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Expanding perfect fluid generalizations of the *C* metric

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Petrov type D gravitational fields, generated by a perfect fluid with spatially homogeneous energy density and with flow lines which form a nonshearing and nonrotating timelike congruence, are reexamined. It turns out that the anisotropic such spacetimes, which comprise the vacuum C metric as a limit case, can have *nonzero* expansion, contrary to the conclusion in the original investigation by Barnes [A. Barnes, Gen. Relativ. Gravit. **4**, 105 (1973).]. Apart from the static members, this class consists of cosmological models with precisely one symmetry. The general line element is constructed and some important properties are discussed. It is also shown that purely electric Petrov type D vacuum spacetimes admit shear-free normal timelike congruences everywhere, even in the nonstatic regions. This result incited to deduce intrinsic, easily testable criteria regarding shear-free normality and staticity of Petrov type D spacetimes in general, which are added in an appendix.

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I. INTRODUCTION

The C metric is a well-known exact solution of Einstein's vacuum equation with zero cosmological constant. The static region of the corresponding spacetime was first described by Weyl [1]. At about the same time Levi-Civita [2] constructed its line element in closed form, arriving at essentially one cubic polynomial with two parameters as the metric structure function. The C metric is a Petrov type D solution for which at each spacetime point both Weyl principal null directions (PNDs) are geodesic, nonshearing, nontwisting but diverging; it thus belongs to the Robinson-Trautman class of solutions and was rediscovered as such [3]. The label C derives from the invariant classification of static degenerate Petrov type D vacuum spacetimes by Ehlers and Kundt [4]. The importance of this solution as summarized by Kinnersley and Walker [5] is threefold. First, the C metric describes a spacetime with only two independent Killing vector fields (KVFs) which can be fully analyzed. Next, it is an "example of almost everything," most notably it describes a radiative, locally asymptotically flat spacetime, while containing a static region. The C metric is contained in the class of boost-rotation-symmetric spacetimes [6,7], which are the only axially symmetric, radiative and asymptotically flat spacetimes with two Killing vectors. Finally, the solution has a clear physical interpretation as the anisotropic gravitational field of two Schwarzschild black holes being uniformly accelerated in opposite directions by a cosmic string or strut, provided that $m\alpha < 1/\sqrt{27}$, where the mass m and acceleration α are equivalents of the two essential parameters of Levi-Civita [5,8] (see, however, the end of Sec. II C for a comment).

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Generalizations of the C metric have been widely considered. Adding a cosmological constant Λ is straightforward, and we will henceforth refer with "C metric" to such Einstein spaces. Incorporating electromagnetic charge $|q|^2 \equiv e^2 + g^2$ is equally natural and leads to quartic structure functions [5]. Recently, the question of how to include rotation for the holes received a new answer [9,10], avoiding the closed timelike curves appearing in the previously considered "spinning" C metric [11,12], just as in the Newman-Unti-Tamburino (NUT) solutions [13]. All these generalizations fit in the well-established class $\mathcal{D}$ of Petrov type D Einstein-Maxwell solutions with a non-null electromagnetic field possessing geodesic and nonshearing null directions aligned with the PNDs [14,15], which reduces for zero electromagnetic field to the subclass $\mathcal{D}_0$ of Petrov type D Einstein spaces and which contains all well-known 4D black hole metrics. In fact, all $\mathcal{D}$ metrics can be derived by performing "limiting contractions" [16] from the most general member, the Plebański-Demiański line element [17], which exhibits two quartic structure functions with six essential parameters m, α, $|q|^2$, Λ, NUT parameter l, and angular momentum a. A physically comprehensive and simplified treatment can be found in [18,19], also surveying recent work in this direction.

In this paper we present a new family of Petrov type D, expanding and anisotropic perfect fluid (PF) generalizations of the C metric. The direct motivation and background for this work is the following.

According to the Goldberg-Sachs theorem [20] the two PNDs of any member of $\mathcal{D}_0$ are precisely those null directions which are geodesic and nonshearing. Such a member is purely electric (PE, cf. Appendix B) precisely when both PNDs, as well as the complex null directions orthogonal to them, are nontwisting [nonrotating or hypersurface orthogonal (HO)]. This is, in particular, the case for the C metric. As we will show, it implies the existence of an *umbilical synchronization* (US), i.e., a nonshearing and nonrotating unit *timelike* vector field (tangent to a congruence of observers). The importance of USs in cosmology was stressed in [21]. If a congruence of observers measuring isotropic radiation admits orthogonal hypersurfaces, an US exists. Only small deviations from isotropy are seen in the cosmic microwave background, and scalar perturbations of a Friedmann-Lemaître-Robertson-Walker universe preserve the existence of an US [22]. In general, spacetimes admitting an US have zero magnetic part of the Weyl tensor with respect to it [23] and thus are either of Petrov type O, or PE and of type D or I [16]. Conformally flat spacetimes always admit USs [see e.g. (6.15) in [16]]. Trümper showed that algebraically general vacua with an US are static [24]. Motivated by this result and by his own work [25] on static PFs, Barnes [26] studied PF spacetimes with an US tangent to their flow lines. He was able to generalize Trümper's result to Petrov type I such PFs and recovered Stephani's results on conformally flat PF solutions which are either of a generalized Schwarzschild type or of a generalized Friedmann type (so-called Stephani universes) [27]. The type D solutions were integrated and invariantly partitioned, based on the direction of the gradient of the energy density relative to the PNDs and the flow vector at each point. Class I, characterized by the energy density being constant on the hypersurfaces orthogonal to the flow lines and thus the only class containing Einstein spaces as limit cases, was further subdivided using the gradient of Ψ_2 (cf. Sec. II B for details). By solving the field equations, Barnes concludes that class ID, consisting of the anisotropic class I models, solely contains nonexpanding solutions. Hence, these PF solutions would not be viable as a cosmological model. However, based on an integrability analysis of class I in the Geroch-Held-Penrose (GHP) formalism [28], we found that this conclusion cannot be valid and this led to a detailed reinvestigation.

In this article we construct the general line element of the full ID class, comprising both the known nonexpanding perfect fluid models and the new expanding ones, and discuss some elementary properties. We want to stress the following point. The full class represents a PF generalization of the C metric in the sense that the C metric is contained as the Einstein space limit. The physical interpretation of this fact is however not established. This would require one to exhibit this solution for small masses as a perturbation of a known PF solution, just as the C-metric interpretation of small accelerating black holes has been established in a flat or (anti–)de Sitter background [5,29–32].

However, the mathematical relation with the C metric is useful. As already deduced in [26], the PF solution is, just as the C metric, conformally related to the direct sum of two 2D metrics. The fact that one part is equal for the PF solution and the C metric is helpful in the analysis, e.g. we will show that (a part of) the axis of symmetry can readily be identified as a conical singularity, analogous to the defect of the cosmic string present in the C metric. The nonstatic spacetimes presented are exact PF solutions with only this symmetry, and the analysis appears to be within reach. For the expanding ID PF models both the matter density $w(t)$ and the expansion scalar $\theta(t)$ can be arbitrary functions. This freedom is displayed explicitly in the metric form, and makes the solutions more attractive as a cosmological model.

The paper is organized as follows. In Sec. II we present the GHP approach to class I. We derive a closed set of equations, construct suitable scalar invariants, interpret the invariant subclassification of [26], and start the integration. At the end we provide alternative characterizations for the Einstein space members and identify their static regions and USs. In Sec. III we finish the construction of the general ID line element in a transparent way, and correct the calculative error of [26] in the original approach. Then we deduce basic properties of the ID perfect fluid models. In Sec. IV we summarize the main results and indicate points of further research. The work greatly benefited from the use of the GHP formalism, which at the same time elucidates the deviation from the C metric. In Appendix A we provide a pragmatic survey of this formalism for the nonexpert reader. In Appendix B, finally, we present criteria for deciding when a Petrov type D spacetime admits a (rigid) US or is static.

Notation.—For spacetimes (M, g_{ab}) we take $(+++-)$ as the metric signature and use geometrized units $8\pi G = c = 1$, where G is the gravitational coupling constant and c the speed of light. Λ denotes the cosmological constant. We make consistent use of the abstract Latin index notation for tensor fields, as advocated in [33]. Round (square) brackets denote (anti-)symmetrization, η_{abcd} is the spacetime alternating pseudotensor, and $\nabla_c T_{ab\cdots}$ ($\mathcal{L}_{\mathbf{X}} T_{ab\cdots}$) designates the Levi-Civita covariant derivative (Lie derivative with respect to X^a) of the tensor field $T_{ab\cdots}$. One has

$$d_a f = \nabla_a f, \qquad d_b Y_a = \nabla_{[b} Y_{a]}$$

for the exterior derivative of a scalar field f, respectively, one-form field Y_a, and

$$\mathbf{X}(f) \equiv X^a d_a f$$

denotes the Leibniz action of a vector field X^a on f; when X^a is the x^i-coordinate vector field $\partial_{x^i}{}^a$ we write $\partial_{x^i} f$ or $f_{,x^i}$, and a prime denotes ordinary derivation for functions of one variable, $f'(x) \equiv \partial_x f(x)$. However, we use index-free notation in line elements $ds^2 = g_{ij} dx^i dx^j$. The specific GHP notation is introduced in Appendix A.

II. GHP APPROACH TO CLASS I

A. Definition and integrability

We consider Barnes's class I [26], consisting of spacetimes (M, g_{ab}) with the following properties:

(i) The spacetime admits a unit timelike vector field u^a ($u^a u_a = -1$) which is nonshearing and nonrotating, i.e., its covariant derivative is of the form

$$\nabla_b u_a = \theta h_{ab} - \dot{u}_a u_b, \qquad h_{ab} \equiv g_{ab} + u_a u_b, \quad (1)$$

where the acceleration $\dot{u}^a = u^b \nabla_b u^a$ and expansion rate $\theta = \nabla_a u^a$ are the remaining kinematic quantities of u^a.

(ii) The Einstein tensor has the structure

$$G_{ab} = S u_a u_b + p g_{ab} = w u_a u_b + p h_{ab}, \qquad (2)$$

$$D_a w \equiv h_a{}^b \nabla_b w = 0, \qquad (3)$$

i.e., the spacetime represents the gravitational field of either a perfect fluid with shear-free normal four-velocity u^a, pressure $p + \Lambda$, and spatially homogeneous energy density $w - \Lambda$ (case $S \equiv w + p \neq 0$) or a vacuum (*Einstein space* case $S = 0$, where $w = -p$ may be identified with Λ).

(iii) The Weyl tensor C_{abcd} is degenerate but nonzero, i.e., the spacetime is algebraically special but not conformally flat.

Choose null vector fields k^a and l^a, subject to the normalization condition $k^a l_a = -1$, such that

$$u^a = \frac{1}{\sqrt{2q}}(q k^a + l^a), \qquad q > 0. \qquad (4)$$

Within the GHP formalism (cf. Appendix A) based on the complex null tetrad $(k^a, l^a, m^a, \bar{m}^a)$, q is $(-2, -2)$ weighted and conditions (i) and (ii) translate into

$$\begin{gathered} \pi + \bar{\tau} = q\bar{\kappa} + q^{-1}\nu, \qquad \lambda = q\bar{\sigma}, \\ \mu - \bar{\mu} = q(\bar{\rho} - \rho), \end{gathered} \qquad (5)$$

$$\text{Þ}' q - q \text{Þ} q = -2q(\mu - q\bar{\rho}), \qquad \eth q = \eth' q = 0 \qquad (6)$$

and

$$\Phi_{01} = \Phi_{12} = \Phi_{02} = 0, \qquad (7)$$

$$\Phi_{11} = \frac{S}{8}, \qquad \Phi_{00} = \frac{S}{4q}, \qquad \Phi_{22} = \frac{qS}{4}, \qquad (8)$$

$$R \equiv 24\Pi = w - 3p = 4w - 3S, \qquad (9)$$

$$\eth w = \eth' w = 0, \qquad \text{Þ}' w - q \text{Þ} w = 0, \qquad (10)$$

respectively. By virtue of condition (i) the magnetic part $H_{ab} \equiv \frac{1}{2}\eta_{acmn} C^{mn}{}_{bd} u^c u^d$ of the Weyl tensor with respect to u^a vanishes [23]. In combination with condition (iii) it follows that the Weyl tensor is purely electric with respect to u^a, $E_{ab} \equiv C_{acbd} u^c u^d \neq 0$, the Weyl-Petrov type is D, and at each point u^a lies in the plane Σ spanned by the Weyl PNDs (cf. Appendix B for a GHP proof of these well-known facts). Hence, choosing k^a and l^a along the PNDs, $(k^a, l^a, m^a, \bar{m}^a)$ is a Weyl principal null tetrad (WPNT) and we have

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \qquad (11)$$

$$\bar{\Psi} = \Psi \neq 0, \qquad \Psi \equiv 2\Psi_2. \qquad (12)$$

Under the restrictions (7) and (11), the GHP Bianchi equations are given by (A31)–(A36) and their prime duals. Combining these with the other equations in (5)–(12) results in

$$\kappa = 0, \qquad \nu = 0, \qquad \sigma = \lambda = 0, \qquad (13)$$

$$\bar{\rho} = \rho, \qquad \bar{\mu} = \mu, \qquad \pi = -\bar{\tau}, \qquad (14)$$

$$\text{Þ}\Psi = 3\rho\Psi, \qquad \text{Þ}'\Psi = -3\mu\Psi, \qquad (15)$$

$$\eth\Psi = 3\tau\Psi, \qquad \eth'\Psi = -3\pi\Psi, \qquad (16)$$

$$\text{Þ}' S - q \text{Þ} S = S(\text{Þ} q - \mu + q\rho), \qquad (17)$$

$$\eth S = \tau S, \qquad \eth' S = \bar{\tau} S, \qquad (18)$$

$$\text{Þ}' w = q \text{Þ} w = -\frac{3S(\mu - q\rho)}{2}. \qquad (19)$$

With (7)–(9) and (11)–(14) the Ricci equations, given by (A25)–(A30) and their prime duals, reduce to

$$\text{Þ}\mu = -\text{Þ}'\rho \qquad (20)$$

$$= -\eth'\tau + \mu\bar{\rho} + \tau\bar{\tau} + \frac{\Psi}{2} + \frac{w}{3} - \frac{S}{4}, \qquad (21)$$

$$\text{Þ}'\mu = -\mu^2 - \frac{qS}{4}, \qquad \eth\mu = \eth'\mu = 0, \qquad (22)$$

$$\text{Þ}\rho = \rho^2 + \frac{S}{4q}, \qquad \eth\rho = \eth'\rho = 0, \qquad (23)$$

$$\text{Þ}\tau = \text{Þ}'\tau = 0, \qquad \eth\tau = \tau^2, \qquad (24)$$

$$-\eth\pi = \eth'\tau = \eth\bar{\tau} \equiv \frac{H}{2} \qquad (25)$$

and the complex conjugates of (24), while the commutator relations applied to a $(\mathrm{w}_p, \mathrm{w}_q)$-weighted scalar η become

$$[\text{Þ}, \text{Þ}']\eta = (\mathrm{w}_p + \mathrm{w}_q)\left(\tau\bar{\tau} - \frac{\Psi}{2} + \frac{w}{6} - \frac{S}{4}\right)\eta, \qquad (26)$$

$$[\eth, \eth']\eta = (\mathrm{w}_p - \mathrm{w}_q)\left(-\mu\rho + \frac{\Psi}{2} - \frac{w}{6}\right)\eta, \qquad (27)$$

$$[\text{Þ}, \eth]\eta = (-\tau\text{Þ} + \rho\eth + \mathrm{w}_q\rho\tau)\eta, \qquad (28)$$

$$[\text{Þ}, \eth']\eta = (-\bar{\tau}\text{Þ} + \rho\eth' + w_p\rho\bar{\tau})\eta, \quad (29)$$

$$[\text{Þ}', \eth]\eta = (-\tau\text{Þ}' - \mu\eth + w_p\mu\tau)\eta, \quad (30)$$

$$[\text{Þ}', \eth']\eta = (-\bar{\tau}\text{Þ}' - \mu\eth' + w_q\mu\bar{\tau})\eta. \quad (31)$$

Then the $[\eth, \eth'](\tau)$, $[\eth, \eth'](\bar{\tau})$, $[\text{Þ}, \eth'](\tau)$, and $[\text{Þ}', \eth'](\tau)$ commutator relations imply

$$\eth H = 2\tau(H + \Psi - G), \qquad \eth' H = 2\bar{\tau}(H + \Psi - G),$$
$$\text{Þ}H = \rho(H + F), \qquad \text{Þ}'H = -\mu(H + F), \quad (32)$$

where

$$F \equiv 2\tau\bar{\tau}, \qquad G \equiv 2\mu\rho + \frac{w}{3}. \quad (33)$$

One checks that the integrability conditions for the system (6)–(33) of partial differential equations (PDEs) are identically satisfied, indicating that corresponding solutions exist. Those for which u^a is nonexpanding additionally satisfy

$$\theta \sim \mu - q\bar{\rho} = 0 \quad (34)$$

[cf. (96) and (100)]. However, (34) does not follow as a consequence of the ansätze; this implies the existence of expanding anisotropic perfect fluid models in class I (Sec. III). Also, the scalar invariant $\mu\rho$ may be strictly negative, which is incompatible with (34); as a consequence, the class I Einstein spaces are not necessarily static (Sec. II C).

B. Metric structure and subclassification

The first, second, and last parts of (13) and (14) precisely account for the hypersurface orthogonality of k^a, l^a, and $m^a \leftrightarrow \bar{m}^a$, respectively. Thus real scalar fields u, v, (zero weighted) and U, V [$(-1,-1)$, respectively, $(1,\ 1)$ weighted], and complex scalar fields ζ (zero weighted) and Z [$(1,-1)$ weighted] exist such that

$$d_a u = \frac{\Psi^{1/3}}{U} k_a, \qquad d_a v = \frac{\Psi^{1/3}}{V} l_a, \qquad d_a \zeta = \frac{\Psi^{1/3}}{Z} m_a. \quad (35)$$

By (A16) this is equivalent to

$$\text{Þ}' u = -\Psi^{1/3}/U, \qquad \text{Þ}u = \eth u = \eth' u = 0, \quad (36)$$

$$\text{Þ}v = -\Psi^{1/3}/V, \qquad \text{Þ}'v = \eth v = \eth' v = 0, \quad (37)$$

$$\eth' \zeta = \Psi^{1/3}/Z, \qquad \text{Þ}\zeta = \text{Þ}'\zeta = \eth\zeta = 0, \quad (38)$$

$$\eth\bar{\zeta} = \Psi^{1/3}/\bar{Z}, \qquad \text{Þ}\bar{\zeta} = \text{Þ}'\bar{\zeta} = \eth'\bar{\zeta} = 0. \quad (39)$$

The commutator relations (28)–(31) applied to u, v, ζ, and $\bar{\zeta}$ then yield

$$\eth U = \eth' U = \eth V = \eth' V = 0, \quad (40)$$

$$\text{Þ}Z = \text{Þ}'Z = \text{Þ}\bar{Z} = \text{Þ}'\bar{Z} = 0. \quad (41)$$

Hence, when we take these fields as coordinates, (35)–(41) imply that the zero-weighted fields UV and $Z\bar{Z}$ only depend on (u, v), respectively, $(\zeta, \bar{\zeta})$, such that all class I metrics are conformally related to direct sums of metrics on two-spaces:

$$g_{ab} = \Psi^{-2/3}(g^{\perp}_{ab} \oplus g^{\Sigma}_{ab}), \quad (42)$$

$$g^{\perp}_{ab} \equiv 2\Psi^{2/3} m_{(a}\bar{m}_{b)} = 2Z\bar{Z}(\zeta, \bar{\zeta}) d_{(a}\zeta d_{b)}\bar{\zeta}, \quad (43)$$

$$g^{\Sigma}_{ab} \equiv -2\Psi^{2/3} k_{(a} l_{b)} = -2UV(u, v) d_{(a} u d_{b)} v. \quad (44)$$

The line elements of $g^{\perp}_{ab}$ and g^{Σ}_{ab} will be denoted by $ds^2_{\perp}$, respectively, ds^2_{Σ}.

In the case where such a two-space is not of constant curvature, however, we will construct more suitable coordinates in the sequel. Inspired by the GHP manipulations of [34] for type D vacua [35], we start this construction by deducing suitable combinations of the scalar invariants F, G, H, and Ψ. From (A16), (10), and (15)–(33), it is found that

$$d_a F = 3\Psi^{1/3}\varphi\alpha_a, \qquad d_a G = 3\Psi^{1/3}\gamma\beta_a, \quad (45)$$

$$d_a \varphi = 2\Psi^{1/3} x\alpha_a, \qquad d_a \gamma = 2\Psi^{1/3} y\beta_a, \quad (46)$$

$$d_a x = \Psi^{1/3}\alpha_a, \qquad d_a y = \Psi^{1/3}\beta_a, \quad (47)$$

where

$$\alpha_a \equiv \bar{\tau} m_a + \tau\bar{m}_a, \qquad \beta_a \equiv \mu k_a - \rho l_a \quad (48)$$

are invariantly defined one-forms and

$$\varphi \equiv \frac{H + F}{3\Psi^{1/3}}, \qquad \gamma \equiv \frac{-H + \Psi + F + 2G}{3\Psi^{1/3}}, \quad (49)$$

$$x \equiv \frac{H + \Psi - G}{3\Psi^{2/3}}, \qquad y \equiv \frac{-H + 2\Psi + G}{3\Psi^{2/3}}. \quad (50)$$

Consequently, the scalar invariants,

$$C \equiv 3(\varphi - x^2) = 3(\gamma - y^2), \quad (51)$$

$$D \equiv -x^3 - Cx + F = y^3 + Cy - G, \quad (52)$$

are *constant* ($d_a C = d_a D = 0$). From (50) and (52) it follows that F, G, H, and Ψ are biunivocally related to x, y, C, and D, where

$$2\tau\bar{\tau} \equiv F = x^3 + Cx + D, \quad (53)$$

$$2\mu\rho \equiv G - \frac{w}{3} = y^3 + Cy - D - \frac{w}{3}, \quad (54)$$

$$2\eth'\tau \equiv H = 2x^3 + 3x^2 y + Cy - D, \quad (55)$$

$$\Psi = (x+y)^3 \neq 0. \tag{56}$$

Barnes [26] partitioned class I according to the position of the gradient $\nabla^a\Psi$ relative to Σ and $\Sigma^\perp$. This relates to the vanishing of the invariants $\tau\bar{\tau} = -\pi\tau$ or $\mu\rho$, maximal symmetry of $g^\perp_{ab}$ or g^Σ_{ab}, and spatial rotation or boost isotropy of g_{ab}, as follows.

First assume $\tau = 0$. In this case (25) and the first parts of (33) and (49)–(52) imply

$$\begin{aligned} H = F = \varphi = 0, \qquad \Psi - G = 3x\Psi^{2/3}, \\ C = -3x^2, \qquad D = 2x^3, \end{aligned} \tag{57}$$

such that x is constant. In combination with the last part of (14) and (16) and the first parts of (47) and (48) one gets

$$\tau\bar{\tau} = 0 \Leftrightarrow \pi = \tau = 0 \Leftrightarrow x = \text{const} \Leftrightarrow \nabla^a\Psi \in \Sigma. \tag{58}$$

The $[\eth, \eth']$ commutator relation applied to ζ, $\bar{\zeta}$, and Z imply $\eth Z = \eth'\bar{Z} = 0$ and $\eth\eth' Z = 3x\Psi^{2/3}Z$. Herewith the Gaussian curvature of the two-space with metric $g^\perp_{ab}$ becomes

$$\begin{aligned} K^\perp &= -(Z\bar{Z})^{-1}(\ln(Z\bar{Z}))_{,\zeta\bar{\zeta}} = -\Psi^{-2/3}\eth\eth'(\ln Z\bar{Z}) \\ &= -\Psi^{-2/3}\eth\left(\frac{\eth' Z}{Z}\right) = -3x, \end{aligned} \tag{59}$$

where the dual of (35) was used in the calculation. In conjunction with the results of Goode and Wainwright [36], we conclude that (58) yields the class I solutions which are locally rotationally symmetric (LRS) of label II in the Stewart-Ellis classification [37], characterized by $g^\perp_{ab}$ having constant curvature $K^\perp = -3x$. As is well known (see e.g. the appendix of [38]) the coordinates ζ and $\bar{\zeta}$ may then be adapted such that $Z\bar{Z}(\zeta, \bar{\zeta}) = (1 + K^\perp\zeta\bar{\zeta}/2)^{-1}$ in (43), or an alternative form may be taken:

$$\begin{aligned} ds^2_\perp &= \frac{2d\zeta d\bar{\zeta}}{1 + \frac{K^\perp}{2}\zeta\bar{\zeta}} = Y^2_\perp(dx_1^2 + \cos(\sqrt{k_\perp}x_1)^2 dx_2^2), \\ K^\perp &= k_\perp Y_\perp^{-2}, \qquad k_\perp \in \{-1, 0, 1\}. \end{aligned} \tag{60}$$

Now assume $\mu\rho = 0$. It follows from (20)–(23), (33), and (55) and the second parts of (49)–(52) that

$$\begin{aligned} S = \gamma = 0, \qquad G = \frac{w}{3} \equiv \frac{\Lambda}{3}, \\ -H + 2\Psi + \frac{\Lambda}{3} = 3y\Psi^{2/3}, \qquad C = -3y^2, \\ D = -2y^3 - \frac{\Lambda}{3}, \end{aligned} \tag{61}$$

such that y is constant. In combination with (15) and the second parts of (47) and (48) this implies

$$\mu\rho = 0 \Leftrightarrow \mu = \rho = 0 \Leftrightarrow y = \text{const} \Leftrightarrow \nabla^a\Psi \in \Sigma^\perp. \tag{62}$$

By a similar reasoning as in the case of $\tau = 0$ one concludes that (62) yields the locally boost-isotropic Einstein spaces of Petrov type D, characterized by g^Σ_{ab} having constant curvature

$$K^\Sigma = -3y, \tag{63}$$

such that in this case one may take $UV(u, v) = (1 - K^\Sigma uv/2)^{-1}$ in (44) and we have

$$\begin{aligned} ds^2_\Sigma &= -\frac{2dudv}{1 - \frac{K^\Sigma}{2}uv} = Y^2_\Sigma(dx_3^2 - \cos(\sqrt{k_\Sigma}x_3)^2 dx_4^2), \\ K^\Sigma &= k_\Sigma Y_\Sigma^{-2}, \qquad k_\Sigma \in \{-1, 0, 1\}. \end{aligned} \tag{64}$$

With (42) and ds^2_Σ written in the second form, it is clear that

$$\partial_{x_4}{}^a = -\Psi^{-2/3}Y^2_\Sigma\cos(\sqrt{k_\Sigma}x_3)^2 d^a x_4 \tag{65}$$

is a HO timelike Killing vector field.

Four subclasses of class I thus arise, which were labeled by Barnes as follows:

$$\begin{aligned} \text{IA}: \tau = 0 = \mu\rho, \qquad \text{IB}: \tau = 0 \neq \mu\rho, \\ \text{IC}: \tau \neq 0 = \mu\rho, \qquad \text{ID}: \tau \neq 0 \neq \mu\rho. \end{aligned} \tag{66}$$

We proceed with the respective integrations. Notice that in the joint case $\mu\rho\tau = 0$ one has

$$2(\tau\bar{\tau} + \mu\rho) = (x+y)^3 + K(x+y)^2 - \frac{w}{3}, \tag{67}$$

with $K = K^\perp$ for $\tau = 0$ and $K = K^\Sigma$ for $\mu\rho = 0$. When $\tau \neq 0$ or $\mu\rho \neq 0$ we may take x, respectively, y as a coordinate, where (47), (48), and (56) imply

$$(x+y)(\bar{\tau}m_a + \tau\bar{m}_a) = d_a x, \tag{68}$$

$$(x+y)(\mu k_a - \rho l_a) = d_a y. \tag{69}$$

In view of (42)–(44) and (56) it then remains to determine suitable complementary coordinates for x in $g^\perp_{ab}$ or y in g^Σ_{ab}.

For $\tau \neq 0$, Frobenius's theorem and (68) suggest to examine whether zero-weighted functions ϕ and f exist such that

$$i\frac{x+y}{2\tau\bar{\tau}}(\tau\bar{m}_a - \bar{\tau}m_a) = f d_a\phi. \tag{70}$$

By (A16) this amounts to calculating the integrability conditions of the system

$$Þ\phi = Þ'\phi = 0, \qquad \bar{\tau}\eth\phi = -\tau\eth'\phi = i\frac{x+y}{2f}, \tag{71}$$

which turn out to be

$$Þf = Þ'f = 0, \qquad \alpha_a\nabla^a f = 0. \tag{72}$$

These last equations have the trivial solution $f = 1$, for which a solution ϕ of (71) is determined up to an irrelevant constant. Herewith the invariantly defined one-form on the left-hand side in (70) is exact, and we take ϕ as the coordinate complementary to x. On solving (68) and (70) with $f = 1$ for m_a and $\bar{m}_a$ and using (53) we conclude from (43) that

$$ds_\perp^2 = \frac{dx^2}{2\tau\bar{\tau}} + 2\tau\bar{\tau}d\phi^2, \qquad 2\tau\bar{\tau} = x^3 + Cx + D \quad (73)$$

for classes IC and ID. Clearly, the metric solutions should be restricted to spacetime regions where $x^3 + Cx + D > 0$ for consistency, while

$$\partial_\phi{}^a = i\frac{\tau\bar{m}^a - \bar{\tau}m^a}{x+y} = \frac{2\tau\bar{\tau}}{(x+y)^2}d^a\phi, \quad (74)$$

is a HO spacelike KVF.

For $\mu\rho \neq 0$ one analogously considers

$$ð\psi = ð'\psi = 0, \qquad \mu Þ\psi = \rho Þ'\psi = \frac{x+y}{2g}, \quad (75)$$

but the integrability conditions of this system are now

$$ðg = ð'g = 0, \qquad \beta_a\nabla^a g = -gS\frac{\mu^2 + q^2\rho^2}{q\mu\rho}. \quad (76)$$

So $g = 1$ is *only* a solution in the Einstein subcase $S = 0$, for which we then get

$$ds_\Sigma^2 = \frac{dy^2}{2\mu\rho} - 2\mu\rho d\psi^2, \qquad 2\mu\rho = y^3 + Cy - D - \frac{\Lambda}{3} \quad (77)$$

from (44), (54), (69), and (75), with KVF

$$\partial_\psi{}^a = \frac{\mu k^a + \rho l^a}{x+y} = -\frac{2\mu\rho}{(x+y)^2}d^a\psi, \quad (78)$$

which is timelike for $\mu\rho > 0$ and spacelike for $\mu\rho < 0$. In general, the second vector field in (78) is always HO: the integrability conditions of (76) are checked to be identically satisfied, such that solutions g and a corresponding solution ψ of (75) exist. However, taking ψ as a complementary coordinate of y eventually leads to a very complicated system of coupled PDEs for $g = g(y, \psi)$, which is impossible to solve explicitly. We shall remedy this in Sec. III A but now discuss characterizing features of the Einstein space limit cases.

C. Characterizations of PE Petrov type *D* Einstein spaces

Petrov type D Einstein spaces constitute the class $\mathcal{D}_0$ (cf. the Introduction) and are all explicitly known. The line elements are obtained by putting the electromagnetic charge parameter Φ_0 or $e^2 + g^2$ equal to zero in the $\mathcal{D}$ metrics given by Debever *et al.* [14], respectively, García [15]. These coordinate forms generalize and streamline those found by Kinnersley [39] in the $\Lambda = 0$ case.

Recently, a manifestly invariant treatment of $\mathcal{D}_0$, making use of the GHP formalism, was presented [34]. Within GHP, $\mathcal{D}_0$ metrics are characterized by the existence of a complex null tetrad with respect to which (11) and $\Phi_{ij} = 0$ hold [i.e., the tetrad is a WPNT and (7) and (8) with $S = 0$ hold]. According to the Goldberg-Sachs theorem [20], (13) holds and characterizes WPNTs as well. The scalar invariant identities (see [34,40])

$$\mu\bar{\rho} = \bar{\mu}\rho, \qquad \pi\bar{\pi} = \tau\bar{\tau}, \quad (79)$$

just as (15), (16), (20), and (21) and the first equation of (25) are also valid in general. From these relations it follows that

$$(12) \Leftrightarrow (14), \quad (80)$$

i.e., a Petrov type D Einstein space is PE if and only if the WPNT directions are HO. In fact, it can readily be shown by a more detailed analysis than in [34] that if the spacetime belongs to Kundt's class, i.e., if one of the PNDs is moreover nondiverging, one has

$$\mu = 0 \Rightarrow \rho = 0 \quad \text{or} \quad \rho - \bar{\rho} \neq 0 \neq \pi + \bar{\tau}. \quad (81)$$

Equations (4), (5), and (31) in [34] then imply

$$\rho = \bar{\rho} \neq 0 \Rightarrow \mu = \bar{\mu} \neq 0 = \pi + \bar{\tau}, \quad (82)$$

$$\pi = -\bar{\tau} \neq 0 \Rightarrow \mu = \bar{\mu}, \qquad \rho = \bar{\rho}. \quad (83)$$

One concludes that the Kundt and Robinson-Trautman subclasses of $\mathcal{D}_0$ have empty intersection, and that the latter consists of PE spacetimes for which *both* PNDs are nontwisting but diverging. These results—which remain valid for the electrovac class $\mathcal{D}$, just as the two theorems below—are implicit in [14], where the concerning PE metrics form the Einstein space subclasses of the classes labeled by

$$\begin{aligned} &C^{00}\colon \tau = 0 = \mu\rho, \qquad C^0{}_+\colon \tau = 0 \neq \mu\rho, \\ &C^0{}_-\colon \tau \neq 0 = \mu\rho, \qquad C^*\colon \tau \neq 0 \neq \mu\rho. \end{aligned} \quad (84)$$

By the Einstein space specifications $S = 0$ and $w = \Lambda =$ const, the boost field q disappears from the equations (7)–(33) and is not determined by the geometry, in contrast to the situation for perfect fluids $S \neq 0$ (cf. Sec. III A). Moreover, Eqs. (5) and (6), i.e. the requirement that an US given by (4) exists, are decoupled from (7)–(33) and is not needed to derive (15)–(33) from (7)–(14). From the integrability of the complete set (6)–(33) and (66) and the above we conclude:

Theorem 2.1: The closed set (7)–(33) characterizes the class $\mathcal{D}_0$ of PE Petrov type D Einstein spaces, which are precisely those Einstein spaces for which the WPNT directions are HO or, alternatively, those which belong to Barnes's class I, all admitting a 1-degree freedom of USs in all regions of spacetime. Barnes's boost-isotropic Kundt classes IA and IC coincide with C^{00}, respectively, $C^0{}_-$, while the Robinson-Trautman members of $\mathcal{D}_0$ constitute ${C^0}_+$ and C^*, form the Einstein space subclasses of IB, respectively, ID, and possess nontwisting but diverging PNDs at each point.

The result is in agreement with Proposition B.1, which provides criteria for deciding when a Petrov type D spacetime allows for an US, regardless of the structure of the energy-momentum tensor. The hypersurface orthogonality (13) and (14) of the WPNT directions corresponds to criterion 5 and is actually equivalent to a 1-degree freedom of USs. It is worth mentioning that all LRS II spacetimes, i.e. those exhibiting (pseudo-) spherical or plane symmetry, share this property with the $\mathcal{D}_0$ and $\mathcal{D}$ metrics. On the other hand, certainly not all PE spacetimes admit an US. For instance, the Gödel solution is an LRS I PE perfect fluid of Petrov type D, described in GHP by (7)–(13) and

$$S/2 = w = p = -3\Psi_2 = -2\mu\rho = \text{const} > 0, \tag{85}$$

$$\pi = \tau = 0, \qquad \mu = q\rho, \qquad \bar{\mu} = -\mu, \qquad \bar{\rho} = -\rho, \tag{86}$$

with $q > 0$ being annihilated by all weighted GHP derivatives; hence the invariant $(\mu - \bar{\mu})(\rho - \bar{\rho}) = 4\mu\rho = 4q\rho^2$ appearing in criterion 2 of Proposition B.1 is strictly negative, and it follows that the Gödel solution does not admit an US. As another example, the spatially homogeneous $\Lambda = 0$ vacuum metrics

$$ds^2 = t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 - dt^2, \tag{87}$$

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1, \qquad p_1p_2p_3 \neq 0 \tag{88}$$

attributed to Kasner [41] are PE [42]. If the p_i are all different, there is a complete group G_3I of isometries and the Petrov type is I. In this case ${\partial_t}^a$ is the up to reflection unique Weyl principal vector field and hence the only possible US candidate; however, its shear tensor has the nonzero eigenvalues $(1/3 - p_i)/t$ and hence the spacetime does not admit an US. On the other hand, if two p_i's are equal it follows that $p_2 = p_3 = -2p_1 = 2/3$ (without loss of generality). Then the line element represents a Petrov type D, nonstationary, plane-symmetric vacuum which, according to Theorem 2.1, admits a 1-degree freedom of USs (cf. the end of this section).

In theorem 3 of [26] it is claimed that all vacuum spacetimes admitting an US are static, which would generalize Trümper's result [24] by including Petrov type D. However, this conclusion only holds when $\mu\rho \geq 0$. Indeed, a static member of $\mathcal{D}_0$ necessarily admits a rigid (i.e. nonexpanding) US, such that $\mu\rho \geq 0$, cf. (34). Conversely, when $\mu\rho = 0$ or $\mu\rho > 0$ for a PE member, it admits the HO timelike KVF (65), respectively, (78) and is thus static. This is in agreement with Proposition B.3: regarding $\mu = \rho = 0$ criterion 6'' tells that in fact all boost-isotropic spacetimes, with $\pi = -\bar{\tau}$ with respect to a WPNT, are static, while for $\mu\rho > 0$ one checks that criterion 2'' is satisfied by virtue of (11)–(33). In Appendix B the freedom of the rigid USs and HO timelike KVF directions (static observers) in these cases is also specified, which is in accordance with a result by Wahlquist and Estabrook [43]. In summary we have the following:

Theorem 2.2: A Petrov type D Einstein space is static if it admits a rigid US. This is precisely the case when the spacetime is PE and has a positive or zero scalar invariant $\mu\rho$, being the product of the divergences of (nontwisting) Weyl principal null vectors k^a and l^a subject to $k^a l_a = -1$. For $\mu\rho > 0$ there is an up to reflection unique rigid US, defined from the geometry by (4) and $q = \mu/\rho$ and parallel to the unique HO timelike KVF direction. For $\mu\rho = 0 \Rightarrow \mu = \rho = 0$ (classes IA and IC) all USs are rigid USs and have a 1-degree freedom, while the HO timelike KVF directions are parametrized by two constants.

For completeness we display standard coordinate forms of the PE Petrov type D Einstein space metrics, as recovered here by (42), (56), (60), (64), (73), and (77).

C^{00} corresponds to (60) and (64). From (57), (59), (61), and (63) one deduces that

$$K^\perp = -3x = -3y = K^\Sigma, \qquad \Psi_2 = -\frac{\Lambda}{3} = 4x^3 \neq 0.$$

Rescaling ζ, u, and v by a factor $(2x)^{-1}$ one arrives at

$$ds^2 = \frac{2d\zeta d\bar{\zeta}}{1 + \frac{\Lambda}{2}\zeta\bar{\zeta}} - \frac{2dudv}{1 - \frac{\Lambda}{2}uv}, \qquad 1 + \frac{\Lambda}{2}\zeta\bar{\zeta} > 0, \qquad \Lambda \neq 0.$$

This represents the Einstein space limit $\Phi_0 = 0$ of Bertotti's static and homogeneous electrovac family with cosmological constant [44], exhibiting spatial rotation and boost isotropy (complete group G_6 of isometries). The $\Lambda = 0$ limit yields flat Minkowski spacetime.

${C^0}_+$ and ${C^0}_-$ correspond to (60) and (77), respectively, (64) and (73). Making use of (67), replacing in the ${C^0}_+$ (${C^0}_-$) case the coordinate y (x) by $r = -(2m)^{1/3}/(x + y)$, rescaling the remaining coordinates by a factor $(2m)^{-1/3}$ and writing $Y \equiv (2m)^{1/3} > 0$ one finds

$$ds^2 = r^2(d\xi^2 + \delta\cos(\sqrt{k}\xi)^2 d\eta^2) + \frac{dr^2}{g_k(r)} - \delta g_k(r)d\chi^2,$$

$$g_k(r) = k - \frac{2m}{r} - \frac{\Lambda}{3}r^2, \qquad k = K(2m)^{1/3} \in \{-1, 0, 1\},$$

with $\delta = 1$ for ${C^0}_+$ ($\delta = -1$ for ${C^0}_-$). These solutions have a complete group G_4 of isometries acting on spacelike (timelike) three-dimensional orbits, and for $\Lambda = 0$ correspond to Kinnersley's case I (IV) with $l = 0$. The static region of ${C^0}_+$ $[g_k(r) > 0]$ yields class A in the classification of static Petrov type D vacua by Ehlers and Kundt [4]; ${C^0}_-$ is static everywhere and corresponds to class B. Regarding C^0_+, the subcase $k = 1$ reproduces after $\xi \mapsto \pi/2 - \xi$ the well-known forms of the spherically symmetric Schwarzschild-Kottler interior and exterior metrics [45]; the subcase $k = \Lambda = 0$, $r > 0$ [$g_k(r) < 0$] gives another form of the plane-symmetric Kasner metrics (cf. supra).

C^* corresponds to (73) and (77), which gives the line element

$$ds^2 = \frac{1}{(x+y)^2}\left(\frac{dx^2}{f(x)} + f(x)d\phi^2 + \frac{dy^2}{g(y)} - g(y)d\psi^2\right),$$
$$f(x) = x^3 + Cx + D > 0, \qquad g(y) = -f(-y) - \frac{\Lambda}{3}. \tag{89}$$

The KVFs $\partial_\phi{}^a$ and $\partial_\psi{}^a$ generate the complete, Abelian group G_2 of isometries. For $\Lambda = 0$, (89) is the form of the C metric obtained by Levi-Civita and recovered by Ehlers and Kundt, and corresponds to Kinnersley's case IIIA. It is generally assumed—and suggested in the original paper [5]—that the Kinnersley-Walker form

$$ds^2 = \frac{1}{\alpha^2(\xi+\eta)^2}\left(\frac{d\xi^2}{h(\xi)} + h(\xi)d\phi^2 + \frac{d\eta^2}{k(\eta)} - k(\eta)d\psi^2\right),$$
$$h(\xi) = 1 - \xi^2 - 2m\alpha\xi^3 > 0, \qquad k(\eta) = -h(-\eta) \tag{90}$$

equivalently describes the gravitational field of the $\Lambda = 0$ C metric. However, this is not entirely correct. Equating the Lorentz invariants appearing in the right-hand sides of the equations in (50)–(52), calculated for the metrics (89) and (90), yields

$$\begin{aligned} x &= -(2m)^{1/3}\left(\alpha\xi + \frac{1}{6m}\right),\\ y &= -(2m)^{1/3}\left(\alpha\eta - \frac{1}{6m}\right),\\ C &= -\frac{1}{3(2m)^{4/3}},\\ D &= \alpha^2 - \frac{1}{54m^2}. \end{aligned} \tag{91}$$

Hence (90) only covers the range $C < 0$, $D > -2(-\frac{C}{3})^{3/2}$, whereas in general the constant scalar invariants C and D are allowed to take any real value. Yet, the cubic $f(x)$ has discriminant $-4C^3 - 27D^2$; thus it has three distinct real roots if and only if

$$C < -3\left(\frac{D}{2}\right)^{2/3} \quad \Leftrightarrow C < 0, \qquad |D| < 2\left(\frac{-C}{3}\right)^{3/2}. \tag{92}$$

Thus (91) *is* compatible for this case, and by further rescaling ϕ and ψ with a factor $\alpha(2m)^{-1/3}$ one arrives at (90); (92) is equivalent with $m\alpha < 1/\sqrt{27}$, leading to the physical interpretation of two uniformly accelerating masses. Recently, Hong and Teo [46] introduced a normalized factored form for this situation, which greatly simplifies certain analyses of the C metric. A further coordinate transformation can be made such that the Schwarzschild metric is comprised as the subcase $\alpha = 0$. This was further exploited for the full $\mathcal{D}$ class in [19].

Finally, we write down the equations which determine all USs for a member of C^*, in the coordinates y and ψ of (89). Let

$$U^a = -\frac{x+y}{\sqrt{g(y)}}\partial_t{}^a, \qquad V^a = -(x+y)\sqrt{g(y)}\partial_y{}^a$$

in the static region and

$$U^a = -(x+y)\sqrt{-g(y)}\partial_y{}^a, \qquad V^a = \frac{x+y}{\sqrt{-g(y)}}\partial_t{}^a$$

in the nonstatic region, and gauge fix $k^a = (U^a + V^a)/\sqrt{(2)}$, $l^a = (U^a - V^a)/\sqrt{(2)}$. The unit timelike field (4) is an US if and only if (6) holds; this translates to $q = q(y, \psi)$ and

$$g(y)(q \pm 1)q_{,y} + (q \mp 1)(q_{,\psi} + g'(y)q) = 0. \tag{93}$$

Here and below the upper (lower) signs should be taken in the static (nonstatic) region. For solutions $q = q(y)$, i.e. $q_{,\psi} = 0$, direct integration of (93) yields

$$g(y)(q(y) \mp 1)^2 = E_\pm q(y), \tag{94}$$

with $E_+ \geq 0$ and $E_- < 0$ constants of integration. Notice that in the static region the solution $q(y) = 1 \Leftrightarrow E_+ = 0$ yields the unique static observer. In the case $q_{,\psi} \neq 0$ the solutions get implicitly determined by an equation of the form $\psi = \psi(y, q)$, on applying the method of characteristics for first-order PDEs (see e.g. [47]). In the subcase where $C = D = \Lambda = 0$, $g(y)$ reduces to y^3 and this equation reads

$$\psi = -\left(\frac{y(q \mp 1)^{2/3}}{q^{1/3}}\right)^2 \int \frac{(q \mp 1)^{4/3}dq}{3q^{5/3}} + Z\left(\frac{y(q \mp 1)^{2/3}}{q^{1/3}}\right). \tag{95}$$

Here Z is a free function of its argument, making the 1-degree freedom of USs more explicit. Replacing (73) by (60) does not alter these equations, i.e., the above remains valid for $C^0{}_+$. Then $C = D = 0$ is equivalent to $x = K^\perp = 0$, cf. (57) and (59); for $\Lambda = 0$ the nonstatic region $y < 0$ corresponds to the plane-symmetric Kasner vacuum metrics, where $y = -(3t/2)^{-2/3}$ and a rescaling of the other coordinates recovers (87), the USs being determined by (94) and (95) with the lower sign.

III. PERFECT FLUID GENERALIZATIONS OF THE C METRIC

A. Line element

We resume the integration of class I started at the end of Sec. II C. We thereby focus on the subclass ID characterized by $\tau \neq 0 \neq \mu\rho$. Let us first summarize what we did so far. We started off with the closed set (6)–(33) of first-order GHP equations in the seven (weighted) real variables

Ψ, S, w, μ, ρ, q, $\eth'\tau$ and the complex variable τ. These variables are equivalent to two dimensionless spin and boost gauge fields, e.g. $\tau/\bar{\tau}$ and μ/ρ, and seven real scalar invariants. The boost and spin gauge fields could serve to invariantly fix the tetrad—the ID members being therefore anisotropic—but can be further ignored. For the C^*-Einstein spaces, $S = 0$ and $w = \Lambda = \text{const}$, and we remarked that q is not a part of the intrinsic describing set of variables. Hence we end up with four real scalar invariants in this subcase. These invariants are equivalent to the two constants C and D and two independent functions x and y, which we took as coordinates and in terms of which, on adding two coordinates ϕ and ψ related to the symmetries, the corresponding C^* metric can be expressed. In the perfect fluid case $S \neq 0$, (8) gives the boost field q which, starting from an arbitrary gauge (k^a, l^a), turns u^a given by (4) into the invariantly defined fluid four-velocity. The four invariants and their use persist, just as the coordinate ϕ. However, the scalar invariants w and S are no longer constant and ψ is no longer a suitable coordinate. Thus we need one more scalar invariant for our description and one remaining coordinate complementary to y.

For the first purpose it is natural to look at the kinematics of the fluid, which are fully determined by

$$b \equiv 2\nabla_{(a}u_{c)}m^a\bar{m}^c = \nabla_a u_c v^a v^b = \frac{\theta}{3}, \tag{96}$$

$$\dot{u}_{\parallel} \equiv v_a\dot{u}^a, \qquad \dot{u}^{\perp}_a \equiv 2\bar{m}_{(a}m_{c)}\dot{u}^c. \tag{97}$$

Here

$$v^a \equiv \frac{1}{\sqrt{2q}}(qk^a - l^a) \tag{98}$$

is the intrinsic spacelike vector field, which determines at each point the up to reflection unique normalized vector orthogonal to u^a and lying in the PND plane Σ, while $\dot{u}_{\parallel}$ and $\dot{u}^{\perp}_a$ are the component along v^a, respectively, projection onto $\Sigma^{\perp}$ of the acceleration $\dot{u}^a$. In analogy with (96) we define the invariant

$$\tilde{b} \equiv 2\nabla_{(a}v_{c)}m^a\bar{m}^c. \tag{99}$$

The relation with GHP quantities is

$$b = \frac{\mu - q\rho}{\sqrt{2q}}, \qquad \tilde{b} = -\frac{\mu + q\rho}{\sqrt{2q}}, \tag{100}$$

$$\dot{u}_{\parallel} = (2q)^{-3/2}(\text{Þ}'q + q\text{Þ}q) = \frac{\text{Þ}q}{\sqrt{2q}} - b, \tag{101}$$

$$-\dot{u}^{\perp}_a = \bar{\tau}m_a + \tau\bar{m}_a \equiv \alpha_a = \frac{d_a x}{x+y}. \tag{102}$$

Notice that (100) is equivalent to

$$bu_a - \tilde{b}v_a = \mu k_a - \rho l_a \equiv \beta_a = \frac{d_a y}{x+y}. \tag{103}$$

In combination with (53) and (54), Eqs. (102) and (103) imply

$$2\tau\bar{\tau} = \dot{u}^{\perp}_a\dot{u}^{\perp a} = x^3 + Cx + D, \tag{104}$$

$$2\mu\rho = \tilde{b}^2 - b^2 = y^3 + Cy - D - \frac{w}{3}. \tag{105}$$

We choose b as the final describing invariant and use $\tilde{b}$ and $\dot{u}_{\parallel}$ as auxiliary variables. In view of (101) and (102) one deduces that the differential information for S, w, and b comprised in (6)–(33) is precisely

$$D_a S = -S\dot{u}_a, \tag{106}$$

$$d_a w = -\mathbf{u}(w)u_a, \qquad \mathbf{u}(w) = -3bS, \tag{107}$$

$$d_a b = -\mathbf{u}(b)u_a, \qquad \mathbf{u}(b) = -\mathbf{v}(\tilde{b}) + \tilde{b}(\dot{u}_{\parallel} - \tilde{b}) - \frac{S}{2}, \tag{108}$$

$$\mathbf{v}(\tilde{b}) = -\frac{x+y}{2}(3y^2 + C). \tag{109}$$

From (109) it follows that $\tilde{b}$ is nonconstant, such that we may see the second part of (108) as a definition of $\dot{u}_{\parallel}$. Equation (107) is nothing but the energy, respectively, momentum conservation equations for a perfect fluid subject to $D_a w = 0$. The first part of Eq. (108) confirms that $D_a\theta = 0$ [23,26], while the second implies again that the expansion scalar does not vanish in general (cf. the end of Sec. II A and below).

For the second purpose we rely on the hypersurface orthogonality of u^a by assumption: zero-weighted real scalar fields t and I exist such that

$$d_a t = Iu_a. \tag{110}$$

The integrability condition hereof is

$$D_a I = -I\dot{u}_a = -I(\dot{u}^{\perp}_a + \dot{u}_{\parallel}v_a), \tag{111}$$

which is equivalent to

$$\eth I = \tau I, \qquad \mathbf{v}(I) = -\dot{u}_{\parallel}I. \tag{112}$$

From $\tilde{b} \neq 0$, (103) and (110) it follows that t is functionally independent of y (and of x and ϕ) and we take it as the fourth coordinate. With the aid of (110) and (111), Eq. (106) and the first parts of (107) and (108) precisely tell that $A \equiv \frac{S}{2I}$, w and b only depend on t. Hence $\tilde{b} = \tilde{b}(y, t)$ from (105). On using (102) and (103), the first part of (112) is equivalent to $J = J(y, t)$, where

$$J \equiv \frac{x+y}{I\tilde{b}}. \tag{113}$$

Eliminating $\dot{u}_{\parallel}$ between the second parts of (108) and (112), and using $\mathbf{v}(x+y) = -\tilde{b}(x+y)$ implied by (102) and (103), yields

$$\tilde{b}^2\mathbf{v}(J) = \tilde{b}J\mathbf{u}(b) + A(x+y). \quad (114)$$

Inverting (103) and (110) we get

$$(x+y)u_a = \tilde{b}Jd_at, \qquad (x+y)v_a = bJd_at - \frac{d_ay}{\tilde{b}}, \quad (115)$$

or dually

$$-\frac{u^a}{x+y} = \frac{\partial_t{}^a}{\tilde{b}J} + b\partial_y{}^a, \qquad -\frac{v^a}{x+y} = \tilde{b}\partial_y{}^a. \quad (116)$$

Thus in the chosen coordinates (114) reads

$$\partial_yJ(y,t) = \tilde{b}(y,t)^{-3}[b'(t) - A(t)]. \quad (117)$$

From (42) and (73), $g^{\Sigma}_{ab} = (v_av_b - u_au_b)/(x+y)^2$ and the only remaining Eq. (107) we obtain the line element

$$ds^2 = (x+y)^{-2}[ds^2_{\perp} + ds^2_{\Sigma}], \quad (118)$$

$$ds^2_{\perp} = \frac{dx^2}{f(x)} + f(x)d\phi^2, \qquad f(x) = x^3 + Cx + D, \quad (119)$$

$$ds^2_{\Sigma} = \left(bJdt - \frac{dy}{\tilde{b}}\right)^2 - (\tilde{b}Jdt)^2, \quad (120)$$

where

$$b = b(t), \qquad w = w(t), \qquad A \equiv \frac{\tilde{b}JS}{2(x+y)} = A(t), \quad (121)$$

$$w'(t) = 6b(t)A(t) \Leftrightarrow d_aw = 6bAd_at = 3bSu_a, \quad (122)$$

$$\tilde{b} = \tilde{b}(y,t) = \sqrt{y^3 + Cy - D + b(t)^2 - w(t)/3}, \quad (123)$$

$$J = J(y,t) = [b'(t) - A(t)]\int\frac{dy}{\tilde{b}(y,t)^3} + L(t), \quad (124)$$

with $L(t)$ a free function of integration. The solutions are defined and regular in the coordinate regions

$$2\tau\bar{\tau} \equiv x^3 + Cx + D > 0, \quad (125)$$

$$\tilde{b}(y,t)^2 \equiv y^3 + Cy - D + b(t)^2 - w(t)/3 > 0. \quad (126)$$

Notice that we nowhere used $S \neq 0$ explicitly in the above integration procedure. Therefore, the above line element describes the *complete* class ID, including the C^*-vacuum limits which correspond to $w(t) = \Lambda$ and $A(t) = 0$, cf. (121). In this case the coordinate transformation $(t, y, x, \phi) \mapsto (\psi, y, x, \phi)$, which connects (118)–(124) to the original form (89), eliminates $b(t)$ and $L(t)$ and follows from (4), (78), (98), (100), and (115), giving

$$d_a\psi = -\frac{x+y}{2\mu\rho}(\mu k_a + \rho l_a) = \frac{x+y}{2\mu\rho}(\tilde{b}u_a - bv_a)$$
$$= Jd_at + \frac{b}{\tilde{b}(\tilde{b}^2 - b^2)}d_ay. \quad (127)$$

Hence, $\psi = \psi(y,t)$ and it is the solution of the consistent system

$$\partial_t\psi = J, \qquad \partial_y\psi = \frac{b}{\tilde{b}(\tilde{b}^2 - b^2)}, \quad (128)$$

the integrability condition hereof being precisely (117) with $A(t) = 0$. The transformation is singular at degenerate roots of $\tilde{b}^2$ and at the union of the black hole and acceleration horizons [19,48] $\tilde{b}^2 - b^2 \equiv f(-y) + \Lambda/3 = 0$, which separate the static from the nonstatic regions. Let us emphasize that the $b(t)$ freedom is essentially a freedom in the choice of coordinates. The form (89) describes the full C-metric manifold; y can take any value, and the sign of $f(-y) + \frac{\Lambda}{3}$ is positive in the static region and negative in the nonstatic region. In the form (118)–(120) y is always spacelike and we have constructed t as a synchronized timelike coordinate corresponding to an US u^a, with associated expansion rate $\theta(t) = 3b(t)$; for fixed $b(t)$ the range of y is constrained by (126) and only this subregion of the manifold is described by the coordinates. For example, (118)–(124) with $A(t) = 0$, $w(t) = \Lambda/3$, $b(t) = 0$, and $L(t) = 1$ [which formally reduces to (89) on putting $t = \psi$ additionally] only describes the static part of the C metric, the vector field u^a then lying along the unique HO timelike KVF direction. However, in the neighborhood of any point with coordinate label y, the metric can be described by (118)–(124), by choosing $b(t)^2 > f(-y) + \Lambda/3$.

We neither used $\tau \neq 0$. This implies that the line element of the complete class IB, characterized by $\mu\rho \neq 0 = \tau$ and constituted by all LRS II Einstein spaces and shear-free perfect fluids with $D_aw = 0$, is described by (118)–(124), with (119) replaced by (60). This class was first described by Kustaanheimo [49] and rediscovered by Barnes [26], both using different coordinates [see also (16.49) and (16.51) in [16]].

Of course, the result (118)–(124) could have been obtained without referring to GHP calculus. Barnes [26] showed that the metric can be written in the form

$$ds^2 = (x+Y)^{-2}(f^{-1}dx^2 + fd\phi^2 + dz^2 - e^{2Z}dt^2), \quad (129)$$

with $f = f(x)$. Indeed, from (115), (117), and (123) it follows that $(x+y)v_a$ is exact: $(x+y)v_a = d_az$; z is used as a coordinate instead of y, and one puts $J\tilde{b} \equiv e^Z$, $Z = Z(z,t)$. Notice from (103), (110), and (113) that now

$$y = Y(z,t), \qquad Y_{,z} = -\tilde{b}, \qquad \theta = 3Y_{,t}e^{-Z}. \quad (130)$$

Let us directly attack the field equations in these coordinates, thereby correcting [26]. One can check that only four of the field equations are not identically satisfied [the indices 1–4 label the Weyl principal tetrad vectors naturally associated with (129)]:

$$G_{34} = 0 = -Y_{,tz} + Y_{,t}Z_{,z}, \tag{131}$$

$$\begin{aligned} G_{11} - G_{33} &= 0 \\ &= f' - 2Y_{,zz} + (x+Y)(Z_{,z}^2 + Z_{,zz} - f''/2), \end{aligned} \tag{132}$$

$$\begin{aligned} G_{11} &= p \\ &= 2(e^{-2Z}(Y_{,tt} - Y_{,t}Z_{,t}) - Y_{,z}Z_{,z} - f'/2)(x+Y) \\ &\quad + (Z_{,zz} + Z_{,z}^2)(x+Y)^2 + 3(Y_{,z}^2 + f - Y_{,t}^2 e^{-2Z}), \end{aligned} \tag{133}$$

$$\begin{aligned} G_{33} + G_{44} &= S \\ &= 2(x+Y)[Y_{,zz} - Y_{,z}Z_{,z} + e^{-2Z}(Y_{,tt} - Y_{,t}Z_{,t})]. \end{aligned} \tag{134}$$

Hence, *if supplemented with* $\theta \sim Y_{,t} = 0$, these equations are the ones obtained by Barnes in [26]: Eq. (131) $\equiv \mathbf{v}(\theta) = 0$ was missed out, and both Eqs. (133) and (134) differ from Eqs. (4.23), respectively, (4.24) in [26] by a term $2(x+Y)Y_{,tt}e^{-2Z}$. Thus, it is clear that with these differences a correct nonexpanding solution can be found, but the analysis of expanding solutions will be incorrect.

Differentiating (133) twice with respect to x yields $d^4 f(x)/dx^4 = 0$, whence

$$f(x) = ax^3 + bx^2 + cx + d. \tag{135}$$

Substituting this in Eq. (133), and equating coefficients of powers of x, leads to

$$Z_{,zz}(z,t) + Z_{,z}(z,t)^2 = 3aY(z,t) - b, \tag{136}$$

$$Y_{,zz}(z,t) = \frac{c}{2} - Y(z,t)b + \frac{3}{2}aY(z,t)^2. \tag{137}$$

The solutions $Y(z,t)$ of the last equation are defined by

$$\int^{Y(z,t)} \frac{dr}{\sqrt{ar^3 - br^2 + cr + f_1(t)}} - z + f_2(t) = 0, \tag{138}$$

which can be solved for z in terms of Y. This eventually suggests to transform coordinates from (z,t) into (y,t), with $y = Y(z,t)$. Rescaling and translating coordinates allows us to set $a = 1$ and $b = 0$. One can check that the remaining equations lead exactly to Eqs. (117) and (122), recovering solutions (118)–(124).

B. Properties

Consider the metric (118)–(124), for which we assume henceforth that it describes a perfect fluid [$A(t) \neq 0$]. In contrast to the Einstein subcase, u^a is now the unique invariantly defined fluid velocity, and the expansion rate $\theta(t)$ and energy density $w(t) - \Lambda$ of the fluid are scalar invariants. Expressions for the pressure $p + \Lambda$ and the components $\dot{u}^{\perp a}$ and $\dot{u}_{\|}$ of the acceleration follow from (102), (108), (109), (116), and (119)–(124):

$$p = \frac{2(x+y)A(t)}{(\tilde{b}J)(y,t)} - w(t), \qquad \dot{u}^{\perp a} = -(x+y)f(x)\partial_x{}^a,$$

$$\dot{u}_{\|} = \tilde{b}(y,t) - \frac{x+y}{\tilde{b}(y,t)}\left(\frac{3y^2 + C}{2} + \frac{b'(t) - A(t)}{(\tilde{b}J)(y,t)}\right).$$

The fluid is nonshearing and nonrotating, i.e. u^a is an US. Because of (13) and (14) criterion 5 of Proposition B.1 is satisfied, such that there is a 1-degree freedom of USs. These can be found by taking $q = 1$ in (4) and (98), hereby fixing the (k^a, l^a) gauge geometrically, and solving (6) with q replaced by Q, the USs then being $(Qk^a + l^a)/\sqrt{2Q}$. This yields $Q = Q(y,t)$ and

$$\begin{aligned} &\tilde{b}J[\tilde{b}(Q+1) + b(Q-1)]Q_{,y} + (Q-1)Q_{,t} \\ &\quad = -2Q(Q-1)\tilde{b}(\tilde{b}J)_{,y} \\ &\quad = -\frac{Q(Q-1)}{\tilde{b}}[(3y^2 + C)\tilde{b}J + 2(b' - A)]. \end{aligned}$$

If the class is to be used as a cosmological model, it is interesting to discuss the intrinsic freedom. By (115) and (122) we have that $2A(t)d_a t = Su_a$ and $J(y,t)d_a t = (x + y)u_a/\tilde{b}$ are invariantly defined one-forms, and hence so is $L(t)d_a t$ because of (124). It follows that $\frac{L}{A}(t)$ is a scalar invariant. Moreover, as $A(t)d_a t$ is exact we may remove the only remaining coordinate freedom on t by putting $A(t) = 1$, such that the conservation of energy equation (122) can be considered as a definition $\theta(t) = w'(t)/2$. Hence, in this most general picture for $S \neq 0$, the scalar constants C, D and invariants $\frac{L}{A}(t)$, $w(t)$ characterize the model within the class. Notice that the presence of two invariantly defined, distinguishing free functions could have been predicted, since after elimination of $\dot{u}_{\|}$, there are two scalar invariants $\mathbf{u}(b)$ and $\mathbf{u}(S)$ remaining unprescribed in the system of Eqs. (106)–(109).

In this fashion however, the physical implications remain obscure: it would be nice to have a free function, with a clear physical interpretation, instead of L/A. Spacetimes with $L(t) = 0$ have $w(t)$ as the only free function. If $L(t) \neq 0$, $L(t)$ can alternatively be fixed to 1 by a t-coordinate transformation. In this case the metric structure functions display the expansion scalar, the energy density, and the pressure [since $2A(t)d_a t = (w+p)u_a$]; these are related by energy conservation (122), where $w(t)$ and $A(t)$ can be chosen freely. Alternatively, one can subdivide further in $\theta = 0$ and $\theta \neq 0$. In the case $\theta = 0$, the energy density $w - \Lambda$ is constant because of (122) and can be chosen freely, just as $A(t)$. In the most interesting case $\theta \neq 0$, $w(t)$ and $\theta(t)$ can be chosen freely, determining Su_a via (122).

Thus class ID provides a class of anisotropic cosmological models with arbitrary evolution of energy density and (nonzero) expansion.

Regarding symmetry, all perfect fluid ID models admit at least one KVF $\partial_\phi{}^a$ given by (74), which at each point yields an invariantly defined spacelike vector orthogonal to $\dot{u}^{\perp a}$ and lying in $\Sigma^\perp$. If ϕ is chosen to be a periodic coordinate, with range given by $[-\pi E, \pi E[$, the spacetime is cyclically symmetric. We will then refer to the region $F \equiv f(x) = 0$, where the norm of $\partial_\phi{}^a$ vanishes, as the axis of symmetry [48,50]. Finding the complete group of isometries and their nature is trivial in our approach. The functions x, y, w, and L/A are invariant scalars, such that $K^a d_a x = K^a d_a y = K^a d_a w = K^a d_a (L/A) = 0$ for any KVF K^a. As the ID models are anisotropic, it follows that the complete isometry group is at most G_2, and if it is G_2, both w and L/A are constant. Conversely, when w and L/A are constant we have $\theta \equiv 3b = 0$ from (122), $\tilde{b} = \tilde{b}(y)$ from (123), and $J(y,t) = -A(t)F_2(y)$ from (124). By redefining the time coordinate such that $A(t) = 1$ one sees from (118)–(124) that $\partial_t{}^a$ is a HO timelike KVF. We conclude that the ID perfect fluid models have at least one spacelike KVF $\partial_\phi{}^a$, which may be interpreted as the generator of cyclic symmetry. They admit a second independent KVF if and only if both scalar invariants w and L/A are constant, in which case the spacetimes are static and the complete group of isometries is Abelian G_2, generated by $\partial_\phi{}^a$ and $\partial_t{}^a$.

Consider the case where $f(x)$ has 3 real nondegenerate roots x_i, i.e. (92) holds. If $x_1 < x_2 < x_3$ then $f(x) > 0$ for all $x \in]x_1, x_2[$. Furthermore, we let ϕ be a periodic coordinate. The ratio between circumference and radius of a small circle around the axis, $x = x_1$ or $x = x_2$, is given by

$$\lim_{\substack{x\to x_2\\ <}} \frac{2\pi E\sqrt{f(x)}}{\int_x^{x_2}\sqrt{f^{-1}(x)}dx} = -\pi E(3x_2^2 + C), \tag{139}$$

respectively,

$$\lim_{\substack{x\to x_1\\ >}} \frac{2\pi E\sqrt{f(x)}}{\int_{x_1}^{x}\sqrt{f^{-1}(x)}dx} = \pi E(3x_1^2 + C). \tag{140}$$

It is only possible to choose the parameter E such that the complete axis is regular, if $3x_1^2 + C = -(3x_2^2 + C)$. However, eliminating C and D between this equation and $f(x_1) = f(x_2) = 0$ implies $x_1 = x_2$. Consequently, if $f(x)$ has three real nondegenerate roots, the spacetime contains a conical singularity. This echoes the properties of the C metric [5,48], and suggests the presence of a cosmic string.

IV. CONCLUSIONS AND DISCUSSION

A new class of Petrov type D exact solutions of Einstein's field equation in a perfect fluid with spatially homogeneous energy density has been presented. It consists of all anisotropic such fluids with shear-free normal four-velocity, and generalizes a previously found class to include nonzero expansion. The analysis and integration was rooted in the 2 + 2 structure of the metric and use of invariant quantities. This approach clarified the link with the vacuum C-metric limit, and certain properties of the vacuum case are inherited. However, the presence of the perfect fluid defines generically two extra invariants. For the expanding solutions, this translates into an evolution of energy density and expansion which can be chosen freely. This subclass contains only one (potentially cyclic) symmetry.

The viability of these solutions as a low-symmetry class of cosmological models is subject to further research. More in particular, it should be clarified whether a thermodynamic interpretation of the perfect fluid can be made [51]—it is certainly not possible to prescribe a barotropic equation of state $p = p(w)$. The relation with the C metric also suggests to further examine the arising coordinate ranges, properties of horizons, and whether an interpretation as a perturbation for small masses of a known PF solution exists for certain members.

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APPENDIX A: GEROCH-HELD-PENROSE FORMALISM

The GHP formalism [16,28] is a complex, scalar formalism, which is a "weighted" version of the Newman-Penrose (NP) tetrad formalism. Use is made of a complex null tetrad $(\mathbf{e}_1{}^a, \mathbf{e}_2{}^a, \mathbf{e}_3{}^a, \mathbf{e}_4{}^a) \equiv (m^a, \bar{m}^a, l^a, k^a)$, where

$$k^a l_a = -1, \qquad m^a \bar{m}_a = 1 \tag{A1}$$

and all other inner products vanish. To put it in other words, at each point one takes a timelike plane, two vectors k^a and l^a lying along its real null directions, and two vectors m^a and $\bar{m}^a$ lying along the complex conjugate null directions of the orthogonal spacelike plane, these pairs of vectors satisfying the normalization conditions (A1). We use the labels $\hat{a}$, $\hat{b}$, etc. for the tetrad indices. The basic variables of the formalism are the spin coefficients ($\Gamma_{\hat{a}\hat{b}\hat{c}} \equiv \mathbf{e}_{\hat{a}}{}^a \nabla_c (\mathbf{e}_{\hat{b}})_a \mathbf{e}_{\hat{c}}{}^c = -\Gamma_{\hat{b}\hat{a}\hat{c}}$)

$$\kappa = \Gamma_{414}, \qquad \tau = \Gamma_{413}, \qquad \sigma = \Gamma_{411}, \qquad \rho = \Gamma_{412}, \tag{A2}$$

$$\nu = \Gamma_{233}, \qquad \pi = \Gamma_{234}, \qquad \lambda = \Gamma_{232}, \qquad \mu = \Gamma_{231}, \tag{A3}$$

the 9 independent components of the traceless part of the

Ricci tensor $S_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}$,

$$\Phi_{00} = \tfrac{1}{2}S_{ab}k^a k^b, \qquad \Phi_{22} = \tfrac{1}{2}S_{ab}l^a l^b, \tag{A4}$$

$$\Phi_{01} = \tfrac{1}{2}S_{ab}k^a m^b, \qquad \Phi_{12} = \tfrac{1}{2}S_{ab}l^a m^b, \tag{A5}$$

$$\Phi_{02} = \tfrac{1}{2}S_{ab}m^a m^b, \qquad \Phi_{11} = \tfrac{1}{2}S_{ab}(k^a l^b + m^a \bar{m}^b), \tag{A6}$$

with $\Phi_{ji} = \bar{\Phi}_{ij}$, the multiple

$$\Pi \equiv \frac{R}{24} \tag{A7}$$

of the Ricci scalar, and the 10 independent components of the Weyl tensor C_{abcd},

$$\Psi_0 = C_{abcd}k^a m^b k^c m^d, \qquad \Psi_4 = C_{abcd}l^a \bar{m}^b l^c \bar{m}^d, \tag{A8}$$

$$\Psi_1 = C_{abcd}k^a l^b k^c m^d, \qquad \Psi_3 = C_{abcd}l^a k^b l^c \bar{m}^d, \tag{A9}$$

$$\Psi_2 = C_{abcd}k^a m^b \bar{m}^c l^d. \tag{A10}$$

Changes of the tetrad leaving the null directions spanned by k^a, l^a, m^a, and $\bar{m}^a$ invariant, and at the same time preserving the normalization conditions (A1), consist of boosts

$$k^a \to Ak^a, \qquad l^a \to A^{-1}l^a \tag{A11}$$

and spatial rotations

$$m^a \to e^{i\theta}m^a. \tag{A12}$$

Quantities transforming under (A11) and (A12) as

$$\eta \to A^{(\mathrm{w}_p+\mathrm{w}_q)/2}e^{i(\mathrm{w}_p-\mathrm{w}_q)\theta/2}\eta \tag{A13}$$

are called *well weighted* of type $(\mathrm{w}_p, \mathrm{w}_q)$ or $(\mathrm{w}_p, \mathrm{w}_q)$ *weighted* [*zero weighted* in the case of type (0, 0)]. They have boost weight $\mathrm{w}_B(\eta) = \frac{\mathrm{w}_p+\mathrm{w}_q}{2}$ and spin weight $\mathrm{w}_S(\eta) = \frac{\mathrm{w}_p-\mathrm{w}_q}{2}$. One can check that the GHP basic variables are well weighted, their weights following from the definitions (A2)–(A13)—see also equation (7.36) in [16]. For example, $\mathrm{w}_B(\nu) = -2$, $\mathrm{w}_S(\nu) = -1$, implying ν is of type $(-3, -1)$. The following derivative operators are defined such that a well-weighted quantity η is transformed in a well-weighted quantity:

$$D_{\hat{a}}\eta = \mathbf{e}_{\hat{a}}(\eta) + \Gamma_{34\hat{a}}\mathrm{w}_B(\eta)\eta + \Gamma_{12\hat{a}}\mathrm{w}_S(\eta)\eta. \tag{A14}$$

When η is of type $(\mathrm{w}_p, \mathrm{w}_q)$ one can check that

$$\mathrm{w}_B(D_{\hat{a}}\eta) = \mathrm{w}_B(\eta) + \tilde{w}_B(\hat{a}),$$
$$\mathrm{w}_S(D_A\eta) = \mathrm{w}_S(\eta) + \tilde{w}_S(\hat{a}),$$

where

$$\tilde{w}_B(\hat{a}) = \begin{cases} 1, & \hat{a}=4, \\ -1, & \hat{a}=3, \\ 0, & \hat{a}=1,2 \end{cases}, \qquad \tilde{w}_S(\hat{a}) = \begin{cases} 1, & \hat{a}=1 \\ -1, & \hat{a}=2 \\ 0, & \hat{a}=3,4. \end{cases}$$

One uses the notation

$$\eth \equiv D_1, \qquad \eth' \equiv D_2, \qquad \text{Þ}' \equiv D_3, \qquad \text{Þ} \equiv D_4. \tag{A15}$$

Notice that the differential of zero-weighted scalars f can be expressed as

$$d_a f = -\text{Þ}' f k_a - \text{Þ} f l_a + \eth' f m_a + \eth f \bar{m}_a \tag{A16}$$

$$= -\boldsymbol{l}(f)k_a - \mathbf{k}(f)l_a + \bar{\mathbf{m}}(f)m_a + \mathbf{m}(f)\bar{m}_a \tag{A17}$$

$$= -\mathbf{u}(f)u_a + \mathbf{v}(f)v_a + \bar{\mathbf{m}}(f)m_a + \mathbf{m}(f)\bar{m}_a, \tag{A18}$$

where u^a and v^a are related to k^a and l^a according to (4), respectively, (98).

The basic (or "structure") equations of the GHP formalism are the following:

(a) the commutator relations of the weighted derivatives, in the joint $D_{\hat{a}}$ notation given by

$$\begin{aligned}[D_{\hat{a}}, D_{\hat{b}}]\eta = {}& 2\Gamma^{\hat{c}}_{[\hat{a}\hat{b}]}D_{\hat{c}}\eta + \mathrm{w}_B(\eta)(R_{34\hat{a}\hat{b}} \\ & + 2\Gamma_{3\hat{c}[\hat{a}}\Gamma^{\hat{c}}_{|4|\hat{b}]})\eta + \mathrm{w}_S(\eta)(R_{12\hat{a}\hat{b}} \\ & + 2\Gamma_{1\hat{c}[\hat{a}}\Gamma^{\hat{c}}_{|2|\hat{b}]})\eta + \tilde{w}_B(\hat{b})\Gamma_{34\hat{a}}D_{\hat{b}}\eta \\ & + \tilde{w}_S(\hat{b})\Gamma_{12\hat{a}}D_{\hat{b}}\eta - \tilde{w}_B(\hat{a})\Gamma_{34\hat{b}}D_{\hat{a}}\eta \\ & - \tilde{w}_S(\hat{a})\Gamma_{12\hat{b}}D_{\hat{a}}\eta;\end{aligned}$$

(b) 12 complex Ricci identities (or equations), namely,

$$\begin{aligned}\mathbf{e}_{\hat{c}}(\Gamma_{\hat{a}\hat{b}\hat{d}}) - \mathbf{e}_{\hat{d}}(\Gamma_{\hat{a}\hat{b}\hat{c}}) = {}& R_{\hat{a}\hat{b}\hat{c}\hat{d}} - 2\Gamma_{\hat{a}\hat{e}[\hat{c}|}\Gamma^{\hat{e}}_{\hat{b}|\hat{d}]} \\ & - 2\Gamma_{\hat{a}\hat{b}\hat{e}}\Gamma^{\hat{e}}_{[\hat{c}\hat{d}]}\end{aligned}$$

with $[\hat{a}\hat{b}] = [14], [23]$ (the complex conjugates corresponding to $[\hat{a}\hat{b}] = [24], [13]$);

(c) the Bianchi identities (or equations)

$$\begin{aligned}\mathbf{e}_{[\hat{f}}(R_{\hat{c}\hat{d}]\hat{a}\hat{b}}) = {}& -2R_{\hat{a}\hat{b}\hat{e}[\hat{c}}\Gamma^{\hat{e}}_{\hat{d}\hat{f}]} + \Gamma^{\hat{e}}_{\hat{a}[\hat{c}}R_{\hat{d}\hat{f}]\hat{e}\hat{b}} \\ & - \Gamma^{\hat{e}}_{\hat{b}[\hat{c}}R_{\hat{d}\hat{f}]\hat{e}\hat{a}}.\end{aligned}$$

One can show that, after writing the directional derivatives $\mathbf{e}_{\hat{a}}$ in terms of the weighted derivatives $D_{\hat{a}}$, these basic equations (a)-(c) form a consistent, closed system of PDEs in the variables (A2)–(A10) and with formal derivative operators $D_{\hat{a}}$. Compared to the NP formalism, the 6 complex Ricci identities which concern directional derivatives of the non-well-weighted NP spin coefficients α, β, γ, and ϵ (corresponding to $[\hat{a}\hat{b}] = [12], [34]$) have been absorbed in the commutator relations. Explicitly, for a $(\mathrm{w}_p, \mathrm{w}_q)$-weighted scalar one gets

$$\begin{aligned}[\text{Þ}, \text{Þ}'](\eta) = {}& (\pi + \bar{\tau})\eth(\eta) + (\bar{\pi} + \tau)\eth'(\eta) \\ & + (\kappa\nu - \pi\tau + \Pi - \Phi_{11} - \Psi_2)\mathrm{w}_p\eta \\ & + (\overline{\kappa\nu} - \overline{\pi\tau} + \Pi - \Phi_{11} - \bar{\Psi}_2)\mathrm{w}_q\eta,\end{aligned} \tag{A19}$$

$$[\eth, \eth'](\eta) = (\mu - \bar{\mu})\text{Þ}(\eta) + (\rho - \bar{\rho})\text{Þ}'(\eta) + (\lambda\sigma - \mu\rho - \Pi - \Phi_{11} + \Psi_2)\mathrm{w}_p\eta - (\overline{\lambda\sigma} - \overline{\mu\rho} - \Pi - \Phi_{11} + \bar{\Psi}_2)\mathrm{w}_q\eta, \quad \text{(A20)}$$

$$[\text{Þ}, \eth](\eta) = \bar{\pi}\text{Þ}(\eta) - \kappa\text{Þ}'(\eta) + \bar{\rho}\eth(\eta) + \sigma\eth'(\eta) + (\kappa\mu - \sigma\pi - \Psi_1)\mathrm{w}_p\eta + (\overline{\kappa\lambda} - \overline{\pi\rho} - \Phi_{01})\mathrm{w}_q\eta, \quad \text{(A21)}$$

together with the equations obtained by applying the complex conjugate and/or prime dual operation to (A21). This *prime dual operation* is generated by interchanging $k^a \leftrightarrow l^a$ and $m^a \leftrightarrow \bar{m}^a$, which comes down to

$$\kappa \leftrightarrow -\nu, \qquad \tau \leftrightarrow -\pi, \qquad \sigma \leftrightarrow -\lambda, \qquad \rho \leftrightarrow -\mu, \quad \text{(A22)}$$

$$\Phi_{ij} \leftrightarrow \Phi_{2-i2-j}, \qquad \Psi_i \leftrightarrow \Psi_{4-i}, \quad \text{(A23)}$$

$$\text{Þ} \leftrightarrow \text{Þ}', \qquad \eth \leftrightarrow \eth'. \quad \text{(A24)}$$

The interchange (A24) means that $(\text{Þ}'\eta)' = \text{Þ}\eta'$, etc., and is due to (A14) and

$$\mathrm{w}_B(\eta') = -\mathrm{w}_B(\eta),$$
$$\mathrm{w}_S(\eta') = -\mathrm{w}_S(\eta), \quad \text{i.e.,}$$
$$\mathrm{w}_p(\eta') = -\mathrm{w}_p(\eta),$$
$$\mathrm{w}_q(\eta') = -\mathrm{w}_q(\eta).$$

Regarding complex conjugation one has $\overline{\text{Þ}\eta} = \text{Þ}\bar{\eta}$, $\overline{\eth\eta} = \eth'\bar{\eta}$ and

$$\mathrm{w}_B(\bar{\eta}) = \mathrm{w}_B(\eta),$$
$$\mathrm{w}_S(\bar{\eta}) = -\mathrm{w}_S(\eta), \quad \text{i.e.,}$$
$$\mathrm{w}_p(\bar{\eta}) = \mathrm{w}_q(\eta),$$
$$\mathrm{w}_q(\bar{\eta}) = \mathrm{w}_p(\eta).$$

Explicitly, the 12 complex Ricci identities read

$$\text{Þ}\tau - \text{Þ}'\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + \Phi_{01} + \Psi_1, \quad \text{(A25)}$$

$$\eth\rho - \eth'\sigma = (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa + \Phi_{01} - \Psi_1, \quad \text{(A26)}$$

$$\text{Þ}\sigma - \eth\kappa = (\rho + \bar{\rho})\sigma + (\bar{\pi} - \tau)\kappa + \Psi_0, \quad \text{(A27)}$$

$$\text{Þ}\rho - \eth'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau + \kappa\pi + \Phi_{00}, \quad \text{(A28)}$$

$$\text{Þ}'\sigma - \eth\tau = -\sigma\mu - \bar{\lambda}\rho - \tau^2 + \kappa\bar{\nu} - \Phi_{02}, \quad \text{(A29)}$$

$$\text{Þ}'\rho - \eth'\tau = -\bar{\mu}\rho - \lambda\sigma - \tau\bar{\tau} + \kappa\nu - 2\Pi - \Psi_2 \quad \text{(A30)}$$

and their prime duals (A25′)–(A30′). Finally, the Bianchi identities involve weighted derivatives of the Riemann tensor components. In full generality they are given in Ref. [16], (7.32a-k), or [33], (4.12.36-41).

The formalism is especially suited for situations where two null directions are singled out by the geometry, such that k^a and l^a can be chosen along them. In particular, the Weyl tensor of a Petrov type D spacetime has precisely two PNDs; choosing k^a and l^a along them is equivalent to condition (11), and a complex null tetrad realizing this condition is called a *Weyl principal null tetrad*. When (7) and (11) are both satisfied, the Bianchi identities reduce to

$$0 = \sigma(2\Phi_{11} + 3\Psi_2) - \bar{\lambda}\Phi_{00}, \quad \text{(A31)}$$

$$\text{Þ}\Psi_2 + \text{Þ}'\Phi_{00} + 2\text{Þ}\Pi = \rho(2\Phi_{11} + 3\Psi_2) - \bar{\mu}\Phi_{00}, \quad \text{(A32)}$$

$$\text{Þ}\Phi_{11} + \text{Þ}'\Phi_{00} + 3\text{Þ}\Pi = 2(\rho + \bar{\rho})\Phi_{11} - (\mu + \bar{\mu})\Phi_{00}, \quad \text{(A33)}$$

$$\eth\Psi_2 + 2\eth\Pi = -\tau(2\Phi_{11} - 3\Psi_2) + \bar{\nu}\Phi_{00}, \quad \text{(A34)}$$

$$\eth\Phi_{11} - 3\eth\Pi = 2(\tau - \bar{\pi})\Phi 11 - \bar{\nu}\Phi_{00} + \kappa\Phi_{22}, \quad \text{(A35)}$$

$$\eth\Phi_{00} = \kappa(2\Phi_{11} - 3\Psi_2) - \bar{\pi}\Phi_{00} \quad \text{(A36)}$$

and their prime duals (A31′)–(A36′).

In general, the GHP formalism may be used to find a class of solutions, defined by a particular set of properties. One first translates these properties in terms of GHP variables, yielding (algebraic or differential) constraints on the system of basic equations, then recloses the resulting extended system (integrability analysis), and finally describes the corresponding metrics in terms of coordinates (integration). These coordinates are four suitable, functionally independent zero-weighted scalars f; they may be combinations of (derivatives of) basic variables, appearing in the reclosed system $\mathcal{S}$ itself, or "external" coordinates associated to HO vector fields due to Frobenius's theorem. The geometric duals of the null tetrad vectors, and hence the metrics $g_{ab} = -2k_{(a}l_{b)} + 2m_{(a}\bar{m}_{b)}$, are obtained by inverting (A16) for the chosen f's. Eventually the remaining equations of $\mathcal{S}$ are written in terms of these coordinates and the resulting PDEs are solved as far as possible. We refer to [52] for enlightening discussions, and to e.g. [53] or this work for illustrations. In particular for Petrov type D spacetimes, notice that zero-weighted combinations of WPNT spin coefficients and their weighted derivatives (e.g. $\mu\rho$ or $\eth'\tau$) are *scalar (Lorentz) invariants* x, which are thus annihilated by any present KVF K^a, $\mathbf{K}(x) = \mathcal{L}_{\mathbf{K}}x = 0$. This facilitates the detection of KVFs. More generally, zero-weighted tensor fields $T_{ab\cdots}$, algebraically constructed from the Riemann tensor, WPNT vectors and covariant derivatives thereof, are *invariantly defined* by the geometry, and $\mathcal{L}_{\mathbf{K}}T_{ab\cdots} = 0$.

APPENDIX B: (RIGID) SHEAR-FREE NORMALITY AND STATICITY OF PETROV TYPE D SPACETIMES

Consider (an open region of) a spacetime and a unit timelike vector field u^a defined on it. Choose a null vector field k^a. At each point, k^a and u^a span a timelike plane Σ, the first null direction of which is spanned by k^a. Construct the null vector field l^a by taking at each point the unique vector lying along the second null direction and satisfying $k^a l_a = -1$. Then u^a is decomposed as in (4), where $q = A^2$, $A = -(\sqrt{2}k^a u_a)^{-1}$. The field v^a defined in (98) determines at each point the up to reflection unique unit spacelike vector lying in Σ and orthogonal to u^a. The electric and magnetic parts of the Weyl tensor with respect to u^a can be decomposed as

$$\begin{aligned} E_{ab} &\equiv C_{acbd}u^c u^d \\ &= (\Psi_2 + \bar{\Psi}_2)[v_a v_b - \bar{m}_{(a}m_{b)}] \\ &\quad + \left[\frac{\Psi_4 + q^2\bar{\Psi}_0}{2q} m_a m_b + 2\frac{\Psi_3 - q\bar{\Psi}_1}{\sqrt{2q}} m_{(a}v_{b)}\right] \\ &\quad + \text{c.c,} \end{aligned} \tag{B1}$$

$$\begin{aligned} H_{ab} &\equiv \frac{\eta_{acmn}}{2} C^{mn}{}_{bd}u^c u^d \\ &= i(\Psi_2 - \bar{\Psi}_2)[v_a v_b - \bar{m}_{(a}m_{b)}] \\ &\quad + i\left[\frac{\Psi_4 - q^2\bar{\Psi}_0}{2q} m_a m_b + 2\frac{\Psi_3 + q\bar{\Psi}_1}{\sqrt{2q}} m_{(a}v_{b)}\right] \\ &\quad + \text{c.c.} \end{aligned} \tag{B2}$$

If u^a exists such that $H_{ab} = 0$, the Weyl tensor is PE with respect to u^a, the spacetime itself being also called PE. A criterion in terms of Weyl tensor concomitants, deciding whether this is the case, follows from the flow diagram 9.1 in [16] and Theorem 1 in [54].

Suppose now that the spacetime admits a unit timelike vector field u^a satisfying (1)—corresponding to an US, i.e. forming the tangent field of a shear-free and vorticity-free cloud of test particles. Within the GHP formalism based on k^a and l^a as introduced above, this is the case if and only if a $(-2,-2)$-weighted field q exists satisfying (5) and (6). By virtue of these relations, the $\overline{[\eth, \eth'](q)}$ commutator relation yields (12), adding $2q\overline{[(A26)} - (A26)']$ to the $\overline{[\text{Þ}' - q\text{Þ}, \eth'](q)}$ commutator relation gives $\Psi_3 + q\bar{\Psi}_1 = 0$ and the combination $q^2\overline{(A27)} - (A27)' + q[(A29)' - \overline{(A29)}]$ produces $\Psi_4 - q^2\bar{\Psi}_0 = 0$. Hence $H_{ab} = 0$ from (B2), and if we choose k^a to be a (multiple) PND, $\Psi_0 = 0$ ($\Psi_0 = \Psi_1 = 0$), then also l^a is a (multiple) PND, $\Psi_4 = 0$ ($\Psi_4 = \Psi_3 = 0$). Hence the spacetime must be either conformally flat (all Ψ_i zero), and then USs are always admitted [see e.g. (6.15) in [16]], or PE and of Petrov type D or I, the Weyl tensor being PE with respect to u^a. For Petrov type I, there are 4 distinct PNDs, and u^a is the up to reflection unique timelike vector lying along the intersection of the planes spanned by two particular pairs of PNDs. For Petrov type D, k^a and l^a can be taken to be the multiple PNDs, and u^a lies in the plane Σ spanned by them.

Propositions 4 in [55] and 16 in [56] imply intrinsic, easily testable criteria for deciding when a Petrov type I spacetime admits an US, respectively, is static. Here we present likewise criteria in the Petrov type D case. These criteria are invariant statements, in terms of GHP basic variables and weighted derivatives associated to an arbitrary WPNT. Given a Petrov type D spacetime in coordinates, the determination of the PNDs, and hence the WPNTs, is straightforward and can be performed covariantly [56]. It then suffices to fix one WPNT and calculate the appearing spin-boost covariant expressions by using definitions (A2)–(A10) and (A14). For complex $(2k, 2k)$-weighted scalars ($k \in \mathbb{Z}$) $z = \text{Re}(z) + i\,\text{Im}(z)$ we mean with $z > 0$ ($z < 0$) that z is real and strict positive (negative) in the sequel.

It turns out that, given (11) and (12), the integrability conditions of (6) are identically satisfied. Thus we find:

Proposition B.1: A Petrov type D spacetime admits an US if and only if, with respect to an arbitrary WPNT, Ψ_2 is real and one of the following sets of conditions holds:

(1) $\sigma \neq 0$, the scalar invariant $\lambda\sigma > 0$, and $q_0 \equiv \lambda/\bar{\sigma}$ satisfies (5) and (6);
(2) $\rho \neq \bar{\rho}$, the real scalar invariant $(\mu - \bar{\mu})(\rho - \bar{\rho}) > 0$, and $q_0 \equiv -(\mu - \bar{\mu})/(\rho - \bar{\rho})$ satisfies (5) and (6);
(3) $\lambda = \sigma = \mu - \bar{\mu} = \rho - \bar{\rho} = 0$, the scalar invariant $\kappa\nu \neq 0$ and one of the following situations occurs, where q_0 defined in each subcase satisfies (6) and where $b \equiv (\pi + \bar{\tau})/\kappa$, $c \equiv \nu/\kappa$:
   (a) $\text{Im}(b)\,\text{Im}(c) > 0$ and $q_0 \equiv \text{Im}(c)/\text{Im}(b)$ also satisfies $q_0^2 - \text{Re}(b)q_0 + \text{Re}(c) = 0$;
   (b) $b = \text{Re}(b)$, $c < 0$, and $q_0 \equiv (b + \sqrt{b^2 - 4c})/2$;
   (c) $b > 0$, $c > 0$, $b^2 \geq 4c$, and $q_0 \equiv (b + \sqrt{b^2 - 4c})/2$ or $q_0 \equiv (b - \sqrt{b^2 - 4c})/2$;
(4) $\lambda = \sigma = \mu - \bar{\mu} = \rho - \bar{\rho} = 0$, and either
   (a) $\kappa = 0 \neq \nu$, $(\bar{\pi} + \tau)\nu > 0$, and $q_0 = \nu/(\pi + \bar{\tau})$ satisfies (6), or
   (b) $\kappa \neq 0 = \nu$, $(\pi + \bar{\tau})\kappa > 0$, and $q_0 = (\pi + \bar{\tau})/\bar{\kappa}$ satisfies (6);
(5) the WPNT directions are HO, i.e., (13) and (14) holds.

The subdivision of case 3 stems from a straightforward analysis of the first equation of (5). In cases 1, 2, 3a, 3b, and 4 there is a unique US, whereas there may be one or two USs in case 3c. Because of the number and nature of Eq. (6) there is a 1-degree freedom of USs in case 5, where the condition $\Psi_2 = \bar{\Psi}_2$ can be dropped since it is implied by the imaginary part of (A30) + (A30′) and (13) and (14). Important examples of spacetimes satisfying criterion 5 are the Petrov type D purely electric Einstein spaces and their "electrovac" generalizations (see [40,57] and Sec. II C) and all spacetimes with (pseudo-)spherical or planar symmetry (which constitute the LRS II Lorentzian spaces, see [16,37,58]). These examples all satisfy (7) on top of (13) and (14) and are further characterized by $\Phi_{00} = \Phi_{22} = (\Phi_{11} =)0$, respectively, $\pi = \tau = \eth R = 0$ (cf. [36]).

The spacetime will admit a unit timelike vector field u^a satisfying

$$u_{a;b} = -\dot{u}_a u_b, \tag{B3}$$

corresponding to a rigid US or modeling a rigid nonrotating cloud of test particles, if and only if a $(-2, -2)$-weighted field q exists satisfying (5), (6), and (34). Notice that, given (34), the third equation of (5) is identically satisfied. Hence we have

Proposition B.2: A Petrov type D spacetime admits a rigid US if and only if, with respect to an arbitrary WPNT, Ψ_2 is real and one of the following sets of conditions holds:

(1′) condition 1 with the third equation of (5) replaced by (34);

(2′) the scalar invariant $\mu\rho > 0$ and $q_0 \equiv \mu/\bar{\rho}$ satisfies (5) and (6);

(3′)-(5′) conditions 3–5 with $\mu - \bar{\mu} = \rho - \bar{\rho} = 0$ replaced by $\mu = \rho = 0$.

In case 5′, the spacetime possesses geodesic, shear-free, and nondiverging PNDs ($\kappa = \sigma = \rho = 0$, $\nu = \lambda = \mu = 0$)—thus belonging to Kundt's class—and HO Weyl principal complex null directions ($\lambda = \sigma = \pi + \bar{\tau} = 0$), and admits a 1-degree freedom of rigid USs.

The spacetime is static if and only if it admits a HO timelike KVF. An equivalent characterization was given by Ehlers and Kundt [4]: the spacetime is static if and only if a unit timelike vector field u^a exists for which shear, vorticity, and expansion scalar vanish, i.e. (B3) holds, and for which the acceleration $\dot{u}^a$ is Fermi propagated along the integral curves of u^a:

$$\ddot{u}_{[a} u_{b]} = 0. \tag{B4}$$

The field u^a is then parallel to a (HO and timelike) KVF and identified with a congruence of static observers. By a long but straightforward calculation, thereby simplifying expressions by means of (5), (6), (34), and (A25), (A25′), and the $[\text{Þ}, \text{Þ}'](q)$ commutator relation, one shows that the extra condition (B4) is equivalent to

$$(q\kappa + q^{-1}\bar{\nu})(\text{Þ}q + \sqrt{2q}) - 2\text{Þ}\bar{\nu} + 2q\text{Þ}\tau + \Phi_{12} - q\Phi_{01} = 0, \tag{B5}$$

$$\text{Þ}\text{Þ}q = \pi\tau + \overline{\pi\tau} - q(\kappa\pi + \overline{\kappa\pi}) - q^{-1}(\nu\bar{\pi} + \bar{\nu}\pi) + 2\Phi_{11} - \frac{R}{12} + 2\Psi_2. \tag{B6}$$

In case 5′ above, the Ricci equations (A25) and (A28), and (A28′) yield $\text{Þ}\tau = \Phi_{01}$ and $\Phi_{00} = \Phi_{22} = 0$, and so (B5) and (B6) reduce to

$$\Phi_{12} + q\Phi_{01} = 0, \tag{B7}$$

$$\text{Þ}\text{Þ}q = -2\tau\bar{\tau} + 2\Phi_{11} - \frac{R}{12} + 2\Psi_2. \tag{B8}$$

In the subcase $\Phi_{01} = \Phi_{12} = 0$ of (B7), the $[\text{Þ}, \text{Þ}']$, $[\text{Þ}, \eth]$, and $[\text{Þ}, \eth']$ commutators applied to q yield

$$\text{Þ}'\text{Þ}q = -q\text{Þ}\text{Þ}q + (\text{Þ}q)^2, \tag{B9}$$

$$\eth\text{Þ}q = \tau\text{Þ}q, \qquad \eth'\text{Þ}q = \bar{\tau}\text{Þ}q. \tag{B10}$$

The compatibility requirement of (B8)–(B10) with the commutator relations for $\text{Þ}q$ gives the single condition

$$\text{Þ}'R + q\text{Þ}R = 0. \tag{B11}$$

According to the Sach's star dual [28] of the LRS criterion in [36], the subcase $\text{Þ}'R = \text{Þ}R = 0$ of (B11) precisely corresponds to a boost-isotropic spacetime with $\pi + \bar{\tau} = 0$. From the above we conclude the following:

Proposition B.3: A Petrov type D spacetime is static if and only if, with respect to an arbitrary WPNT, one of the following sets of conditions holds:

(1″)-(4″) Ψ_2 is real, conditions 1′–4′ hold and q_0 additionally satisfies (B5) and (B6);

(5″a) condition 5′ holds, the scalar invariant $\Phi_{01}\Phi_{21} < 0$ and $q_0 \equiv -\Phi_{12}/\Phi_{01}$ satisfies (6) and (B8);

(5″b) condition 5′ holds, $\Phi_{01} = \Phi_{21} = 0$, the scalar invariant $(\text{Þ}'R)(\text{Þ}R) < 0$ and $q_0 \equiv -\text{Þ}'R/\text{Þ}R$ satisfies (6) and (B8);

(6″) the spacetime is (locally) boost isotropic and $\pi + \bar{\tau} = 0$.

The HO timelike KVF directions are parametrized by two constants in case 6″ [59], are 1 or 2 in number in case 3″c, and are unique in all other cases.

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