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Noncommutative relativistic particles

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We present a relativistic formulation of noncommutative mechanics where the object of noncommutativity $\theta^{\mu\nu}$ is considered as an independent quantity. Its canonical conjugate momentum is also introduced so that it permits one to obtain an explicit form for the generators of the extended Poincaré group in the noncommutative case. The theory, which is invariant under reparametrization, generalizes recent nonrelativistic results. Free noncommutative bosonic particles satisfy an extended Klein-Gordon equation depending on two parameters.

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I. INTRODUCTION

More than 60 years ago the first paper on space-time noncommutativity was written by Snyder [1]. There, the space-time coordinates[1] x^μ have been promoted to operators $\mathbf{x}^\mu$ satisfying the algebra

$$\begin{aligned}[\mathbf{x}^\mu, \mathbf{x}^\nu] &= ia^2\mathbf{M}^{\mu\nu},\\ [\mathbf{M}^{\mu\nu}, \mathbf{x}^\lambda] &= i(\mathbf{x}^\mu\eta^{\nu\lambda} - \mathbf{x}^\nu\eta^{\mu\lambda}),\\ [\mathbf{M}^{\mu\nu}, \mathbf{M}^{\alpha\beta}] &= i(\mathbf{M}^{\mu\beta}\eta^{\nu\alpha} - \mathbf{M}^{\mu\alpha}\eta^{\nu\beta} + \mathbf{M}^{\nu\alpha}\eta^{\mu\beta}\\ &\quad - \mathbf{M}^{\nu\beta}\eta^{\mu\alpha}),\end{aligned} \tag{1.1}$$

which is consistent with the identification $\mathbf{x}^\mu = a\mathbf{M}^{4\mu}$, M^{AB} representing the generators of the group $SO(1,4)$. That work was not very successful in its original motivation, which was the introduction of a natural cutoff for quantum field theories. However, in present times, space-time noncommutativity has been a very studied subject, associated with strings [2–4] and noncommutative field theories (NCFT's) [5]. In NCFT's, usually the first of relations (1.1) is written as

$$[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}, \tag{1.2}$$

but in most situations, and contrary to what occurs in (1.1), the object of noncommutativity $\theta^{\mu\nu}$ is considered as a constant matrix, which implies in the violation of the Lorentz symmetry [5]. A constant θ is indeed a consequence of the adopted theory. When strings have their end points on D-branes, in the presence of a constant antisymmetric tensor field background, this kind of canonical noncommutativity effectively arises. It is possible, however, to consider $\theta^{\mu\nu}$ as an independent operator [6], resulting in a true Lorentz invariant theory. Reference [6] has some of its consequences explored, for instance, in [7–11]. These last works are based on some contraction of the algebra (1.1), or equivalently, in the so-called Doplicher, Fredenhagen, and Roberts (DFR) algebra [12], that assumes, besides (1.2), the structure

$$[\mathbf{x}^\mu, \theta^{\alpha\beta}] = 0, \qquad [\theta^{\mu\nu}, \theta^{\alpha\beta}] = 0. \tag{1.3}$$

An important point of the DFR algebra is that the Weyl representation of noncommutative operators obeying (1.2) and (1.3) keeps the usual form of the Moyal product, and consequently the form of the usual NCFT's, although the fields have to be considered as depending not only on $\mathbf{x}^\mu$ but also on $\theta^{\alpha\beta}$. The DFR algebra has been proposed based on arguments coming from general relativity and quantum mechanics. The construction of a noncommutative theory which keeps Lorentz invariance is an important matter, since there is no experimental evidence to assume Lorentz symmetry violation [13].

In noncommutative quantum mechanics [14–32], as in NCFT, a similar framework with constant θ is usually employed, leading also to the violation of the Lorentz symmetry in the relativistic case or of the rotation symmetry for nonrelativistic formulations. In some recent works [33–36] one of the authors has explored some consequences of considering the object of noncommutativity as an independent quantity, respectively, as an operator acting in Hilbert space, in the quantum case [33–35], or as a phase space coordinate, in the case of nonrelativistic classical mechanics [36]. In both situations it was introduced as a canonical conjugate momentum for θ. It has been shown that the nonrelativistic theories described by [33,36] are related through the Dirac quantization procedure, once a proper second class constraint structure is postulated. Both theories are invariant under the action of $SO(D)$. The relativistic quantum treatment has been presented in [34] for bosons and in [35] for fermions. In another treatment the second quantized model proposed in [37] is studied by two of the present authors.

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[1]$A, B = 0, 1, 2, 3, 4$; $\mu, \nu = 0, 1, 2, 3$. The parameter a has dimension of length and $\hbar = c = 1$.

In the present work we generalize the formalism appearing in [36] (in its free limit) to the relativistic case, constructing in such a way a noncommutative relativistic classical theory which, under quantization, furnishes the theories presented in [34,37]. Our results are invariant under the action of the Lorentz group $SO(1, D)$ as well as under some generalization of the Poincaré group, previously discussed in [34,35]. We also show that our formalism is related to the one of Ref. [23], after eliminating some auxiliary variables.

As an introduction to the subject, we first present a brief review of the ordinary free relativistic particle in Sec. II. In Sec. III the algebraic structure for the noncommutative case is derived, by using the Dirac theory for Hamiltonian constrained systems. The first class constraint that generates the reparametrization transformations is introduced in Sec. IV. The corresponding first order action which generates the constraint structure is also presented in that section, and its reparametrization invariance is proved. In Sec. V we present some equivalent actions, not explicitly depending on the momenta. We discuss aspects related to the quantization of such a model, where a generalized Klein-Gordon equation is derived, depending on two parameters. Concluding remarks are left for Sec. VI.

II. THE COMMUTATIVE RELATIVISTIC PARTICLE

The commutative free relativistic particle can be described by the first order action

$$S = \int d\tau L_{FO}, \tag{2.1}$$

where τ is an arbitrary evolution parameter and[2]

$$L_{FO} = p \cdot \dot{x} - \lambda \chi. \tag{2.2}$$

In (2.2), $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$, λ is a Lagrange multiplier, and χ is a first class constraint expressing the mass shell condition

$$\chi = \tfrac{1}{2}(p^2 + m^2) = 0. \tag{2.3}$$

The equation of motion for p^μ is just $\dot{x} - \lambda p = 0$. If that solution is reintroduced in (2.1) one obtains the einbein form of the action, where

$$L_e = \frac{\dot{x}^2}{2\lambda} - \frac{\lambda}{2} m^2. \tag{2.4}$$

Now the equation of motion for λ gives

$$\lambda^2 = -\frac{\dot{x}^2}{m^2} \tag{2.5}$$

and when this is introduced in L_e, one gets the explicit reparametrization invariant action $\int d\tau L_0$, where

$$L_0 = -m \frac{\dot{x}^2}{\sqrt{-\dot{x}^2}}. \tag{2.6}$$

All three actions are equivalent and are invariant under reparametrization or redefinition of the evolution parameter τ. Let us consider in some detail the first order action. Under the equal τ Poisson bracket structure given by

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \tag{2.7}$$

the reparametrization invariance is generated by $G = \epsilon\chi$, where $\epsilon = \epsilon(\tau)$ is an arbitrary infinitesimal parameter and χ is given by (2.3). It is well known that the phase space variables y are transformed according to

$$\delta y = \{y, G\}, \tag{2.8}$$

giving, in our case,

$$\delta x^\mu = \epsilon p^\mu, \qquad \delta p_\mu = 0. \tag{2.9}$$

As can be verified,

$$\begin{aligned} \delta L_{FO} &= p \cdot \delta\dot{x} - \delta\lambda\chi = p \cdot \frac{d}{d\tau}\delta x - \delta\lambda\chi \\ &= \epsilon p \cdot \dot{p} + \dot{\epsilon} p^2 - \delta\lambda\chi, \end{aligned} \tag{2.10}$$

and if $\delta\lambda = \dot{\epsilon}$, δL_{FO} turns into a total derivative and the variation of the action (2.1) vanishes if ϵ vanishes in the extremes. This is characteristic of the so-called covariant systems [38]. Under quantization, the phase space variables become operators acting in Hilbert space, the brackets (2.7) become commutators and this permits, for instance, that in the coordinate representation, the momenta acquire the usual derivative realization. In this situation, the constraint (2.3) acting over a state vector gives just the Klein-Gordon equation

$$(\Box - m^2)\Psi(x) = 0, \tag{2.11}$$

which selects the physical states in Hilbert space. This guarantees that a state represented by Ψ is invariant under a unitary gauge transformation generated by χ.

III. THE NONCOMMUTATIVE ALGEBRAIC STRUCTURE

In this section, we present a relativist generalization of the algebraic structure found in [34]. To achieve this goal, it is introduced as a constrained Hamiltonian system living in a phase space spanned by the quantities x^μ, Z^μ, and $\theta^{\mu\nu}$ and their conjugate momenta p_μ, K_μ, and $\pi_{\mu\nu}$. x^μ represents the usual coordinates, as in Sec. II. $\theta^{\mu\nu}$ is the object of noncommutativity which appears, as an operator, in (1.2), and Z^μ represents auxiliary variables introduced in order to properly implement space-time noncommutativity. After introducing the second class constraints necessary to generate the adequate Dirac brackets, Z^μ and K_μ can be eliminated from the final results, once the con-

[2] From this point, we adopt $\mu, \nu = 0, 1, 2, \ldots, D$, with arbitrary $D \geq 1$. $\eta^{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$.

straints can be used in a strong way. According to the discussion above, the fundamental nonvanishing equal τ Poisson brackets involving all the phase space variables are given by

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{\theta^{\mu\nu}, \pi_{\rho\sigma}\} = \delta^{\mu\nu}{}_{\rho\sigma}, \qquad \{Z^\mu, K_\nu\} = \delta^\mu_\nu, \tag{3.1}$$

where $\delta^{\mu\nu}{}_{\rho\sigma} = \delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho$. The second class constraints $\Xi^a = 0$, $a = 1, \ldots, 2D + 2$, appearing in [34], are here generalized to

$$\Psi^\mu = Z^\mu - \tfrac{1}{2}\theta^{\mu\nu}p_\nu, \qquad \Phi_\mu = K_\mu - p_\mu, \tag{3.2}$$

with the associated constraint matrix

$$(\Delta^{ab}) = \begin{pmatrix} \{\Psi^\mu, \Psi^\nu\} & \{\Psi^\mu, \Phi^\nu\} \\ \{\Phi^\mu, \Psi^\nu\} & \{\Phi^\mu, \Phi^\nu\} \end{pmatrix} = \begin{pmatrix} 0 & \eta^{\mu\nu} \\ -\eta^{\mu\nu} & 0 \end{pmatrix} \tag{3.3}$$

with inverse

$$(\Delta^{-1}_{ab}) = \begin{pmatrix} 0 & -\eta_{\mu\nu} \\ \eta_{\mu\nu} & 0 \end{pmatrix}. \tag{3.4}$$

Now the Dirac brackets between any two phase space functions A and B are given by [38]

$$\{A, B\}_D = \{A, B\} - \{A, \Xi^a\}\Delta^{-1}_{ab}\{\Xi^b, B\}. \tag{3.5}$$

As one can verify, the algebraic structure above permits one to derive the Dirac brackets

$$\begin{aligned}
&\{x^\mu, p_\nu\}_D = \delta^\mu_\nu, \qquad \{x^\mu, x^\nu\}_D = \theta^{\mu\nu},\\
&\{p_\mu, p_\nu\}_D = 0, \qquad \{\theta^{\mu\nu}, \pi_{\rho\sigma}\}_D = \delta^{\mu\nu}{}_{\rho\sigma},\\
&\{\theta^{\mu\nu}, \theta^{\rho\sigma}\}_D = 0, \qquad \{\pi_{\mu\nu}, \pi_{\rho\sigma}\}_D = 0,\\
&\{x^\mu, \theta^{\rho\sigma}\}_D = 0, \qquad \{x^\mu, \pi_{\rho\sigma}\}_D = -\tfrac{1}{2}\delta^{\mu\nu}{}_{\rho\sigma}p_\nu,\\
&\{p_\mu, \theta^{\rho\sigma}\}_D = 0, \qquad \{p_\mu, \pi_{\rho\sigma}\}_D = 0,
\end{aligned} \tag{3.6}$$

involving the physical variables x^μ, p_μ, $\theta^{\mu\nu}$, and $\pi_{\mu\nu}$. The brackets listed above generalize the algebra found in Refs. [33,34]. It is also interesting to display the remaining Dirac brackets where the auxiliary variables Z^μ and K_μ appear as

$$\begin{aligned}
&\{Z^\mu, K_\nu\}_D = 0, \qquad \{Z^\mu, Z^\nu\}_D = 0,\\
&\{K_\mu, K_\nu\}_D = 0, \qquad \{Z^\mu, x^\nu\}_D = -\tfrac{1}{2}\theta^{\mu\nu},\\
&\{K_\mu, x^\nu\}_D = -\delta^\nu_\mu, \qquad \{Z^\mu, p_\nu\}_D = 0,\\
&\{K_\mu, p_\nu\}_D = 0, \qquad \{Z^\mu, \theta^{\sigma\rho}\}_D = 0,\\
&\{Z^\mu, \pi_{\sigma\rho}\}_D = \tfrac{1}{2}\delta^{\mu\nu}{}_{\sigma\rho}p_\nu, \qquad \{K^\mu, \theta^{\sigma\rho}\}_D = 0,\\
&\{K_\mu, \pi_{\sigma\rho}\}_D = 0.
\end{aligned} \tag{3.7}$$

As in the nonrelativistic case, the shifted coordinate

$$X^\mu = x^\mu + \tfrac{1}{2}\theta^{\mu\nu}p_\nu \tag{3.8}$$

also plays a fundamental role. As can be verified,

$$\begin{aligned}
&\{X^\mu, X^\nu\}_D = 0, \qquad \{X^\mu, p_\nu\}_D = \delta^\mu_\nu,\\
&\{X^\mu, x^\nu\}_D = \tfrac{1}{2}\theta^{\mu\nu}, \qquad \{X^\mu, \theta^{\rho\sigma}\}_D = 0,\\
&\{X^\mu, \pi_{\rho\sigma}\}_D = 0, \qquad \{X^\mu, Z^\nu\}_D = -\tfrac{1}{2}\theta^{\mu\nu},\\
&\{X^\mu, K_\nu\}_D = \delta^\mu_\nu,
\end{aligned} \tag{3.9}$$

and so the generator of the Lorentz group

$$M^{\mu\nu} = X^\mu p^\nu - X^\nu p^\mu - \theta^{\mu\sigma}\pi_\sigma{}^\nu + \theta^{\nu\sigma}\pi_\sigma{}^\mu \tag{3.10}$$

actually closes in the $SO(1, D)$ algebra, with the use of the Dirac brackets given above. Actually

$$\{M^{\mu\nu}, M^{\rho\sigma}\}_D = \eta^{\mu\sigma}M^{\rho\nu} - \eta^{\nu\sigma}M^{\rho\mu} - \eta^{\mu\rho}M^{\sigma\nu} + \eta^{\nu\rho}M^{\sigma\mu}. \tag{3.11}$$

From a different point of view, a similar structure has been postulated in [39].

The Lorentz transformation of any phase space function A is generated by the action of $M^{\mu\nu}$. Actually, a generalized Poincaré transformation can be introduced through the action of the generator [34]

$$G = \tfrac{1}{2}\omega_{\mu\nu}M^{\mu\nu} - a^\mu p_\mu + \tfrac{1}{2}b^{\mu\nu}\pi_{\mu\nu} \tag{3.12}$$

by defining

$$\delta A = -\{A, G\}_D, \tag{3.13}$$

which defines the action of the generalized Poincaré group $\mathcal{P}'$ with generators $M^{\mu\nu}$, p_μ, and $\pi^{\mu\nu}$. Specifically one arrives at

$$\begin{aligned}
\delta\mathbf{x}^\mu &= \omega^\mu{}_\nu\mathbf{x}^\nu + a^\mu + \tfrac{1}{2}b^{\mu\nu}p_\nu,\\
\delta\mathbf{X}^\mu &= \omega^\mu{}_\nu\mathbf{X}^\nu + a^\mu,\\
\delta\mathbf{p}_\mu &= \omega_\mu{}^\nu\mathbf{p}_\nu,\\
\delta\theta^{\mu\nu} &= \omega^\mu{}_\rho\theta^{\rho\nu} + \omega^\nu{}_\rho\theta^{\mu\rho} + b^{\mu\nu},\\
\delta\pi_{\mu\nu} &= \omega_\mu{}^\rho\pi_{\rho\nu} + \omega_\nu{}^\rho\pi_{\mu\rho},\\
\delta\mathbf{M}^{\mu\nu} &= \omega^\mu{}_\rho\mathbf{M}^{\rho\nu} + \omega^\nu{}_\rho\mathbf{M}^{\mu\rho} + a^\mu\mathbf{p}^\nu - a^\nu\mathbf{p}^\mu + b^{\mu\rho}\pi_\rho{}^\nu + b^{\nu\rho}\pi^\mu{}_\rho.
\end{aligned} \tag{3.14}$$

The defining brackets of $\mathcal{P}'$ are those in (3.11) and supplemented by the following brackets:

$$\begin{aligned}
&\{M^{\mu\nu}, \pi^{\rho\sigma}\}_D = \eta^{\mu\sigma}\pi^{\rho\nu} - \eta^{\mu\rho}\pi^{\sigma\nu} - \eta^{\nu\sigma}\pi^{\rho\mu} + \eta^{\nu\rho}\pi^{\mu\sigma},\\
&\{p_\mu, p_\nu\}_D = 0, \quad \{\pi^{\mu\nu}, p^\rho\}_D = 0, \quad \{\pi^{\mu\nu}, \pi^{\rho\sigma}\}_D = 0.
\end{aligned} \tag{3.15}$$

We see that the generalized Poincaré group $\mathcal{P}'$ is a semidirect product of the Lorentz group generated by M's acting on the $D + 1 + \frac{1}{2}D(D - 1)$ dimensional commutative subgroup given by p's and π's.

The form of transformations (3.14) guarantees the Poincaré invariance of the theory. This is only possible because of the introduction of the canonical pair $\theta^{\mu\nu}$, $\pi_{\mu\nu}$ as independent phase space variables, which permits the existence of an object like $M^{\mu\nu}$ in (3.10). We observe that $\mathcal{P}'$ has 4 Casimir invariants which have been discussed in Ref. [34]. Also it is important to notice that (3.14) closes in an algebra, i.e., for two independent infinitesimal transformations δ_1 and δ_2, we see that $[\delta_1, \delta_2]A = \delta_3 A$, with parameter composition rules given by

$$\begin{aligned} \omega_{3\ \nu}^{\mu} &= \omega_{1\ \alpha}^{\mu}\omega_{2\ \nu}^{\alpha} - \omega_{2\ \alpha}^{\mu}\omega_{1\ \nu}^{\alpha}, \\ a_3^{\mu} &= \omega_{1\ \nu}^{\mu}a_2^{\nu} - \omega_{2\ \nu}^{\mu}a_1^{\nu}, \\ b_3^{\mu\nu} &= \omega_{1\ \rho}^{\mu}b_2^{\rho\nu} - \omega_{2\ \rho}^{\mu}b_1^{\rho\nu} - \omega_{1\ \rho}^{\nu}b_2^{\rho\mu} + \omega_{2\ \rho}^{\nu}b_1^{\rho\mu}. \end{aligned} \tag{3.16}$$

IV. THE FIRST ORDER ACTION

The structure presented in the last section is almost identical to that found in Ref. [36], replacing spatial indices by space-time indices, and δ's by η's in convenient places. Other points can be more subtle.

Usually relativistic classical systems as relativistic particles, strings, or branes are invariant under reparametrization. This is associated with two related facts [38]: there are M first class constraints that generate the reparametrization when the parameter space has dimension M, and the associated canonical Hamiltonian usually vanishes. This is just the case treated in Sec. II, where $M = 1$. For the free noncommutative relativistic particle, this is also the case. So, it is necessary to introduce some first class constraint. A first candidate to be the desired constraint is the one given by the mass shell condition (2.3), since it has vanishing Poisson brackets with the second class constraints (3.2) and represents a suitable physical condition. One of the consequences of adopting (2.3) as the reparametrization generator is that only the physical coordinate x^μ transforms, among all the phase space variables. Actually, accordingly to the prescription (2.8), G has vanishing Poisson brackets with all the remaining phase space variables. The reparametrization invariance is just the invariance of the action under the redefinition of τ, and there is no apparent reason to explain such an asymmetric behavior between the ordinary coordinates and the tensor ones, given by the objects of noncommutativity.

It is tempting to add to χ a term like $\frac{1}{2}\pi^2$, but not only its dimension is L^{-4}, when the dimension of χ is L^{-2}, as it is not first class, in the sense that it has nonvanishing Poisson brackets with Ψ^μ, as defined in (3.2) [by construction any quantity has vanishing Dirac brackets with the second class constraints, as can be verified from (3.5)]. A related quantity, however, is first class:

$$\chi' = \tfrac{1}{2}[\pi^2 + K_\mu \pi^{\mu\nu} p_\nu + \tfrac{1}{4}(K^2 p^2 - (K \cdot p)^2)]. \tag{4.1}$$

Its form has been achieved by inspection. In the above expression internal products are implicitly understood. On the second class constraint surface, however, $\chi' \approx \frac{1}{2}\pi^2$. As can be verified,

$$\{\chi', \Psi^\mu\} = 0, \qquad \{\chi', \Phi_\mu\} = 0. \tag{4.2}$$

In a broad sense, the bracket structure is generated by the second class constraints and the dynamics is generated by the first class constraint, for covariant systems. By considering the quantum systems which appear in [34,37], it is possible to write the desired first class constraint as

$$\Upsilon = \frac{1}{\lambda^2}\chi' + \chi, \tag{4.3}$$

which, with (3.2), completes the set of constraints. In (4.3), χ is given by (2.3) and χ' by (4.1). λ is a parameter with length dimension, as the Plank's length.

After these points, it is possible, also in the present case, to construct an action that generates all the algebraic structure displayed above. It is written as in (2.1), but now

$$L_{FO} = p \cdot \dot{x} + K \cdot \dot{Z} + \pi \cdot \dot{\theta} - \lambda_a \Xi^a - \lambda \Upsilon. \tag{4.4}$$

Constraints (3.2) and (4.3) are generated as secondary constraints associated with the conservation of the trivial ones that express that the canonical momenta conjugate to the Lagrange multipliers vanish identically. It is not necessary to display this procedure here since it is quite trivial. We observe that in the commutative limit where $\theta^{\mu\nu}$ and $\pi_{\mu\nu}$ vanish, Z^μ and $K_\mu - p_\mu$ also vanish due to (3.2) and Υ goes to χ. So (2.2) is recovered from (4.4).

Now, the reparametrization generator is assumed to be Υ, and if one defines $G = \epsilon\Upsilon$, prescription (2.8) gives

$$\begin{aligned} \delta x^\mu &= \epsilon\Big[p^\mu + \frac{1}{\lambda^2}\Big(-\pi^{\mu\nu}K_\nu + \frac{1}{2}(K^2 p^\mu - K \cdot p K^\mu)\Big)\Big], \\ \delta p_\mu &= 0, \\ \delta\theta^{\mu\nu} &= \frac{\epsilon}{\lambda^2}\Big[\pi^{\mu\nu} - \frac{1}{2}(p^\mu K^\nu - p^\nu K^\mu)\Big], \\ \delta\pi_{\mu\nu} &= 0, \\ \delta Z^\mu &= \frac{\epsilon}{\lambda^2}\Big[\pi^{\mu\nu}p_\nu + \frac{1}{2}(p^2 K^\mu - K \cdot p p^\mu)\Big], \\ \delta K_\mu &= 0. \end{aligned} \tag{4.5}$$

It is not hard to verify that under (4.5)

$$\delta L_{FO} = \dot{\Gamma} - \epsilon\dot{\Upsilon} - \delta\lambda\Upsilon - \delta\lambda_a \Xi^a \tag{4.6}$$

and so the first order action is invariant if $\delta\lambda_a = 0$ and $\delta\lambda = \dot{\epsilon}$, ϵ vanishing in the extremes. In (4.6),

$$\Gamma = p \cdot \delta x + \pi \cdot \delta\theta + K \cdot \delta Z, \tag{4.7}$$

where δx, $\delta\theta$, and δZ are given by (4.5).

V. SOME ADDITIONAL POINTS

In the previous section the first order action has been used to derive the constraint structure necessary to generate the Dirac brackets and the reparametrization transformations. As in the ordinary case, it is also possible here to eliminate some of the variables in favor of the others, by using for instance the second class constraints in a strong way. By starting from the first order Lagrangian (4.4), we arrive at

$$L_1 = p \cdot (\dot{x} + \dot{Z}) + \pi \cdot \dot{\theta} - \lambda_1 \cdot \left(Z - \frac{1}{2}\theta \cdot p\right) - \gamma\left(\frac{1}{\lambda^2}\pi^2 + p^2 + m^2\right), \tag{5.1}$$

if one uses the equation of motion for λ_2^μ, which is $p^\mu - K^\mu = 0$. We observe that the form of the first class constraint is also simplified due to symmetry. If we now use the equation of motion for $\lambda_{1\mu}$, which is just $Z^\mu - \frac{1}{2}\theta^{\mu\nu}p_\mu = 0$, we arrive at

$$L_2 = p \cdot \dot{x} + \pi \cdot \dot{\theta} - \gamma\left(\frac{1}{\lambda^2}\pi^2 + p^2 + m^2\right) + \frac{1}{2}p \cdot \theta \cdot \dot{p}. \tag{5.2}$$

We observe that the last term in (5.2) has already appeared in [23]. In Deriglazov's treatment there is the introduction of a factor of θ^{-2} in the corresponding term in order to introduce an additional gauge invariance which can be fixed by imposing constant θ's. In those works there is no term in π and any dynamics for the θ sector, which is a necessary ingredient to implement the quoted symmetry. Also that symmetry is broken if any interaction is introduced via minimal coupling procedures. As can be verified, (5.2) can be the starting point for essentially the same structure described in the last two sections.

An important point is that it is possible to quantize the classical structure displayed so far. As a first step the phase space variables y^A are promoted to the operators $\mathbf{y}^A$ acting on some Hilbert space, and the Dirac quantization prescription is consistently adopted, where

$$\{y^A, y^B\}_D \rightarrow \frac{1}{i}[\mathbf{y}^A, \mathbf{y}^B]. \tag{5.3}$$

As the canonical quantization is following the rule given above, all the second class constraints can be taken in a strong way. So, from (3.6) it follows the equal τ commutator structure

$$\begin{aligned}
&[\mathbf{x}^\mu, \mathbf{p}_\nu] = i\delta^\mu_\nu, \qquad [\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu},\\
&[\mathbf{p}_\mu, \mathbf{p}_\nu] = 0, \qquad [\theta^{\mu\nu}, \pi_{\rho\sigma}] = i\delta^{\mu\nu}{}_{\rho\sigma},\\
&[\theta^{\mu\nu}, \theta^{\rho\sigma}] = 0, \qquad [\pi_{\mu\nu}, \pi_{\rho\sigma}] = 0,\\
&[\mathbf{x}^\mu, \theta^{\rho\sigma}] = 0, \qquad [\mathbf{x}^\mu, \pi_{\rho\sigma}] = -\frac{i}{2}\delta^{\mu\nu}{}_{\rho\sigma}p_\nu,\\
&[\mathbf{p}_\mu, \theta^{\rho\sigma}] = 0, \qquad [\mathbf{p}_\mu, \pi_{\rho\sigma}] = 0,
\end{aligned} \tag{5.4}$$

and it is not necessary to consider the auxiliary variables Z^μ and K_μ, since the constraints (3.2) are to be taken strongly. By the same reason, χ' in (4.1) reduces to $\frac{1}{2}\pi^2$ and so, the first class constraint Υ reduces to the simpler form

$$\Upsilon = \frac{1}{2}\left(\mathbf{p}^2 + \frac{1}{\lambda^2}\pi^2 + m^2\right). \tag{5.5}$$

For a theory that presents gauge degrees of freedom, the physical states are selected by imposing that they have to be annihilated by the first class constraints [38]. This fact assures that a unitary gauge transformation, generated by the first class constraints, keeps the physical states unchanged, as it should be. This procedure is in the foundations of several quantization procedures of gauge theories [38]. In our case, if $|\Psi\rangle$ represents a physical state in Hilbert space, it must satisfy the condition

$$\left(\mathbf{p}^2 + \frac{1}{\lambda^2}\pi^2 + m^2\right)|\Psi\rangle = 0. \tag{5.6}$$

Observe that this constraint condition does not represent what would be obtained if we were describing a particle in a space-time with $D + 1 + \frac{D(D+1)}{2}$ dimensions. This is so because p^2 and π^2 are independent Casimir invariants [34]. This anticipates the fact that in this model the bosonic particle is classified by two parameters and not by one, given by the rest mass, as in the ordinary case.

As in the nonrelativistic case [33], it is necessary to choose a basis for the Hilbert space associated with such a system. Because of the noncommutativity between the coordinate operators, their eigenvectors cannot form that basis. Again the shifted coordinate operator

$$\mathbf{X}^\mu = \mathbf{x}^\mu + \tfrac{1}{2}\theta^{\mu\nu}\mathbf{p}_\nu \tag{5.7}$$

plays a fundamental role. As one can verify,

$$\begin{aligned}
&[\mathbf{X}^\mu, \mathbf{X}^\nu] = 0, \qquad [\mathbf{X}^\mu, \mathbf{p}_\nu] = i\delta^\mu_\nu, \qquad [\mathbf{X}^\mu, \theta^{\rho\sigma}] = 0,\\
&[\mathbf{X}^\mu, \pi_{\rho\sigma}] = 0, \qquad [\mathbf{p}_\mu, \mathbf{p}_\nu] = 0, \qquad [\theta^{\mu\nu}, \theta^{\rho\sigma}] = 0,\\
&[\pi^{\mu\nu}, \pi^{\rho\sigma}] = 0, \qquad [\pi^{\mu\nu}, \theta_{\rho\sigma}] = \delta^{\mu\nu}_{\rho\sigma}.
\end{aligned} \tag{5.8}$$

This permits one to adopt

$$\mathbf{M}^{\mu\nu} = \mathbf{X}^\mu\mathbf{p}^\nu - \mathbf{X}^\nu\mathbf{p}^\mu - \theta^{\mu\sigma}\pi_\sigma{}^\nu + \theta^{\nu\sigma}\pi_\sigma{}^\mu \tag{5.9}$$

as the generators of the Lorentz group $SO(1, D)$, since it closes in the appropriate algebra

$$[\mathbf{M}^{\mu\nu}, \mathbf{M}^{\rho\sigma}] = i\eta^{\mu\sigma}\mathbf{M}^{\rho\nu} - i\eta^{\nu\sigma}\mathbf{M}^{\rho\mu} - i\eta^{\mu\rho}\mathbf{M}^{\sigma\nu} + i\eta^{\nu\rho}\mathbf{M}^{\sigma\mu} \tag{5.10}$$

and generates the Lorentz transformations, as in Sec. III, but now with the use of a commutator structure. The eigenvectors of the shifted coordinate operator (5.7) also can be used in the construction of a basis in Hilbert space. Generalizing what has been done in [33], it is possible to choose a coordinate basis $|X', \theta'\rangle$ in such a way that

$$\mathbf{X}^\mu|X', \theta'\rangle = X'^\mu|X', \theta'\rangle, \qquad \theta^{\mu\nu}|X', \theta'\rangle = \theta'^{\mu\nu}|X', \theta'\rangle, \tag{5.11}$$

and also for $|X', \pi'\rangle$, $|p', \theta'\rangle$, or $|p', \pi'\rangle$, satisfying usual orthonormality and completeness relations. This quantum structure has been discussed with some detail in [34], leading to a generalized Klein-Gordon equation, which can also be derived from a Lagrangian formalism. The complex version of this field theory has been (secondly) quantized in [37], and it has been shown that the same algebra represented by (3.16) can be derived via Heisenberg relations.

VI. CONCLUSIONS

To close this work, we observe that it has been possible to consistently treat the object of noncommutativity $\theta^{\mu\nu}$ as a phase space coordinate, once its conjugate momentum is also considered. The classical and the corresponding quantum theory constructed are invariant under the action of the extend Poincaré group $\mathcal{P}'$, and the results are very simple, at least in the free case. The physical states are selected by a condition that implies in a modified Klein-Gordon equation with an extended derivative operator, involving the objects of noncommutativity. The second quantization of this model has also been constructed and presents interesting features [37].

Another point that must be considered is the introduction of interactions, for instance by using some minimal coupling procedure with extended covariant derivatives. This program follows a route that is not the usual one found in NCFT's. Contrary to what occurs here, the usual formulations of NCFT's do not introduce modifications in the ordinary field theories, in the free case. As it is well known [5], the only interaction terms capture noncommutativity through Moyal products. These modifications seem to be relevant because we expect that unusual geometrical structures may arise at very high energies, and this new physics probably should occur even for a free particle. We comment that in string theory, however, an approach similar to the one found here could present drastic consequences. This is so not only because the dynamics associated with θ, π could not be disregarded but, more important, because the counting of the bosonic degrees of freedom would be different from the one appearing in ordinary string theory. Here the idea is that if tensor operators are included, as the objects of noncommutativity, the counting of the string bosonic degrees of freedom is not $D + 1$ but $D + 1 + \frac{D(D+1)}{2}$, due to the existence of $\theta^{\mu\nu}$. This implies that in $D + 1 = 4$, the number of bosonic degrees of freedom would be 10. So, in a supersymmetric scheme, the string anomaly cancellation would occur just for $D + 1 = 4$. Related ideas appeared earlier in Ref. [40], without involving noncommutativity.

[1] H. S. Snyder, Phys. Rev. **71**, 38 (1947).
[2] M. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987); J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, England, 1998); R. Szabo, *An Introduction to String Theory and D-Brane Dynamics* (Imperial College Press, London, 2004).
[3] M. R. Douglas and C. Hull, J. High Energy Phys. 02 (1998) 008.
[4] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.
[5] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001); R. J. Szabo, Phys. Rep. **378**, 207 (2003); R. Banerjee, B. Chakraborty, S. Ghosh, P. Mukherjee, and S. Samanta, Found. Phys. **39**, 1297 (2009).
[6] C. E. Carlson, C. D. Carone, and N. Zobin, Phys. Rev. D **66**, 075001 (2002).
[7] M. Haghighat and M. M. Ettefaghi, Phys. Rev. D **70**, 034017 (2004).
[8] C. D. Carone and H. J. Kwee, Phys. Rev. D **73**, 096005 (2006).
[9] M. M. Ettefaghi and M. Haghighat, Phys. Rev. D **75**, 125002 (2007).
[10] H. Kase, K. Morita, Y. Okumura, and E. Umezawa, Prog. Theor. Phys. **109**, 663 (2003); K. Imai, K. Morita, and Y. Okumura, Prog. Theor. Phys. **110**, 989 (2003).
[11] S. Saxell, Phys. Lett. B **666**, 486 (2008).
[12] S. Doplicher, K. Fredenhagen, and J. E. Roberts, Phys. Lett. B **331**, 39 (1994); Commun. Math. Phys. **172**, 187 (1995).
[13] J. Jaeckel, V. V. Khoze, and A. Ringwald, J. High Energy Phys. 02 (2006) 028.
[14] M. M. Sheikh-Jabbari, Phys. Lett. B **450**, 119 (1999).
[15] C. Durval and P. Horvathy, Phys. Lett. B **479**, 284 (2000).
[16] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001).
[17] J. Gamboa, M. Loewe, and J. C. Rojas, Phys. Rev. D **64**, 067901 (2001).
[18] V. P. Nair and A. P. Polychronakos, Phys. Lett. B **505**, 267 (2001).
[19] M. Chaichian, A. Demichev, P. Presnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Lett. B **527**, 149 (2002).
[20] R. Banerjee, Mod. Phys. Lett. A **17**, 631 (2002).
[21] Stefano Bellucci and A. Nersessian, Phys. Lett. B **542**, 295 (2002).
[22] P.-M. Ho and H.-C. Kao, Phys. Rev. Lett. **88**, 151602 (2002).
[23] A. A. Deriglazov, Phys. Lett. B **555**, 83 (2003); J. High Energy Phys. 03 (2003) 021; arXiv:0911.2592.
[24] A. Smailagic and E. Spallucci, J. Phys. A **36**, L467 (2003); J. Phys. A **36**, L517 (2003).
[25] L. Jonke and S. Meljanac, Eur. Phys. J. C **29**, 433 (2003).
[26] A. Kokado, T. Okamura, and T. Saito, Phys. Rev. D **69**, 125007 (2004).

[27] A. Kijanka and P Kosinski, Phys. Rev. D **70**, 127702 (2004).
[28] I. Dadic, L. Jonke, and S. Meljanac, Acta Phys. Slovaca **55**, 149 (2005).
[29] S. Bellucci and A. Yeranyan, Phys. Lett. B **609**, 418 (2005).
[30] X. Calmet, Phys. Rev. D **71**, 085012 (2005); X. Calmet and M. Selvaggi, Phys. Rev. D **74**, 037901 (2006).
[31] F. G. Scholtz, B. Chakraborty, J. Govaerts, and S. Vaidya, J. Phys. A **40**, 14581 (2007).
[32] M. Rosenbaum, J. David Vergara, and L. R. Juarez, Phys. Lett. A **367**, 1 (2007).
[33] R. Amorim, Phys. Rev. Lett. **101**, 081602 (2008).
[34] R. Amorim, Phys. Rev. D **78**, 105003 (2008).
[35] R. Amorim, J. Math. Phys. (N.Y.) **50**, 022303 (2009).
[36] R. Amorim, J. Math. Phys. (N.Y.) **50**, 052103 (2009).
[37] R. Amorim and E. M. C. Abreu, Phys. Rev. D **80**, 105010 (2009).
[38] P. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964); K. Sundermeyer, *Constrained Dynamics*, Lecture Notes in Physics Vol. 169 (Springer-Verlag, Berlin, 1982); M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).
[39] A similar operator can also be found in M. Chaichian, K. Nishijima, and A. Turanu, Phys. Lett. B **633**, 129 (2006); J. M. Gracia-Bondía, F. Ruiz Ruiz, F. Lizzi, and P. Vitale, Phys. Rev. D **74**, 025014 (2006).
[40] R. Amorim and J. Barcelos-Neto, Z. Phys. C **58**, 513 (1993).