Canonical structure of higher derivative gravity in 3D

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We give an explicitly gauge-invariant canonical analysis of linearized quadratic gravity theories in three dimensions for both flat and de Sitter backgrounds. In flat backgrounds, we also study the effects of the gravitational Chern-Simons term, include the sources, and compute the weak field limit as well as scattering between spinning massive particles.

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I. INTRODUCTION

Recently, Bergshoeff *et al.* [1] found that, in three dimensions, among the class of higher-derivative theories defined by the Lagrangian $\kappa^{-1}R + \alpha R^2 + \beta R^2_{\mu\nu}$, a special case $8\alpha + 3\beta = 0$ and $\kappa^{-1} < 0$ (let us call it BHT gravity) and its parity-violating extension, with a gravitational Chern-Simons term, have massive ghost-free spin-2 particles in their free spectrum around both flat and (anti)-de Sitter [(a)dS] spacetimes. Perhaps, the most interesting feature of the BHT model is that it is the first and (apart from some bimetric theories) the only known example of a (parity-invariant) theory that provides a nonlinear extension to the Pauli-Fierz mass term for spin-2 particles. In addition, being a three-dimensional theory, it is power-counting superrenormalizable whose four-dimensional cousin is renormalizable [2]. Therefore, it is possible that the BHT model may turn out to be a perturbatively well-defined quantum gravity in three dimensions. But of course, unitarity of the model beyond tree level is yet to be checked.

Various aspects of the theory such as its ghost-freedom and tree-level unitarity [1,3–5] and Newtonian limits [5] have been explored. Also, classical solutions and related issues were studied in [1,6–10], and supergravity extensions were given in [11].

In this paper, we give an explicitly gauge-invariant, detailed analysis of the canonical structure of the generic quadratic models in $2 + 1$ dimensions for both flat and de Sitter (dS) backgrounds. In flat space, we also include the gravitational Chern-Simons term in our analysis. It is interesting to see how at the linearized level BHT theory is singled out as a unique regular "harmonic oscillator" (massive free field), which avoids the infamous Ostragradskian instability that ruins every higher-time derivative theory [12]. (It was claimed that adding interactions might yield stable higher-time derivative theories [13].) All the other quadratic theories are ghost-ridden higher-derivative Pais-Uhlenbeck [14] oscillators at the linearized level. In addition, we discuss the Newtonian limits, weak fields, and the tree-level scattering of particles with mass and spin in these models.

The layout of the paper is as follows: Section II is devoted to a flat spacetime analysis, which includes the canonical structure of both the parity-invariant and parity-violating quadratic gravity, in addition to the effects of static sources and weak field solutions with circular symmetry. In Sec. III, canonical structure analysis is extended to de Sitter space. Some of the computations are relegated to the Appendices. Tree-level scattering amplitude between spinning massive particles is given in Appendix A. In Appendix B, generic quadratic action is written in terms of two auxiliary fields. Finally, we list some results that may be helpful in the analysis of field equations.

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II. HIGHER-DERIVATIVE SPIN-2 IN FLAT SPACETIME

We start our analysis of the higher-derivative spin-2 fields in flat space, which is considerably simpler than the de Sitter background, which we deal with in the next section. The action

$$I = \int d^3x\sqrt{-g}\left(\frac{1}{\kappa}R + \alpha R^2 + \beta R^2_{\mu\nu}\right), \qquad (1)$$

gives the desired spin-2 model when expanded as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the usual flat spacetime metric with mostly plus signature. (Actually, spin-2 here is a misnomer. It should be symmetric rank-2 tensor, since without any constraints in addition to spin-2, it has spin-1 and spin-0 components. But, in what follows, we will call $h_{\mu\nu}$ a spin-2 field.) Below, we will also add the parity-violating gravitational Chern-Simons term to this action. In practice, to actually get the action for $h_{\mu\nu}$, it is somewhat more convenient to linearize the full nonlinear field equations and then integrate them (after carefully taking care of the overall sign, which will be relevant for the discussion of ghosts). Then, the action (1) up to boundary terms becomes

$$I = -\frac{1}{2}\int d^3x h_{\mu\nu}\left[\frac{1}{\kappa}\mathcal{G}_L^{\mu\nu} + (2\alpha+\beta)(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)R_L + \beta\Box\mathcal{G}_L^{\mu\nu}\right]. \tag{2}$$

Here, the linearized Einstein and Ricci tensors, and curvature scalar read

$$\mathcal{G}_L^{\mu\nu} = R_L^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}R_L, \qquad R_L = \partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h,$$
$$R_L^{\mu\nu} = \tfrac{1}{2}(\partial_\sigma\partial^\mu h^{\nu\sigma} + \partial_\sigma\partial^\nu h^{\mu\sigma} - \Box h^{\mu\nu} - \partial^\mu\partial^\nu h), \tag{3}$$
$$h = \eta^{\mu\nu}h_{\mu\nu},$$

where $\Box = \partial_\mu\partial^\mu = -\partial_0^2 + \nabla^2$. Raising and lowering operations are carried out with $\eta_{\mu\nu}$. To explore the canonical structure and identify the free fields, $h_{\mu\nu}$ can be decomposed in terms of six *a priori* free functions of $(t, \vec{x})$:

$$h_{ij} \equiv (\delta_{ij} + \hat{\partial}_i\hat{\partial}_j)\phi - \hat{\partial}_i\hat{\partial}_j\chi + (\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i)\xi,$$
$$h_{0i} \equiv -\epsilon_{ij}\partial_j\eta + \partial_i N_L, \qquad h_{00} \equiv N, \tag{4}$$

where $\hat{\partial}_i \equiv \partial_i/\sqrt{-\nabla^2}$. From these, one can compute $\mathcal{G}^L_{\mu\nu}$ in terms of three functions:

$$\mathcal{G}^L_{00} = -\tfrac{1}{2}\nabla^2\phi, \qquad \mathcal{G}^L_{0i} = -\tfrac{1}{2}(\epsilon_{ik}\partial_k\sigma + \partial_i\dot{\phi}),$$
$$\mathcal{G}^L_{ij} = -\tfrac{1}{2}[(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j)q - \hat{\partial}_i\hat{\partial}_j\ddot{\phi} - (\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i)\dot{\sigma}],$$

where $\dot{\phi} = \partial\phi/\partial t$, etc. Here, q, σ, and ϕ are invariant under gauge transformations $\delta_\zeta h_{\mu\nu} = \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu$ and are defined as

$$q \equiv \nabla^2 N - 2\nabla^2\dot{N}_L + \ddot{\chi}, \qquad \sigma \equiv \dot{\xi} - \nabla^2\eta. \tag{5}$$

Note also that ϕ is gauge invariant unlike the other components of $h_{\mu\nu}$. Linearized scalar curvature is computed to be

$$R_L = q - \Box\phi.$$

Therefore, as required by the Bianchi identity, $\partial_\mu\mathcal{G}_L^{\mu\nu} = 0$, the number of arbitrary functions reduces from six to three. One can use either ϕ, σ, q; or ϕ, σ, R_L combinations. The Einstein-Hilbert part of the action can be computed as

$$I_{\text{EH}} = -\frac{1}{2\kappa}\int d^3x h_{\mu\nu}\mathcal{G}_L^{\mu\nu} = \frac{1}{2\kappa}\int d^3x(\phi q + \sigma^2),$$

which clearly shows that there is no propagating degree of freedom in the pure Einstein theory. To compute the quadratic part, its better to use the self-adjointness of the involved operators to rewrite the action as explicitly gauge invariant not just gauge invariant up to a boundary term, which will simplify the computations in a great deal:

$$I_{2\alpha+\beta} = -\frac{2\alpha+\beta}{2}\int d^3x h_{\mu\nu}(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)R_L = \frac{2\alpha+\beta}{2}\int d^3x R_L^2,$$
$$I_\beta = -\frac{\beta}{2}\int d^3x h_{\mu\nu}\Box\mathcal{G}_L^{\mu\nu} = -\frac{\beta}{2}\int d^3x\left(-2\mathcal{G}^L_{\mu\nu}\mathcal{G}_L^{\mu\nu} + \frac{1}{2}R_L^2\right) = \frac{\beta}{2}\int d^3x(q\Box\phi + \sigma\Box\sigma).$$

In the I_β action, the second equality follows after one moves the $\Box$ term to $h_{\mu\nu}$, and then uses (3) and the Bianchi identity. Collecting all the terms, the total action in terms of the gauge-invariant combinations is

$$I = \frac{1}{2}\int d^3x\left[\frac{1}{\kappa}\phi q + (2\alpha+\beta)(q - \Box\phi)^2 + \beta q\Box\phi\right] + \frac{\beta}{2}\int d^3x\left(\sigma\Box\sigma + \frac{1}{\kappa\beta}\sigma^2\right). \tag{6}$$

σ describes a single scalar field with mass $m_g^2 \equiv -\frac{1}{\kappa\beta}$, which is nontachyonic for $\kappa\beta < 0$ and a nonghost for $\beta > 0$, therefore $\kappa < 0$. For the ϕ and q part of the action, the discussion bifurcates whether $2\alpha + \beta = 0$, or not. Let us first consider the $2\alpha + \beta \neq 0$ case, for which the nondynamical field q can be eliminated, yielding the action

$$I_\phi = \frac{1}{2}\int d^3x\left[\frac{\beta(8\alpha+3\beta)}{4(2\alpha+\beta)}(\Box\phi)^2 + \frac{(4\alpha+\beta)}{2\kappa(2\alpha+\beta)}\phi\Box\phi - \frac{1}{4\kappa^2(2\alpha+\beta)}\phi^2\right]. \tag{7}$$

There are apparently several special points, one of which is the BHT limit $8\alpha + 3\beta = 0$, for which the higher-derivative term disappears. (The $4\alpha + \beta = 0$ theory seems special, but it has a tachyonic excitation; on the other hand, the $\beta = 0$ model is ghost and tachyon-free for $\kappa > 0$.) Therefore, at the linearized level, the BHT model is actually not a higher-derivative theory, so it escapes the Ostragradski instability. The ϕ field part of the BHT action reads

$$I_{\text{BHT},\phi} = -\frac{1}{2\kappa}\int d^3x\left(\phi\Box\phi + \frac{1}{\kappa\beta}\phi^2\right),$$

which again describes a single degree of freedom with the same mass as σ. This is to be expected in this parity-invariant theory, since σ and ϕ are two helicity degrees of freedom of the massive spin two field in three dimensions. Also, observe that for ϕ to be a nonghost, κ has to be negative.

For generic α and β, except for $2\alpha + \beta \neq 0$, (7) describes a higher-derivative Pais-Uhlenbeck [14] oscillator which can be rewritten in terms of simple oscillators in the following way. Defining new fields as

$$\varphi_1 \equiv \phi - \frac{\Box\phi}{m_g^2}, \qquad \varphi_2 \equiv \phi - \frac{\Box\phi}{m_s^2},$$

(7) becomes

$$I_\phi = \frac{1}{64\kappa(2\alpha+\beta)^2}\int d^3x[(8\alpha+3\beta)^2\varphi_1(\Box - m_s^2)\varphi_1 - \beta^2\varphi_2(\Box - m_g^2)\varphi_2], \tag{8}$$

with m_g given as above and m_s as

$$m_s^2 = \frac{1}{\kappa(8\alpha+3\beta)}.$$

For $8\alpha + 3\beta < 0$, φ_1 is nontachyonic just like φ_2, but unlike φ_2, it describes a ghostlike excitation since its kinetic energy comes with the wrong sign.

A. $2\alpha + \beta = 0$ theory

We have seen in the above discussion that the $2\alpha + \beta = 0$ case is a somewhat singular theory. If one naively takes the $\epsilon \equiv 2\alpha + \beta \to 0$ limit in (8), one gets

$$I_\phi = \frac{1}{8\kappa\epsilon}\int d^3x\left\{\frac{\beta}{m_g^2}\left[\left(\Box - m_g^2\right)\phi\right]^2 - 4\epsilon\phi(\Box - m_g^2)\phi + O(\epsilon^2)\right\},$$

which is a degenerate (equal mass) Pais-Uhlenbeck oscillator after a divergent rescaling of ϕ. But, more properly, suppose from the onset at the level of the action, we set $2\alpha + \beta = 0$ to get (apart from the decoupled σ field)

$$I_\phi = \frac{\beta}{2}\int d^3x(q\Box\phi - m_g^2 q\phi).$$

Variation with respect to ϕ gives a massive wave equation for q, and *vice versa*. But, these equations do not reveal the ghost structure of the theory. So, let us define $q \equiv m_g^2(\Psi_1 + \Psi_2)$, $\phi \equiv \Psi_1 - \Psi_2$, which turns the action to

$$I = \frac{m_g^2\beta}{2}\int d^3x[(\Psi_1\Box\Psi_1 - m_g^2\Psi_1^2) - (\Psi_2\Box\Psi_2 - m_g^2\Psi_2^2)].$$

Since $\beta > 0, \Psi_2$ is a ghost excitation. The Newtonian limit of this theory is quite interesting: From the general tree-level scattering amplitude computation given in [5], one sees that as in the pure Einstein-Hilbert theory, the $2\alpha + \beta = 0$ case has a vanishing Newtonian potential between static sources: the spin-0 ghost excitation gives a repulsive component which cancels the attractive one coming from the spin-2 part.

B. Adding static sources

Up to now, we have studied the free field spectrum of higher-derivative gravity. Let us remedy this by adding matter with the usual gravity-matter coupling:

$$I_{\text{source}} = \frac{1}{2}\int d^3x h_{\mu\nu}T^{\mu\nu}.$$

In the case of a static source, $T^{00} = \rho(\vec{x})$, $T^{0i} = 0$, $T^{ij} = 0$, (in a related context, we somewhat generalize this in Appendix A), I_{source} becomes

$$I_{\text{source}} = \frac{1}{2}\int d^3x N\rho(\vec{x}) = \frac{1}{2}\int d^3x\left(\frac{1}{\nabla^2}q + 2\dot{N}_L - \frac{1}{\nabla^2}\ddot{\chi}\right)\rho(\vec{x}),$$

where in the second equality, we have used the definition of q in (5). After dropping the boundary terms and using the symmetry of the Green's function, we have

$$I_{\text{source}} = \frac{1}{2}\int d^3x q\frac{1}{\nabla^2}\rho.$$

Redefining $\varphi \equiv \phi + \kappa\frac{1}{\nabla^2}\rho$ and $\tilde{q} \equiv q + \kappa\rho$, the total action reduces to

$$I = \frac{1}{2}\int d^3x\left[\frac{1}{\kappa}(\varphi\tilde{q} - \kappa\varphi\rho + \sigma^2) + (2\alpha+\beta)(\tilde{q} - \Box\varphi)^2 + \beta(\tilde{q}\Box\varphi - \kappa\rho\Box\varphi - \kappa\tilde{q}\rho + \kappa^2\rho^2 + \sigma\Box\sigma)\right].$$

Specifically, for $8\alpha + 3\beta = 0$, integrating out $\tilde{q}$, one ends up with

$$I = \frac{1}{2}\int d^3x\left[\beta(\sigma\Box\sigma - m_g^2\sigma^2) - \frac{1}{\kappa}(\varphi\Box\varphi - m_g^2\varphi^2) + \varphi\rho\right].$$

The last term is the interaction part which gives the attractive (for $\kappa < 0$) potential energy

$$U = \frac{\kappa}{4}\int d^2x\rho_1\frac{1}{\nabla^2 - m_g^2}\rho_2 = \frac{\kappa}{8\pi}m_1 m_2 K_0(m_g r), \tag{9}$$

where we took point sources, $\rho_1(\vec{x}) = m_1\delta^2(\vec{x} - \vec{x}_1)$, $\rho_2(\vec{x}) = m_2\delta^2(\vec{x} - \vec{x}_2)$, and K_0 is the modified Bessel function. This result matches that of [5].

C. Weak field approximation

It is also highly instructive to capture some of the above results from the nonlinear theory (1). But, even in the circularly symmetric case, nontrivial exact solutions for which $g_{00} \neq g^{rr}$ are not known, and we have not been able to find one. Nevertheless, since we just need the weak field approximation, we can do the following: The ansatz

$$ds^2 = -f(r)dt^2 + \frac{b^2(r)}{f(r)}dr^2 + r^2d\theta^2$$

can be inserted into the action (1), which is to be varied with respect $f(r)$ and $b(r)$ (See the details of this Weyl trick in [15]) . For the sake of simplicity, let us just consider the BHT theory. Then, an approximate solution can be found by setting $f(r) = 1 + \int^r dr a(r)$, $b(r) = 1 + \int^r dr v(r)$, where a and v are small. At first order, we have

$$\frac{4}{\kappa}v + 2\beta v'' + 2\beta a'' + r\beta a''' = 0, \tag{10}$$

$$\beta r^2 a'' + \frac{2}{\kappa} r^2 a + 2r\beta v' - 2\beta v = 0. \tag{11}$$

Here, $'$ denotes differentiation with respect to r. v can be determined as $v = a + \frac{r}{2}a'$. Putting it back to (11) gives

$$r^2 a'' + ra' - a(m_g^2 r^2 + 1) = 0, \tag{12}$$

which is solved by $a(r) = c_1 I_1(m_g r) + c_2 K_1(m_g r)$. Recall that $g_{00} \approx -1 - \int^r dr a(r)$, and $g_{rr} \approx 1 + \int^r dr(2v(r) - a(r))$. Thus, for decaying fields c_1 vanishes, and the metric components become

$$g_{00} \approx -1 + cK_0(m_g r), \qquad g_{rr} \approx 1 + dK_1(m_g r),$$

where c and d are constants related to the mass of the source. This is consistent with our earlier result (9).

D. Higher-derivative gravity plus a Chern-Simons term

We will now extend the preceding discussion in flat space by adding a gravitational Chern-Simons term [16]

$$I = \int d^3x\sqrt{-g}\bigg[\frac{1}{\kappa}R + \alpha R^2 + \beta R_{\mu\nu}^2 - \frac{1}{2\mu}\epsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}\bigg(\partial_\mu\Gamma^\sigma{}_{\rho\nu} + \frac{2}{3}\Gamma^\sigma{}_{\mu\beta}\Gamma^\beta{}_{\nu\rho}\bigg)\bigg], \tag{13}$$

where $\epsilon_{012} = 1$, and μ is the Chern-Simons coupling with an arbitrary sign. (Without the α, β terms, but with a Pauli-Fierz mass term, a canonical analysis was carried out in [17,18].) Linearization of the Chern-Simons part yields

$$I_{\rm CS} = -\frac{1}{2\mu}\int d^3x \epsilon_{\mu\alpha\beta}\mathcal{G}_L^{\alpha\nu}\partial^\mu h^\beta{}_\nu = \frac{1}{2\mu}\int d^3x\sigma(q + \Box\phi).$$

The total action in terms of the gauge-invariant combinations becomes

$$I = \frac{1}{2}\int d^3x\bigg[\frac{1}{\kappa}(\phi q + \sigma^2) + (2\alpha + \beta)(q - \Box\phi)^2 + \beta(q\Box\phi + \sigma\Box\sigma) + \frac{1}{\mu}\sigma(q + \Box\phi)\bigg].$$

Assuming that $2\alpha + \beta \neq 0$, q can be eliminated to yield the action

$$I = \frac{1}{2}\int d^3x\bigg\{\beta\bigg[\sigma\Box\sigma + \bigg(\frac{1}{\kappa\beta} - \frac{1}{4\mu^2\beta(2\alpha+\beta)}\bigg)\sigma^2\bigg] + \bigg[\frac{1}{\mu} + \frac{(4\alpha+\beta)}{2\mu(2\alpha+\beta)}\bigg]\sigma\Box\phi - \frac{1}{2\kappa\mu(2\alpha+\beta)}\sigma\phi + \frac{1}{\kappa}\bigg[\frac{\beta\kappa(8\alpha+3\beta)}{4(2\alpha+\beta)}(\Box\phi)^2 + \frac{(4\alpha+\beta)}{2(2\alpha+\beta)}\phi\Box\phi - \frac{1}{4\kappa(2\alpha+\beta)}\phi^2\bigg]\bigg\}.$$

For generic α, β one can diagonalize this action, but it is rather cumbersome and not particularly illuminating, so we just consider the $8\alpha + 3\beta = 0$ case,

$$I_{\rm BHT-CS} = \frac{\beta}{2}\int d^3x\bigg\{\bigg[\sigma\Box\sigma - \bigg(m_g^2 + \frac{1}{\mu^2\beta^2}\bigg)\sigma^2\bigg] + \frac{2m_g^2}{\beta\mu}\sigma\phi + m_g^2(\phi\Box\phi - m_g^2\phi^2)\bigg\}.$$

To decouple the σ, ϕ fields, one possible route is to take the Fourier transform of the fields, put the Lagrangian in a matrix form, and then diagonalize the matrix. This procedure yields

$$I_{\rm BHT-CS} = \frac{\beta}{2}\int d^3x(\Psi_+\Box\Psi_+ - m_+^2\Psi_+^2 + \Psi_-\Box\Psi_- - m_-^2\Psi_-^2),$$

where the masses read

$$m_\pm^2 = m_g^2 + \frac{1}{2\mu^2\beta^2} \pm \frac{1}{\mu\beta}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}},$$

and the new fields are defined as

$$\begin{pmatrix}\Psi_- \\ \Psi_+\end{pmatrix} = \begin{bmatrix} N_+ & (m_+^2 - m_g^2)N_+ \\ N_- & (m_-^2 - m_g^2)N_- \end{bmatrix}\begin{pmatrix}\sigma \\ m_g\phi\end{pmatrix},$$

$$N_\pm = \sqrt{1 + \bigg[\frac{\mu\beta}{m_g}(m_\pm^2 - m_g^2)\bigg]^2}.$$

$m_\pm$ agree with those of [1,11]. As the $+2$ and -2 helicity modes have different masses, it is a parity-violating theory as expected. In the $\beta \to 0$ limit, which is the topologically massive gravity with a single degree of freedom [16], m_+ diverges and drops out, $m_- = -|\mu|/\kappa$.

III. HIGHER-DERIVATIVE SPIN-2 IN A DE SITTER BACKGROUND

Now, we will study the canonical structure of the higher-derivative theory in an (anti)-de Sitter background defined by the action

$$I = \int d^3x\sqrt{-g}\bigg[\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}^2\bigg],$$

whose linearization about an (a)dS background yields

$$I = -\frac{1}{2}\int d^3x\sqrt{\bar{g}}h_{\mu\nu}\Big[a\mathcal{G}_L^{\mu\nu} + (2\alpha+\beta)\Big(\bar{g}^{\mu\nu}\Box - \nabla^\mu\nabla^\nu + \frac{2}{\ell^2}\bar{g}^{\mu\nu}\Big)R_L + \beta\Big(\Box\mathcal{G}_L^{\mu\nu} - \frac{1}{\ell^2}\bar{g}^{\mu\nu}R_L\Big)\Big],$$

where $a \equiv \frac{1}{\kappa} + \frac{12}{\ell^2}\alpha + \frac{2}{\ell^2}\beta$, and $1/\ell^2$ is the cosmological constant which is related to α, β, κ and the bare cosmological constant Λ_0 of the full theory as $\frac{1}{\ell^2} = \frac{1}{4\kappa(3\alpha+\beta)} \times [1 \pm \sqrt{1-8\kappa\Lambda_0(3\alpha+\beta)}]$ [19]. For the sake of simplicity, we will consider the background to be a de Sitter spacetime, but since our results will be analytic in ℓ, in the final expressions one can take $\ell \to i\ell$ to obtain the results in anti-de Sitter spacetime. (To keep the signature intact, one also needs to Wick rotate a space coordinate). For dS, we take the metric, $\bar{g}_{\mu\nu}$, with which all the covariant derivatives and raising-lowering operations should be made, to be in the Poincaré form

$$ds^2 = \frac{\ell^2}{t^2}(-dt^2 + dx^2 + dy^2),$$

and define the perturbation as

$$g_{\mu\nu} = \frac{\ell^2}{t^2}\eta_{\mu\nu} + h_{\mu\nu}.$$

Linearized forms of Einstein and Ricci tensors, and Ricci scalar are given as

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_L - \frac{2}{\ell^2}h_{\mu\nu},$$
$$R^L_{\mu\nu} = \frac{1}{2}(\nabla^\sigma\nabla_\mu h_{\nu\sigma} + \nabla^\sigma\nabla_\nu h_{\mu\sigma} - \Box h_{\mu\nu} - \nabla_\mu\nabla_\nu h),$$
$$R_L = \nabla_\alpha\nabla_\beta h^{\alpha\beta} - \Box h - \frac{2}{\ell^2}h, \tag{14}$$

where $\Box \equiv \nabla_\mu\nabla^\mu = \frac{t^2}{\ell^2}\eta^{\mu\nu}\nabla_\mu\nabla_\nu$. Decomposition of $h_{\mu\nu}$ into "spatial" tensor h_{ij}, spatial vector h_{0i}, and "scalar" h_{00} is

$$h_{ij} \equiv \frac{\ell^2}{t^2}[(\delta_{ij} + \hat{\nabla}_i\hat{\nabla}_j)\phi - \hat{\nabla}_i\hat{\nabla}_j\chi + (\tilde{\epsilon}_i{}^k\hat{\nabla}_k\hat{\nabla}_j + \tilde{\epsilon}_j{}^k\hat{\nabla}_k\hat{\nabla}_i)\xi]$$
$$= \frac{\ell^2}{t^2}\Big[(\delta_{ij} + \hat{\nabla}_i\hat{\nabla}_j)\phi - \hat{\nabla}_i\hat{\nabla}_j\chi + \frac{t^2}{\ell^2}(\tilde{\epsilon}_{ik}\hat{\nabla}_k\hat{\nabla}_j + \tilde{\epsilon}_{jk}\hat{\nabla}_k\hat{\nabla}_i)\xi\Big],$$
$$h_{0i} \equiv \frac{\ell^2}{t^2}(-\tilde{\epsilon}_i{}^k\nabla_k\eta + \partial_i N_L) = \frac{\ell^2}{t^2}\Big(-\frac{t^2}{\ell^2}\tilde{\epsilon}_{ij}\nabla_j\eta + \partial_i N_L\Big),$$
$$h_{00} \equiv \frac{\ell^2}{t^2}N,$$

where $\hat{\nabla}_i \equiv \nabla_i/\sqrt{-\nabla_k^2}$ and the covariant derivative is for two-dimensional space with metric $\gamma_{ij} = \frac{\ell^2}{t^2}\delta_{ij}$. Since the two-dimensional space is flat, then $\nabla_i \to \partial_i$ and $\hat{\partial}_i \equiv \partial_i/\sqrt{-\partial_k^2}$. $\tilde{\epsilon}_{ik}$ is the Levi-Civita tensor for two-dimensional space, which is related with the corresponding tensor density ϵ_{ik} by

$$\tilde{\epsilon}_{ik} = \sqrt{\gamma}\epsilon_{ik} \Rightarrow \tilde{\epsilon}_{ik} = \frac{\ell^2}{t^2}\epsilon_{ik}.$$

The convention for ϵ_{ik} is $\epsilon_{12} = 1$ (the convention for Levi-Civita tensor density for the upper indices is $\epsilon^{12} = 1$ naturally with the induced metric). As a result, the final form of the decomposition is

$$h_{ij} = \frac{\ell^2}{t^2}[(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j)\phi - \hat{\partial}_i\hat{\partial}_j\chi + (\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i)\xi],$$
$$h_{0i} = \frac{\ell^2}{t^2}(-\epsilon_{ij}\partial_j\eta + \partial_i N_L), \qquad h_{00} = \frac{\ell^2}{t^2}N,$$

with the convention for Levi-Civita tensor *density* $\epsilon_{12} = 1$. Here, all the spatial indices are raised and lowered by δ_{ij}. A further note on this specific choice of decomposition is about the ℓ^2/t^2 coefficients: With this coefficients, at every step the flat space limit $\ell \to \infty$, $\ell/t \to 1$ will be clear.

Unlike the flat space case, ϕ is not gauge invariant anymore. In fact, under the gauge transformations $\delta_\zeta h_{\mu\nu} = \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu$, where ζ_μ can be decomposed as $\zeta_\mu = (\zeta_0, -\epsilon_{ij}\partial_j\zeta + \partial_i\kappa)$, the components of $h_{\mu\nu}$ transform as

$$\delta_\zeta\phi = 2\frac{t}{\ell^2}\zeta_0, \qquad \delta_\zeta\chi = 2\frac{t^2}{\ell^2}\Big(\partial_i^2\kappa + \frac{1}{t}\zeta_0\Big),$$
$$\delta_\zeta\xi = \frac{t^2}{\ell^2}\partial_i^2\zeta, \qquad \delta_\zeta\eta = \frac{t^2}{\ell^2}\Big(\dot{\zeta} + \frac{2}{t}\zeta\Big),$$
$$\delta_\zeta N_L = \frac{t^2}{\ell^2}\Big(\dot{\kappa} + \zeta_0 + \frac{2}{t}\kappa\Big), \qquad \delta_\zeta N = 2\frac{t^2}{\ell^2}\Big(\dot{\zeta}_0 + \frac{1}{t}\zeta_0\Big).$$

Again, from the linearized Bianchi identity, $\nabla_\mu\mathcal{G}_L^{\mu\nu} = 0$, we know that there should be three independent gauge-invariant combinations constructed out of the (derivatives of) six scalar fields. By inspection, one can find these combinations, but the quickest way would be to look at the independent components of the gauge-invariant tensor $\mathcal{G}_L^{\mu\nu}$. This led us to the following *four* gauge-invariant functions:

$$f \equiv \frac{\ell}{t}\Big[\phi - \frac{2}{t}N_L + \frac{1}{t}\frac{1}{\nabla^2}\Big(\dot{\phi} + \dot{\chi} - \frac{2}{t}N\Big)\Big],$$
$$p \equiv \frac{\ell}{t}\Big(\dot{\phi} - \frac{1}{t}N\Big),$$
$$q \equiv \frac{\ell}{t}\Big[\nabla^2 N + \ddot{\chi} - 2\nabla^2\dot{N}_L - \frac{1}{t}(\dot{N} - 2\nabla^2 N_L + \dot{\chi}) + \frac{2}{t^2}N\Big],$$
$$\sigma \equiv \frac{\ell}{t}(\dot{\xi} - \nabla^2\eta),$$

and a relation between them coming from the Bianchi identity

$$t\nabla^2\left(\dot{f} - p + \frac{f}{t}\right) - \dot{p} - q = 0. \qquad (15)$$

In terms of these, the components of the linearized Einstein tensor can be found as

$$\mathcal{G}^L_{00} = -\frac{t}{2\ell}\nabla^2 f, \qquad \mathcal{G}^L_{0i} = -\frac{t}{2\ell}(\partial_i p + \epsilon_{ik}\partial_k\sigma),$$
$$\mathcal{G}^L_{ij} = -\frac{t}{2\ell}[(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j)q - \hat{\partial}_i\hat{\partial}_j\dot{p} - (\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i)\dot{\sigma}].$$

The linearized curvature scalar follows as

$$R_L = \frac{t^3}{\ell^3}(q - \nabla^2 f + \dot{p}) = \frac{t^4}{\ell^3}\nabla^2(\dot{f} - p),$$

where in the second line we used the Bianchi identity.

Using the above, the Einstein-Hilbert action can be reduced to the following form:

$$I_{\rm EH} = -\frac{a}{2}\int d^3x\sqrt{\bar{g}}h_{\mu\nu}\mathcal{G}^{\mu\nu}_L = \frac{a}{2}\int d^3x\left[\frac{\ell^2}{t^2}fR_L + \frac{t}{\ell}(f\nabla^2 f + p^2 + \sigma^2)\right].$$

As in the flat space case, computations get a lot simpler if the higher-derivative parts of the Lagrangian are organized in such a way that $h_{\mu\nu}$ is replaced by some gauge-invariant combinations. This can be done again upon use of the self-adjointness of the involved operators as follows:

$$I_{2\alpha+\beta} = -\frac{(2\alpha+\beta)}{2}\int d^3x\sqrt{\bar{g}}h_{\mu\nu} \times\left(\bar{g}^{\mu\nu}\Box - \nabla^\mu\nabla^\nu + \frac{2}{\ell^2}\bar{g}^{\mu\nu}\right)R_L = \frac{(2\alpha+\beta)}{2}\int d^3x\sqrt{\bar{g}}R_L^2.$$

For the β term, one has

$$I_\beta = -\frac{\beta}{2}\int d^3x\sqrt{\bar{g}}h_{\mu\nu}\left(\Box\mathcal{G}^{\mu\nu}_L - \frac{1}{\ell^2}\bar{g}^{\mu\nu}R_L\right) = -\frac{\beta}{2}\int d^3x\sqrt{\bar{g}}\left[(\Box h_{\mu\nu})\mathcal{G}^{\mu\nu}_L - \frac{1}{\ell^2}hR_L\right].$$

After organizing $R^L_{\mu\nu}$ (14) into a form where the indices μ and ν in the covariant derivatives stay at the far left, and using the Bianchi identity, $\nabla_\mu\mathcal{G}^{\mu\nu}_L = 0$, one arrives at

$$I_\beta = -\frac{\beta}{2}\int d^3x\sqrt{\bar{g}}\left(-2\mathcal{G}^L_{\mu\nu}\mathcal{G}^{\mu\nu}_L + \frac{1}{2}R_L^2 + \frac{2}{\ell^2}h_{\mu\nu}\mathcal{G}^{\mu\nu}_L\right).$$

Note that, had we not done this and instead computed $h_{\mu\nu}\Box\mathcal{G}^{\mu\nu}_L$ directly, putting the result into an explicitly gauge-invariant form would be somewhat time-consuming. Not worrying about the correct canonical dimensions for the fields, one can collect all the parts computed above to end up with

$$I = \frac{1}{2}\int d^3x\left\{\left(a + \frac{2\beta}{\ell^2}\right)\left[\frac{\ell^2}{t^2}fR_L + \frac{t}{\ell}(f\nabla^2 f + p^2 + \sigma^2)\right] + (2\alpha+\beta)\frac{\ell^3}{t^3}R_L^2 + \beta\frac{t^3}{\ell^3}\left[\dot{\sigma}^2 + \sigma\nabla^2\sigma + \dot{p}^2 + p\nabla^2 p + (\nabla^2 f)^2 + \frac{\ell^3}{t^3}R_L\nabla^2 f - \frac{\ell^3}{t^3}R_L\dot{p} - \dot{p}\nabla^2 f\right]\right\}.$$

The flat space limit of this action gives (6). In this form, not all the fields are independent: After defining $\varphi \equiv \nabla^2 f$, and using the Bianchi identity (15), we can further simplify the action to

$$I = \frac{1}{2}\int d^3x\left\{\left(a + \frac{2\beta}{\ell^2}\right)\frac{t}{\ell}(-tp\varphi + p^2) + (2\alpha+\beta)\frac{t^5}{\ell^3}(\dot{\varphi} - \nabla^2 p)^2 + \beta\frac{t^3}{\ell^3}(\dot{p}^2 - p\nabla^2 p - \varphi^2 - t\varphi\nabla^2 p - t\dot{p}\,\dot{\varphi} - \varphi\dot{p})\right\} + I_\sigma, \qquad (16)$$

where the σ field decouples from the rest

$$I_\sigma = \frac{1}{2}\int d^3x\left[\beta\frac{t^3}{\ell^3}(\dot{\sigma}^2 + \sigma\nabla^2\sigma) + \left(a + \frac{2\beta}{\ell^2}\right)\frac{t}{\ell}\sigma^2\right]. \qquad (17)$$

For vanishing α and β, cosmological Einstein theory does not have any propagating degrees of freedom just like its flat space partner. For generic α and β, there are 3 degrees of freedom. Recall that a minimally coupled scalar field with the correct canonical dimension is in the following form:

$$I = -\frac{1}{2}\int d^3x\sqrt{\bar{g}}(\partial_\mu\Phi\partial^\mu\Phi + m^2\Phi^2) = -\frac{1}{2}\int d^3x\left\{\frac{\ell}{t}[-\dot{\Phi}^2 + (\partial_i\Phi)^2] + \frac{\ell^3}{t^3}m^2\Phi^2\right\}.$$

Therefore, after rescaling $\sigma \to \frac{\ell^2}{t^2}\sigma$ in (17), one finds the mass of the σ field as

$$m_g^2 = -\frac{a}{\beta} - \frac{2}{\ell^2} = -\frac{1}{\kappa\beta} - \frac{12\alpha}{\ell^2\beta} - \frac{4}{\ell^2}. \qquad (18)$$

For generic α and β, unlike the flat space case, diagonalizing the φ, p action is highly nontrivial. But, there are various ways to see the basic oscillators in this model. One such method is to Fourier transform the fields just in the $\vec{x}$ space and then consider the zero two-momentum limit. That would be equivalent to dropping the ∇^2 terms in the action. Note that this construction does not change the number of degrees of freedom, of course as long as ∇^2 (field) is not the lowest order term. Another way is to directly study the equations of motion. We shall employ both of these methods below.

A. Masses from the nonrelativistic limit

Apart from the decoupled σ part, the generic α, β theory (16) reads in the nonrelativistic limit as

$$I = \frac{1}{2}\int d^3x\bigg[\bigg(a + \frac{2\beta}{\ell^2}\bigg)\frac{t}{\ell}(-tp\varphi + p^2) + (2\alpha + \beta) \times \frac{t^5}{\ell^3}\dot{\varphi}^2 + \beta\frac{t^3}{\ell^3}(\dot{p}^2 - \varphi^2 - t\dot{p}\,\dot{\varphi} - \varphi\dot{p})\bigg].$$

To decouple the fields, first note that $2\alpha + \beta = \frac{\beta}{4} + \frac{8\alpha+3\beta}{4}$, and rescale φ as $\varphi \to \frac{1}{t}\varphi$ to get the action

$$I = \frac{1}{2}\int d^3x\bigg[\bigg(a + \frac{2\beta}{\ell^2}\bigg)\frac{t}{\ell}(-p\varphi + p^2) + \frac{\beta}{4}\frac{t^3}{\ell^3}\bigg(\dot{\varphi}^2 - \frac{\varphi^2}{t^2} + 4\dot{p}^2 - 4\dot{p}\,\dot{\varphi}\bigg) + \frac{(8\alpha + 3\beta)}{4}\frac{t^3}{\ell^3}\bigg(\dot{\varphi}^2 + \frac{3\varphi^2}{t^2}\bigg)\bigg].$$

Then, define a new field as $\Phi \equiv \varphi - 2p$, which leads to the decoupled actions for the Φ and φ fields. As the spin-2 helicity partner of the σ field, the Φ action is exactly like the σ action with the same mass m_g (18);

$$I_\Phi = \frac{\beta}{8}\int d^3x\bigg[\frac{t^3}{\ell^3}\dot{\Phi}^2 + \frac{t}{\ell}\bigg(\frac{a}{\beta} + \frac{2}{\ell^2}\bigg)\Phi^2\bigg],$$

and the spin-0 mode has the action

$$I_\varphi = \frac{(8\alpha + 3\beta)}{8}\int d^3x\bigg[\frac{t^3}{\ell^3}\dot{\varphi}^2 - \frac{1}{(8\alpha + 3\beta)}\frac{t}{\ell} \times \bigg(a - \frac{24\alpha}{\ell^2} - \frac{6\beta}{\ell^2}\bigg)\varphi^2\bigg],$$

which after putting into the canonical form by rescaling $\varphi \to \frac{\ell^2}{t^2}\varphi$ yields the mass

$$m_s^2 = \frac{1}{\kappa(8\alpha + 3\beta)} - \frac{4}{\ell^2}\bigg(\frac{3\alpha + \beta}{8\alpha + 3\beta}\bigg).$$

In the $8\alpha + 3\beta = 0$ case, the φ field freezes out and m_g^2 matches the result of [1] obtained with the help of an auxiliary field, not via canonical analysis. For generic α and β, in accordance with the analysis of [1], one can introduce two auxiliary fields to rewrite the action (1), but decoupling of the scalar mode from the spin-2 mode is not immediately clear. This is done in Appendix B.

B. Equations of motions in the BHT case

The above nonrelativistic analysis reveals the canonical structure of the generic α, β theory. But here let us consider the relativistic equations of motion for the $8\alpha + 3\beta = 0$ case. Dropping the σ field in (16), we have

$$I = \frac{\beta}{2}\int d^3x\bigg\{m_g^2\frac{t}{\ell}(tp\varphi - p^2) + \frac{t^5}{4\ell^3}(\dot{\varphi} - \nabla^2 p)^2 + \frac{t^3}{\ell^3} \times (\dot{p}^2 - p\nabla^2 p - \varphi^2 - t\varphi\nabla^2 p - t\dot{p}\,\dot{\varphi} - \varphi\dot{p})\bigg\}.$$

It appears that there are 2 degrees of freedom in this action (which would conflict our earlier result, and the result of [1]), but this is a red herring, there is only a single degree of freedom. A quick way to see this is to look at the Hessian matrix, $\mathcal{H} = \frac{\partial^2\mathcal{L}}{\partial\dot{q}_i\partial\dot{q}_j}$,

$$\mathcal{H} = \frac{\beta t^3}{4\ell^3}\begin{pmatrix} t^2 & -2t \\ -2t & 4 \end{pmatrix}.$$

Since $\det\mathcal{H} = 0$, there is a constraint in the model. Therefore, "velocities" $\dot{\varphi}$ and $\dot{p}$ cannot be separately expressed in terms of the canonical momenta

$$\Pi_\varphi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \frac{\beta t^5}{4\ell^3}\bigg(\dot{\varphi} - \nabla^2 p - \frac{2}{t}\dot{p}\bigg),$$

$$\Pi_p \equiv \frac{\partial\mathcal{L}}{\partial\dot{p}} = \frac{\beta t^3}{2\ell^3}(2\dot{p} - t\dot{\varphi} - \varphi).$$

One can use the Dirac's constraint analysis method to obtain the Hamiltonian for this singular Lagrangian, but here it suffices to consider just the field equations. Taking the variations with respect to φ and p yield

$$\delta\varphi\!: \frac{m_g^2 t^2}{\ell}p - \frac{t^3}{\ell^3}(2\varphi + t\nabla^2 p + \dot{p}) - \frac{1}{2\ell^3}\partial_0[t^5(\dot{\varphi} - \nabla^2 p) - 2t^4\dot{p}] = 0,$$

and

$$\delta p\!: \frac{m_g^2 t}{\ell}(t\varphi - 2p) - \frac{t^5}{2\ell^3}\nabla^2\bigg(\dot{\varphi} - \nabla^2 p + \frac{4}{t^2}p + \frac{2}{t}\varphi\bigg) - \frac{1}{\ell^3}\partial_0[t^3(2\dot{p} - t\dot{\varphi} - \varphi)] = 0.$$

By inspection, and with a hint from the field equations which give $R_L = 0$, one observes that $\dot{\varphi} = \nabla^2 p$ and the other equation reduces to

$$\frac{\ell}{t}\bigg(-\ddot{\varphi} - \frac{1}{t}\dot{\varphi} + \nabla^2\varphi\bigg) - \frac{\ell^3}{t^3}\bigg(m_g^2 - \frac{1}{\ell^2}\bigg)\varphi = 0,$$

which is not yet in the canonical wave equation form in dS. To put in the canonical form, $(\Box - m^2)\ \phi = 0$, rescale $\varphi \to \varphi/t$ to obtain

$$\frac{\ell}{t}\bigg(-\ddot{\varphi} + \frac{1}{t}\dot{\varphi} + \nabla^2\varphi\bigg) - \frac{\ell^3}{t^3}m_g^2\varphi = 0,$$

$$\Rightarrow (\Box - m_g^2)\varphi = 0,$$

which is exactly like the σ field.

IV. CONCLUSIONS

We have studied the canonical structure of the linearized quadratic gravity models in an explicitly gauge-invariant way for both flat and dS backgrounds in three dimensions. In flat spacetime, the general action is decoupled into three harmonic oscillators. After considering the signs and various limits of the parameters κ, α, β, the BHT case is singled out as the unique unitary and nontachyonic theory (namely, a regular massive free spin-2 field, not a higher-time derivative one), while the others are all higher-derivative Pais-Uhlenbeck oscillators. Sources are also added to the theory, and Newtonian potentials for both static and spinning particles are calculated. Moreover, we have computed the weak field limit of the circularly symmetric spacetime. We extended our flat space analysis to include the gravitational Chern-Simons term and investigated the oscillator structure for the BHT limit: We have seen that in this limit the oscillators decouple with different masses, violating parity as expected. In dS, we have also found the most general action in terms of three gauge-invariant functions constructed from the (derivatives of the) components of the metric perturbation and carried out the decoupling of the fields in the nonrelativistic limit at the level of the action and in a relativistic form at the level of the field equations. For future work, to go beyond the free field level and introduce nonlinearities, such as $O(h^3)$ and interactions, our gauge-invariant actions will be of great use. Another interesting point about the models that we discussed here is that, especially in (anti)-de Sitter backgrounds, for certain tuned values of the parameters novel phenomena such as partial masslessness or chiral gravity arise. These topics will be addressed in a separate work.

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APPENDIX A: SPINNING MASSES

It is also of some interest to understand how spinning point particles interact in the generic higher-derivative model. This can be done as follows: First, note that the energy-momentum tensor for a massive (m) spinning (j) pointlike source is

$$T_{00}=m\delta^{(2)}(\vec{r}-\vec{r}_1),\quad T^i{}_0=\frac{1}{2}j\epsilon^{ij}\partial_j\delta^{(2)}(\vec{r}-\vec{r}_1),\quad T_{ij}=0.$$

For two such conserved sources scattering amplitude was computed in [5] as

$$4A=\int d^3x\Big\{-2T'_{\mu\nu}\Big[\beta\Box^2+\frac{1}{\kappa}\Box\Big]^{-1}T^{\mu\nu}+T'\Big[\beta\Box^2+\frac{1}{\kappa}\Box\Big]^{-1}T-T'\Big[(8\alpha+3\beta)\Box^2-\frac{1}{\kappa}\Box\Big]^{-1}T\Big\}.$$

From the nonspinning case, the only added part will be

$$-4T'_{i0}\Big(\beta\Box^2+\frac{1}{\kappa}\Box\Big)^{-1}T^{i0}=-\frac{j_1j_2}{\beta m_g^2}\partial_i\delta^{(2)}(\vec{r}-\vec{r}_1)\times\Big(\frac{1}{\Box}-\frac{1}{\Box-m_g^2}\Big)\times\partial_i\delta^{(2)}(\vec{r}-\vec{r}_2).$$

After carrying out the space integrations, it reads

$$-4T'_{i0}\Big(\beta\Box^2+\frac{1}{\kappa}\Box\Big)^{-1}T^{i0}=-\frac{j_1j_2}{2\pi\beta}K_0(m_g|\vec{r}_1-\vec{r}_2|),$$

for $\vec{r}_1\neq\vec{r}_2$. Then, the total Newtonian potential energy, $U=A/\text{time}$, becomes

$$U=\frac{\kappa}{8\pi}(m_1m_2+4m_g^2j_1j_2)K_0(m_g|\vec{r}_1-\vec{r}_2|)-\frac{\kappa}{8\pi}m_1m_2K_0(m_s|\vec{r}_1-\vec{r}_2|).$$

Since j_1 and j_2 could be of any sign, the part coming from the spin-spin interaction can be repulsive or attractive. In the BHT limit the last term disappears.

APPENDIX B: THE α, β THEORY WITH AUXILIARY FIELDS

Consider the quadratic Lagrangian (1) in three dimensions. Using two auxiliary fields ϕ and $f_{\mu\nu}$, one can rewrite it as

$$\mathcal{L}=\frac{1}{\kappa}\sqrt{-g}\Big[R-f^{\mu\nu}G_{\mu\nu}-\phi R+\frac{m_1^2}{2}\phi^2+\frac{m_2^2}{4}(f^{\mu\nu}f_{\mu\nu}-f^2)\Big],$$

where $m_1^2=-\frac{4}{\kappa(8\alpha+3\beta)}$ and $m_2^2=-\frac{1}{\kappa\beta}$. After linearization around flat spacetime, we have

$$\kappa\mathcal{L}_{\text{linearized}}=-\Big(\frac{1}{2}h^{\mu\nu}+f^{\mu\nu}\Big)\mathcal{G}^L_{\mu\nu}-\phi R_L-\frac{2}{\kappa(8\alpha+3\beta)}\phi^2-\frac{1}{4\kappa\beta}(f^{\mu\nu}f_{\mu\nu}-f^2).$$

For $8\alpha+3\beta=0$, ϕ decouples, and $f_{\mu\nu}$ can be eliminated to yield the action describing spin-2 field with a Pauli-Fierz mass [1]. But, for generic α and β, one has to find a way to decouple ϕ, $f_{\mu\nu}$, and $h_{\mu\nu}$ keeping in mind that there should be a kinetic term for the ϕ field. This is possible by rescaling $h_{\mu\nu}$, but we have not pursued this [20].

APPENDIX C: LINEARIZED FIELD EQUATIONS IN THE DE SITTER BACKGROUND

In the body of the text, we worked mostly at the level of the action. To check our results at the level of the field equations, some of the computations in this Appendix are needed. The trace of the linearized field equation is

$$(8\alpha+3\beta)\Box R_L+\Big[\frac{6(4\alpha+\beta)}{\ell^2}-a\Big]R_L=0,$$

where $\bar{g}^{\mu\nu}\mathcal{G}^L_{\mu\nu} = -\frac{R_L}{2}$ was used. Without further ado, let us list the results of somewhat tedious, yet relevant computations:

$$\Box\mathcal{G}^L_{00} = \frac{t^3}{2\ell^3}\Big[\Big(\nabla^2\ddot{f} + \frac{5}{t}\nabla^2\dot{f} - \nabla^2\nabla^2 f\Big) - \frac{4}{t}\nabla^2 p - \frac{3}{t^2}\nabla^2 f - \frac{2\ell^3}{t^5}R_L\Big],$$

$$\Box\mathcal{G}^L_{0i} = \frac{t^3}{2\ell^3}\partial_i\Big(\ddot{p} + \frac{3}{t}\dot{p} - \nabla^2 p - \frac{2}{t^2}p - \frac{2}{t}\nabla^2 f\Big) + \frac{t^3}{2\ell^3}\epsilon_{ij}\partial_j\Big(\ddot{\sigma} + \frac{3}{t}\dot{\sigma} - \nabla^2\sigma - \frac{2}{t^2}\sigma\Big),$$

$$\Box\mathcal{G}^L_{ij} = \frac{t^3}{2\ell^3}(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j)\Big(\ddot{q} + \frac{5}{t}\dot{q} + \frac{1}{t^2}q - \nabla^2 q - \frac{2}{t^2}\nabla^2 f\Big) - \frac{t^3}{2\ell^3}\hat{\partial}_i\hat{\partial}_j\Big(\dddot{p} + \frac{5}{t}\ddot{p} + \frac{1}{t^2}\dot{p} - \nabla^2\dot{p} - \frac{4}{t}\nabla^2 p - \frac{2}{t^2}\nabla^2 f\Big) - \frac{t^3}{2\ell^3}(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i) \times \Big(\dddot{\sigma} + \frac{5}{t}\ddot{\sigma} + \frac{1}{t^2}\dot{\sigma} - \nabla^2\dot{\sigma} - \frac{2}{t}\nabla^2\sigma\Big).$$

$\mathcal{G}^L_{\mu\nu}$, and R_L, computed in the body of the text, together with the Bianchi identity (15), and the above results are sufficient to study the field equations.

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