(2\alpha^2 + \beta^2)}{3\sqrt{3}(|\alpha|^2 + |\beta|^2)^{3/2}} \tag{127}$$

while the other T components and $\mathrm{sDet} a_{XYZ}$ vanish.

An example of a normalized physical superentangled state is provided by

$$\begin{aligned} |\Psi\rangle = {} & \frac{1}{\sqrt{8}}(|000\rangle + |\bullet\bullet 0\rangle + |\bullet 0\bullet\rangle + |0\bullet\bullet\rangle + |111\rangle \\ & + |\bullet\bullet 1\rangle + |\bullet 1\bullet\rangle + |1\bullet\bullet\rangle) \end{aligned} \tag{128}$$

for which

$$\mathrm{sDet}\, a_{XYZ} = \frac{1}{64} \tag{129}$$

and

$$\tau_{XYZ} = 4\sqrt{\text{sDet}a_{XYZ}(\text{sDet}a_{XYZ})^{\#}} = \frac{1}{16}. \tag{130}$$

VI. CONCLUSION

In this paper we have taken the first steps toward generalizing quantum information theory to super quantum information theory. We introduced the superqubit defined over an appropriate super Hilbert space. We acknowledge that there are still important issues to address, notably how to interpret "physical" states with nonvanishing soul for which probabilities are no longer real numbers but elements of a Grassman algebra. (The sum of the probabilities still add up to one, however.) The examples of V avoided this problem, being pure body. DeWitt advocates retaining only such pure body states in the Hilbert space [23], but this may be too draconian. See [27] for an alternative approach.

Nevertheless, for the SLOCC equivalence group $[SL(2,\mathbb{C})]^n$ and the LOCC equivalence group $[SU(2)]^n$, we presented their minimal supersymmetric extensions, $[OSp(1|2)]^n$ and $[uOSp(1|2)]^n$, respectively, and showed explicitly how superqubits would transform under these groups for $n = 1, 2, 3$. Furthermore, we found supersymmetric invariants that are the obvious candidates for supersymmetric entanglement measures for $n = 2, 3$. We hope in future work to classify fully the 2 and 3 superqubit entanglement classes and their corresponding orbits as was done for the 2 and 3 qubit entanglement classes in [2,4,6].

As noted in the Introduction, a physical realization of our superqubit is more likely to be found in condensed-matter physics than high-energy physics. While the polarizations of a photon or the spins of an electron provide examples of a qubit, the inclusions of photinos or selectrons do not obviously provide examples of a superqubit, since the supersymmetrization of the (S)LOCC equivalence groups is distinct from the supersymmetrization of the spacetime Poincaré group.

We would also like to point out that this work is part of the ongoing correspondence between ideas in string and M-theory and ideas in quantum information theory. See [28] for a review. This paper continues the trend of using mathematical tools from one side to describe phenomena on the other.

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Note added.—A very interesting paper has recently appeared [32], which analyzes quantum computing with superqubits.

APPENDIX A: SUPERLINEAR ALGEBRA

Grassmann numbers are the 2^n-dimensional vectors populating the Grassmann algebra Λ_n, which is generated by n mutually anticommuting elements $\{\theta^i\}_{i=1}^n$.

Any Grassmann number z may be decomposed into "body" $z_{\mathcal{B}} \in \mathbb{C}$ and "soul" $z_{\mathcal{S}}$ viz.

$$z = z_{\mathcal{B}} + z_{\mathcal{S}}, \qquad z_{\mathcal{S}} = \sum_{k=1}^{\infty} \frac{1}{k!} c_{a_1\cdots a_k}\theta^{a_1}\cdots\theta^{a_k}, \tag{A1}$$

where $c_{a_1\cdots a_k} \in \mathbb{C}$ are totally antisymmetric. For finite dimension n the sum terminates at $k = 2^n$ and the soul is nilpotent $z_{\mathcal{S}}^{n+1} = 0$.

One may also decompose z into even and odd parts u and v

$$\begin{aligned} u &= z_{\mathcal{B}} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} c_{a_1\cdots a_{2k}}\theta^{a_1}\cdots\theta^{a_{2k}},\\ v &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} c_{a_1\cdots a_{2k+1}}\theta^{a_1}\cdots\theta^{a_{2k+1}}, \end{aligned} \tag{A2}$$

which may also be expressed as the direct sum decomposition $\Lambda_n = \Lambda_n^0 \oplus \Lambda_n^1$. Furthermore, analytic functions f of Grassmann numbers are defined via

$$f(z) := \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(z_{\mathcal{B}}) z_{\mathcal{S}}^k, \tag{A3}$$

where $f^{(k)}(z_{\mathcal{B}})$ is the kth derivative of f evaluated at $z_{\mathcal{B}}$ and is well defined if f is nonsingular at $z_{\mathcal{B}}$ [23].

One defines the *grade of a Grassmann number* as

$$\deg x := \begin{cases} 0 & x \in \Lambda_n^0 \\ 1 & x \in \Lambda_n^1, \end{cases} \tag{A4}$$

where the grades 0 and 1 are referred to as even and odd, respectively.

Define the star * and superstar $^\#$ operators [25,26,29] satisfying the following properties:

$$\begin{aligned} &(\Lambda_n^0)^\star = \Lambda_n^0, \qquad (\Lambda_n^1)^\star = \Lambda_n^1,\\ &(\Lambda_n^0)^\# = \Lambda_n^0, \qquad (\Lambda_n^1)^\# = \Lambda_n^1,\\ &(x\theta_i)^\star = x^*\theta_i^\star, \qquad \theta_i^{\star\star} = \theta_i, \qquad (\theta_i\theta_j)^\star = \theta_j^\star\theta_i^\star,\\ &(x\theta_i)^\# = x^*\theta_i^\#, \qquad \theta_i^{\#\#} = -\theta_i, \qquad (\theta_i\theta_j)^\# = \theta_i^\#\theta_j^\#, \end{aligned} \tag{A5}$$

where $x \in \mathbb{C}$ and * is ordinary complex conjugation, which means

$$\alpha^{\star\star} = \alpha, \qquad \alpha^{\#\#} = (-)^{\deg\alpha}\alpha \tag{A6}$$

for pure even/odd Grassmann α. The impure case follows by linearity.

Following [23] one may, if so desired, take the formal limit $n \to \infty$ defining the infinite dimensional vector space Λ_∞. Elements of Λ_∞ are called *supernumbers*. Our results are independent of the dimension of the underlying Grassmann algebra and one can use supernumbers throughout, but for the sake of simplicity we restrict to finite dimensional algebra by assigning just one Grassmann generator θ and its superconjugate $\theta^\#$ to every superqubit.

The grade definition applies to the components $T_{X_1\cdots X_k}$ of any k-index array of Grassmann numbers T, but one may also define $\deg X_i$, the *grade of an index*, for such an array by specifying a characteristic function from the range of the index X_i to the set $\{0, 1\}$. In general the indices can have different ranges and the characteristic functions can be arbitrary for each index. It is then possible to define $\deg T$, the *grade of an array*, as long as the compatibility condition

$$\deg T \equiv \deg(T_{X_1\cdots X_k}) + \sum_{i=1}^{k} \deg X_i \bmod 2 \;\; \forall \;\; X_i \tag{A7}$$

is satisfied. In precisely such cases the entries of T satisfy

$$\begin{aligned} \deg(T_{X_1\cdots X_k}) &= \deg T + \sum_{i=1}^{k} \deg X_i \bmod 2, \\ &\Rightarrow \deg T = \deg(T_{\underbrace{1\cdots 1}_{k}}), \\ \deg(T_1 T_2) &= \deg T_1 + \deg T_2 \bmod 2, \end{aligned} \tag{A8}$$

so that in other words T is partitioned into blocks with definite grade such that the nearest neighbors of any block are of the opposite grade to that block. The array grade simply distinguishes the two distinct ways of accomplishing such a partition (i.e. the two possible grades of the first element $T_{1\cdots 1}$). Grassmann numbers and the Grassmann number grade may be viewed as special cases of arrays and the array grade.

Special care must be taken not to confuse this notion of array grade with whether the array entries at even/odd index positions vanish. An array T may be decomposed as

$$T = T_E + T_O, \tag{A9}$$

where the pure even part T_E is obtained from T by setting to zero all entries satisfying $\deg(T_{X_1\cdots X_k}) = 1$, and similarly *mutatis mutandis* for T_O. The property of being pure even or pure odd is therefore independent of the array grade as defined above.

The various grades commonly appear in formulas as powers of -1 and the shorthand

$$(-)^X := (-1)^{\deg X} \tag{A10}$$

is often used. The indices of superarrays may be supersymmetrized as follows:

$$\begin{aligned} &T_{X_1\cdots[[X_i|\cdots|X_j]]\cdots X_k} \\ &:= \tfrac{1}{2}[T_{X_1\cdots X_i\cdots X_j\cdots X_k} + (-)^{X_i X_j} T_{X_1\cdots X_j\cdots X_i\cdots X_k}]. \end{aligned} \tag{A11}$$

While we require these definitions for some of our considerations, one typically only uses arrays with 0, 1, or 2 indices where the characteristic functions are monotonic: supernumbers, supervectors, and supermatrices, respectively. Functions of grades extend to mixed superarrays (with nonzero even *and* odd parts) by linearity.

A $(p|q) \times (r|s)$ supermatrix is just an $(p + q) \times (r + s)$-dimensional block partitioned matrix

$$M = \begin{matrix} & r & s \\ p & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$}} \\ q & & \end{matrix}, \tag{A12}$$

where entries in the A and D blocks are grade $\deg M$, and those in the B and C blocks are grade $\deg M + 1 \bmod 2$. The special cases $s = 0$ or $q = 0$ can be permitted to make the definition encapsulate row and column supervectors. Supermatrix multiplication is defined as for ordinary matrices; however, the trace, transpose, adjoint, and determinant have distinct super versions [25,30].

The supertrace str M of a supermatrix is M defined as

$$\mathrm{str}\, M := \sum_X (-)^{(X+M)X} M_{XX} \tag{A13}$$

and is linear, cyclic modulo sign, and insensitive to the supertranspose

$$\begin{aligned} \mathrm{str}(M + N) &= \mathrm{str}(M) + \mathrm{str}(N), \\ \mathrm{str}(MN) = (-)^{MN}\mathrm{str}(NM), &\qquad \mathrm{str} M^{\mathrm{st}} = \mathrm{str} M. \end{aligned} \tag{A14}$$

The supertranspose M^{st} of a supermatrix M is defined componentwise as

$$M^{\mathrm{st}}{}_{X_1X_2} := (-)^{(X_2+M)(X_1+X_2)} M_{X_2X_1}. \tag{A15}$$

Unlike the transpose the supertranspose is not idempotent; instead,

$$\begin{aligned} M^{\mathrm{st\,st}}{}_{X_1X_2} &= (-)^{(X_1+X_2)} M_{X_1X_2}, \\ M^{\mathrm{st\,st\,st}}{}_{X_1X_2} &= (-)^{(X_1+M)(X_1+X_2)} M_{X_2X_1}, \\ M^{\mathrm{st\,st\,st\,st}}{}_{X_1X_2} &= M_{X_1X_2}, \end{aligned} \tag{A16}$$

so that it is of order 4. The supertranspose also satisfies

$$(MN)^{\mathrm{st}} = (-)^{MN} N^{\mathrm{st}} M^{\mathrm{st}}. \tag{A17}$$

The adjoint † and superadjoint ‡ of a supermatrix are defined as

$$M^\dagger := M^{\star t}, \qquad M^\ddagger := M^{\#\mathrm{st}}, \tag{A18}$$

and satisfy

$$\begin{aligned} M^{\dagger\dagger} = M, &\qquad M^{\ddagger\ddagger} = (-)^M M, \\ (MN)^\dagger = N^\dagger M^\dagger, &\qquad (MN)^\ddagger = (-)^{MN} N^\ddagger M^\ddagger. \end{aligned} \tag{A19}$$

The preservation of anti-super-Hermiticity, $M^{\ddagger} = -M$, under scalar multiplication by Grassmann numbers, as required for the proper definition of $\mathfrak{uosp}(1|2)$ [31], necessitates the left/right multiplication rules:

$$\begin{aligned}(\alpha M)_{X_1X_2} &= (-)^{X_1\alpha}\alpha M_{X_1X_2},\\ (M\alpha)_{X_1X_2} &= (-)^{X_2\alpha}M_{X_1X_2}\alpha.\end{aligned} \tag{A20}$$

The Berezinian is defined as

$$\begin{aligned}\mathrm{Ber}\,M &:= \det(A - BD^{-1}C)/\det(D)\\ &= \det(A)/\det(D - CA^{-1}B)\end{aligned} \tag{A21}$$

and is multiplicative, insensitive to the supertranspose, and generalizes the relationship between trace and determinant

$$\begin{aligned}\mathrm{Ber}(MN) &= \mathrm{Ber}(M)\mathrm{Ber}(N),\\ \mathrm{Ber}M^{\mathrm{st}} &= \mathrm{Ber}M,\\ \mathrm{Ber}e^{M} &= e^{\mathrm{str}\,M}.\end{aligned} \tag{A22}$$

The direct sum and super tensor product are unchanged from their ordinary versions. As such, the dimension of the tensor product of two superqubits is given by

$$(2|1) \otimes (2|1) = (2|1|2|3|1), \tag{A23}$$

while the threefold product is

$$(2|1)^{\otimes 3} = (2|1|2|3|3|1|2|3|1|2|1|2|3|1), \tag{A24}$$

with similar results holding for the associated density matrices. In analogy with the ordinary case we have

$$\begin{aligned}(M\otimes N)^{t} &= M^{t}\otimes N^{t},\\ (M\otimes N)^{\mathrm{st}} &= M^{\mathrm{st}}\otimes N^{\mathrm{st}},\\ \mathrm{str}(M\otimes N) &= \mathrm{str}\,M\mathrm{str}\,N.\end{aligned} \tag{A25}$$

These definitions are manifestly compatible with Hermiticity and super-Hermiticity.

Denoting the total number of bosonic elements in the product of n superqubits by B_n, and similarly the total number of fermionic elements by F_n, we know that B_n (F_n) is given by the total number of basis kets with an even (odd) number of $\bullet$'s:

$$\begin{aligned}B_n &= \binom{n}{0}2^{n} + \binom{n}{2}2^{n-2} + \cdots = \frac{3^n+1}{2},\\ F_n &= \binom{n}{1}2^{n-1} + \binom{n}{3}2^{n-3} + \cdots = \frac{3^n-1}{2},\end{aligned} \tag{A26}$$

so that, in particular, $B_n - F_n = 1$: the number of bosonic elements is always one more than the number of fermionic ones.

In supermatrix representations of superalgebras, one may represent the superbracket of generators M and N as

$$[[M, N]] := MN - N_E M - N_O(M_E - M_O). \tag{A27}$$

One may also consider supermatrices M and N whose components are themselves supermatrices. Provided the component supermatrices are pure even (odd) at even (odd) index positions (e.g. M_{11} is a pure even supermatrix for even M), one may write the superbracket of such supermatrices as

$$[[M_{X_1X_2}, N_{X_3X_4}]] = M_{X_1X_2}N_{X_3X_4} - (-)^{(X_1+X_2)(X_3+X_4)}N_{X_3X_4}M_{X_1X_2}, \tag{A28}$$

where the final two indices are suppressed. This grouping of supermatrices into supermatrices is useful for summarizing the superbrackets of superalgebras.

APPENDIX B: ORTHOSYMPLECTIC SUPERALGEBRAS

Supermatrix representations of the orthosymplectic supergroup $OSp(p|2q)$ consist of supermatrices $M \in GL(p|2q)$ satisfying

$$M^{\mathrm{st}}EM = E, \tag{B1}$$

but for convenience we choose instead to use supermatrices $M \in GL(2q|p)$ satisfying (B1). In this convention, the invariant supermatrix E is defined by

$$E := \begin{pmatrix}\mathbb{J}_{2q} & 0\\ 0 & \mathbb{1}_p\end{pmatrix}, \qquad \mathbb{J}_{2q} := \begin{pmatrix}0 & \mathbb{1}_p\\ -\mathbb{1}_p & 0\end{pmatrix}. \tag{B2}$$

Definitions of supermatrices, the supertranspose, and further details of superlinear algebra may be found in Appendix A.

Writing a generic supermatrix $\mathfrak{M}$ of the super Lie algebra $\mathfrak{osp}(p|2q)$ as

$$\mathfrak{M} = \begin{pmatrix}A & B\\ C & D\end{pmatrix} \tag{B3}$$

permits (B1) to be rewritten as the following conditions on the blocks of the algebra supermatrices:

$$A^{t}\mathbb{J} = -\mathbb{J}A, \qquad C = B^{t}\mathbb{J}, \qquad D^{t} = -D. \tag{B4}$$

Depending on the value of p, the superalgebra falls into one of three basic, "classical" families

$$\mathfrak{osp}(p|2q) = \begin{cases}B(r,q) & p = 2r+1, r \geq 0\\ C(q+1) & p = 2\\ D(r,q) & p = 2r, r \geq 2.\end{cases} \tag{B5}$$

Clearly it is the first case that will concern us, in particular, with $r = 0$, $q = 1$. $B(r,q)$ has rank $q + r$, dimension $2(q+r)^2 + 3q + r$, and even part $\mathfrak{so}(p) \oplus \mathfrak{sp}(2q)$, which for $\mathfrak{osp}(1|2)$ are 1, 5, and $\mathfrak{sl}(2)$, respectively.

One generates $\mathfrak{osp}(p|2q)$ as a matrix superalgebra by defining the supermatrices U and G

$$(U_{X_1X_2})_{X_3X_4} := \delta_{X_1X_4}\delta_{X_2X_3}, \qquad G := \begin{pmatrix}\mathbb{J}_{2q} & 0\\ 0 & H_p\end{pmatrix}, \tag{B6}$$

where

$$H_p := \begin{cases} \sigma_1 \otimes \mathbb{1}_r & p = 2r \\ [\sigma_1 \otimes \mathbb{1}_r] \oplus (1) & p = 2r + 1 \end{cases} \tag{B7}$$

with σ_1 being the first Pauli matrix. Here the indices X_i range from 1 to $2q + p$ and are partitioned as $X_i = (\bar{X}_i, \dot{X}_i)$ with $\bar{X}_i$ ranging from 1 to $2q$, and $\dot{X}_i$ taking on the remaining p values. Note that under (B6), G has the following symmetry properties:

$$G_{\bar{X}_1\bar{X}_2} = -G_{\bar{X}_2\bar{X}_1}, \qquad G_{\dot{X}_1\dot{X}_2} = +G_{\dot{X}_2\dot{X}_1}, \qquad G_{\bar{X}_1\dot{X}_2} = 0 = G_{\dot{X}_2\bar{X}_1}, \tag{B8}$$

which are shared with the invariant supermatrix E. In the special case $p = 1$, G reduces to E.

The generators T are obtained as

$$T_{X_1X_2} = 2G_{[[X_1|X_3}U_{X_3|X_2]]}, \tag{B9}$$

where T has array grade zero and the index grades are monotonically increasing:

$$\deg X := \begin{cases} 0 & X \in \{1, \ldots, 2q\} \\ 1 & X \in \{2q + 1, \ldots, 2q + p\}. \end{cases} \tag{B10}$$

Clearly T has symmetry properties $T_{X_1X_2} = T_{[[X_1X_2]]}$. The $2q(2q + 1)/2$ generators $T_{\bar{X}_1\bar{X}_2}$ generate $\mathfrak{sp}(2q)$, the $p(p - 1)/2$ generators $T_{\dot{X}_1\dot{X}_2}$ generate $\mathfrak{so}(p)$, and both are even (bosonic), while the $2pq$ generators $T_{\bar{X}_1\dot{X}_2}$ are odd (fermionic). These supermatrices yield the $\mathfrak{osp}(p|2q)$ superbrackets

$$[[T_{X_1X_2}, T_{X_3X_4}]] := 4G_{[[X_1[[X_3}T_{X_2]]X_4]]}, \tag{B11}$$

where the supersymmetrization on the right-hand side is over pairs X_1X_2 and X_3X_4 as on the left-hand side. The action of the generators on $(2q|p)$-dimensional supervectors a_X is given by

$$(T_{X_1X_2})_{X_3X_4}a_{X_4} \equiv (T_{X_1X_2}a)_{X_3} = 2G_{[[X_1|X_3}a_{X_2]]}. \tag{B12}$$

This action may be generalized to an N-fold super tensor product of $(2q|p)$ supervectors by labeling the indices with integers $k = 1, 2, \ldots, N$

$$(T_{X_kY_k}a)_{Z_1\cdots Z_k\cdots Z_N} = (-)^{(X_k+Y_k)\sum_{i=1}^{k-1}|Z_i|}2G_{[[X_k|Z_k}a_{Z_1\cdots|Y_k]]\cdots Z_N}. \tag{B13}$$

In our special case $p = 1$ we denote the lone dotted index $\dot{X}_i$ by a bullet $\bullet$ and start counting the barred indices at zero so that $X_i = (0, 1, \bullet)$. Obviously the $T_{\bullet\bullet}$ generator vanishes identically, leaving only the following superbrackets:

$$[T_{A_1A_2}, T_{A_3A_4}] = 4E_{(A_1(A_3}T_{A_2)A_4)}, \qquad [T_{A_1A_2}, T_{A_3\bullet}] = 2E_{(A_1|A_3}T_{A_2)\bullet}, \qquad \{T_{A_1\bullet}, T_{A_2\bullet}\} = T_{A_1A_2}, \tag{B14}$$

which are written out in Table V with $T_A \equiv T_{A\bullet} \equiv T_{\bullet A}$. Explicitly the generators are

TABLE V. $\mathfrak{osp}(1|2)$ superbrackets.

	T_{01}	T_{00}	T_{11}	T_0	T_1
T_{01}	0	$-2T_{00}$	$2T_{11}$	$-T_0$	T_1
T_{00}	$2T_{00}$	0	$4T_{01}$	0	$2T_0$
T_{11}	$-2T_{11}$	$-4T_{01}$	0	$-2T_1$	0
T_0	T_0	0	$2T_1$	T_{00}	T_{01}
T_1	$-T_1$	$-2T_0$	0	T_{01}	T_{11}

$$T_{01} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T_{00} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T_{11} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B15a}$$

appearing with Grassmann even coefficients with complex parameters, and

$$T_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \tag{B15b}$$

appearing with Grassmann odd coefficients with complex parameters. In particular, (B15a) with complex coefficients generate the bosonic subalgebra $\mathfrak{sl}(2, \mathbb{C})$. In order to make contact with [7], we rescale the generators into a new supermatrix P

$$P_{X_1X_2} := \tfrac{1}{2}T_{X_1X_2} \equiv E_{[[X_1|X_3}U_{X_3|X_2]]} \tag{B16}$$

to yield the superbrackets

$$[P_{A_1A_2}, P_{A_3A_4}] = 2\varepsilon_{(A_1(A_3}P_{A_2)A_4)}, \qquad [P_{A_1A_2}, Q_{A_3}] = \varepsilon_{(A_1|A_3}Q_{A_2)}, \qquad \{Q_{A_1}, Q_{A_2}\} = \tfrac{1}{2}P_{A_1A_2}, \tag{B17}$$

where $Q_A \equiv P_A$, which are summarized as

$$[[P_{X_1X_2}, P_{X_3X_4}]] = 2E_{[[X_1[[X_3}P_{X_2]]X_4]]}. \tag{B18}$$

The rescaled generators have the action

$$(P_{X_1X_2}a)_{X_3} = E_{[[X_1|X_3}a_{X_2]]}, \qquad (P_{X_kY_k}a)_{Z_1\cdots Z_k\cdots Z_N} = (-)^{(X_k+Y_k)\sum_{i=1}^{k-1}Z_i}E_{[[X_k|Z_k}a_{Z_1\cdots|Y_k]]\cdots Z_N}, \tag{B19}$$

which summarizes Tables II, III, and IV.

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