Linear growth rate of structure in parametrized post-Friedmannian universes

Pedro G. Ferreira*
Oxford Astrophysics, Physics, DWB, Keble Road, Oxford, OX1 3RH, UK

Constantinos Skordis†
School of Physics and Astronomy, University of Nottingham, University Park, Nottingham, NG7 2RD, UK
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A possible solution to the dark energy problem is that Einstein's theory of general relativity is modified. A suite of models have been proposed that, in general, are unable to predict the correct amount of large scale structure in the distribution of galaxies or anisotropies in the cosmic microwave background. It has been argued, however, that it should be possible to constrain a *general* class of theories of modified gravity by focusing on properties such as the growing mode, gravitational slip, and the effective, time-varying Newton's constant. We show that assuming certain physical requirements such as stability, metricity, and gauge invariance, it is possible to come up with consistency conditions between these various parameters. In this paper we focus on theories which have, at most, second derivatives in the metric variables and find restrictions that shed light on current and future experimental constraints without having to resort to a (as yet unknown) complete theory of modified gravity. We claim that future measurements of the growth of structure on small scales (i.e. from 1–200h^{-1} Mpc) may lead to tight constraints on both dark energy and modified theories of gravity.

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I. INTRODUCTION

The *dark energy* problem, i.e. the possibility that 70% of the Universe seems to be permeated by an invisible fluid which behaves repulsively under gravity and does not cluster, has been the focus of research in cosmology for over a decade. There are a host of proposals [1] and a battery of experiments are under way, or on the drawing board, to characterize the nature of this elusive source of energy [2–5].

In recent years, an alternative possibility has emerged, that Einstein's general theory of relativity is incorrect on cosmological scales and must be modified. Although the idea that general relativity is incomplete has been around since the early 1960s [6–9], there are now a number of proposals for what this theory of modified gravity might be [10]. The Einstein-Hilbert action, $S_g \propto \int d^4x\sqrt{-g}R$ (where g is the metric determinant and R is the scalar curvature of a metric g_{ab}) can be replaced by a more general form $S_g \propto \int d^4x\sqrt{-g}F(R)$, where F is an appropriately chosen function of R [11,12]; the dynamics of the gravitational field can emerge from a theory in higher dimensions such as one might encounter in brane worlds [13]; a preferred reference frame may emerge from the spontaneous symmetry breaking of local Lorentz symmetry [14–17]; the metric that satisfies the Einstein equation is not necessarily the one that defines geodesic motion [18] but is related to a second metric via additional fields [19–21] or connections [22–24]; the Einstein-Hilbert action may be deformed by choosing as fundamental variables of gravity, $SU(2)$ connections [25–27].

Many of these models have been successful in reproducing, for example, the observed relation between redshift and luminosity distances from distant supernovae. They have, however, generally failed to reproduce the observed clustering of galaxies on large scales as well as the anisotropies in the CMB unless the modified theory becomes effectively equivalent to general relativity (i.e. the Einstein-Hilbert action and a cosmological constant), e.g. [28–30]. The general problem that seems to plague most theories is an excess of power on the very largest scales, which manifests itself through the integrated Sachs-Wolfe effect and a mismatch between the normalization of the power spectrum of fluctuations on the largest and smallest scales. As yet, a truly compelling and viable model of modified theory of gravity has yet to be put forward, which may resolve the dark energy problem.

All is not lost, however, and progress can be made in learning about potential modifications to gravity by extracting phenomenogical properties that can be compared to observations, with the "parametrized post-Friedmannian" approach [31]. In this paper we focus on a key observable characterizing the evolution of large scale structure: the growing mode of gravitational collapse.

The time evolution of the density field can be a sensitive probe of not only the expansion rate of the Universe but also its matter content. In a flat, matter dominated universe we have that δ_M, the density contrast of matter, evolves as $\delta_M \propto a$ where a is the scale factor of the Universe. We can parametrize deviations from this behavior in terms of γ [32–34] through

*pgf@astro.ox.ac.uk
†skordis@nottingham.ac.uk

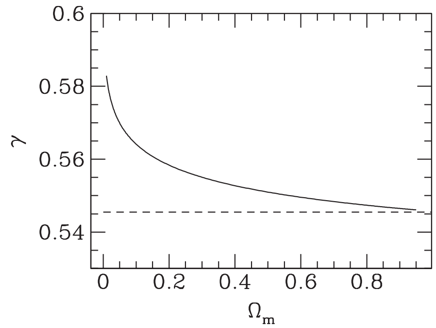

FIG. 1. The solid line is the growth parameter, γ, for a ΛCDM universe, as a function of Ω_M. For small values of Ω_Λ, γ is well approximated by $6/11$ (dashed line) but there are deviations as Ω_Λ grows; we find errors of 0.7%, 3.3%, and 4.2% when $\Omega_M =$ 0.7, 0.3, and 0.05.

$$\gamma \equiv \frac{\ln[\frac{\dot{\delta}_M}{\mathcal{H}\delta_M}]}{\ln\Omega_M}, \tag{1}$$

where Ω_M is the fractional density of matter, $\mathcal{H} = \frac{\dot{a}}{a}$ and overdots are derivatives with regard to conformal time, τ. For standard growth in the presence of a cosmological constant, one has that $\gamma \approx 6/11$ to a very good approximation. This is not true over a wide range of values for Ω_M. In fact, in Fig. 1 we can see that γ deviates from its early-universe asymptotic value as $\Omega_M \to 0$. A natural question to ask is how γ depends on different aspects of the Universe and how one might use it to constrain dark energy and modifications to gravity. In this paper we will focus on a few of these properties.

One important property of the Universe is the *equation of state* of dark energy, characterized by the constant (or function of time), w:

$$P_E = w\rho_E. \tag{2}$$

P_E and ρ_E are the pressure and energy densities of dark energy. The function w may be time varying and is related to the adiabatic speed of sound c_a^2 as

$$c_a^2 = w - \frac{\dot{w}}{3\mathcal{H}(1+w)}. \tag{3}$$

Another important property is *gravitational slip*, ζ, which is normally defined to be

$$\Phi - \Psi \equiv \zeta\Phi, \tag{4}$$

where we are taking a linearly perturbed metric in the conformal Newtonian gauge,

$$ds^2 = -a^2(1+2\Psi)d\tau^2 + a^2(1-2\Phi)d\vec{x}^2. \tag{5}$$

Such a parametrization has been advocated in a number of papers on modified gravity [35–41], and it has been shown that it can lead to a number of observational effects. Albeit simple, and appealing, such a parametrization of slip is not necessarily general and, as we shall see in the next section, necessarily implies other nontrivial modifications to the gravitational sector. Such modifications are, in general, not explicitly acknowledged but may correspond to unexpected assumptions about any putative, underlying theory. Hence, a more general assumption (at least within the context of second order theories) would be that gravitational slip would depend on Φ and $\dot{\Phi}$ (this is explained in more detail in Sec. II D and in [42])

Finally, we can define an *effective Newton's constant* in the relativistic Newton-Poisson equation

$$\nabla^2\Phi = 4\pi a^2 G_{\rm eff}\sum_X \rho_X\left[\delta_X + 3(1+w_X)\frac{\dot{a}}{a}\theta_X\right],$$

where δ_X is the density contrast, and θ_X is the momentum of the cosmological fluid X, which has an equation of state w_X. We can define the dimensionless function

$$\mu^2 \equiv \frac{G}{G_{\rm eff}}, \tag{6}$$

where G is the "bare" Newton constant. It then makes sense to try to constrain (γ, w, ζ, and μ) in the hope that it may be possible to shed light on a possible theory of modified gravity.

Although there have been alternative proposals [43], a number of groups have pioneered the use of this simple parametrization of modified gravity (in terms of ζ, μ^2 or both): in [35,36] it was argued that gravitational slip might be a generic prediction for modified theories of gravity, in [37,38] it was shown that it would be possible to constrain it through cross correlations of the CMB with galaxy surveys and in [44] from the integrated Sachs-Wolfe effect; weak lensing has been proposed as a possible route for constraining these parameters [45–47] with a tentative detection of modification being proposed in [40]. Much is expected from applying these methods to future ambitious experiments that will map out the large scale structure of the Universe. Indeed constraints of general relativity are a core element of the science that could be extracted from the Euclid experiment [3].

Given that such an approach is phenomenological, the general attitude has been to leave these parameters completely free. There is merit to such an approach in that one is not restricting oneself to a particular theory and hence constraints will be general. It is true however that is possible to identify (reasonably general) consistency conditions for (γ, w, ζ, and μ), contingent on specific physical assumptions. In this paper we state these assumptions and present restrictions on (γ, w, ζ, and μ). We shall use the formalism first proposed by one of us [42], which spells out how to build consistent modifications to gravity.

This paper is structured as follows: In Sec. II, we recap the formalism presented in [42] and relate it to the parameters we wish to study phenomenologically. We discuss how the consistency conditions reduce the freedom to choose arbitrary (γ, ζ, μ^2). In Sec. III, we implement the consistency conditions and find a relationship between the parameters by looking at the evolution equation for the density contrast in matter for small wavelengths. In doing so, we find analytic expressions for the relationships and briefly assess the range of scale to which they are applicable. In Sec. IV, we find analytic expressions for γ to second order for a general parametrization which is consistent with the parametrized post-Newtonian (PPN) approximation on small scales. In Sec. V, we discuss the generality of the results and how they may be extended to other, more exotic models.

II. THE FORMALISM

We now summarize the formalism, the details of which can be found in [42]: we present the field equations, the evolution equations for the fluid components, and the consistency conditions for modifications to the field equations. We shall further assume a spatially flat universe, but our results can be easily generalized to include curvature.

A. The background cosmology

As discussed in [42], the background equations for any theory of gravity for which the metric is Friedmann-Robertson-Walker (FRW) can be recast in the usual form used in general relativity. The Friedmann equation simply reads

$$3\mathcal{H}^2 = 8\pi G a^2 \sum_X \rho_X. \tag{7}$$

In addition to the Friedmann equation we also have the Raychaudhuri equation $-2\frac{\ddot{a}}{a} + \mathcal{H}^2 = 8\pi G a^2 \sum_X P_X$. With the help of the Friedmann equation, in a universe containing only pressureless matter and dark energy (as is approximately the case in the late universe) the Raychaudhuri equation may be rewritten as

$$\dot{\mathcal{H}} = -\tfrac{1}{2}\mathcal{H}^2(1 + 3w\Omega_E). \tag{8}$$

The dark energy density ρ_E and w, may be in general a function of additional degrees of freedom, the scale factor a or $\mathcal{H}$. For example, for $F(R)$ one gets $\rho_E = \frac{1}{2}(RF_R - F) - \frac{3\mathcal{H}}{a^2}\dot{F}_R - \frac{3\mathcal{H}^2}{a^2}F_R$. But this explicit dependence of ρ_E (or of w) is irrelevant. One may always treat ρ_E as a standard fluid with a time-varying equation of state w subject to energy conservation $\dot{\rho}_E + 3\mathcal{H}(1 + w)\rho_E = 0$ (but note that there may be additional field equations that determine the time dependence of w). In a universe containing only pressureless matter and dark energy, the energy conservation equation for dark energy can be rewritten as

$$\dot{\Omega}_E = -3\mathcal{H} w \Omega_M \Omega_E. \tag{9}$$

Our discussion above has one important consequence: that one cannot distinguish modifications of gravity from ordinary fluid dark energy using observables based on FRW alone. As discussed in [42], and further below, the situation changes drastically once we consider linear fluctuations.

B. The field equations

The idea is to parameterize deviations from Einstein gravity at a linear level. Schematically we can write the modified Einstein equations in the form

$$\delta G^{\rm mod}_{\mu\nu} = \delta G_{\mu\nu} - \delta U_{\mu\nu} = 8\pi G \delta T_{\mu\nu} + 8\pi G \delta T^{\rm DE}_{\mu\nu}. \tag{10}$$

Note that for a tensor F we use δF to indicate a linear perturbation of F, and we assume that δU_{ab} is made of the scalar metric perturbations and their derivatives. Let us also stress that the background tensor corresponding to U_{ab}, i.e. $\bar{U}_{ab}$ vanishes. We assume that "normal matter" (i.e. baryons, dark matter, neutrinos, and photons) are contained in $T_{\mu\nu}$ and that dark energy, or any nonmetric degrees of freedom that behave like dark energy (such as a scalar field- quintessence- or a dark fluid), are contained in $T^{\rm DE}_{\mu\nu}$. In this paper we will restrict ourselves to two fluids: $T_{\mu\nu}$ is the energy-momentum tensor for a pressureless fluid with density ρ_M, density contrast δ_M, and momentum θ_M, while $T^{\rm DE}_{\mu\nu}$ is the energy-momentum tensor of a fluid with density $\rho_{\rm DE}$, density contrast $\delta_{\rm DE}$ and $\theta_{\rm DE}$, which can be characterized defined in terms of (possibly time varying) equation of state and sound speed (we shall use the approach of [48] to model a quintessence-like fluid with a constant equation of state.

The field equations can be rewritten in the following form:

$$-2k^2\Phi = 8\pi G a^2 \sum_X \rho_X[\delta_X + 3(1 + w_X)\mathcal{H}\theta_X] + U_\Delta + 3\mathcal{H}U_\theta, \tag{11}$$

$$2(\dot{\Phi} + \mathcal{H}\Psi) = 8\pi G a^2 \sum_X (\rho_X + P_X)\theta_X + U_\theta, \tag{12}$$

$$\frac{d}{d\tau}(\dot{\Phi} + \mathcal{H}\Psi) = 4\pi G a^2 \rho_E \Pi_E + \tfrac{1}{6}U_P + \tfrac{1}{3}\nabla^2 U_\Sigma - 2\mathcal{H}(\dot{\Phi} + \mathcal{H}\Psi) + (\mathcal{H}^2 - \dot{\mathcal{H}})\Psi,$$
$$\Phi - \Psi = U_\Sigma. \tag{13}$$

As advertised, the U terms contain modifications to gravity, and we have used the notation from [42]: $U_\Delta \equiv -a^2 U^0{}_0$, $\vec{\nabla}_i U_\theta = -a^2 U^0{}_i$, $U_P = a^2 U^i{}_i$, and $[\vec{\nabla}^i\vec{\nabla}_j - \frac{1}{3}\vec{\nabla}^2\delta^i{}_j]U_\Sigma = a^2(U^i{}_j - \frac{1}{3}U^k{}_k\delta^i{}_j)$. Further, we have parameterized the dark energy pressure perturbation $\Pi_E \equiv \delta P_E/\rho_E$ as

$$\Pi_E = c_s^2\delta_E + 3(c_s^2 - c_a^2)(1 + w)\mathcal{H}\theta_E. \tag{14}$$

For adiabatic fluids, as is the case of radiation and cold dark matter, $c_s = c_a$, and we get $\Pi_E = c_s^2 \delta_E$. In general, however, $c_s \neq c_a$ and may in fact be a function of space as well as time.

C. The fluid equations

It is convenient to define $\Delta_X = \delta_X - 3(1 + w_X)\Phi$ for $X = M$, dark energy. We then have that the equations of motion for the fluids [48] are

$$\begin{aligned}
\dot{\Delta}_M &= -k^2\theta_M, \\
\dot{\theta}_M &= -\mathcal{H}\theta_M + \Psi, \\
\dot{\Delta}_E &= 3\mathcal{H}(w - c_s^2)\Delta_E - (1 + w)k^2\theta_E \\
&\quad - 9(1 + w)\mathcal{H}(c_s^2 - c_a^2)[\Phi + \mathcal{H}\theta_E], \\
\dot{\theta}_E &= (3c_s^2 - 1)\mathcal{H}\theta_E + c_s^2\left(\frac{1}{1 + w}\Delta_E + 3\Phi\right) + \Psi.
\end{aligned} \tag{15}$$

D. The consistency conditions

In principle, one should be able to choose arbitrary combinations of metric perturbations to go in the tensor U. Yet in [42] it was argued that by assuming a set of general properties, it is possible to restrict the form of U. For the purpose of this paper, we shall choose the general theory of gravity to satisfy the following restrictions:

(1) *The fundamental geometric degree of freedom is the metric:* This encompasses most modified theories of gravity, including the first order Palatini (torsionless) formulations (where the connection Γ^c_{ab} is independent of the metric at the level of the action) or purely affine theories [49], provided a metric can be defined.

(2) *The field equations are at most second order:* This does restrict the class of acceptable theories (for example $F(R)$ theories are generally higher derivative but it has become clear that it is these higher order terms that lead (again) to instabilities in the generation of large scale structure.

(3) *The field equations are gauge-form invariant:* Gauge-form invariance is the linearized version of the full diffeomorphism invariance of any gravitational theory with a manifold structure. It is the unbroken symmetry of the field equations under gauge transformations. After a gauge transformation, the field equations retain their exact form: they are *form* invariant (see [50–53] for further discussion).

It is of course possible to relax some of these conditions, and we will discuss how in the conclusions.

Armed with these conditions we can construct U in Fourier space solely out of Φ and $\dot{\Phi}$ such that $U_\Delta = k^2 A\Phi$, $U_\theta = kB\Phi$, $U_P = k^2 C_1\Phi + kC_2\dot{\Phi}$, and $U_\Sigma = D_1\Phi + D_2\dot{\Phi}/k$. All operators above (e.g. A) are dimensionless. Terms such as $\ddot{\Phi}$, $\dot{\Psi}$, and $\ddot{\Psi}$ (and higher derivatives) are forbidden, while terms proportional to Ψ are allowed but their coefficients vanish as a result of the Bianchi identities. This last fact is a consequence of the consistency conditions outlined above, and the reader is referred to [42] for a more thorough treatment.

We can rearrange (11) in the form of (6) to immediately read off

$$\mu^2 = 1 + \frac{A + 3\mathcal{H}_k B}{2},$$

where we let $\mathcal{H}_k \equiv \mathcal{H}/k$.

Furthermore, we have that the shear field equation becomes

$$\Phi - \Psi = \zeta\Phi + \frac{g}{k}\dot{\Phi},$$

where we have let $\zeta = D_1$ and $g = D_2$.

We see that the gravitational slip $\Phi - \Psi$ will generally depend on both Φ *and its time derivative*, $\dot{\Phi}$, and not solely on Φ. Defining $\Phi = \gamma_{\rm PPF}\Psi$ as has been in proposed in [37,38], we find that

$$\gamma_{\rm PPF}[\Phi] = \frac{1}{1 - \zeta - \frac{g}{k}\frac{d\ln\Phi}{d\tau}} \approx 1 + \zeta + \frac{g}{k}\frac{d\ln\Phi}{d\tau}. \tag{16}$$

Thus, unless $g = 0$, $\gamma_{\rm PPF}$ is an explicit functional of Φ, introducing interesting environmental dependence on the matter distribution. All parameterizations of the slip used so far, for which $\Phi - \Psi \propto \Phi$, have ignored this possibility, which suggests that they were over simplistic (although see [31]).

As shown in [42], as a result of the Bianchi identities we have $\nabla_a U^a{}_b = 0$, leading to a series of restrictions on the coefficients:

$$\begin{aligned}
A &= \mathcal{H}\frac{2\mathcal{H}_k\dot{g} + 2k(2\mathcal{H}_k^2 + \frac{1}{3})g - 2\mathcal{H}\zeta}{\dot{\mathcal{H}} - \mathcal{H}^2 - \frac{k^2}{3}}, \\
B &= -\frac{k}{3\mathcal{H}}A - \frac{2}{3}g, \\
C_1 &= \frac{3}{k}(\dot{B} + 2\mathcal{H}B) + 2\zeta = -A - \frac{1}{\mathcal{H}}[\dot{A} + kB], \\
C_2 &= -\frac{k}{\mathcal{H}}A.
\end{aligned} \tag{17}$$

This means a consistent modification to the Einstein equations is uniquely determined by two arbitrary free functions, $\zeta(\tau, k)$ and $g(\tau, k)$.

Finally, we can combine the expression for A and B appropriately to find that g has a simple interpretation as a perturbation of the effective gravitational constant

$$\mu^2 = 1 - \frac{\mathcal{H}}{k}g. \tag{18}$$

Hence, the consistency conditions lead to an important relationship between a generalized form of the gravitational slip. In particular, *if we consider time variations of the Newtonian potential, it is inconsistent to consider a restricted parametrization of the gravitational slip of the form* $\Phi - \Psi = \zeta\Phi$ *on all scales* [54].

E. Parameterizing ζ and g

We are not assuming any particular underlying theory of modified gravity and hence do not have a specific model for ζ and g. Our interest is in theories that may mimic the behavior of dark energy so we expect deviations from Einstein gravity to emerge as Ω_E begins to diverge from 0. A simple assumption is to assume that the gravitational slip is analytic at $\Omega_E = 0$ and Taylor expand it:

$$\zeta = \zeta_1\Omega_E + \zeta_2\Omega_E^2 + \mathcal{O}(\Omega_E^3). \tag{19}$$

We can do the same for g so that

$$g = g_1\Omega_E + g_2\Omega_E^2 + \mathcal{O}(\Omega_E^3). \tag{20}$$

In Sec. III, we find how the growth depends on such a parametrization and, in particular, determine analytic expressions for γ.

This way of parameterizing ζ and g has three major advantages:

(i) It is in the spirit of the PPN formalism where the PPN parameters are isolated from the potentials, which are dependent on the density profiles and thus the solutions; the role of the "potential" in this case is taken by $\Omega_E(\tau)$, which depends on the background cosmology.

(ii) Expanding in powers of Ω_E isolates the background effects of dark energy from the genuine effects of the perturbations. In particular, the dark energy relative density Ω_{0E} or the dark matter relative density Ω_{0m} would have no effect on the parameters ζ_i and g_i.

(iii) This expansion makes mathematical sense for any analytic function, as the function $\Omega_E(\tau)$ is always bounded to be $0 \leq \Omega_E \leq 1$, i.e. it is a naturally small parameter.

Note that we have dropped any k dependence from this parametrization. There are two ways that k dependence can enter, either relative to a fixed scale k_0 (which may be part of some theory of gravity) or relative to the Hubble scale $\mathcal{H}$. If we wish to see how our results are affected by a scale dependence relative to the temporal changes introduced by the FRW background we can extend the parametrization to

$$\zeta = \zeta^{(0)} + \zeta^{(1)}\mathcal{H}_k, \qquad g = g^{(0)} + g^{(1)}\mathcal{H}_k, \tag{21}$$

where, as above, we have

$$\zeta^{(0)} = \zeta_{01}\Omega_E + \mathcal{O}(\Omega_E^2), \qquad \zeta^{(1)} = \zeta_{11}\Omega_E + \mathcal{O}(\Omega_E^2), \tag{22}$$

and likewise with $g^{(0)}$ and $g^{(1)}$.

It turns out that, if we attempt to, on one hand generalize our parametrization of ζ and g, but, on the other hand pin it down so as to be consistent with PPN method used on much smaller scales, we need to change our previous approach. It is entirely possible that there are other scales in the system. Furthermore, as we show in the Appendix, to leading order we may have $g \propto \mathcal{H}_k^m$ where m is negative. These can complicate the simple model we considered above. For example, consider the function $f = e^{\ell\mathcal{H}}$ where ℓ is a fixed scale. How does one expand this on small scales? One would be tempted to write $f = e^{\ell k\mathcal{H}_k} \approx 1 + \ell k\mathcal{H}_k = 1 + \ell\mathcal{H}$ but this clearly makes no sense. The scale k was artificially introduced and leads to erroneous conclusions.

In general we have a function $\zeta(\tau, k)$. Since ζ is dimensionless while τ and k are not, the functional dependence on τ and k must come in dimensionless combinations. It is convenient to exchange τ with either $\tau(\mathcal{H})$ or with $\tau(\Omega_E)$. Thus, the most general function ζ will have the form $\zeta = \zeta(\Omega_E, \mathcal{H}_k, \ell\mathcal{H}, k/k_0)$ for constants ℓ and k_0, and there may be additional dimensionless parameters entering. We can thus isolate the leading-order dependence of ζ on $\mathcal{H}_k$ and write

$$\zeta = \zeta_L(\Omega_E, k)\mathcal{H}_k^n \tag{23}$$

for a constant n. We expand g in a similar way as

$$g = g_L(\Omega_E, k)\mathcal{H}_k^m \tag{24}$$

for a constant m. In the Appendix we show that a consistent PPN limit fixes $n = 0$ and $m = -1$.

Thus, we arrive at our general expansion of ζ and g in the small-scale limit, which is consistent with PPN:

$$\zeta = \zeta^{(0)} \qquad g = g^{(-1)}\frac{1}{\mathcal{H}_k}, \qquad \zeta^{(0)} = \zeta_1\Omega_E + \zeta_2\Omega_E^2 + \zeta_3\Omega_E^3, \tag{25}$$

$$g^{(-1)} = g_1\Omega_E + g_2\Omega_E^2 + g_3\Omega_E^3. \tag{26}$$

The parameters ζ_i and g_i may in principle be k dependent, e.g. $\zeta_1 \propto (k/k_0)^N$ for a fixed scale k_0 and power index N. We shall not investigate this further in this work but we stress it as a possibility and note that our results would include these cases. In Sec. IV, we conclude by presenting and analyzing the resulting growth rate due to such a parametrization.

III. THE GROWTH RATE ON SMALL SCALES FOR A SIMPLIFIED MODEL OF ζ AND g

The definition of γ originally arose when characterizing the evolution of small-scale density perturbations. We expect it to be particularly useful when characterizing the growth of structure on small scales (by which we mean roughly between 1 and $200h^{-1}$ Mpc) as would be probed by galaxy redshift surveys (through redshift measurements of the power spectrum, for example, or redshift space distortions [55–57]) and weak lensing surveys.

In this section we focus on the behavior of this system in the limit in which $\mathcal{H}_k \equiv \mathcal{H}/k \ll 1$, i.e. on scales deep inside the horizon. We can then assume that $\Delta_{\rm DE} \simeq \theta_{\rm DE} \simeq 0$. This is true for $c_s^2 \sim O(1)$ or larger. Since in this paper we are concerned with modifications of gravity rather than the speed of sound we will leave the full treatment of small c_s^2 for a future investigation. We shall, however, show the effect of small c_s^2 numerically further below.

In what follows we will present a modified evolution equation, find analytic expressions for the growth factor and compare to numerical results for the full system.

A. Evolution of density perturbations

Combining the fluid Eqs. (15) in one second order equation, we find that Δ_M obeys

$$\ddot{\Delta}_M + \mathcal{H}\mathcal{U}\dot{\Delta}_M - \tfrac{3}{2}\Omega_M\mathcal{H}^2\mathcal{V}\Delta_M = 0, \qquad (27)$$

with the damping coefficient modified by

$$\mathcal{U}[\zeta, g] = 1 + \frac{3\Omega_M\mathcal{H}_k}{2(1-\mathcal{H}_k g)} \times \left\{g + \frac{3\mathcal{H}_k[1-\zeta-gB/2]}{1-\mathcal{H}_k g + 9\mathcal{H}_k^2\Omega_M/2}\right\} \qquad (28)$$

and the response term modified by

$$\mathcal{V}[\zeta, g] = \frac{1-\zeta-gB/2}{(1-\mathcal{H}_k g)(1+9\mathcal{H}_k^2\Omega_M/2-\mathcal{H}_k g)}. \qquad (29)$$

Specifying ζ and g completely fixes $\mathcal{U}$ and $\mathcal{V}$.

If we further take the provisional small-scale limit $\mathcal{H}_k \ll 1$ (i.e. without assuming anything about ζ and g) we find that

$$\mathcal{U}[\zeta, g] = 1 + \frac{3\Omega_M\mathcal{H}_k}{2(1-\mathcal{H}_k g)}\left\{g + \frac{3\mathcal{H}_k[1-\zeta-gB/2]}{1-\mathcal{H}_k g}\right\} \qquad (30)$$

and

$$\mathcal{V}[\zeta, g] = \frac{1-\zeta-gB/2}{(1-\mathcal{H}_k g)^2}. \qquad (31)$$

The full small-scale limit, including ζ and g is presented in the Appendix.

B. Analytic expressions for the growing mode

We expect modifications of gravity to kick in when the expansion rate starts to deviate from matter domination. In this section we will work with the parametrization of ζ and g proposed in Eqs. (19) and (20). We can immediately see from Eqs. (30) and (31) that the effects from g will only come in at order $\mathcal{H}_k$. We shall also restrict ourselves to constant w and leave varying w for Sec. IV. In this section we shall present the derivation and result to first order in Ω_E and then present the result to second order in Ω_E.

The starting point is

$$\ddot{\Delta}_M + \mathcal{H}\dot{\Delta}_M - \frac{3\mathcal{H}^2\Omega_M}{2}(1-\zeta_1\Omega_E)\Delta_M = 0.$$

Changing variables to $\ln a$ and defining $\Delta_M = aY$, we can rewrite this equation as

$$Y'' + \frac{5-3w\Omega_E}{2}Y' + \frac{3}{2}\Omega_E[1-w+\zeta_1\Omega_M]Y = 0,$$

where we have used the Raychauduri Eq. (8) and where we set $()' = \frac{d}{d\ln a}$.

Changing variables to Ω_E and using the 0th order fluid conservation equation, rewritten as $\Omega_E' = -3w\Omega_M\Omega_E$, we find

$$3w^2\Omega_M^2\Omega_E^2\frac{d^2Y}{d\Omega_E^2} + \frac{w}{2}\Omega_M\Omega_E[3w(2-3\Omega_E)-5]\frac{dY}{d\Omega_E} + \frac{1}{2}[\Omega_E(1-w)+\zeta_1\Omega_M\Omega_E]Y = 0. \qquad (32)$$

Equation (32) has a three regular singular points (at $\Omega_E = 0$ and $\Omega_E = 1$) and can therefore be transformed into the hypergeometric equation. We wish to find its behavior around $\Omega_E = 0$ and do so by expanding Y, $Y = 1 + Y_1\Omega_E$ to find (to lowest order in Ω_E)

$$Y_1 = \frac{1-w+\zeta_1}{w(5-6w)},$$

and hence

$$\Delta_M = a\left[1 + \frac{1-w+\zeta_1}{w(5-6w)}\Omega_E\right].$$

Therefore to $\mathcal{O}(\Omega_E)$

$$\ln\Delta_M = \ln a + \frac{1-w+\zeta_1}{w(5-6w)}\Omega_E,$$

which we can use to find the logarithmic derivative of the growth factor $f \equiv d\ln\Delta_M/d\ln a$:

$$f = 1 - \frac{3(1-w+\zeta_1)}{5-6w}\Omega_E.$$

As stated above, we are parametrizing the growth factor using $f = \Omega_M^\gamma$ and so we have that

$$\gamma = \gamma_0 = \frac{3(1-w+\zeta_1)}{5-6w}, \qquad (33)$$

where the subscript 0 is due to the fact that this is the lowest order approximation.

We can easily see that, for $w = -1$ and $\zeta_1 = 0$ we retrieve $\gamma = 6/11 \simeq 0.54\,545\ldots$, the approximation first proposed in [32] and subsequently rederived and advocated in [33,58]. If we assume a more general (but constant) equation of state, we improve on the approximation advocated in [33].

It is possible to further improve the approximation by going to next order in Ω_E. For this we need Y and ζ to $\mathcal{O}(\Omega_E^2)$, i.e. $Y = 1 + Y_1\Omega_E + Y_2\Omega_E^2$ and $\zeta = \zeta_1\Omega_E + \zeta_2\Omega_E^2$. We now have that

$$\gamma = \gamma_0 + \gamma_1\Omega_E, \tag{34}$$

where γ_0 is as derived above and given in (33) and

$$\gamma_1 = \frac{3w}{2}[-3Y_1 - (2-3w)Y_1^2 + 4Y_2], \tag{35}$$

where we have defined

$$Y_2 = \frac{(1-w)(15w^2 - 4w - 1) + \zeta_1(9w^2 + 2w - 2)}{2w^2(12w-5)(5-6w)} - \frac{\zeta_1^2 + w(5-6w)\zeta_2}{2w^2(12w-5)(5-6w)}. \tag{36}$$

In the case of $w = -1$ (i.e. the cosmological constant) and $\zeta = 0$ we find that expression (35) reduces to

$$\gamma_1 = \tfrac{15}{2057}.$$

As stated above assuming that g can be parametrized in the same way as ζ, independently of $\mathcal{H}_k$, we find that it does not affect the growth of structure on small scales. The situation is of course different if we consider expanding $1 - \mu^2 = g\mathcal{H}_k$ in powers of Ω_E with coefficients that are $\mathcal{H}_k$ independent.

C. Comparison with numerical results

We can solve (13), (15), and (17) to assess the quality of this analytic approximation. We first restrict ourselves to a dark energy like fluid with a large sound speed, $c_s^2 \sim O(1)$, (such as a quintessence model or most other field like models) and assume no modifications to gravity. We see a number of features in Fig. 2. First of all, γ is very clearly not independent of Ω_M as has been generally assumed. In fact as $\Omega_M \to 0$, γ deviates substantially from its asymptotic value at $\Omega_M = 1$. Nevertheless, we find that (35) is a good approximation to the true behavior. In Fig. 2 we plot the true and approximate behaviors of γ for $w = -1$, -0.8, -0.6 and -0.4: we find deviations of at most 10% for $w = -0.4$ at $\Omega_M = 0.1$.

We may relax the condition $c_s^2 = 1$ substantially before the dark energy perturbations affect the growing mode in the density field. This is clearly illustrated in Fig. 3 where, for $w = -0.6$, three different values of the sound speed are chosen: $c_s^2 = 5\times 10^{-4}$, $c_s^2 = 10^{-5}$ $c_s^2 = 10^{-6}$ and $c_s^2 = 0$. For $c_s^2 = 5\times 10^{-4}$, the growth is still indistinguishable from $c_s^2 = 1$ and only once the Jeans scale for dark energy falls substantially below the cosmological horizon is the effect noticeable. In particular our results hold for values c_s such that $c_s > 10 \sim \mathcal{H}_k$, which approximately translates to $c_s^2 > \sim 3\times 10^{-4}$.

We note in passing that the effects of the speed of sound without modifications of gravity, have been studied in [59–61]. In particular, [61] have found fitting formulas for γ, which interpolate between $c_s = 0$ and $c_s \sim O(1)$. However, those fitting formulas do not account for modifications of gravity and are in fact quite model dependent (they depend on the background cosmology). The current constraints on the speed of sound do not rule out small values [62] and in fact they are consistent with $c_s = 0$.

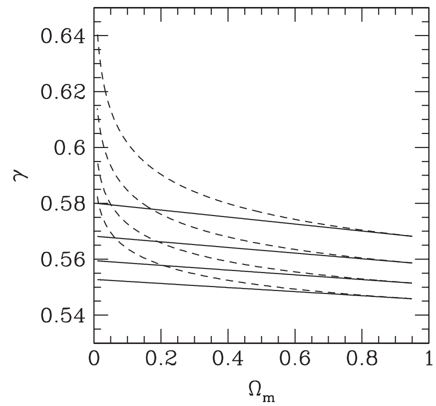

FIG. 2. The growth parameter, γ, for a selection of dark energy models, as a function of Ω_M. The dashed curves are the numerical results for $w = -1$, -0.8, -0.6, and -0.4 in ascending order, and the corresponding analytic approximations are plotted in the solid line.

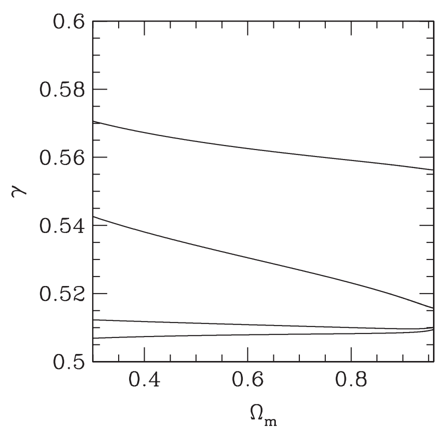

FIG. 3. The growth parameter, γ, for a selection of dark energy models where $w = -0.6$ and the sound speed is chosen to be $c_s^2 = 5\times 10^{-4}$, $c_s^2 = 10^{-5}$, $c_s^2 = 10^{-6}$ and $c_s^2 = 0$ in descending order from the top of the figure, as a function of Ω_M.

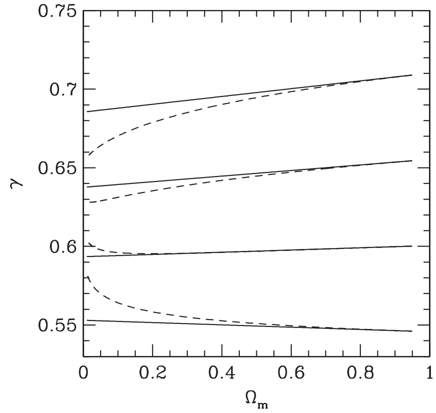

FIG. 4. The growth parameter, γ, for a selection of gravitational slip parameters of the form $\zeta = \zeta_1 \Omega_E$, as a function of Ω_M. The dashed curves are the numerical results for $\zeta_1 = 0$, 0.2, 0.4, and 0.6 in ascending order, and the corresponding analytic approximations are plotted in the solid line.

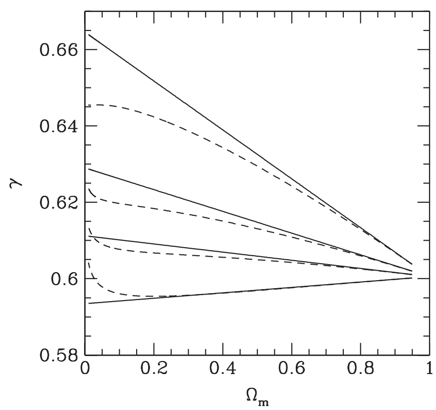

FIG. 5. The growth parameter, γ, for a selection of gravitational slip parameters of the form $\zeta = \zeta_1 \Omega_E + \zeta_2 \Omega_E^2$, as a function of Ω_M. The dashed curves are the numerical results for $\zeta_1 = 0.2$ and $\zeta_2 = 0$, 0.125, 0.2 and 0.4, and the corresponding analytic approximations are plotted in the solid line.

Since this work concerns the effects of modifications of gravity rather than effects coming from the speed of sound, however, we leave the case for small c_s for a future investigation.

Let us now introduce modifications to gravity and assume a non-negligible gravitational slip. In Figs. 4 and 5 we show how well the analytic approximation fares in comparison to different values of ζ_1 and ζ_2 (in these figures we restrict ourselves to $w = -1$ but the agreement between the numerical and approximate estimates of γ is generic). In Fig. 4, we restrict ourselves to a gravitational slip which is linear in Ω_E and identify the two main effects. First of all, the quicker the onset of slip, the more effective the suppression of growth due to the onset of dark energy-γ_0 increases with ζ_1. Furthermore, the dependence of γ on Ω_E, through γ_1, changes sign so that for larger ζ_1, γ becomes smaller as Ω_M decreases.

This last effect is further affected by ζ_2. Indeed, we find that the slope of γ as function of Ω_M can greatly be affected by higher order terms in ζ.

D. Intermediate and large scales

While the γ parametrization is particularly useful on small scales where terms dependent on $\mathcal{H}_k$ can be discarded, this is not true once we look at horizon crossing, i.e. $\mathcal{H}_k \simeq 1$. This regime will be of particular importance for measurements of the cosmic microwave background and using the Integrated Sachs-Wolfe effect to look for the presence of dark energy or modified gravity [37,38,44,63]. Let us now consider the parametrization presented in Eqs. (21)

We expand Eq. (27) to include the first order term in $\mathcal{H}_k$, which gives

$$\ddot{\Delta}_M + \mathcal{H}\left[1 + \frac{3\mathcal{H}_k \Omega_M}{2} g^{(0)}\right]\dot{\Delta}_M - \frac{3}{2}\Omega_M \mathcal{H}^2 \{1 - \zeta^{(0)} + \mathcal{H}_k[g^{(0)}(2 - \zeta^{(0)}) - \zeta^{(1)}]\}\Delta_M = 0, \quad (37)$$

where the functions $\zeta^{(0)}$, $\zeta^{(1)}$and $g^{(0)}$ are further expanded in powers of Ω_E using Eq. (22). We apply the same techniques as in Sec. III B above, and expand γ as

$$\gamma = \gamma^{(0)} + \gamma^{(1)}\mathcal{H}_k,$$

where the coefficients $\gamma^{(0)}$ and $\gamma^{(1)}$ are further expanded in powers of Ω_E, i.e. $\gamma^{(0)} = \gamma_{00} + O(\Omega_E)$. Carrying through the expansion we find that

$$\gamma^{(0)} = 3\frac{1 - w + \zeta_{01}}{5 - 6w} + O(\Omega_E), \quad (38)$$

$$\gamma^{(1)} = \frac{27}{4}\frac{3\zeta_{01}^2 + 2\zeta_{11} - 2g_{01}}{3w - 1} + O(\Omega_E). \quad (39)$$

There are now three free modified gravity constants: ζ_{01}, ζ_{11} and g_{01}. Notice how the first order correction in $\mathcal{H}_k$ to γ only depends on the modified gravity parameters and is zero for wCDM, i.e. corrections in $\mathcal{H}_k$ to γ for wCDM come to second order.

We find that scale-dependent corrections in powers of $\mathcal{H}_k$ are always subdominant compared with corrections in powers of Ω_E, or corrections with respect to a fixed scale, e.g. $\sim(\ell k)^N$. They are thus effectively negligible. We find that at $k \sim 0.04h$ Mpc^{-1} corrections in powers of $\mathcal{H}_k$ are around 1% at redshift $z = 1$ for $\zeta_{11} = -0.6$ and become smaller at larger k (smaller scales), or lower redshift ($z \sim 0$). Hence, it is perfectly reasonable to discard scale-dependent corrections that come in powers of $\mathcal{H}_k$.

IV. GENERAL EVOLUTION OF ζ AND g

We now wish to address a more realistic expansion of the ζ and g proposed in Sec. II, which is consistent with the

PPN approximation on small scales, namely, (25) and (26). Having in mind our findings in Sec. III D on intermediate scales, we disregard any dependence of Δ_M (and hence of γ) on $\mathcal{H}_k$, and therefore to this order we can set $\Delta_M = \delta_M$.

We perform the calculation in steps. First, we solve the second order differential equation obeyed by δ by applying a Taylor series expansion in Ω_E, resulting in a set of coefficients $\{Y_1, Y_2, Y_3\}$, which are functions of the expansion coefficients of w, $\mathcal{U}$ and $\mathcal{V}$. Then we relate the a perturbative expansion coefficients for δ, namely, $\{Y_1, Y_2, Y_3\}$, to the γ parameter. Finally, we relate $\{Y_1, Y_2, Y_3\}$ for general $\mathcal{U}$ and $\mathcal{V}$ to the specific case of our $g\zeta$CDM model.

A. Perturbative solution of the δ equation

We start with the equation for the matter density contrast in the absence of dark energy perturbations, namely,

$$\ddot{\delta}_M + \mathcal{H}\mathcal{U}\dot{\delta}_M - \frac{3\mathcal{H}^2\Omega_M}{2}\mathcal{V}\delta_M = 0. \tag{40}$$

Changing the independent variable from τ to $\ln a$ and the dependent variable from δ_M to Y defined by $Y \equiv \delta_M/a$, we get

$$Y'' + \left[\mathcal{U} + \frac{3}{2}(1 - w\Omega_E)\right]Y' + \left[\mathcal{U} + \frac{1}{2}(1 - 3w\Omega_E) - \frac{3\Omega_M}{2}\mathcal{V}\right]Y = 0. \tag{41}$$

On small scales we may expand $Y(k,\tau) = Y^{(0)}(k,\tau) + Y^{(1)}(k,\tau)\mathcal{H}_k + O(\mathcal{H}_k^2)$ [see the Appendix)], where the functional coefficients $Y^{(i)}(k,\tau)$ have no dependence on $\mathcal{H}_k$ but may still be k or τ dependent through combinations of the form ℓk or τ/ℓ (where ℓ is some scale, not necessarily the same scale for all such combinations). We are interested in the small-scale limit $\mathcal{H}_k \to 0$, and we shall work with $Y(k,\tau) = Y^{(0)}(\ell k, \Omega_E)$ only. As discussed in Sec. III D above, at the scale of validity of the γ parameterization $\mathcal{H}_k$ corrections are always small and irrelevant. Since time dependence only comes through τ/ℓ for some scale ℓ we may further exchange τ/ℓ with a function of Ω_E, thus we let $Y(k,\tau) = Y(k,\Omega_E)$.

We now Taylor expand $Y(k,\Omega_E)$ in powers of Ω_E. To get γ to $O(\Omega_E^2)$ we need to expand Y to $O(\Omega_E^3)$ as

$$Y = 1 + Y_1\Omega_E + Y_2\Omega_E^2 + Y_3\Omega_E^3. \tag{42}$$

We then use the above expansion (42) into (41) and match orders [64]. To be able to do that we need to expand the functions w, $\mathcal{U}$, and $\mathcal{V}$. Since w always appears in the combination $w\Omega_E$ we only need it to $O(\Omega_E^2)$. The functions $\mathcal{U}$ and $\mathcal{V}$, however, are needed to $O(\Omega_E^3)$. Thus, we expand

$$w = w_0 + w_1\Omega_E + w_2\Omega_E^2, \tag{43}$$

$$\mathcal{U} = 1 + \mathcal{U}_1\Omega_E + \mathcal{U}_2\Omega_E^2 + \mathcal{U}_3\Omega_E^3, \tag{44}$$

$$\mathcal{V} = 1 + \mathcal{V}_1\Omega_E + \mathcal{V}_2\Omega_E^2 + \mathcal{V}_3\Omega_E^3. \tag{45}$$

While w_i are constants, the $\mathcal{U}_i$ and $\mathcal{V}_i$ coefficients may be k dependent, for example $\mathcal{U}_1 = \mathcal{U}_{01}(\frac{k}{k_0})^N$ for some index N and scale k_0. Using the expansions (42)–(45) into (41) and equating orders in Ω_E we find

$$\begin{aligned}
Y_1 &= \frac{-1 + w_0 + \mathcal{V}_1 - \frac{2}{3}\mathcal{U}_1}{w_0(6w_0 - 5)},\\
Y_2 &= \frac{1}{2w_0(12w_0 - 5)}\left\{w_1 + \mathcal{V}_2 - \mathcal{V}_1 - \frac{2}{3}\mathcal{U}_2 + \left[-1 - \frac{2}{3}\mathcal{U}_1 + \mathcal{V}_1 + 5w_1 + w_0(-4 + 2\mathcal{U}_1 + 15w_0 - 18w_1)\right]Y_1\right\},\\
Y_3 &= \frac{1}{3w_0(18w_0 - 5)}\left\{\mathcal{V}_3 - \frac{2}{3}\mathcal{U}_3 - \mathcal{V}_2 + w_2 + \left[\mathcal{V}_2 - \frac{2}{3}\mathcal{U}_2 - \mathcal{V}_1 - 4w_1 + 5w_2 + 2w_1(\mathcal{U}_1 - 6w_1)\right.\right.\\
&\quad \left. + w_0(-2\mathcal{U}_1 + 2\mathcal{U}_2 - 9w_0 + 42w_1 - 24w_2)\right]Y_1 + \left[-1 - \frac{2}{3}\mathcal{U}_1 + \mathcal{V}_1 - 9w_0 + 10w_1\right.\\
&\quad \left.\left. + 2w_0(2\mathcal{U}_1 + 27w_0 - 30w_1)\right]Y_2\right\}.
\end{aligned} \tag{46}$$

We notice that in general there are nine initial coefficients appearing in (40) that determine only three final coefficients Y_i for the solution to (40).

Having found the coefficients Y_i we proceed to relate them to γ.

B. From δ to γ

We can now use the definition of the logarithmic change of the growth rate $f = \frac{d\ln\delta}{d\ln a}$ and then get γ from $\gamma = \ln f/\ln\Omega_M$. We find that γ is expanded as

$$\gamma = \gamma_0 + \gamma_1\Omega_E + \gamma_2\Omega_E^2, \tag{47}$$

where

$$\begin{aligned}\gamma_0 &= 3w_0Y_1,\\ \gamma_1 &= -\gamma_0(\tfrac{3}{2} + Y_1 - \tfrac{1}{2}\gamma_0) + 3w_1Y_1 + 6w_0Y_2,\\ \gamma_2 &= \gamma_0(-\tfrac{11}{6} - \tfrac{1}{2}\gamma_0 + \tfrac{1}{3}\gamma_0^2) - \tfrac{3}{2}\gamma_1 + \tfrac{1}{2}[6w_1\gamma_0 + 6w_2 - \gamma_0^2\\ &\quad - 3\gamma_0 - 2\gamma_1]Y_1 + [-(1-6w_0)\gamma_0 + 6w_1]Y_2\\ &\quad + 9w_0Y_3.\end{aligned} \tag{48}$$

Given a set of coefficients $\{Y_1, Y_2, Y_3, w_0, w_1, w_2\}$ we can get $\gamma(\Omega_E)$. Note that $\{Y_1, Y_2, Y_3\}$ may be k dependent, for example $Y_1 = Y_{01}(\frac{k}{k_0})^N$ for some index N and scale k_0.

One important point is in order. What we have done so far is more general than the approach we discussed in the main part of the article. In particular, the derivation of the γ coefficients in this section would hold for any theory for which the density contrast obeys (40). One such theory is Dvali-Gabadadze-Porrati gravity (DGP), even though strictly speaking DGP does not fit within our framework of the main part of the article.

To connect the γ coefficients above with our framework we must perform a third step: relate the $\mathcal{U}_i$ and $\mathcal{V}_i$ coefficients with expansions of g and ζ.

C. Relating to the $g\zeta$CDM model

As shown in the Appendix, to be consistent with ultra-small-scale quasistatic limit the functions ζ and g must have the form $\zeta = \zeta^{(0)} + O(\mathcal{H}_k)$ and $\mathcal{H}_k g = g^{(-1)} + O(\mathcal{H}_k)$. As discussed in the last part of Sec. II E, we further expand $\zeta^{(0)}$ and $g^{(-1)}$ as

$$\zeta^{(0)} = \zeta_1\Omega_E + \zeta_2\Omega_E^2 + \zeta_3\Omega_E^3, \tag{49}$$

and

$$g^{(-1)} = g_1\Omega_E + g_2\Omega_E^2 + g_3\Omega_E^3, \tag{50}$$

respectively, where the coefficients may once again be k dependent. To lowest order in $\mathcal{H}_k$ we find

$$\begin{aligned}\mathcal{U}_1 &= \tfrac{3}{2}g_1,\\ \mathcal{U}_2 &= \tfrac{3}{2}[g_2 - g_1 + g_1^2],\\ \mathcal{U}_3 &= \tfrac{3}{2}[g_3 - g_2 + g_1^2(g_1 - 1) + 2g_1g_2],\\ \mathcal{V}_1 &= 2g_1 - \zeta_1,\\ \mathcal{V}_2 &= 2g_2 + (2 + 3w_0)g_1^2 - g_1\zeta_1 - \zeta_2,\\ \mathcal{V}_3 &= 2g_3 + 2(1 + 3w_0)g_1^3 + (3w_1 - 3w_0 - \zeta_1)g_1^2\\ &\quad + g_1g_2(4 + 9w_0) - g_2\zeta_1 - g_1\zeta_2 - \zeta_3.\end{aligned} \tag{51}$$

These expressions may then be used with (46) and (48) to find the γ coefficients.

D. Comparison with numerical results

The fitting formulas we have derived map theoretical properties of the gravitational field onto the observable, γ. This allows us to circumvent the use of a full cosmological perturbation code when trying to observationally constrain the μ^2, ζ and g via the growth of structure. We have already seen how well such an approach fares for the simplified model we used in Sec. III. We now briefly show how the general framework fares- note that the model for g and ζ matches onto the PPN and we have found the expansion of γ to second order in Ω_E. Both of these properties should allow us to span a wide range of possible models.

If we focus first on the gravitational slip, we can see in Figs. 6–8 that the analytic fit works exceptionally well, to

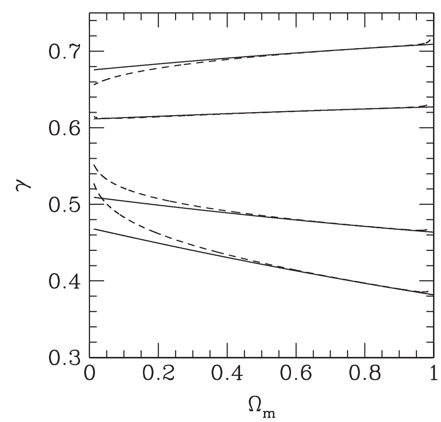

FIG. 6. The growth parameter, γ, for a selection of dark energy models, as a function of Ω_M. The dashed curves are the numerical results for $\zeta_1 = -0.6, -0.3, 0.3$, and 0.6 in ascending order, and the corresponding analytic approximations are plotted in the solid line.

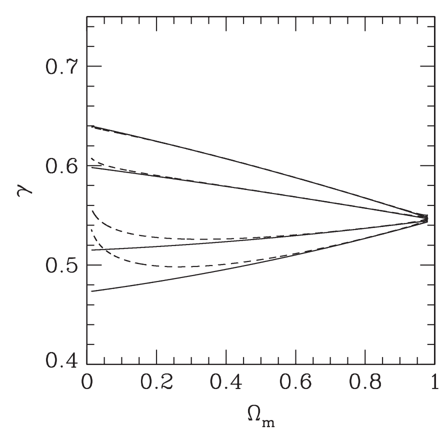

FIG. 7. The growth parameter, γ, for a selection of dark energy models, as a function of Ω_M. The dashed curves are the numerical results for $\zeta_2 = -0.6, -0.3, 0.3$ and 0.6 in ascending order, and the corresponding analytic approximations are plotted in the solid line.

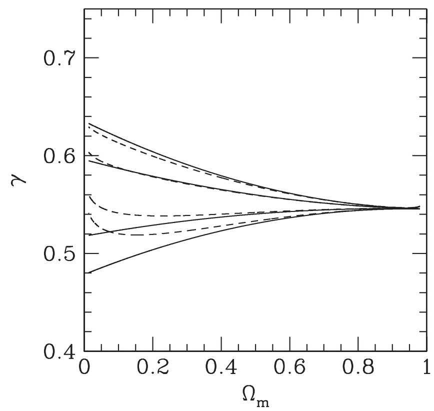

FIG. 8. The growth parameter, γ, for a selection of dark energy models, as a function of Ω_M. The dashed curves are the numerical results for $\zeta_3 = -0.6, -0.3, 0.3$, and 0.6 in ascending order, and the corresponding analytic approximations are plotted in the solid line.

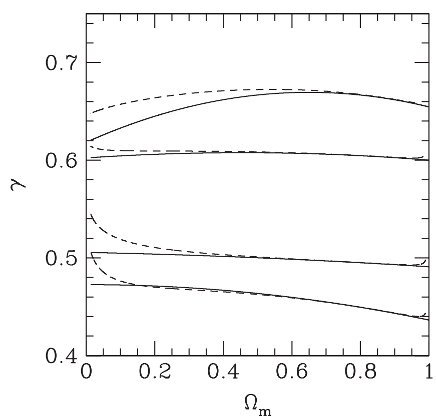

FIG. 9. The growth parameter, γ, for a selection of dark energy models, as a function of Ω_M. The dashed curves are the numerical results for $g_1 = -0.4, -0.2, 0.2$, and 0.4 in descending order and the corresponding analytic approximations are plotted in the solid line.

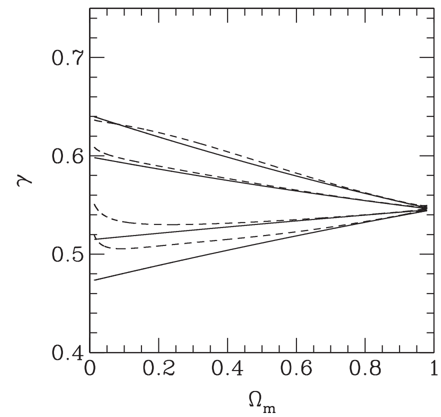

FIG. 10. The growth parameter, γ, for a selection of dark energy models, as a function of Ω_M. The dashed curves are the numerical results for $g_2 = -0.4, -0.2, 0.2$, and 0.4 in descending order, and the corresponding analytic approximations are plotted in the solid line.

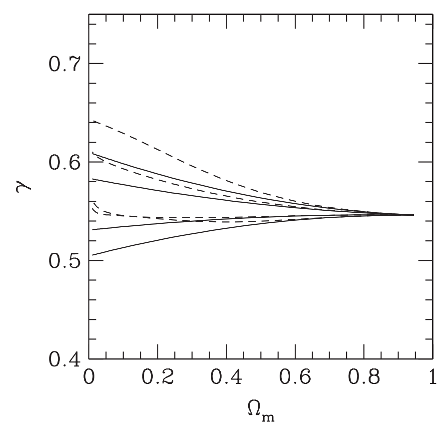

FIG. 11. The growth parameter, γ, for a selection of dark energy models, as a function of Ω_M. The dashed curves are the numerical results for $g_3 = -0.4, -0.2, 0.2$, and 0.4 in descending order, and the corresponding analytic approximations are plotted in the solid line.

within a few percent out to $\Omega_M = 0.1$. The fit starts to diverge in the upturn of γ for low Ω_M a universal feature that becomes more pronounced for certain ranges of ζ. From inspecting these figures, it is clear that the behavior of γ near $\Omega_E = 1 - \Omega_M$ is telling something qualitative about ζ. The parameter ζ_1 drives the offset of γ (i.e.γ_0), ζ_2 the slope (i.e. γ_1) and ζ_3 the curvature (i.e. γ_2). Measuring these three coefficients can give a direct handle on the time evolution of ζ.

In Figs. 9–11, we find similar effects when looking at the time-varying Newton constant, i.e. g (or μ^2). Again there is a direct mapping between g_1, g_2 and g_3 and γ_0, γ_1 and γ_2. The accuracy of the approximation breaks down for smaller values of Ω_M yet is still excellent in the range of interest for observational cosmology. For small values of Ω_E he accuracy is less than a percent and really only becomes large (of order 5–10%) for $\Omega_M < 0.1$.

V. DISCUSSION

Let us briefly recap what we have done. The main point of this paper is that, when introducing modifications to gravity in linear perturbation theory, one must take into account the consistency conditions in the field equations. These necessarily lead to restrictions in the form of the modifications that can be introduced. Most notably, and within the context of second order theories, this means that if one wishes to include modifications to the Newton-Poisson equation, then one *cannot consider the simplified gravitational slip*, $\Psi = \zeta\Phi$, and must include an extra term such that $\Psi = \zeta\Phi + (g/k)\dot{\Phi}$ where

$\frac{g}{k}\mathcal{H} = 1 - G_{\rm eff}/G_0$. If we wish to construct a proper parametrized post Friedmannian approach to modified gravity, any parameter we introduce must be independent of the environment or initial conditions in the perturbations. The only way to do this is to use the parametrization we are advocating. To our knowledge, all attempts at studying cosmological deviations from general relativity have ignored this and hence it is unclear what class of theories they map onto and which types of theories are being constrained.

Having taken this point on board, we have found the expression for the growth parameter on small scales in terms of both the gravitational slip, ζ and the modified Newton constant, $\mu^2 = 1 - \frac{g}{k}\mathcal{H}$. Given a set of cosmological constraints on γ and its dependence on Ω_M, it is now straightforward to calculate constraints on ζ and g. The growth parameter is given by Eq. (47) which can be seen as a Taylor expansion in terms of $1 - \Omega_M$. The coefficients in this expansion, γ_0, γ_1 and γ_2 can be expressed in terms of the equation of state, w [see Eq. (43)], ζ [see Eq. (25)] and g [see Eq. (26)] by using Eqs. (51), followed by Eqs. (46) and finally Eqs. (48).

With these relations in hand, it is now possible to use cosmological observations to place constraints on theories of modified gravity. In this paper we have focused on small scales (by which we mean between 1 and $200h^{-1}$ Mpc), scales that should be probed by redshift space distortion measurements, galaxy power spectra and weak lensing. Furthermore, we can now do this consistently, relating modifications in the growth rate with changes in the gravitational slip. This is of particular importance when considering weak lensing where observations probe $\Phi + \Psi$. It is also clear from our analysis that we have come up against the limitations of the γ parametrization: it is useful and effective on very small scales but not on scales comparable to the cosmological horizon. On those scales, one should be using the full set of field equations. We therefore do not advocate using our fitting formula to the growth on larger scales such as would be probed by the integrated Sach-Wolfe.

How general is this method? We have declared from the outset, the class of theories that we are considering. They must be metric, with second order equations and satisfy gauge-form invariance. From what we have learnt about modifications of gravity, these seem a reasonable set of conditions to apply- they lead to theories which are less likely to be marred by gross instabilities either at the classical or quantum level. We should point out that all other attempts at developing such a parametrization have *implicitly* made these assumptions although have not necessarily done so self-consistently. It is possible to extend this analysis beyond the scope of these theories. If we are to go beyond second order, one must include terms in $\ddot{\Phi}$ or even higher. The Bianchi conditions will, again, impose a set of constraints on the coefficients of these terms and should allow a similar type of analysis.

Two well-studied theories are worth mentioning. $F(R)$, $F(R^{\mu\nu}R_{\mu\nu})$, etc., theories come with up-to four time derivatives in the field equations. Thus they do not fall directly within the methods of this paper but do under the general scheme outlined in [42]. In this case one would have to include terms involving $\dot{\Phi}$, $\ddot{\Phi}$, Ψ and $\dot{\Psi}$ in to the G_{00} and G_{0i} Einstein equations, while the G_{ij} equations would need $\dddot{\Phi}$ and $\ddot{\Psi}$ in addition. Theories with higher derivatives are a subject that warrants further investigation and have yet to be properly incorporated in any parametrized modifications of standard general relativity.

One other theory, studied extensively is the DGP theory [13]. In this case, only two time derivatives are present in the field equations and just like our frame, DGP contains two nonmetric dynamical degrees of freedom, which can be effectively written ad δ_E and θ_E. However, our framework cannot encompass DGP because DGP cannot be written as a generalized fluid as we have assumed of dark energy in this work (hence it is not a failure of our use of δU_{ab}). Nevertheless, our γ parameterization in powers of Ω_E is still valid, and indeed needed. In the case of DGP we find that

$$\gamma = \frac{11}{16} + \frac{7}{5632}\Omega_E - \frac{93}{4096}\Omega_E^2 \tag{52}$$

gives an error on γ around 5% at $\Omega_M < 0.1$, dropping to 2% at $\Omega_M \sim 0.2$ and <1% for larger values of Ω_M. The error on the corresponding density contrast at those values of Ω_M is <2%, ~1% and <0.5%, respectively. Notice how the coefficients are entirely fixed and *do* not depend on the only free parameter of the theory, namely, the scale r_c. Rather r_c comes to play a role only through $\Omega_E = \frac{1}{Hr_c}$.

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APPENDIX: THE QUASISTATIC LIMIT—CONNECTING TO THE PPN APPROACH

We reduce our equations to small scales and slow expansions. To make contact with the PPN expansion, we will write the Einstein equations in powers of the 3-velocity v. The 3-velocity is related to the 4-velocity by $u^i = a^{-1}v^i$. In this gauge, for scalar perturbations we get $u^i = a^{-1}\vec{\nabla}^i\theta$, so that $v^i = \vec{\nabla}^i\theta$. Hence, $\theta = -k^{-2}\vec{\nabla}_i v^i$. Letting $v = \sqrt{v^i v_i}$ we get on dimensional grounds $k\theta = v$.

The PPN order bookkeeping is $k\frac{\partial}{\partial\tau} \equiv ()' \sim O(v)$ and $\Phi \sim \Psi \sim \delta\rho \sim O(v^2)$. The same bookkeeping prescription holds in our case, and in addition, we also have $\mathcal{H}_k \sim O(v)$ and $\Delta_M \sim \delta_M \sim O(0)$.

We are now ready to find the small-scale limit which is consistent with PPN. We start from the operators A, B, C_1

and C_2. Since $\mathcal{H}_k' = -\frac{1}{2}\mathcal{H}_k^2(1+3w\Omega_E)$ and $\Omega_E' = -3\mathcal{H}_k w\Omega_M\Omega_E$, and letting $J = \mathcal{H}_k^2(1+w\Omega_E)$, we get

$$A = -3\mathcal{H}_k \frac{2\mathcal{H}_k(g'-\zeta) + 2(2\mathcal{H}_k^2 + \frac{1}{3})g}{1+\frac{9}{2}J} \rightarrow 6\mathcal{H}_k\left[\mathcal{H}_k\zeta - \frac{1}{3}g\right],$$

$$B = \frac{2\mathcal{H}_k(g'-\zeta) + \mathcal{H}_k^2(1-3w\Omega_E)g}{1+\frac{9}{2}J} \rightarrow \mathcal{H}_k[-2\zeta + 2g' + \mathcal{H}_k(1-3w\Omega_E)g],$$

$$C_1 = -6\mathcal{H}_k\frac{1}{1+\frac{9}{2}J}\zeta' + \frac{1}{(1+\frac{9}{2}J)^2}\{2 + 9\mathcal{H}_k^2(1+3w\Omega_E) - 27\mathcal{H}_k^4[1 + w\Omega_E + 3w^2\Omega_M\Omega_E]\}\zeta + 6\mathcal{H}_k\frac{1}{1+\frac{9}{2}J}g'' + \frac{3\mathcal{H}_k^2}{(1+\frac{9}{2}J)^2}\{4 - 6w\Omega_E + 27\mathcal{H}_k^2[1 + w\Omega_E + w^2\Omega_M\Omega_E]\}g' + \frac{3\mathcal{H}_k^3}{(1+\frac{9}{2}J)^2}\{1 - 6w\Omega_E + 9w^2\Omega_E + 9\mathcal{H}_k^2[1 - 2w\Omega_E + 6w^2\Omega_E - 9w^2\Omega_E^2]\}g \rightarrow 2\zeta + 3\mathcal{H}_k[2g'' + 2\mathcal{H}_k(2-3w\Omega_E)g' + \mathcal{H}_k^2(1 - 6w\Omega_E + 9w^2\Omega_E)g],$$

and

$$C_2 = 3\frac{2\mathcal{H}_k(g'-\zeta) + 2(2\mathcal{H}_k^2 + \frac{1}{3})g}{1+\frac{9}{2}J} \rightarrow 6\left[-\mathcal{H}_k\zeta + \frac{1}{3}g\right],$$

where $\rightarrow$ denotes taking only the lowest order terms that can contribute to the small-scale limit.

Now consider the Einstein equations. In the small-scale limit we get

$$-2k^2\Phi - 6\mathcal{H}(\dot{\Phi} + \mathcal{H}\Psi) = 8\pi G a^2\rho\delta + 6\mathcal{H}[\mathcal{H}\zeta - \tfrac{1}{3}kg]\Phi, \quad \text{(A1)}$$

$$2(\dot{\Phi} + \mathcal{H}\Psi) = 8\pi G a^2\rho\theta + \mathcal{H}[-2\zeta + 2g' + \mathcal{H}_k(1-3w\Omega_E)g]\Phi, \quad \text{(A2)}$$

$$6\frac{d}{d\tau}(\dot{\Phi} + \mathcal{H}\Psi) + 12\mathcal{H}(\dot{\Phi} + \mathcal{H}\Psi) + 2k^2(\Phi - \Psi) = -3(E_F + E_R)\Psi + k^2\{2\zeta + 3\mathcal{H}_k[2g'' + \mathcal{H}_k^2(1 - 6w\Omega_E + 9w^2\Omega_E)g] + 2\mathcal{H}_k(2-3w\Omega_E)g'\}\Phi + 2[-3\mathcal{H}\zeta + kg]\dot{\Phi}, \quad \text{(A3)}$$

and

$$\Phi - \Psi = \zeta\Phi + \frac{g}{k}\dot{\Phi}. \quad \text{(A4)}$$

As argued in Sec. II, we expand ζ in powers of $\mathcal{H}_k$ and write

$$\zeta = \zeta_L(\Omega_E, k)\mathcal{H}_k^n \quad \text{(A5)}$$

to leading order. We expand g in a similar way as

$$g = g_L(\Omega_E, k)\mathcal{H}_k^m. \quad \text{(A6)}$$

The goal now is to find the smallest powers m and n that can be consistent with the Einstein equations as $\mathcal{H}_k \rightarrow 0$.

It is easily seen that $\zeta' = \zeta_1\mathcal{H}_k^{n+1}$ for some function $\zeta_1(\Omega_E, k)$, which is found to be

$$\zeta_{L1} = -\frac{n}{2}(1+3w\Omega_E)\zeta_L - 3w\Omega_M\Omega_E\frac{\partial\zeta_L}{\partial\Omega_E}. \quad \text{(A7)}$$

Similarly we have $g' = g_{L1}\mathcal{H}_k^{m+1}$ and $g'' = g_{L2}\mathcal{H}_k^{m+2}$ and similar expressions to (A7) can be found for g_{L1} and g_{L2}.

Consider again the Einstein equations and now keep on the lowest orders for each variable. For example Φ is $O(2)$, while $(E_F + E_R)\Psi = O(4)$ and $\ddot{\Phi} = O(4)$, etc. We get

$$-2k^2\Phi = 8\pi G a^2\rho\delta + 6k^2\mathcal{H}_k[\zeta_L\mathcal{H}_k^{n+1} - \tfrac{1}{3}g_L\mathcal{H}_k^m]\Phi, \quad \text{(A8)}$$

$$2(\dot{\Phi} + \mathcal{H}\Psi) = 8\pi G a^2\rho\theta + \mathcal{H}[-2\zeta_L\mathcal{H}_k^n + 2g_{L1}\mathcal{H}_k^{m+1} + (1-3w\Omega_E)g_L\mathcal{H}_k^{m+1}]\Phi, \quad \text{(A9)}$$

and

$$\Phi - \Psi = \{\zeta_L\mathcal{H}_k^n + 3\mathcal{H}_k^{m+3}[g_{L2} + (2-3w\Omega_E)g_{L1} + \tfrac{1}{2}(1 - 6w\Omega_E + 9w^2\Omega_E)g_L]\}\Phi + [-3\zeta_L\mathcal{H}_k^{n+1} + g_L\mathcal{H}_k^m]\Phi', \quad \text{(A10)}$$

while the shear Eq. (A4) remains unchanged. Clearly the choice $n = 0$ and $m = -1$ is consistent with all of the above equations.

Let us investigate whether smaller numbers are possible. Suppose that $n < 0$. Then the shear Eq. (A4) implies $\zeta_L\mathcal{H}_k^n\Phi + g_L\mathcal{H}_k^m\Phi' = 0$, hence if $n < 0$ then $m < -1$ and in particular $m = n - 1$. Note that this last relation includes the choice $n = 0$ as a special case. If on the other hand $m < -1$ then the term $g_L\mathcal{H}_k^m\Phi'$ in the shear Eq. (A4) is of order less than two which forces automatically $n < 0$. Thus, without loss of generality we may set

$m = n - 1$. With this choice the Einstein Eq. (A8) becomes

$$-2k^2\Phi = 8\pi G a^2 \rho\delta + 6k^2 \mathcal{H}_k^n[\zeta_L \mathcal{H}_k^2 - \tfrac{1}{3} g_L]\Phi.$$

Therefore, if $n < 0$ we must have $\zeta_L \mathcal{H}_k^2 - \frac{1}{3} g_L = 0$ and since both ζ_L and g_L are independent of $\mathcal{H}_k$, this forces $\zeta_L = g_L = 0$. Thus, the only consistent leading-order choice is $n = 0$ and $m = -1$.

To summarize, a consistent small-scale limit imposes the expansions

$$\zeta = \zeta_L(\Omega_E, k) + O(\mathcal{H}_k), \tag{A11}$$

$$g = g_L(\Omega_E, k)\frac{1}{\mathcal{H}_k} + O(\mathcal{H}_k^0). \tag{A12}$$

Note that there may be additional constraints on the k dependence of ζ_L and g_L.

One source of worry is the $\mathcal{H}_k^{-1}$ term that persists in the shear Eq. (A4). This is not a problem, however. We may write the potentials in terms of the matter variables in a way that no ambiguity arises. We find

$$\Phi = -\frac{4\pi G a^2 \rho_M}{k^2(1 - g_L)}\delta_M, \tag{A13}$$

$$\Psi = \frac{1 - \zeta_L + g_L(\zeta_L - g_L + 3w\Omega_M\Omega_E \frac{\partial g_L}{\partial \Omega_E})}{1 - g_L}\Phi - \frac{3}{2}\mathcal{H}\Omega_M \frac{g_L}{1 - g_L}\theta_M, \tag{A14}$$

$$\dot{\Phi} = \frac{4\pi G a^2 \rho_M}{1 - g_L}\theta_M - \mathcal{H}\left[1 + \frac{3w\Omega_M\Omega_E \frac{\partial g_L}{\partial \Omega_E}}{1 - g_L}\right]\Phi, \tag{A15}$$

which are perfectly consistent equations.

Finally, (A14) has a further interesting reduction (in this small-scale limit). We replace θ_M by $-\dot{\delta}/k^2$, then use $\dot{\delta}_M = \mathcal{H} f \delta_M$ and finally use (A13) to write $\Phi = \gamma_{\rm PPN}\Psi$ where

$$\frac{1}{\gamma_{\rm PPN}} = \frac{1}{1 - g_L}\Big\{1 - \zeta_L - g_L(1 - g_L)f + g_L\Big(\zeta_L - g_L + 3w\Omega_M\Omega_E\frac{\partial g_L}{\partial \Omega_E}\Big)\Big\}, \tag{A16}$$

while the measured Newton's constant on the Earth, G_N is

$$G_N = \frac{G}{\gamma_{\rm PPN}(1 - g_L)}. \tag{A17}$$

Expanding ζ_L and g_L in powers of Ω_E, we find

$$\gamma_{\rm PPN} \approx 1 + \zeta_1\Omega_E \tag{A18}$$

and

$$\frac{G_N}{G} \approx 1 + (g_1 - \zeta_1)\Omega_E; \tag{A19}$$

hence,

$$\frac{\dot{G}_N}{G_N} \approx -3w(g_1 - \zeta_1)\Omega_E\mathcal{H}. \tag{A20}$$

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