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Emergent gravity at a Lifshitz point from a Bose liquid on the lattice

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We propose a model with quantum bosons on the fcc lattice, which has a stable algebraic Bose liquid phase at low energy. We show that this phase is described by emergent quantum gravity at the Gaussian $z = 3$ Lifshitz fixed point in $3 + 1$ dimensions. The stability of this algebraic Bose liquid phase is guaranteed by the gauge symmetry of gravitons and self-duality of the low-energy field theory. By tuning one parameter in the lattice boson model we can drive a phase transition between the $z = 3$ Lifshitz gravity and another algebraic Bose liquid phase, described by gravity at the $z = 2$ Lifshitz point.

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The challenge of finding a satisfactory theory of quantum gravity has stimulated theoretical physics for many decades. We expect that such a theory should make sense of the quantum fluctuations of the space-time metric at low-energy scales, while providing a quantum mechanical completion of the system at short distances. Recently, a new approach to this long-standing puzzle has been proposed [1,2]. In this approach, gravity is treated using the traditional path integral methods of quantum field theory, but without assuming Lorentz invariance as a fundamental symmetry at short distances. The gauge symmetries are those space-time diffeomorphisms that preserve a preferred foliation of space-time by fixed time slices, generated by

$$\delta x^i = \xi^i(x^j, t), \qquad \delta t = \zeta(t). \tag{1}$$

In terms of the spatial metric $g_{ij}(x^k, t)$, the shift vector $N_i(x^j, t)$ and the lapse function $N(t)$, the action is

$$\begin{aligned} S &= \frac{1}{\kappa^2}\int d^D x dt N\sqrt{g}(L_K - L_V), \\ L_K &= \sum_{i,j} K_{ij}K^{ij} - \lambda\Big(\sum_i K_i^i\Big)^2, \end{aligned} \tag{2}$$

where

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - D_i N_j - D_j N_i) \tag{3}$$

is the extrinsic curvature of the constant time slices in the space-time foliation, D_i is the covariant derivative defined by g_{ij}, κ and λ are coupling constants, and the L_V can be an arbitrary local Lagrangian built from g_{ij}, its Riemann tensor $R^i{}_{jk\ell}$, and the covariant derivative D_i, without the use of time derivatives.

In this broader framework, new Gaussian fixed points of the renormalization group (RG) are possible. These novel fixed points are characterized by a dynamical scaling exponent $z \neq 1$ (which depends on the choices made in L_V), and they exhibit a Lifshitz-type scaling. Given the possibility of such Lifshitz gravity fixed points, it is natural to ask whether they can serve to provide an ultraviolet (UV) completion of gravity, and whether the pattern of the RG flow can restore $z = 1$ and Lorentz invariance in the infrared (IR) regime, where the theory can be tested against general relativity.

In this paper, we show that this new class of Lifshitz gravity fixed points [1,2] with $z = 2$ and $z = 3$ in $3 + 1$ dimensions can also emerge as *infrared* fixed points, in the low-energy limit describing certain lattice systems. The microscopic degrees of freedom of these systems are of the conventional type familiar in condensed matter, but their collective behavior leads at low energies to novel gapless phases, described by the Lifshitz gravity fixed points. The microscopic lattice degrees of freedom exhibit no gauge symmetry: The foliation-preserving gauge invariance at the Lifhitz gravity fixed points is entirely emergent in the low-energy description of the lattice system. The more fundamental physical objects on short length scales give rise to the Lifshitz gravitons as their low-energy and long-distance collective excitations.

An analogous mechanism, with an emergent U(1) gauge symmetry, was realized previously in the quantum dimer model on the cubic lattice. In this model, the short distance excitations are spin singlet valence bond fluctuations, while the long length-scale excitations are photons [3,4]. A similar photon phase can also be constructed in spin models on the pyrochlore lattice [5,6]. Given these examples, it is natural to look for a lattice model whose infrared behavior is controlled by the Lifshitz gravity fixed points of [1,2].

It should be noted that there are two distinct ways in which one can attempt to relate Lifshitz gravity to lattice models. In this paper, we work on a fixed rigid lattice, and find degrees of freedom whose long-distance dynamics is captured by the Gaussian fixed points of Lifshitz gravity. The lattice is nondynamical. Alternatively, one can try to obtain Lifshitz gravity as a continuum limit of a lattice model defined as a sum over a suitable class of random triangulations of space-time geometries. Here it is the lattice itself that is dynamical, and no extra degrees of freedom are invoked. The most promising candidate for such a random lattice model is offered by the causal dynamical triangulations (CDT) approach to quantum gravity. In the CDT approach (see Ref. [7] for a review), a summation over random lattices constrained to respect a preferred foliation structure of space-time serves as a nonperturbative definition of quantum gravity, and yields a continuum limit with four macroscopic space-time dimensions at long distances. It has been suggested in [8] (see also the recent paper [9]) that the CDT approach might be viewed as a lattice regularization of Lifshitz gravity. Further evidence for this scenario comes from the qualitative behavior of the spectral dimension of space-time, which indicates that the model flows from a $z = 3$ UV fixed point to an $z = 1$ fixed point at long distances [8].

In condensed matter systems, gapless bosonic excitations are usually Goldstone modes of certain spontaneously broken continuous symmetry. For instance, the phonon modes of solids are the Goldstone modes of translation symmetry, and the magnons are the Goldstone modes of the broken spin rotation symmetry in the magnetic ordered phase. There has been a lot of interest recently in finding bosonic phases with gapless excitations which do not originate from breaking a symmetry, and which do not require fine-tuning. Such phases are referred to as the *algebraic Bose liquid* (ABL) phases, because the boson density correlation falls off algebraically in space and time. An ABL phase is defined and characterized by its low-energy field theory. Searching for stable ABL phases, or more generally Bose metal phases, is one of the active fields in condensed matter theory [10–15]. The photon phase of the quantum dimer model mentioned above is a well-understood stable ABL phase which was first studied in a high-energy community and then widely applied to condensed matter systems, especially fractionalized states of strongly correlated systems [16,17].

Can we find other examples of ABL phases, especially in three spatial dimensions? In Refs. [10,11], a novel ABL phase with $z = 2$ dispersion was proposed in a quantum boson model on the face centered cubic (fcc) lattice with only local boson hopping and density repulsion. Although the microscopic model only has the lattice point group symmetry and the U(1) global symmetry corresponding to the conservation of the total boson number, a gauge symmetry similar to linearized diffeomorphisms emerges at low energies. We will review the low-energy effective field theory of this model below, and show that it is given by the Lifshitz gravity of Ref. [1] at the $z = 2$ Gaussian fixed point. In the lattice construction, the $z = 2$ dispersion is protected by the gauge symmetry which emerges at low energy and by the microscopic discrete symmetries of the underlying degrees of freedom, and the stability of this ABL phase is guaranteed by its self-duality. In this work, we show that by turning on one extra density repulsion term in the model proposed in Refs. [10,11], one can drive a phase transition between the $z = 2$ phase mentioned above and another stable $z = 3$ phase. We will show that this $z = 3$ phase is also a stable ABL phase, with a self-dual structure at low energy. The field theory of this ABL phase is identical to the $z = 3$ Lifshitz gravity of Ref. [2] at the Gaussian fixed point.

We start with describing the full Hamiltonian of our lattice boson model. This model is defined on the fcc lattice: The physical quantities will be defined on both the sites and the unit square faces of a cubic lattice. We denote each site of the cubic lattice by $\vec{r} = (r_x, r_y, r_z)$, and each unit square in the $(\hat{\imath}, \hat{\jmath})$ plane by $\vec{r} \pm \frac{\hat{\imath}}{2} \pm \frac{\hat{\jmath}}{2}$, with $i, j = x, y, z$. As our dynamical variables, we assign three boson numbers (n_{xx}, n_{yy}, n_{zz}) on each site of a cubic lattice, and one boson number n_{ij} to each face in the $\hat{\imath}\hat{\jmath}$ plane of the cubic lattice. The corresponding creation and destruction operators will be denoted by $b^\dagger_{ii}$, b_{ii}, $b^\dagger_{ij}$ and b_{ij}. The microscopic Hamiltonian contains the following terms:

$$\begin{aligned} H &= H_t + H_v + H_u + H_{v'}, \\ H_t &= -\bar{t}_1 H_{sp} - \bar{t}_2 H_{pp}, \\ H_{v,\hat{x}\mathrm{link}} &= V(2n_{xx,\vec{r}} + 2n_{xx,\vec{r}+\hat{x}} + n_{xy,\vec{r}+(\hat{x}/2)+(\hat{y}/2)} \\ &\quad + n_{xy,\vec{r}+(\hat{x}/2)-(\hat{y}/2)} + n_{xz,\vec{r}+(\hat{x}/2)+(\hat{z}/2)} \\ &\quad + n_{xz,\vec{r}+(\hat{x}/2)-(\hat{z}/2)} - 8\bar{n})^2, \\ H_u &= \sum_{\vec{r}} \sum_{ii} \frac{u_1}{2}(n_{ii,\vec{r}} - \bar{n})^2 \\ &\quad + \sum_{i<j} \frac{u_2}{2}(n_{ij,\vec{r}+(\hat{\imath}/2)+(\hat{\jmath}/2)} - \bar{n})^2, \\ H_{v'} &= \sum_{\vec{r}} V'(n_{xx,\vec{r}} + n_{yy,\vec{r}} + n_{zz,\vec{r}} - 3\bar{n})^2. \end{aligned} \tag{4}$$

H_t include all the local boson hoppings. H_{sp} is the hopping between each site and its adjacent plaquettes, $H_{sp} = \sum_{\vec{r},i,j,k} b^\dagger_{kk,\vec{r}} b_{ij,\vec{r}\pm(\hat{\imath}/2)\pm(\hat{\jmath}/2)} + \mathrm{H.c.}$, while H_{pp} is the hopping between two nearest neighbor plaquettes (Fig. 1a). The exact values of the amplitudes $\bar{t}_1$, $\bar{t}_2$ are unimportant, and we will tentatively take both of them at the same order of magnitude $\bar{t}_1, \bar{t}_2 \sim \bar{t}$. H_v is a large density-density interaction between bosons, $H_{v,\hat{x}\mathrm{link}}$ in Eq. (4) is the part of H_v associated with one link along the $\hat{x}$ direction; it is a sum of the boson numbers on the two sites and the four faces sharing this link, and $\bar{n}$ is the average boson filling on each quantum state. Contributions to H_v for links along $\hat{y}$ and $\hat{z}$ directions are defined analogously. H_u is a small

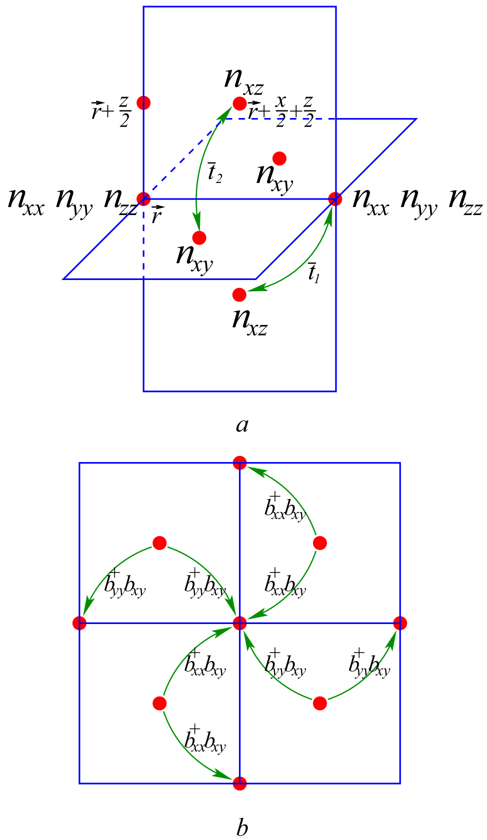

FIG. 1 (color online). (a) The lattice structure of this model, with the coordinates of site, link and plaquette. The term $H_{v,\hat{x}\text{link}}$ in Eq. (4) involves all the sites and plaquettes denoted with solid circles in this figure. The H_t term contains flavor-dependent boson hopping between sites and adjacent plaquettes (the H_{sp} term in Eq. (4)), and also between nearest neighbor plaquettes (the H_{pp} term). (b) the ring exchange term $-t_1\cos(\mathcal{B}_{zz})$ in Eq. (10), which includes eight boson hoppings.

repulsive interaction between each flavor of bosons on both sites and faces, with u_1 generally not equal to u_2. Finally, $H_{v'}$ is another on-site repulsive interaction on each site of the cubic lattice. The lattice structure and the distribution of the boson number is shown in Fig. 1(a).

If we first take $V' = 0$, this model will reduce to the model constructed in Refs. [10,11]. When H_v is dominant, it separates the Hilbert space into a high-energy subspace and a low-energy subspace. The low-energy subspace is subject to a local constraint: The sum in the bracket in H_v vanishes for every link. If we define a symmetric tensor $\mathcal{E}_{ij}$ as $\mathcal{E}_{ii,\vec{r}} = -(-1)^{\vec{r}}2(n_{ii,\vec{r}} - \bar{n})$, $\mathcal{E}_{ij,\vec{r}+(\hat{\imath}/2)+(\hat{\jmath}/2)} = (-1)^{\vec{r}} \times (n_{ij,\vec{r}+(\hat{\imath}/2)+(\hat{\jmath}/2)} - \bar{n})$ $(i \neq j)$ this local constraint can be compactly written as

$$\sum_i \nabla_i \mathcal{E}_{ij} = 0. \qquad (5)$$

Here and throughout the paper, ∇_i denotes the lattice derivative: $\nabla_i f(\vec{r}) = f(\vec{r} + \hat{\imath}) - f(\vec{r})$. Equation (5) simply states that $\mathcal{E}_{ij}$ is covariantly conserved, or divergence-free.

The canonical conjugate variables of n_{ij} on the lattice are the phase angles θ_{ij} of the boson creation operators: $b_{ij} \sim e^{-i\theta_{ij}}$. They satisfy the commutation relation $[n_{ab}, \theta_{cd}] = i\delta_{ac}\delta_{bd}$, $a \leq b$, $c \leq d$. Using θ_{ij}, we introduce a symmetric tensor field A_{ij} as $A_{ii,\vec{r}} = -(-1)^{\vec{r}}\theta_{ii,\vec{r}}$ and $A_{ij,\vec{r}+(\hat{\imath}/2)+(\hat{\jmath}/2)} = (-1)^{\vec{r}}\theta_{ij,\vec{r}+(\hat{\imath}/2)+(\hat{\jmath}/2)}$ for $i \neq j$. Under the discrete symmetries of time reversal T, lattice translations $\mathrm{T}_{\hat{k}}$ along $\hat{k}$, and spatial reflection transformations $\mathrm{P}_{\hat{k},s}$, the components of A_{ij} transform as

$$\begin{aligned}
&\mathrm{T}\colon t \to -t, \qquad A_{ij} \to -A_{ij},\\
&\mathrm{T}_{\hat{k}}\colon \vec{r} \to \vec{r} + \hat{k}, \qquad A_{ij} \to -A_{ij},\\
&\mathrm{P}_{\hat{k},s}\colon r_{\hat{k}} \to -r_{\hat{k}}, \qquad A_{ik} \to -A_{ik}, \qquad i \neq k,\\
&\qquad A_{ij} \to A_{ij}, \qquad i \neq k, \qquad j \neq k,\\
&\qquad A_{ii} \to A_{ii}.
\end{aligned} \qquad (6)$$

Notice that we define $\mathrm{P}_{\hat{k},s}$ to be the site-centered reflection of the lattice, where the origin is located at one of the sites of the cubic lattice. The transformation of $\mathcal{E}_{ij}$ is almost the same as A_{ij}, except that $\mathcal{E}_{ij}$ is even under time reversal.

The local constraint on $\mathcal{E}_{ij}$ Eq. (5) can be interpreted as a Gauss constraint, associated with a partially fixed gauge symmetry. It will generate the following gauge symmetries on A_{ij}:

$$A_{ij} \to A_{ij} + \nabla_i f_j + \nabla_j f_i. \qquad (7)$$

Of course, this gauge symmetry was absent in the microscopic Hamiltonian Eq. (4): It only emerges in the low-energy Hilbert subspace with constraints imposed by H_v. This mechanism is analogous to the way in which the emergent U(1) gauge symmetry appears in the photon phase of the 3d quantum dimer models [3,4] and similar spin models [5,6]. If we interpret A_{ij} as small fluctuations of the metric tensor g_{ij} around the flat background,

$$g_{ij} \approx \delta_{ij} + A_{ij}, \qquad (8)$$

the emergent gauge symmetry in Eq. (7) corresponds to the linearized form of the foliation-preserving diffeomorphisms Eq. (1) of Lifshitz gravity, with $\xi^i(x, t) \approx f_i$.

The low-energy dynamics of this system has to be invariant under the gauge transformation Eq. (7). The lowest-order gauge invariant dynamics is generated at the eighth-order perturbation of $\bar{t}/V$, which can be written as

$$\begin{aligned}
H_{\text{eff}} &= \sum_{\vec{r}} -\sum_{i\neq j} t_1 \cos(R_{ijij})_{\vec{r}} - \sum_{i\neq j, j\neq k, k\neq i} t_2 \cos(R_{ijik})_{\vec{r}},\\
R_{ijij} &= \sum_{a,b,c,d,k} -\epsilon_{ijk}\epsilon_{kab}\epsilon_{kcd}\nabla_a\nabla_c A_{bd}, \qquad i \neq j,\\
R_{ijik} &= \sum_{a,b,c,d} \epsilon_{jab}\epsilon_{kcd}\nabla_a\nabla_c A_{bd}, \qquad i \neq j,\\
&\qquad j \neq k, \qquad k \neq i.
\end{aligned} \qquad (9)$$

With our identification of A_{ij} as the metric fluctuations around the flat metric, R_{ijij} and R_{ijik} represent the six independent components of the linearized curvature tensor of g_{ij}. It is convenient to introduce another rank-two symmetric tensor $\mathcal{B}_{ij}$ as $\mathcal{B}_{ii} = \frac{1}{2}\epsilon_{ijk}R_{jkjk}$ and $\mathcal{B}_{ij} = -R_{ikjk}$. $\mathcal{B}_{ij}$ defined this way is also covariantly constant: $\nabla_i \mathcal{B}_{ij} = 0$. The transformation of $\mathcal{B}_{ij}$ under the discrete lattice symmetries of Eq. (6) is the same as for A_{ij}. The definition of $\mathcal{B}_{ij}$ on the lattice is given in the Appendix. In geometric terms, $\mathcal{B}_{ij}$ are simply the components of the linearized Einstein tensor $R_{ij} - \frac{1}{2}Rg_{ij}$ of the metric (8).

Now the full low-energy Hamiltonian reads

$$\begin{aligned} H &= H_u + H_{\text{eff}} \\ &= \sum_{\vec{r}}\sum_{ii}\frac{u_1}{8}\mathcal{E}^2_{ii,\vec{r}} + \sum_{i<j}\frac{u_2}{2}\mathcal{E}^2_{ij,\vec{r}} - \sum_i t_1\cos(\mathcal{B}_{ii})_{\vec{r}} \\ &\quad - \sum_{i\neq j} t_2\cos(\mathcal{B}_{ij})_{\vec{r}}. \end{aligned} \tag{10}$$

t_1 and t_2 are both at the order of $\sim \bar{t}^8/V^7$. Physically t_1 and t_2 terms correspond to high-order hoppings of bosons. For instance the t_1 term stands for the eighth-order hopping process depicted in Refs. [10,11] and Fig. 1(b). We want to emphasize that the Hamiltonian Eq. (4) is not fine-tuned, in the sense that one is allowed to turn on small local perturbations of any kind that are compatible with the global symmetry of Eq. (4), and the low-energy Hamiltonian will always take the same form as Eq. (10).

The Hamiltonian in Eq. (10) has a continuum limit Gaussian field theory, which characterizes an ABL phase. In the field theory we replace the lattice derivative ∇_i by the continuum limit derivative ∂_i, and expand the cosines in H_{eff} at its minima $-\cos(\mathcal{B}_{ij}) \sim \mathcal{B}^2_{ij}/2$ to the leading nontrivial order (a spin-wave expansion). After we replace $\mathcal{E}_{ij}$ by $\mathcal{E}_{ij} \sim \dot{A}_{ij}$, the resulting Gaussian field theory is described by the following Lagrangian,

$$\mathcal{L} = \sum_i \frac{1}{2u_1}(\dot{A}_{ii})^2 + \frac{t_1}{2}\mathcal{B}^2_{ii} + \sum_{i<j}\frac{1}{2u_2}(\dot{A}_{ij})^2 + \frac{t_2}{2}\mathcal{B}^2_{ij}. \tag{11}$$

The dispersion of the collective modes can be solved straightforwardly, and the result is quadratic, implying $z = 2$. Importantly, the linear dispersion is ruled out by the gauge symmetry and the lattice symmetries; no other terms more relevant than those in Eq. (11) are allowed. The terms $\sum_{i\neq j}\mathcal{E}_{ii}\mathcal{E}_{jj}$ and $\sum_{i\neq j}\mathcal{B}_{ii}\mathcal{B}_{jj}$ are in principle allowed by the symmetry of the system, but it will not change the $z = 2$ dispersion. For instance, the $\sum_{i\neq j}\mathcal{E}_{ii}\mathcal{E}_{jj}$ term can be induced with the V' term in Eq. (4), while $\sum_{i\neq j}\mathcal{B}_{ii}\mathcal{B}_{jj}$ can be induced with even higher-order boson ring exchange.

It is now easy to see that the theory in Eq. (11), together with the constraint of Eq. (5), is equivalent to the Gaussian limit of the $z = 2$ Lifshitz quantum gravity proposed in [1,2]. Indeed, setting $N_i = 0$ as our gauge choice in Lifshitz gravity yields Eq. (5) as the equation of motion, from varying N_i in the action given in Eq. (2). Furthermore, the most general potential term S_V in the $z = 2$ Lifshitz gravity Lagrangian in 3 + 1 dimensions is a sum of two terms,

$$S_V = \sum_{i,j}\kappa_1 R_{ij}R^{ij} + \kappa_2 R^2, \tag{12}$$

while the kinetic term S_K takes the form in Eq. (2). One difference between the lattice system and the field theory Eq. (12) is that, the lattice system does not have the O(3) spatial rotation symmetry, therefore the low-energy field theory deduced from our lattice model does not automatically acquire this O(3) symmetry. However, one can tune the parameters in Eq. (4) to restore the O(3) symmetry in the low-energy limit, and then the field theory will be identical to Eq. (2) plus S_V. Interestingly, if we tune the microscopic parameters to achieve the O(3) symmetry, the ABL phase picks out the special case of the Lifshitz gravity models, satisfying the additional property of detailed balance in the sense of [1,2], with a fixed value of the coupling λ in Eq. (2). Extending the relationship beyond this simplest case requires that the additional terms $\sum_{i\neq j}\mathcal{B}_{ii}\mathcal{B}_{jj}$ and $\sum_{i\neq j}\mathcal{E}_{ii}\mathcal{E}_{jj}$ mentioned above are also generated.

Having established the map between the low-energy continuum Hamiltonian of the lattice boson model and the Hamiltonian of $z = 2$ Lifshitz gravity, a few comments are in order:

(i) In Lifshitz gravity with the full foliation-preserving diffeomorphism invariance of Eq. (1), one additional global constraint follows from the variation of $N(t)$ in the action Eq. (2). Since $N(t)$ is only a function of time, this yields an *integral* constraint, equivalent to the vanishing of the total Hamiltonian on physical states. Not imposing the $\delta t = \zeta(t)$ invariance as a gauge symmetry would eliminate this Hamiltonian constraint. In the lattice approach, the integral Hamiltonian constraint does not seem to be necessary for self-consistency; however, we could consider imposing it in addition to the local constraint given in Eq. (5), thus reproducing the full set of symmetries of the minimal version of Lifshitz gravity.

(ii) In [1,2], a fully interacting nonlinear version of Lifshitz gravity has been proposed, and it was argued that models with $z > 1$ naturally flow in the IR to $z = 1$, under the influence of relevant terms in S_V. The most relevant terms are R—the Einstein-Hilbert term responsible for $z = 1$, and Λ—the cosmological constant term. In the self-interacting Lifshitz gravity theory, such terms are always expected to be generated under RG, and one might wonder why they do not automatically arise in the lattice model. The resolution of this puzzle is simple: The microscopic structure of our lattice model implies that at the $z = 2$ Gaussian fixed point, the discrete symmetries of Eq. (6) will hold. It turns out that these symmetries not only prevent the relevant terms R and Λ from being generated, they are also incompatible with turning on the self-interaction coupling of the full nonlinear Lifshitz gravity.

(iii) One should be more careful with the naive expansion of the cosine functions appearing in Eq. (10). The way we constructed the low-energy Hamiltonian from the microscopic degrees of freedom, the A_{ij} variables are compact, with radius 2π. This compactification does not have a direct analogy in Lifshitz gravity. Moreover, it can lead to topological excitations, which in turn cause tunnelling between different minima of the cosine functions in Eq. (4). Just like in the case of compact QED in 2 + 1 dimensions, the compactification has the potential to destroy the gapless excitations when it is relevant.

The relevance or irrelevance of the compactification and topological excitations can be most naturally studied in the dual picture. The low-energy Hamiltonian of this ABL phase can be schematically written as $H = \mathcal{E}^2 + \mathcal{B}^2$, $\mathcal{E}$ and $\mathcal{B}$ are both symmetric, covariantly constant tensors with six independent components, which suggests a self-dual structure exchanging $\mathcal{E}$ and $\mathcal{B}$. The dual field h_{ij} and the dual momentum π_{ij} are defined as

$$\mathcal{E}_{ij} = \epsilon_{iab}\epsilon_{jcd}\nabla_a\nabla_c h_{bd}, \qquad \mathcal{B}_{ii} = 2\pi_{ii}, \qquad \mathcal{B}_{ij} = \pi_{ij}(i \neq j). \tag{13}$$

Notice that all the derivatives are lattice derivatives, and h_{ij} and π_{ij} are defined on the sites and plaquettes of the cubic lattice (for details, see the Appendix). One can check the commutator and verify that h_{ij} and π_{ij} are canonical conjugate variables, $[\pi_{ij}, h_{kl}] = i\delta_{ik}\delta_{jl}$ $i \leq j$, $k \leq l$, and we can replace π_{ij} by $\dot{h}_{ij}$. Under this duality transformation, $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$ are exchanged, and the dual low-energy continuum limit field theory is precisely the same as the original model. Therefore, this ABL phase is self-dual with the dual gauge symmetry $h_{ij} \to h_{ij} + \partial_i\tilde{f}_j + \partial_j\tilde{f}_i$ in the continuum limit. Because of the compactness of A_{ij}, $\mathcal{E}_{ij}$ and h_{ij} both take discrete eigenvalues, therefore at the microscopic level the dual Lagrangian allows for the periodic potential $\hat{O}_v \sim \cos(2\pi h_{ij})$ which we refer to as the vertex operator. However, this vertex operator violates the dual continuum limit gauge symmetry of the ABL phase, and hence the correlation function between $\hat{O}_v$ vanishes at a distance larger than $V/\bar{t}$. Thus, $\hat{O}_v$ is irrelevant at the ABL Gaussian fixed point.

The degrees of freedom in the $z = 2$ ABL phase indeed match the count of degrees of freedom in $z = 2$ Lifshitz gravity [1]: In addition to the two transverse-traceless polarizations of the graviton, there is an additional "scalar graviton" mode. In the lattice model, these three polarizations appear as independent collective modes of the Hamiltonian in Eq. (10). The scalar mode corresponds to the scale factor of the spatial metric. It was shown in [2] that in classical Lifshitz gravity, the scalar graviton can be eliminated by extending the gauge invariance to include an anisotropic Weyl symmetry, introduced first in [1] (and further studied in [18]). In 3 + 1 dimensions, this requires the dynamical exponent to be $z = 3$, freezes the coupling constant λ in the action (2) to be $\lambda = 1/3$, and requires that the spatial part L_V of the action be conformally invariant.

At the microscopic level of our lattice model, the elimination of the scalar graviton can be arranged simply by turning on $H_{v'}$ in Eq. (4), When both H_v and $H_{v'}$ are dominant, the constraint on $\mathcal{E}_{ij}$ in the low-energy subspace becomes

$$\sum_i \nabla_i\mathcal{E}_{ij} = 0, \qquad \sum_i \mathcal{E}_{ii} = 0. \tag{14}$$

Thus, $\mathcal{E}_{ij}$ is now not only symmetric and covariantly constant, but also traceless. This tracelessness constraint can be interpreted as the Gauss constraint associated with gauge fixing of another gauge symmetry, which acts on A_{ij} via

$$A_{ij} \to A_{ij} + \delta_{ij}\varphi, \tag{15}$$

where φ is an arbitrary scalar field. Now the trace of A_{ij} becomes an unphysical gauge degree of freedom, so there are only two gapless modes at low energy.

The new gauge symmetry in Eq. (15) will turn out to be the anisotropic Weyl invariance of Lifshitz gravity mentioned above, here in the linearized approximation around the Gaussian $z = 3$ fixed point. In order to see that the new constraint is forcing the model to $z = 3$ at low energies, note that the low-energy Hamiltonian is also modified by turning on V'. Let us assume the dynamical term can be written as

$$H_{\mathrm{eff}'} = \sum_{\vec{r}}\sum_i -t_3\cos(a\mathcal{C}_{ii})_{\vec{r}} - \sum_{i\neq j} t_4\cos(b\mathcal{C}_{ij})_{\vec{r}}, \tag{16}$$

where $\mathcal{C}_{ij}(A_{kl})$ is a linear functional of A_{kl}. This new tensor $\mathcal{C}_{ij}$ must be invariant under the gauge transformations of Eq. (7) as well as Eq. (15). The tensor of the lowest order in derivatives satisfying this requirement is the (linearized form of the) Cotton tensor [1,2]:

$$\mathcal{C}_{ij} = \epsilon_{ikl}\nabla_k\left(R_{jl} - \frac{1}{4}R\delta_{jl}\right). \tag{17}$$

Here $R_{jl} = \sum_k R_{jklk}$, $R = \sum_j R_{jj}$ are the linearized Ricci curvature and scalar curvature, respectively. Under the lattice symmetries of Eq. (6), the Cotton tensor transforms as

$$\begin{aligned} &\mathrm{T}: \mathcal{C}_{ij} \rightarrow -\mathcal{C}_{ij}, \\ &\mathrm{T}_{\hat{k}}: \mathcal{C}_{ij} \rightarrow -\mathcal{C}_{ij}, \\ &\mathrm{P}_{\hat{k},s}: \mathcal{C}_{ik} \rightarrow \mathcal{C}_{ik}, \qquad i \neq k, \\ &\qquad \mathcal{C}_{ij} \rightarrow -\mathcal{C}_{ij}, \qquad i \neq k, \qquad j \neq k, \\ &\qquad \mathcal{C}_{ii} \rightarrow -\mathcal{C}_{ii}. \end{aligned} \tag{18}$$

One can straightforwardly verify that $\mathcal{C}_{ij}$ is gauge invariant, symmetric, traceless and covariantly constant:

$$\mathcal{C}_{ij} = \mathcal{C}_{ji}, \qquad \sum_i \mathcal{C}_{ii} = 0, \qquad \sum_i \nabla_i \mathcal{C}_{ij} = 0. \tag{19}$$

The definition of $\mathcal{C}_{ij}$ on the lattice is given in the Appendix. From our microscopic Hamiltonian, the effective low-energy term $H_{\mathrm{eff}'}$ can only emerge at a fairly high order of perturbation. For instance, the t_3 term can be generated at order 32 in $\bar{t}/V$. Now the full low-energy Hamiltonian reads

$$\begin{aligned} H &= H_u + H_{\mathrm{eff}} \\ &= \sum_{\vec{r}} \sum_{ii} \frac{u_1}{2} \mathcal{E}^2_{ii,\vec{r}} + \sum_{i<j} \frac{u_2}{2} \mathcal{E}^2_{ij,\vec{r}} - \sum_i t_3 \cos(\mathcal{C}_{ii})_{\vec{r}} \\ &\quad - \sum_{i \neq j} t_4 \cos(2\mathcal{C}_{ij})_{\vec{r}}. \end{aligned} \tag{20}$$

Since $\mathcal{C}_{ij}$ involves three spatial derivatives, after the spin-wave expansion $-\cos(b\mathcal{C}_{ij}) \sim b^2 \mathcal{C}^2_{ij}/2$, the ABL phase which is described by the continuum limit Gaussian field theory of Hamiltonian in Eq. (20) has collective excitations with the $z = 3$ dispersion. Notice that the lattice symmetry does not require $t_3 = t_4$, but the ratio t_3/t_4 can be tuned by the ratio $\bar{t}_1/\bar{t}_2$ on the lattice.

In [2], a nonlinear self-interacing $z = 3$ Lifshitz gravity has been constructed. In this construction, the nonlinear Cotton tensor plays a central role, with $S_V \propto \mathcal{C}_{ij}\mathcal{C}^{ij}$. When $t_3 = t_4$, the low-energy Hamiltonian of the ABL phase is identical to the Gaussian point of the $z = 3$ Lifshitz gravity of [2]. This emergent theory is gauge invariant not only under linearized foliation-preserving diffeomorphisms, but also under the $z = 3$ version of the anisotropic Weyl transformation [1,2,18].

Are the gapless $z = 3$ modes of this model stable when A_{ij} are treated as compact variables? To address this question, it is again convenient to go to the dual formalism. The continuum limit Hamiltonian takes the schematic form $H = \mathcal{E}^2 + \mathcal{C}^2$, and the fact that $\mathcal{E}_{ij}$ and $\mathcal{C}_{ij}$ are both symmetric, traceless and covariant tensors strongly suggests this theory also has a self-dual structure. To prove the self-duality, we define dual variables $\tilde{h}_{ij}$ and $\tilde{\pi}_{ij}$ as

$$\begin{aligned} \mathcal{E}_{ij} = \mathcal{C}_{ij}(\tilde{h}_{kl}), \qquad \tilde{\pi}_{ii} = \frac{1}{2}\mathcal{C}_{ii}(A_{kl}), \\ \tilde{\pi}_{ij} = \mathcal{C}_{ij}(A_{kl}), \qquad i \neq j. \end{aligned} \tag{21}$$

Here we have treated $\mathcal{C}_{ij}$ as a linear functional of the tensor field A_{ij} or $\tilde{h}_{ij}$. Unlike h_{ij} defined previously, $\tilde{h}_{ii}$ are defined on the dual lattice sites, which are the centers of the unit cubes, and $\tilde{h}_{ij}$ are defined on the dual plaquettes, which are the links of the original lattice (for details, see the Appendix). For instance, a link along the $\hat{z}$ direction is a dual plaquette in the $\hat{x}\,\hat{y}$ plane. Again, one can straightforwardly verify that $\tilde{h}_{ij}$ and $\tilde{\pi}_{ij}$ are a pair of conjugate variables, and $\mathcal{E}_{ij}$ and $\mathcal{C}_{ij}$ are exchanged under duality. The dual graviton $\tilde{h}_{ij}$ of this ABL phase enjoys the same gauge symmetry as A_{ij}, hence the vertex operator $\hat{O}_v \sim \cos(2\pi\tilde{h}_{ij})$ is irrelevant. Therefore the $z = 3$ ABL phase is also a stable gapless phase.

This self-duality structure completes the argument of the stability of both the $z = 2$ and $z = 3$ ABL phases in this model. The self-duality can also be proved in the Euclidean space-time, in the same way as the duality of ordinary classical statistical mechanics models [19]. If the dual theory of a lattice gauge model did not have a large enough gauge symmetry, one would have to fine-tune the system to get a gapless ABL state. The most well-understood example is the compact QED in 2 + 1 dimensions, where the dual theory is a U(1) rotor model without gauge symmetry. In that case, the vertex operator which corresponds to the monopoles in space-time will destroy the gapless photon phase. Another example studied recently is the quantum plaquette model with gauge symmetry $A_{ij} \rightarrow \nabla_i \nabla_j \varphi$. The dual theory of the plaquette model does have a gauge symmetry, but this symmetry is not strong enough to protect the gapless ABL phase [20].

In Ref. [21], gravitonlike collective modes were also obtained through a quantum boson model on the lattice, but the relation of such lattice models to Lifshitz gravity was not noticed. It is worth emphasizing that in our context, the Lifshitz-type graviton Hamiltonian in Eq. (20) can be derived *without any fine-tuning* from a simple boson model of Eq. (4) through perturbation theory, and both the $z = 2$ and $z = 3$ ABL phase are stable due to their self-dual nature.

Besides the difference in the dispersion, the $z = 2$ and $z = 3$ phases also have different algebraic correlations. For instance, in the $z = 2$ phase the equal-time density fluctuation correlation falls off as

$$\langle \delta n(0) \delta n(\vec{r}) \rangle \sim \langle \mathcal{B}(\tilde{h})_0 \mathcal{B}(\tilde{h})_{\vec{r}} \rangle \sim \frac{1}{r^5}, \tag{22}$$

while for the $z = 3$ phase this correlation falls off as

$$\langle \delta n(0) \delta n(\vec{r}) \rangle \sim \langle \mathcal{C}(\tilde{h})_0 \mathcal{C}(\tilde{h})_{\vec{r}} \rangle \sim \frac{1}{r^6}. \tag{23}$$

(In these two equations, the flavor dependence of the correlation has been ignored for simplicity.)

In the original Hamiltonian, the $z = 2$ and $z = 3$ phases were obtained by dialing a small versus large value of the coupling V' in $H_{v'}$. As a result, by increasing V' one can drive a phase transition between these two ABL phases with different dynamical scalings. Compared to the $z = 2$ phase, the $z = 3$ phase has one extra gauge symmetry, of anisotropic Weyl transformations (15). Therefore, if we start with the $z = 3$ phase, this phase transition will be reminiscent of the Higgs transition. In the quantum dimer model, the U(1) gauge symmetry is Higgsed by condensing the dimer vacancies that carry U(1) gauge charge. The phase transition can be described by the rotor Hamiltonian $H_r = -t\cos(\vec{\nabla}\theta - \vec{A})$ [22], where θ is the phase angle of the dimer vacancy creation operator $\psi \sim e^{i\theta}$. After the condensation of ψ, the gauge field A_μ acquires a longitudinal mode by absorbing the Goldstone mode of ψ, which makes the gauge field an ordinary gapped vector field.

In our case, the transition between the two ABL phases can also be intuitively described as closing the gap of the trace mode of $\mathcal{E}_{ij}$, which can be described by condensing φ in Eq. (15). Since all the gauge symmetries in Eq. (7) are still preserved after the transition, the condensate of φ should not violate Eq. (7). With these observations, this transition can be described by the following Lagrangian,

$$L = \frac{1}{\gamma}\dot{\varphi}^2 - \sum_{i\neq j} t_1 \cos(\nabla_i^2\varphi + \nabla_j^2\varphi - R_{ijij}) - \sum_{i\neq j, j\neq k, k\neq i} t_2 \cos(\nabla_j\nabla_k\varphi - R_{ijik}) + \cdots \quad (24)$$

The condensation of φ changes the spectrum from two $z = 3$ modes of A_{ij} and one gapped mode of φ to three $z = 2$ modes of A_{ij}, in a generalization of the ordinary Higgs transition to the scalar mode of Lifshitz gravity. When φ is ordered, this Lagrangian restores the ring exchange terms in Eq. (10) of the $z = 2$ phase; when φ is disordered, after integrating out φ one should recover the phase which is invariant under the transformations in Eqs. (7) and (15). The $z = 3$ ABL phase is then the only candidate. From the condensed matter perspective, the phase transition in Eq. (24) is beyond the ordinary Ginzburg-Landau paradigm, because neither one of the two phases can be characterized by a local order parameter. More thorough RG studies for Eq. (24) are required to determine the nature of this transition.

One interesting question to ask is whether we can obtain relativistic gravitons, with a linear dispersion, from the lattice. In Ref. [11], it was proposed that a long-range interaction can change the dispersion to $z = 1$, but a local theory leading to $z = 1$ gravitons at long distances is still unavailable. As was discussed in Ref. [11] and Ref. [23], a Chern-Simons like term $A_{ij}\mathcal{B}_{ij}$ can lead to a linear dispersion, but this term is only gauge invariant up to a boundary term; therefore, it cannot be generated in the same way as the t_1 and t_2 terms in Eq. (10) through perturbation theory. One possibility is to generate this term by coupling the graviton field A_{ij} to a matter field with a gapless boundary state, just like the CS term for a U(1) gauge field can be generated by coupling the gauge field to a massive Dirac fermion with edge states.

In addition to the linear dispersion, another meaningful goal is to obtain the full nonlinear Lifshitz gravity of self-interacting gravitons, instead of the linearized theory at the Gaussian fixed point. As we pointed out in our comment (ii) below Eq. (12), this difficulty is intimately related to the existence of the discrete symmetries in Eq. (6), implied by the microscopic dynamics of the lattice model. These symmetries do not allow the natural self-interaction coupling of Lifshitz gravity [1,2] to be turned on. It is an interesting challenge to see if our framework can be extended so that its discrete symmetries no longer prevent the self-interaction of gravitons. Note that the discrete symmetries of Eq. (6) act naturally on the A_{ij} variables representing the fluctuations around the fixed flat background, but they do not appear to have a geometrically natural extension to the full metric g_{ij}. Thus, they are associated with the fixed flat spatial geometry, here represented by the fixed flat fcc lattice. These background-dependent discrete symmetries indeed played an important role in our construction of Lifshitz gravity from a lattice system: They prevented the Einstein-Hilbert term and the cosmological constant term from being generated, allowing the $z = 2$ and $z = 3$ ABL phases to be stable at low energies. Since the full nonlinear Lifshitz gravity of [1,2] does not require a choice of a preferred flat background, it is natural to speculate that attempts to turn on the self-interaction of gravitons in our lattice framework may ask for the underlying lattice itself to become dynamical. We will leave these topics to future studies.

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APPENDIX: TENSOR FIELDS ON THE LATTICE

On the lattice, π_{zz} on site $\vec{r}$ is defined as

$$\begin{aligned} 2\pi_{zz,\vec{r}} = (-1)^{\vec{r}}(&\theta_{yy,\vec{r}+\hat{x}} + 2\theta_{yy,\vec{r}} + \theta_{yy,\vec{r}-\hat{x}} + \theta_{xx,\vec{r}+\hat{y}} \\ &+ 2\theta_{xx,\vec{r}} + \theta_{xx,\vec{r}-\hat{y}} - 2\theta_{xy,\vec{r}+(\hat{x}/2)+(\hat{y}/2)} \\ &- 2\theta_{xy,\vec{r}-(\hat{x}/2)+(\hat{y}/2)} - 2\theta_{xy,\vec{r}+(\hat{x}/2)-(\hat{y}/2)} \\ &- 2\theta_{xy,\vec{r}-(\hat{x}/2)-(\hat{y}/2)}) = \mathcal{B}_{zz}. \end{aligned} \quad (A1)$$

The ring exchange $\cos(\mathcal{B}_{zz})$ corresponds to the high order of boson hopping depicted in Fig. 1(b). π_{yz} on plaquette $\vec{r} + \frac{\hat{y}}{2} + \frac{\hat{z}}{2}$ reads

$$\begin{aligned}\pi_{yz,\vec{r}+(\hat{y}/2)+(\hat{z}/2)} &= (-1)^{\vec{r}} \times (\theta_{yz,\vec{r}+\hat{x}+(\hat{y}/2)+(\hat{z}/2)} \\ &\quad + 2\theta_{yz,\vec{r}+(\hat{y}/2)+(\hat{z}/2)} + \theta_{yz,\vec{r}-\hat{x}+(\hat{y}/2)+(\hat{z}/2)} \\ &\quad + \theta_{xx,\vec{r}} + \theta_{xx,\vec{r}+\hat{y}} + \theta_{xx,\vec{r}+\hat{z}} + \theta_{xx,\vec{r}+\hat{y}+\hat{z}} \\ &\quad - \theta_{xy,\vec{r}+(\hat{x}/2)+(\hat{y}/2)} - \theta_{xy,\vec{r}-(\hat{x}/2)+(\hat{y}/2)} \\ &\quad - \theta_{xy,\vec{r}+\hat{z}+(\hat{x}/2)+(\hat{y}/2)} - \theta_{xy,\vec{r}+\hat{z}-(\hat{x}/2)+(\hat{y}/2)} \\ &\quad - \theta_{xz,\vec{r}+(\hat{x}/2)+(\hat{z}/2)} - \theta_{xz,\vec{r}-(\hat{x}/2)+(\hat{z}/2)} \\ &\quad - \theta_{xz,\vec{r}+\hat{y}+(\hat{x}/2)+(\hat{z}/2)} - \theta_{xz,\vec{r}+\hat{y}-(\hat{x}/2)+(\hat{z}/2)}) \\ &= \mathcal{B}_{yz}. \end{aligned} \tag{A2}$$

The tensor field defined this way satisfies $\sum_i \nabla_i \mathcal{B}_{ij} = 0$. The relation between $\mathcal{E}_{ij}$ and h_{ij} is the same as that between $\mathcal{B}_{ij}$ and A_{ij}.

The dual variable $\tilde{\pi}_{ii}$ is defined on the dual sites, which are the centers of the cubes, and $\tilde{\pi}_{ij}$ with $i \neq j$ is located on the links $\vec{r} + \frac{\hat{k}}{2}$, with $k \neq i$, $k \neq j$. $\tilde{\pi}_{xx}$ on the dual site $\vec{r} + \frac{\hat{x}}{2} + \frac{\hat{y}}{2} + \frac{\hat{z}}{2}$ is defined as

$$\begin{aligned} 2\tilde{\pi}_{xx,\vec{r}+(\hat{x}/2)+(\hat{y}/2)+(\hat{z}/2)} &= -\mathcal{B}_{xz,\vec{r}+\hat{y}+(\hat{x}/2)+(\hat{z}/2)} \\ &\quad + \mathcal{B}_{xz,\vec{r}+(\hat{x}/2)+(\hat{z}/2)} \\ &\quad + \mathcal{B}_{xy,\vec{r}+\hat{z}+(\hat{x}/2)+(\hat{y}/2)} \\ &\quad - \mathcal{B}_{xy,\vec{r}+(\hat{x}/2)+(\hat{y}/2)} = \mathcal{C}_{xx}. \end{aligned} \tag{A3}$$

$\tilde{\pi}_{xy}$ on the link $\vec{r} + \frac{\hat{z}}{2}$ is defined as

$$\begin{aligned} 2\tilde{\pi}_{xy,\vec{r}+(\hat{z}/2)} &= -2\mathcal{B}_{yz,\vec{r}+(\hat{y}/2)+(\hat{z}/2)} + 2\mathcal{B}_{yz,\vec{r}-(\hat{y}/2)+(\hat{z}/2)} \\ &\quad + \mathcal{B}_{yy,\vec{r}+\hat{z}} - \mathcal{B}_{yy,\vec{r}} - \mathcal{B}_{zz,\vec{r}+\hat{z}} + \mathcal{B}_{zz,\vec{r}} \\ &\quad - \mathcal{B}_{xx,\vec{r}+\hat{z}} + \mathcal{B}_{xx,\vec{r}} = 2\mathcal{C}_{xy}. \end{aligned} \tag{A4}$$

$\mathcal{C}_{ij}$ defined this way is symmetric, traceless, and covariantly constant. The relation between $\mathcal{E}_{ij}$ and the dual variable $\tilde{h}_{ij}$ is identical to that between $\tilde{\pi}_{ij}$ and A_{ij}, after exchanging sites with cubes, and plaquettes with links. On the lattice, the divergence of $\mathcal{C}_{ij}$ reads

$$\begin{aligned} \sum_i \nabla_i \mathcal{C}_{ix} &= \mathcal{C}_{xx,\vec{r}+(\hat{x}/2)+(\hat{y}/2)+(\hat{z}/2)} - \mathcal{C}_{xx,\vec{r}-(\hat{x}/2)+(\hat{y}/2)+(\hat{z}/2)} \\ &\quad + \mathcal{C}_{xy,\vec{r}+\hat{y}+(\hat{z}/2)} - \mathcal{C}_{xy,\vec{r}+(\hat{z}/2)} + \mathcal{C}_{xz,\vec{r}+\hat{z}+(\hat{y}/2)} \\ &\quad - \mathcal{C}_{xz,\vec{r}+(\hat{y}/2)}. \end{aligned} \tag{A5}$$

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