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We present a comprehensive investigation of cosmological constraints on the class of vector field formulations of modified gravity called generalized Einstein-aether models. Using linear perturbation theory we generate cosmic microwave background and large-scale structure spectra for general parameters of the theory, and then constrain them in various ways. We investigate two parameter regimes: a dark matter candidate where the vector field sources structure formation, and a dark energy candidate where it causes late-time acceleration. We find that the dark matter candidate does not fit the data, and identify five physical problems that can restrict this and other theories of dark matter. The dark energy candidate does fit the data, and we constrain its fundamental parameters; most notably we find that the theory's kinetic index parameter n_{ac} can differ significantly from its Λ CDM value.

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I. INTRODUCTION

Over the past few years, a new suite of models for the dark sector has been proposed. They invoke a vector field which is normally constrained to lie along the timelike direction and may lead to modifications to the gravitational sector [1–4]. Sometimes called Einstein-aether models, they tend to entangle two of the main paradigms currently being considered: on the one hand modified gravity and on the other dark matter and energy.

Vector field models are attractive because they seem to be able to resolve the problem of the dark sector (i.e. dark matter *and* energy) in a unified way. Most of the emphasis has gone into constraining vector field models that lead to accelerated expansion [2–4] although there is a fair amount of work for which the vector field leads to a relativistic version of modified Newtonian dynamics (MOND) [5] or can even play the role of dark matter [1,6]. In fact vector field models seem to incorporate what seems to be a generic feature of relativistic modified gravity models [7]: that it is impossible to construct relativistic models that just modify the gravitational sector without introducing new degrees of freedom, which can then behave like either dark matter or dark energy (although for other approaches see, for example, [8–13]).

There has been significant progress in trying to constrain these models. For example, at a fundamental level it has been shown that a broad class will lead to instabilities and the formation of caustics, signaling a breakdown of the fundamental theory [14]. It has also been shown that for a general choice of kinetic terms, these theories will be plagued by ghosts or tachyons [15–17]. These pathologies are worrying but do not entirely rule out vector field models—it has been shown that modifications to the kinetic term, for example, can cure them.

Substantial work has been done on understanding how these fields in these theories behave on macroscopic scales,

either through their interaction with matter to form galaxies and clusters [18], or on the largest scales, affecting the growth of structure and its effect on the CMB [19]. Indeed for a particular, “vanilla” version of the vector field model, detailed and definitive constraints have been placed on the various coupling constants [20,21].

In [19], it was found that one of the key effects that vectors would have would be to modify the growth rate of structure. This is not surprising—theories that modify gravity tend to have this effect. We also found that it led to a mismatch between the two gravitational potentials, a potentially observable effect [22]. In this paper we wish to pursue this analysis and quantify how strong these effects are. Although we focus on a particular (albeit broad) class of theories, we are interested in extracting *general* lessons from these models. We believe that much of what we learn by looking at these models will shed light on other models of modified gravity (such as, for example, $f(R)$ theories [23] and bimetric theories [24]).

The structure of this paper is as follows. In Sec. II we lay out the essential ingredients for a reasonably broad class of vectorlike models and its background evolution, and specialize to the form used in the remainder of the paper. In Sec. III A we lay out the equations of the perturbed theory, and how we implement them with theoretical constraints. In Sec. IV we discuss the problems with modeling dark matter with the theory. In Sec. V we find and constrain the parameters which let the theory act as dark energy. In Sec. VI we conclude and draw more general lessons about the dark sector.

II. THE THEORY**A. Theory definition**

A general action for a vector field A^a coupled to gravity can be written in the form

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \mathcal{L}(g^{ab}, A^b) \right] + S_M, \quad (1)$$

where g_{ab} is the metric, R the Ricci scalar of that metric, S_M the matter action, and \mathcal{L} is constructed to be generally covariant and local. By construction S_M only couples to the metric g_{ab} and *not* to A^a .

We will restrict ourselves to consider a Lagrangian that only depends on covariant derivatives of A and we will consider a unit timelike A^a . Such a theory can be written in the form

$$\mathcal{L}(g^{ab}, A^a) = \frac{M^2}{16\pi G} F(K) + \frac{1}{16\pi G} \lambda (A^a A_a + 1), \quad (2)$$

$$K = M^{-2} K^{ab}{}_{mn} \nabla_a A^m \nabla_b A^n, \quad (3)$$

$$K^{ab}{}_{mn} = c_1 g^{ab} g_{mn} + c_2 \delta^a{}_m \delta^b{}_n + c_3 \delta^a{}_n \delta^b{}_m, \quad (4)$$

where c_i are dimensionless constants and M has the dimension of mass. We have removed an additional c_4 ‘‘acceleration’’ term in accordance with the transformation described in [25].

As was the case with the tensor-vector-scalar theory (TeVes), the constant G may be a different number from the locally measured value of Newton’s gravitational constant. λ is a nondynamical Lagrange-multiplier field with dimensions of mass-squared.

The gravitational field equations for this theory, obtained by varying the action with respect to g^{ab} (see [1]) are

$$G_{ab} = \tilde{T}_{ab} + 8\pi G T_{ab}^{\text{matter}}, \quad (5)$$

where the stress-energy tensor for the vector field is given by

$$\begin{aligned} \tilde{T}_{ab} = & \frac{1}{2} \nabla_c (F_K (J_{(a}{}^c A_{b)} - J^c{}_{(a} A_{b)} - J_{(ab)} A^c)) - F_K Y_{(ab)} \\ & + \frac{1}{2} g_{ab} M^2 F + \lambda A_a A_b, \end{aligned} \quad (6)$$

$$F_K \equiv \frac{dF}{dK}, \quad (7)$$

$$J^a{}_c = (K^{ab}{}_{cd} + K^{ba}{}_{dc}) \nabla_b A^d. \quad (8)$$

Brackets around indices denote symmetrization [26] and Y_{ab} is the functional derivative

$$Y_{ab} = \nabla_c A^e \nabla_d A^f \frac{\delta(K^{cd}{}_{ef})}{\delta g^{ab}}. \quad (9)$$

The equations of motion for the vector field, obtained by varying with respect to A^b , are

$$\nabla_a (F_K J^a{}_b) + F_K y_b = 2\lambda A_b, \quad (10)$$

where once again we define the functional derivative

$$y_b = \nabla_c A^e \nabla_d A^f \frac{\delta(K^{cd}{}_{ef})}{\delta A^b}. \quad (11)$$

Finally, variations of the action with respect to λ will fix $A^b A_b = -1$. By inspection, contracting both sides of (11) with A^b leads to a solution for λ in terms of the vector field and its covariant derivatives.

These equations allow us to study a general theory of the form presented in Eq. (1) with a unit timelike vector field. For our particular, restricted choice of K we have $Y_{ab} = -c_1 [(\nabla_c A_a (\nabla^c A_b) - (\nabla_a A_c) (\nabla_b A^c))] + y_b = 0$.

B. Background cosmology

In this paper we will restrict ourselves to background cosmologies where the spacetime is of the spatially flat Friedmann-Robertson-Walker (henceforth FRW) form

$$g_{ab} dx^a dx^b = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j. \quad (12)$$

The energy momentum content of each matter fluid (i) in the background is taken to be of the usual form

$$T_{ab}^i = (P^i + \rho^i) u_a^i u_b^i + P^i g_{ab}, \quad (13)$$

where ρ^i and P^i are the comoving density and pressure of the i th fluid, respectively. We assume that all fluids have identical four velocities $u^a = (1, 0, 0, 0)$. Furthermore we define $\rho \equiv \sum_i \rho^{(i)}$.

In spacetimes with FRW symmetries, the vector field must align with the direction ∂_t and so the vector field is entirely fixed to have components (1, 0, 0, 0) in the coordinate system (12). Explicitly, the background value of the scalar K is given by

$$K_{\text{FRW}} = 3 \frac{\alpha H^2}{M^2}, \quad (14)$$

where $H \equiv \partial_t \ln(a(t))$ and

$$\alpha \equiv c_1 + 3c_2 + c_3. \quad (15)$$

The Friedmann equation then takes the form [19]

$$\left[1 - \alpha K^{1/2} \frac{d}{dK} \left(\frac{F}{K^{1/2}} \right) \right] H^2 = \frac{8\pi G}{3} \rho, \quad (16)$$

where ρ still includes only the matter components.

We note that the combination of (16) and the aether stress-energy tensor allows us to write down an effective energy density ρ_{ae} and pressure P_{ae} of the aether. From this we may define the fractional energy density $\Omega_{\text{ae}} \equiv 8\pi G \rho_{\text{ae}} / 3H_0^2$, and the aether equation of state parameter $w_{\text{ae}} \equiv P_{\text{ae}} / \rho_{\text{ae}}$:

$$\Omega_{\text{ae}} = \frac{M^2}{6} \left[\frac{\partial}{\partial H} \left(\frac{F}{H} \right) \right]_{H=H_0}, \quad (17)$$

$$w_{\text{ae}} = -1 - \frac{1}{3H^2} \frac{\frac{d^2}{dH^2} F}{\frac{d}{dH} \left(\frac{F}{H} \right)}. \quad (18)$$

The effect of the vector field on the background expansion may be seen as an expansion rate dependent modification to Newton's constant, i.e., schematically (16) is an equation of the form $3H^2 = 8\pi G(H^2)\rho$. It was found in [19] that various forms of the function F permitted a wide variety of cosmological dynamics: the presence of the vector field variously leading to accelerated expansion, slowed expansion, rescaling of G , and recollapse as summarized in Fig. 1.

C. The functional form of $F(K)$ in cosmology

We must now specify the form of the function $F(K)$ in Eq. (6). There is an obvious set of candidates here—we could attempt to be directly consistent with MOND and use the same branch of F as it uses on small scales. In Appendix A we show that doing so would make it impossible to consistently generate late-time acceleration behavior in the background cosmology. Instead, we will use a simple and reasonable ansatz that works for the regime $|K| \gg 1$ that we consider here.

Existing functional forms for $F(K)$ in the MOND regime typically are dominated for a single monomial term for $|K| \gg 1$ (see for instance [27]) and so it seems reasonable to restrict the function to take this form:

$$\begin{aligned} F &= \gamma(-K)^{n_{ae}} & K < 0 \\ F &= \gamma(+K)^{n_{ae}} & K > 0, \end{aligned} \quad (19)$$

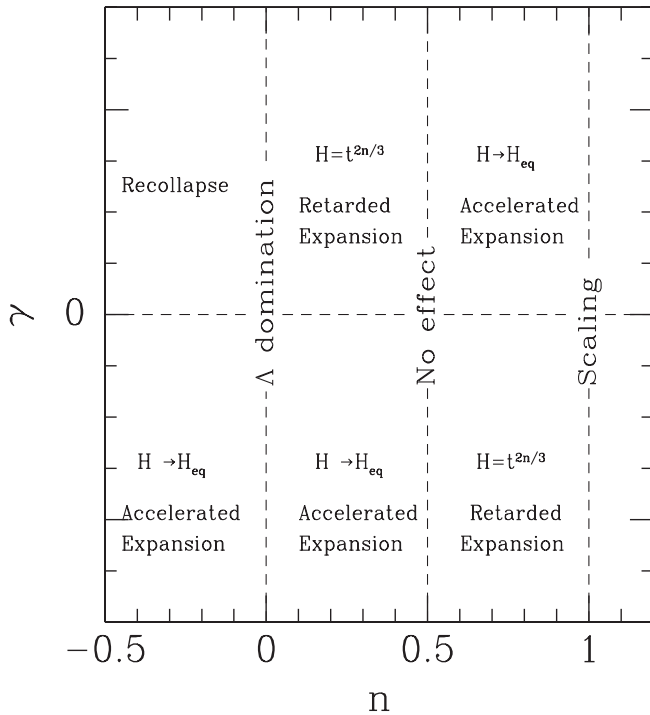


FIG. 1. A schematic representation of the types of the late-time background evolution permitted by the functional form $F = \gamma(-K)^n$ as a function of (n, γ) for $n < 1$.

where $n_{ae} \leq 1$. This form has sufficient power to express a wide variety of behavior, and the parameters γ and n shall be central to our further analysis.

III. LINEAR PERTURBATION THEORY

A. Formalism and Theory

We have seen that the vector field can have a significant effect on the quasistatic, weak field limit and the background cosmological geometry. Significant evidence for nonbaryonic mass persists on the largest cosmological scales [28] therefore it is vital that a relativistic theory of MOND can account for this. As mentioned, it has been argued [29] that even in the quasistatic, weak field limit, a spatial tilt to the vector field may produce significant deviations under some circumstances from the local MOND force law. Similarly it was shown [19] that in the context of linear cosmological perturbations the energy density associated with the projection of the vector field onto surfaces of constant conformal time could, to a degree, act as a cosmological “dark matter.” In this paper we shall comprehensively address the question of whether the Lagrangian (2) represents a viable model of the dark sector in light of precision cosmology.

In the main body of the paper we will consider scalar perturbations. In Appendix B we derive the equations of motion for the vector field's two divergenceless vector modes as well as the gravitational wave tensor modes (the speed of propagation of which is in general modified by the vector field). Requiring the stability of these modes puts constraints on (2) (see Sec. III B).

We shall work in the synchronous gauge (see for instance [30]) and so the metric takes the following form:

$$g_{\mu\nu} dx^\mu dx^\nu = -a^2 d\tau^2 + a^2[\gamma_{ij} + h_{ij}] dx^i dx^j, \quad (20)$$

where τ is conformal time, γ_{ij} is a spatially flat spacelike 3-metric perturbed by h_{ij} which is built from two scalar potentials η and h :

$$\begin{aligned} h_{ij}(\mathbf{x}, \tau) &= \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \left[\hat{\mathbf{k}}_i \hat{\mathbf{k}}_j h(\mathbf{k}, \tau) \right. \\ &\quad \left. + \left(\hat{\mathbf{k}}_i \hat{\mathbf{k}}_j - \frac{1}{3} \delta_{ij} \right) 6\eta(\mathbf{k}, \tau) \right]. \end{aligned} \quad (21)$$

Similarly we will expand the aether field as

$$A^\mu = \frac{1}{a} (1, \partial_i V). \quad (22)$$

The zeroth component of the aether field is, by virtue of the gauge choice and the constraint, fixed as equal to 1 up to second order in perturbations.

A slight complication in the field equations arises because of the presence of the function F which depends nonlinearly on the scalar K . We assume that for modes of interest one may consistently regard the perturbation to K as being much less than unity. Thus one can expand K as

$K = K^0 + K^\epsilon$ and F as $F = F^0 + F_K^0 K^\epsilon$ where $K^\epsilon \ll 1$ where the superscript 0 denotes the quantity corresponding to a function's background value. Explicitly we have that

$$K^0 = 3 \frac{\alpha \mathcal{H}^2}{a^2 M^2}, \quad (23)$$

$$\mathcal{H} K^\epsilon = -\frac{2}{3} K^0 \left(k^2 V - \frac{h'}{2} \right), \quad (24)$$

where primes denote derivatives with respect to conformal time and we have used the conformal Hubble parameter $\mathcal{H} \equiv H/a$. We also make use of the following identity:

$$(K^0)' \mathcal{H} = -4K^0 \left(\mathcal{H}^2 - \frac{1}{2} \frac{a''}{a} \right). \quad (25)$$

Henceforth we will drop the superscripts on K^0 and F^0 , i.e., K and F shall be assumed to represent *background* values of the fields.

Toward simplifying the form of the equations, we will rather use the field ξ instead of V , where ξ is defined as

$$\xi \equiv V - \frac{1}{2k^2} (h + 6\eta)'. \quad (26)$$

For further compactness of notation we define the variables

$$\hat{\alpha} \equiv \left(1 + 2 \frac{F_{KK}}{F_K} K \right) \alpha, \quad (27)$$

$$\hat{c}_1 \equiv \left(1 + 2 \frac{F_{KK}}{F_K} K \right) c_1. \quad (28)$$

The vector field equation of motion (10) becomes

$$\begin{aligned} 0 = & c_1 (1 + c_{13} F_K) \frac{(F_K \xi')'}{F_K} + 2 \mathcal{H} c_1 (1 + c_{13} F_K) \frac{(F_K \xi)'}{F_K} + \left[2c_1 (1 + c_{13} F_K) \left(\frac{a''}{a} - \mathcal{H}^2 \right) + 2(\hat{c}_1 + \hat{\alpha}) \left(\mathcal{H}^2 - \frac{1}{2} \frac{a''}{a} \right) \right. \\ & \left. + c_1 c_{13} (F_{KK} K')' + \frac{1}{3} (\hat{\alpha} + 2c_{13}) k^2 \right] \xi + (c_1 + \hat{\alpha}) \eta' + (\hat{c}_1 + \hat{\alpha}) \frac{1}{k^2} \left(\mathcal{H}^2 - \frac{1}{2} \frac{a''}{a} \right) (h' + 6\eta)' - \frac{3}{2} \frac{c_1}{k^2} \frac{(F_K \Sigma_f)'}{F_K}. \end{aligned} \quad (29)$$

The relevant Einstein equations are

$$\left(1 - \frac{1}{2} \hat{\alpha} F_K \right) k^2 \eta' = 4\pi G a^2 i k^j \delta T^0_j + \frac{1}{6} k^4 (\hat{\alpha} + 2c_{13}) F_K \xi \quad (30)$$

and

$$\begin{aligned} \left(1 + \frac{1}{2} c_1 F_K \right) (\mathcal{H} h' - 2k^2 \eta) = & -8\pi G a^2 \delta T^0_0 - \frac{1}{2} F_K (c_1 + \hat{\alpha}) 6\mathcal{H} \eta' - 2\alpha F_{KK} K \mathcal{H} k^2 \xi + \frac{F_K c_1 k^2}{a^2} (a^2 (1 + c_{13} F_K) \xi)' \\ & - \frac{3}{2} c_1 F_K \Sigma_f, \end{aligned} \quad (31)$$

where we have used the fact that

$$\begin{aligned} (h + 6\eta)'' + 2\mathcal{H} (h + 6\eta)' - 2k^2 \eta \\ = -3\Sigma_f + 2c_{13} k^2 [F_K (2\mathcal{H} \xi + \xi') + F_{KK} K' \xi]. \end{aligned} \quad (32)$$

The functions δT^0_j and δT^0_0 are the first order perturbations to the corresponding components of the matter fields' stress-energy tensors. Summation over field species is assumed. The field Σ_f is the scalar component of the total fluid shear, i.e., $\Sigma_f = -8\pi G a^2 (\hat{k}_i \hat{k}^j - \frac{1}{3} \delta^j_i) \Sigma^i_j$ and $\Sigma^i_j \equiv \delta T^i_j - \frac{1}{3} \delta^i_j \delta T^k_k$. A gauge invariant formulation of the theory's equations may be found in [31].

B. Parameter constraints

We can immediately see a number of constraints on the c_i and the form of the function F . From (29) it can be shown that in the limit of timescales shorter than a Hubble time the quantity

$$C_S^2 = \frac{2}{3} \frac{(\hat{\alpha} + c_{13})}{c_1 (1 + c_{13} F_K)} \quad (33)$$

can be interpreted as the squared sound speed of the field ξ . The avoidance of exponentially growing subhorizon modes dictates that C_S^2 should be positive definite.

Similarly, one may consider the field equations of the two divergenceless ‘‘vector’’ modes of the vector field and the two transverse traceless ‘‘tensor’’ modes of the metric. Each, respectively, has a squared sound speed function (named C_V^2 and C_T^2 respectively) which, as in the scalar case, should be positive definite. These functions are calculated in Appendix B and are as follows:

$$C_T^2 = \frac{1}{1 + c_{13} F_K}, \quad (34)$$

$$C_V^2 = \frac{F_K}{2c_1} \frac{2c_1 + F_K(c_1^2 - c_3^2)}{1 + c_{13}F_K}. \quad (35)$$

Note that the gravitational wave tensor modes now generically have a time-dependent speed of propagation.

Collectively the three positivity constraints imply the following constraints on the c_i and function F parameter space:

$$1 + F_K c_{13} > 0, \quad (36)$$

$$(\hat{\alpha} + 2c_{13})/2c_1 > 0, \quad (37)$$

$$F_K \left(1 + F_K \frac{c_1^2 - c_3^2}{2c_1} \right) > 0. \quad (38)$$

Now we turn to the Einstein equations (30) and (31). Terms in the perturbed vector field stress-energy tensor $\delta\tilde{T}_{ab}$ may contain terms proportional (up to a time-dependent function of the background fields) to δG_{ab} and indeed this can be seen in (30) and (31) through the appearance of terms proportional to the c_i on the left-hand sides of the equations. If the effect ξ is negligible then the effect of the aether stress-energy terms proportional to components of the Einstein tensor may always be absorbed into a redefinition of Newton's constant as a time-dependent effective gravitational coupling. Thus, even if ξ has comparatively little effect, there may be a considerable modification to the link between the matter fields and the gravitational field. The resulting gravitational couplings should be greater than or equal to zero otherwise the gravitational field will interpret normal matter as violating energy conditions, and so risking the appearance of instabilities. This restriction implies the following constraints:

$$\left(1 - \frac{1}{2} \hat{\alpha} F_K \right) > 0, \quad (39)$$

$$\left(1 + \frac{1}{2} c_1 F_K \right) > 0. \quad (40)$$

Throughout our analysis, we will only consider regions of the model's parameter space which satisfy these constraints.

C. Computation

To study the effects of the vector field in detail we have modified the structure formation Boltzmann code CMBEASY [32]. We add a Newton-Raphson solver for the Hubble parameter, with added aether components. The perturbation evolution is also modified to include the aether components ξ and ξ' , and their contribution to the density, pressure, and shear perturbations. We also include the altered metric perturbations in the calculation of the cosmic microwave background (CMB) source function. We use adiabatic initial conditions [33]. Since Boltzmann codes are very highly optimized for Λ CDM models, care must be taken to ensure that modifications are performed in

a consistent manner—for example an unmodified Friedmann equation is often assumed for computational efficiency.

To explore parameter spaces, CMBEASY is coupled to a Monte-Carlo Markov chain (MCMC) engine [34]. We extended this engine to include our new aether parameters: Ω_{ae} , $c_+ = (c_1 + c_3)$, $c_- = (c_1 - c_3)$, C_S^2 , n_{ae} , and M . We include the full ranges of these parameters by allowing the kinetic term to take two branches, for positive and negative K as in (19).

We constrain models using both CMB data from the WMAP experiment [35] and large-scale structure from the SDSS survey [36], though not always at the same time. As we shall see, some aether models are extremely poor fits to the combined data sets; to illustrate such problems we want to find models that fit only the large-scale structure data. In regimes where fits are extremely poor, MCMC does not work particularly well. We ameliorate such situations by running larger numbers of shorter Markov chains and restarting from their best-fit positions, and sometimes by abandoning MCMC altogether and simply performing random searches for good parameter combinations.

IV. THE VECTOR FIELD AS DARK MATTER

A. Background evolution and Doppler peak positions

An unusual property of the model considered here is that at the level of cosmological perturbations the field can mimic a perturbed pressureless fluid in the formation of large-scale structure while behaving entirely differently in the cosmological background [19]. In this paper we would like to consider not just large-scale structure but also other cosmological probes. The anisotropies in the CMB temperature are sensitive to the background dynamics and perturbed dynamics of sources of cosmic mass discrepancies in a largely distinguishable manner.

If the aether plays the role of dark matter only in the perturbations, then the background expansion is dark-energy-dominated at an earlier time. For a given H_0 this reduces the expansion rate of the Universe between recombination and now and so decreases the angle subtended on the sky by given distance at last scattering. This moves the CMB Doppler peaks to higher ℓ . Although the aether can give a suitable time dependence to the effective gravitational coupling G in the Friedmann equations so as to yield the same expansion rate as Λ CDM, the necessary functional forms of F are extremely contrived. For instance these forms essentially contain a new constant scale roughly equal to the Hubble parameter at matter-radiation equality. Even in the event of such a construction, it may not be possible for the squared sound speed of vector field perturbations C_S^2 to remain sufficiently small as to behave like cold dark matter in perturbations [1].

The other input to the peak position is the sound horizon at last scattering. The physics of this is sufficiently robust that changes compensatory to the alteration in the distance

to last scattering are not feasible without exceptional fine-tuning of the perturbational behavior of the aether. Indeed, it has been argued that only additional components that behave like nonrelativistic matter in the background might fix this problem [37].

Could there indeed be such a nonrelativistic matter in the background which allows acoustic peak positions which are consistent with the data while leaving the aether in its role as the seed of the formation of large-scale structure? An appropriate candidate would appear to be massive neutrinos. A suitable mass of such particles so as to account for the right effective contribution to the dust component of the background typically implies that the neutrinos themselves are unable to clump on small enough scales so as to be a good candidate for all of the dark matter. Such a solution as has been proposed in [38,39].

B. Perturbation evolution

We now argue that even with the inclusion of massive neutrinos, one generically expects the aether to have an unacceptable influence on the large-scale CMB anisotropy if it is to also play the dominant role in structure formation.

The first requirement for successful perturbation evolution is that structure can form at all. One necessary condition for this is that the sound speed of the structure seed not be too large, since this would wash out structure. We require that the sound horizon in the model be less than the smallest scales where structure can form linearly: $C_S k_{\max} \tau \lesssim 1$, where $k_{\max} \sim 0.2h/\text{Mpc}$. For matter power observations at $\tau \sim 3 \times 10^4$, the present epoch, this yields $C_S \lesssim 10^{-4}$.

There are two underlying physical processes that further constrain the models.

The first is a change to the growth rate of perturbation amplitude. This can cause discrepancies between the amplitudes we expect in the matter power spectrum and the CMB, since the evolution between the two is different. It can also lead to an integrated Sachs-Wolfe (ISW—see below) effect during the matter era since Φ will accrue a time dependence.

The second is the increased presence of a $\Phi - \Psi$ metric shear. This also leads (directly) to a matter era ISW.

1. Observable 1: ISW

Under the assumption of adiabaticity we have that the anisotropy in the CMB, $\Delta T(\hat{n})/T$ on large scales in a given spatial direction \hat{n} , is given by

$$\begin{aligned} \frac{\Delta T(\hat{n})}{T} &\simeq -\frac{1}{3} \Psi(\tau_*, d_* \hat{n}) - \int_{\tau_*}^{\tau_0} d\tilde{\tau} (\Psi' + \Phi') \\ &\times [\tilde{\tau}, (\tau_0 - \tilde{\tau}) \hat{n}], \end{aligned} \quad (41)$$

where τ_* is the conformal time of last scattering, τ_0 is the conformal time today, d_* is the comoving radius of the surface of last scattering, and Φ and Ψ are the conformal

Newtonian gauge gravitational potentials. The integral in (41) is the integrated Sachs-Wolfe (ISW) effect. Writing the integrand of (41) as $\Psi' + \Phi' = -(\Phi - \Psi)' + 2\Phi'$, we can see this as time derivatives of a shear part and a growth rate part.

In the standard cosmological model, the field Φ has negligible time dependence during matter domination. It gains a time dependence only when the background starts accelerating, and only then can the resulting growth rate ISW contribution be considerable. In the model considered here, the situation may be rather different. It was found in [19] that substantial contributions to the ISW may occur even during the matter era. For the aether field to seed structure formation the field ξ must have a suitable growing mode solution in the matter era. Typically the corresponding spatial curvature perturbation $k^2\Phi$ will then have a time dependence via the Poisson equation.

The shear part can also gain a time dependence in the aether model, which in the ΛCDM is very small even during acceleration.

Each of these effects depend on the functional form of F , the time-dependence of the ξ growing mode and the choice of the parameters c_i . It is extremely challenging to find combinations of the parameters which allow for a realistic growth of structure while making the ISW acceptably low. This is most easily illustrated by considering the theory TeVeS [24] which has many of the same properties as the model considered here. It may be shown that TeVeS can be written as a single metric theory with a timelike vector field of unfixed norm [40,41]. As in the model considered here, the longitudinal component of the vector field can source the growth of structure [28,38]. We will call this field V_T . We consider a matter dominated era where V_T is responsible for the dominant source in the Poisson and shear equations. These equations, respectively, then are [42]

$$k^2\Phi \approx -f_s \mathcal{H} k^2 V_T - \frac{K_B}{2} k^2 V_T', \quad (42)$$

$$k^2(\Psi - \Phi) \approx f_s k^2 (2\mathcal{H} V_T - V_T'), \quad (43)$$

$$f_s(\tau) \equiv \frac{(1 - \bar{A}^4)}{\bar{A}^4}, \quad (44)$$

where K_B is a positive constant of the action and \bar{A}^4 is the norm of the vector squared again (equal to unity in the fixed norm case, but in TeVeS the deviation of $|\bar{A}^2|$ from unity is essentially the background variation of the ‘‘scalar field’’ degree of freedom). Therefore

$$\Psi + \Phi = -(f_s(\tau) + K_B) V_T'. \quad (45)$$

In this era the vector field equation is

$$V_T'' + b_1 \frac{V_T'}{\tau} + b_2 \frac{V_T}{\tau^2} = S[\Phi, \Psi], \quad (46)$$

where

$$b_1 = 2(3 - \bar{A}^4), \quad (47)$$

$$b_2 = 2(2 - \bar{A}^4) + \frac{12\bar{A}^4}{K_B}(1 - \bar{A}^4), \quad (48)$$

and $S[\Phi, \Psi]$ is a source term.

For this situation to arise, there must be a growing mode in V_T [28]. Therefore we require that $b_2 < 0$. The function \bar{A}^4 will be rather close to unity [43] so

$$b_2 \sim 2 + 12f_s(\tau)/K_B. \quad (49)$$

Therefore we require that $f_s(\tau)/K_B \lesssim -1/6$ (recalling that K_B is a positive number). By (45) we see that the contribution to the ISW will be proportional to $f_s(\tau) + K_B$. Meanwhile we see from (42) that the comparative contribution to Poisson's equation is also independently weighted by f_s and K_B . If there is no time dependence in the Poisson Eq. (42) due to the vector field then the vector field will make no contribution to the ISW effect. If there is a time dependence in the Poisson equation, there may yet be no contribution to the ISW if $f_s(\tau) \sim -K_B$ between last scattering and now (though the overdensities of the baryon field will then not generally grow as a , thus contributing to the ISW effect). This would be consistent with the condition for a growing mode in V_T but it does not guarantee that the resulting growing mode would be suitable.

Indeed, it was found [38] that in seeding the growth of large-scale structure in TeVeS there was a significant danger of incurring unacceptably high temperature anisotropies in the CMB on large scales. Although involving a larger number of terms, the same reasoning carries over to the model considered here—i.e., parameters which allow realistic structure formation will typically lead to an unacceptable ISW effect. This is vividly illustrated in Fig. 2 which shows best-fit models as compared to SDSS large-scale structure data. In every case, the corresponding temperature anisotropy displays a dramatically poor fit to the data at low ℓ .

2. Observable 2: Amplitudes

The ratio of the observed amplitudes of the CMB anisotropy and the matter power spectrum is consistent with a growth rate proportional to a [44]. Any uncompensated change to this growth rate in the aether model over this period would lead to a different ratio. The bias parameter between the galaxy distribution and the underlying density field can be used to rectify this difference, but only if the change is relatively small and unphysically large bias parameters (larger than ~ 10) are not required.

C. Summary

We have seen that although the vector field may play a number of the roles that dark matter plays, it seemingly cannot do all at once [45].

The position of the acoustic peaks in the CMB temperature anisotropy should be taken as a strong indication of additional nonbaryonic nonrelativistic matter present in the

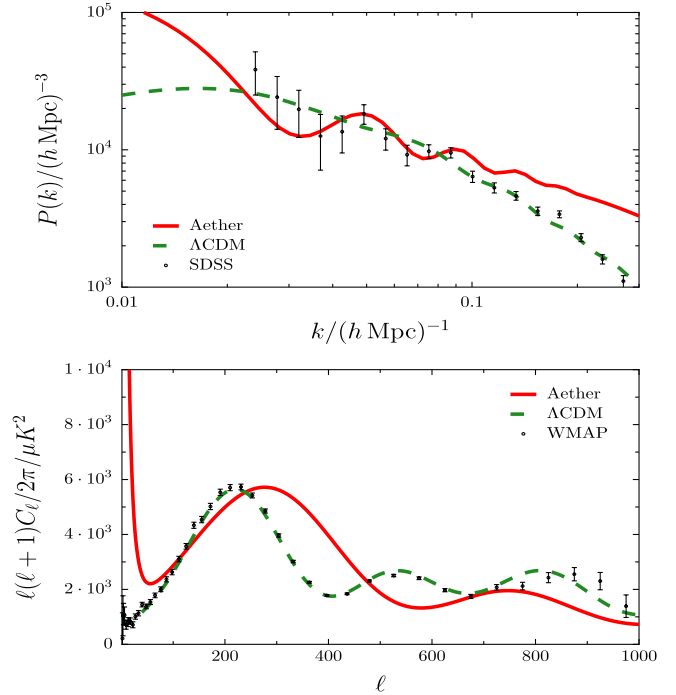


FIG. 2 (color online). Matter power (top) and CMB (bottom) power spectra for the Λ CDM (dashed green) and typical GEA (solid red) models, with WMAP and SDSS constraints.

Universe during matter domination. Such an effect can be achieved in this model by a rescaling of the value of Newton's constant. However, this rescaling cannot persist into the radiation era [47]. Thus, the functional form would have to be approaching the rescaling solution only after one would expect matter-radiation (including cold dark matter) equality to happen.

This implies the presence of a new scale in the theory, roughly corresponding to the Hubble parameter H_{eq} at this time. It seems fair to say in general that a model such as that considered here is more likely to be a cosmologically viable candidate for dark matter if the scale M in the theory is closer to H_{eq} and not H_0 . It is tempting to speculate whether a theory where the scale M itself is dynamical may find more success, but that will not be explored in this work.

Furthermore we have seen that even if the background is consistent with observations, the effect on the evolution of perturbations may be unacceptable, notably either through the ISW effect or comparing the respective amplitudes of the CMB anisotropy and matter power spectrum today.

D. Example problem spectra

It has previously been shown that the Einstein-aether can produce acceptable matter power spectra with certain parameter combinations [19]. Here we show that such combinations do not provide an acceptable fit to CMB measurements. Despite extensive searches we have been

unable to find any parameter set within the model that does fit the WMAP data well; this is entirely in line with the problems discussed above.

We use a parameter set which is consistent with big-bang nucleosynthesis (BBN) limits on $\Omega_B h^2$ and the Hubble Space Telescope key project measurement of H_0 . The standard cosmological parameters are: $\Omega_b h^2 = 0.0193$, $n_s = 0.83$, $H_0 = 89.3$ km/s/Mpc, $\Omega_c h^2 = 0$. The new aether variables are $c_+ = -4.72$, $c_- = -6.11$, $n_{ae} = 0.34$, $\Omega_{ae} = 0.82$, $M_{ae} = 111.3$ km/s/Mpc with c_2 set by requiring zero sound speed. This parameter combination is in no sense optimal, but it does provide an illustration of all the problems that arise here.

Figure 2 shows power spectra from our modified Boltzmann code for this parameter set. The matter power is a realistic fit to the SDSS data (this was the criterion for our choice of parameters). The CMB spectra shows various problems. In the low- ℓ regime a large ISW effect is clearly present, destroying the fit at large scales, as described in Sec. IV B. The positions of the peaks are poorly fit by the model, as expected and discussed in Sec. IV A. Finally, in the plot we have rescaled the amplitude of the matter power spectrum by a factor 0.02, corresponding to a galaxy bias of 0.14 in order to reconcile the relative amplitudes of the two spectra with the data; such a scaling is unphysically

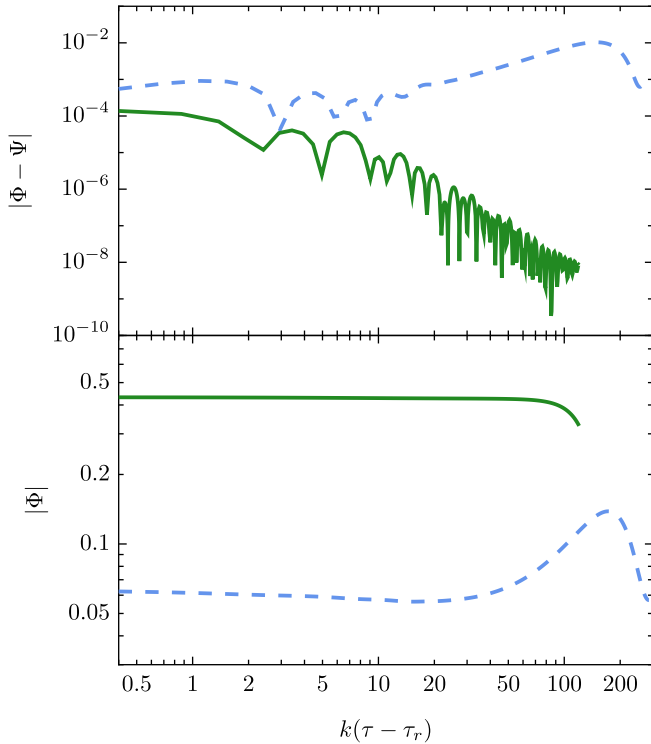


FIG. 3 (color online). Exotic behavior of metric potentials for $k \sim 10^{-2}$ Mpc $^{-1}$. The panels show the fields $|\Phi|$ and $|\Phi - \Psi|$ for a Λ CDM universe (green solid line) and GEA universe (dashed blue line) as a function of $k(\tau - \tau_r)$ where τ_r is the conformal time of recombination.

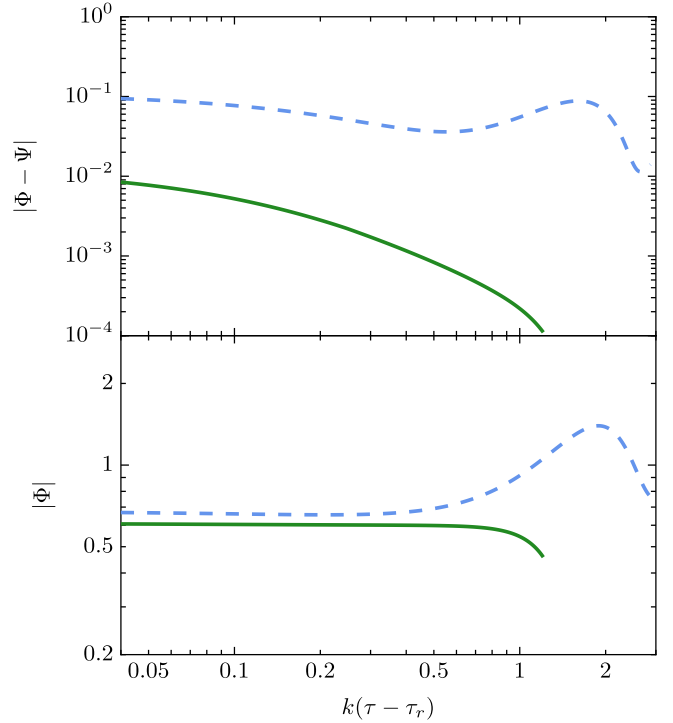


FIG. 4 (color online). Equivalent of Fig. 3 for $k \sim 10^{-4}$ Mpc $^{-1}$.

small. This corresponds to the changed growth rate described in Sec. IV B. All these effects cause severe problems when attempting to simultaneously fit the CMB and large-scale structure.

Figures 3 and 4 illustrate the sources of the extreme ISW effects shown in Fig. 2; the time derivatives of the metric quantities plotted create an ISW effect as shown in Eq. (41). The onset of background acceleration in each case is marked by a turnover in the curves at late time. The generalized Einstein-aether (GEA) universe exhibits a dramatically increased $|\Phi - \Psi|$ and time dependence of $|\Phi|$ during the matter era as compared to the Λ CDM universe. Although the $|\Phi - \Psi|$ has a smaller magnitude its time dependence can be significant for the total ISW effect. Note that values of τ between the two universes do not correspond to the same physical time or redshift since the universes expand at different rates.

V. THE VECTOR FIELD AS DARK ENERGY

A. Dark energy regime

As shown in Fig. 1, our vector field can produce late-time acceleration and so play the role of dark energy. Indeed, for the form of the vector field used here, as the index $n_{ae} \rightarrow 0$ the theory becomes the same as a cosmological constant for both the background and the perturbations. Since the model can fit the data well we can use our MCMC engine to find constraints on the parameters of the vector field, telling us exactly how close to the $n_{ae} = 0$

cosmological constant case the theory must be to fit the CMB and large-scale structure data.

If the model can fit the data only extremely close to $n_{\text{ae}} = 0$ then it does not provide a compelling alternative to the cosmological constant. If, on the other hand, there is significant flexibility in the model and no fine-tuning, or if it can provide a better fit than Λ CDM, then it is somewhat more interesting.

In this section we will consider the resulting background evolution, CMB temperature anisotropy, and matter power spectrum for a universe containing the vector field, cold dark matter, the conventional matter fields, and no cosmological constant. The acceleration will arise solely from the vector field's modification to the Friedmann equation.

B. Constraints on aether dark energy from data

Our MCMC generated the constraints on the vector field parameters shown in Figs. 5–8; these curves are the smoothed histograms from our combined Markov chains.

The most important trend evident in these results is closeness of n_{ae} to the Λ value of zero. We find the best-

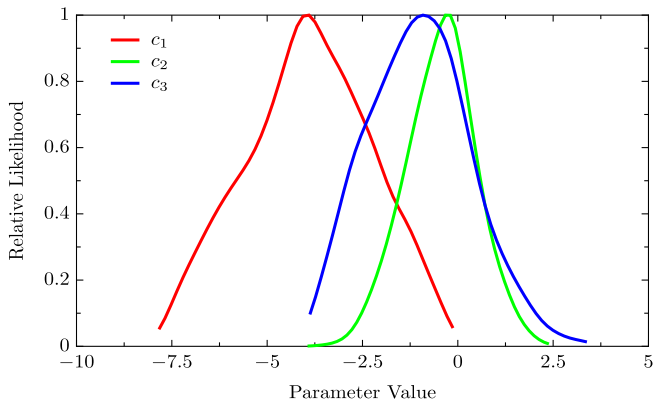


FIG. 5 (color online). Constraints on the three coupling terms of the theory.

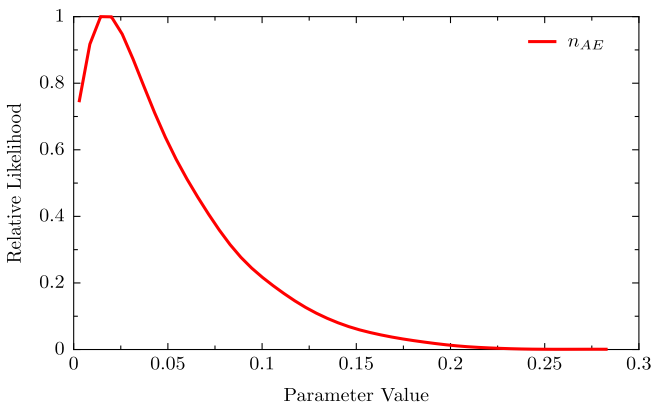


FIG. 6 (color online). Constraints on the kinetic term power law index parameter.

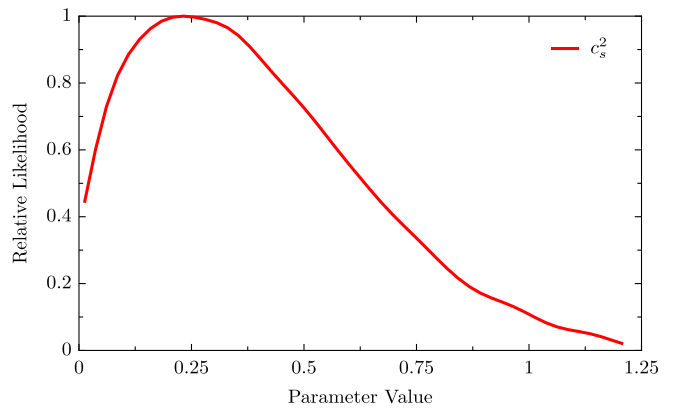


FIG. 7 (color online). Constraints on the vector field sound speed parameter.

fit value $n_{\text{ae}} = 4.2 \times 10^{-3}$, with a 95% upper limit $n_{\text{ae}} < 0.126$.

The best-fit value in the MCMC run is very close to the Λ CDM likelihood of the same data, at the cost of six extra parameters, meaning that it is unlikely to be favored by any model comparison exercise. It does, however, demonstrate the validity of modified gravity-related dark energy candidates.

Having obtained these constraints we can determine their origin. There are two ways in which the vector field must behave like Λ to provide a good fit. The first is that the late-time acceleration should be close to that given by Λ . The second is that any perturbations in the field (which are not present in Λ) should not affect the observable spectra.

C. Constraint origins—acceleration rates

The consistency of the acceleration of the Universe with the cosmological constant equation of state $w = -1$ is being measured with increasing precision in supernova and baryon-acoustic oscillation experiments (which are beyond the scope of this paper). Here, they will be con-

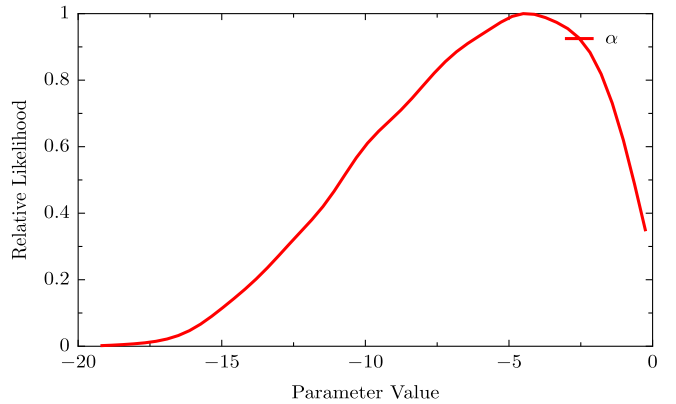


FIG. 8 (color online). Constraints on the parameter $\alpha = c_1 + 3c_2 + c_3$.

strained by the late-time ISW effect induced by dark energy, and by the perturbation growth rate.

We can assess how closely vector-induced acceleration mimics Λ -driven expansion at late times with the equation of state w_{ae} of the vector field in the background:

$$w_{ae} = -1 - \frac{1}{3H^2} \frac{\frac{d^2}{dt^2} F}{\frac{d}{dt} \left(\frac{F}{H}\right)}. \quad (50)$$

For the monomial form of $F(K)$ we have that

$$w_{ae} = -1 - \frac{2n}{3(2n-1)} \frac{\dot{H}}{H^2}. \quad (51)$$

Thus the equation of state will generically deviate from -1 whenever $n \neq 0$ and so the acceleration for these values will not be degenerate with a cosmological constant. We see immediately from Eq. (51) that $w_{ae}(\tau) < -1$ for $0 < n < 1/2$ and $w_{ae}(\tau) > -1$ for $n > 1/2$. This is clearly visible in Fig. 9.

D. Constraint origins—perturbation evolution

Even if the background expansion is rather close to the Λ CDM model, the evolution of perturbations need not be. This is most easily illustrated by considering the Poisson equation on large scales (see [1] for a derivation). On these large scales there is a time-dependent rescaling of the metric perturbation Φ , which we can cast as a modification of the effective gravitational constant G :

$$k^2 \Phi = -4\pi G_{\text{eff}}^{(1)} a^2 \sum_i \bar{\rho}_i \delta_i, \quad (52)$$

$$G_{\text{eff}}^{(1)} \equiv \frac{G}{1 + \frac{c_s}{2} F_K}, \quad (53)$$

where we have assumed that terms proportional to the velocity divergence are ignorable and provisionally considered the effect of the field ξ to be subdominant.

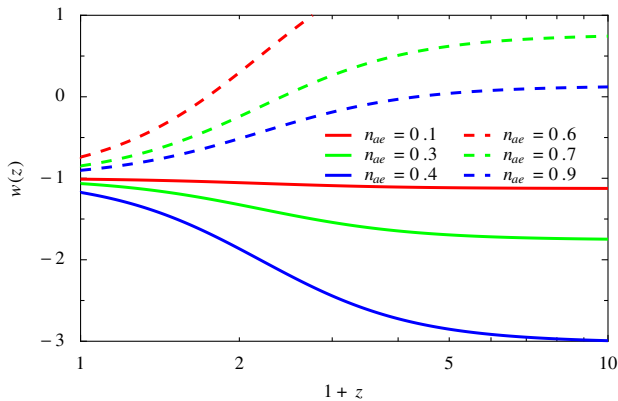


FIG. 9 (color online). The vector field’s equation of state as a function of redshift z , for various values of the kinetic term index n_{ae} .

The Friedmann equation may be used to cast the above equation in a more familiar form by eliminating the background ρ_i in favor of background expansion rate of the Universe and the time-dependent fractional energy density $\Omega_i(\tau) \equiv 8\pi G\rho(\tau)/(3H(\tau)^2)$. This yields

$$k^2 \Phi = -\frac{3}{2} \mathcal{H}^2 \frac{G_{\text{eff}}^{(1)}}{G_{\text{eff}}^{(0)}} \sum_i \Omega_i(\tau) \delta_i, \quad (54)$$

where

$$G_{\text{eff}}^{(0)} \equiv \frac{G}{1 - \alpha K^{1/2} \frac{d}{dK} \left(\frac{F}{K^{1/2}}\right)}. \quad (55)$$

The $n = 0$ Λ CDM Poisson equation may be cast in this form by taking $G_{\text{eff}}^{(1)} = G$ and $G_{\text{eff}}^{(0)} = G/(1 - \Lambda/(3H^2))$. For the case where n differs from 0, the function $G_{\text{eff}}^{(1)}$ will generically possess a time dependence during the background evolution. Therefore the link between the time evolution of the functions $G_{\text{eff}}^{(0)}$, δ_i , and Φ will differ from the case where acceleration is caused by a cosmological constant. We may thus expect the ISW effect to be of a nonstandard form. This is vividly illustrated in Fig. 10 where it can be seen that for a given set of (c_s, γ, M) , variation of n results in a considerable variation in the large-scale CMB temperature anisotropy. Also evident is a variation in the matter power spectrum amplitude with n , evident on all scales.

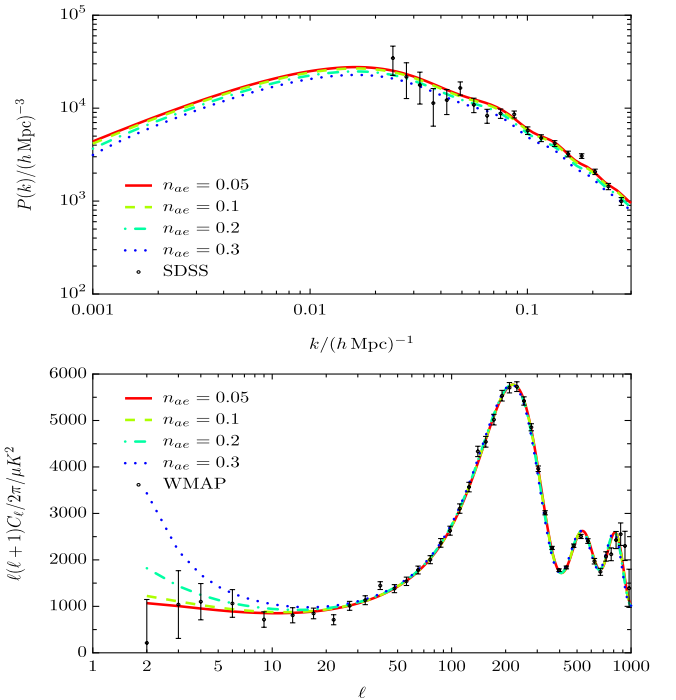


FIG. 10 (color online). Matter power (top) and CMB (bottom) power spectra for the various GEA dark energy models, with WMAP and SDSS constraints. The power law function’s exponent n is varied.

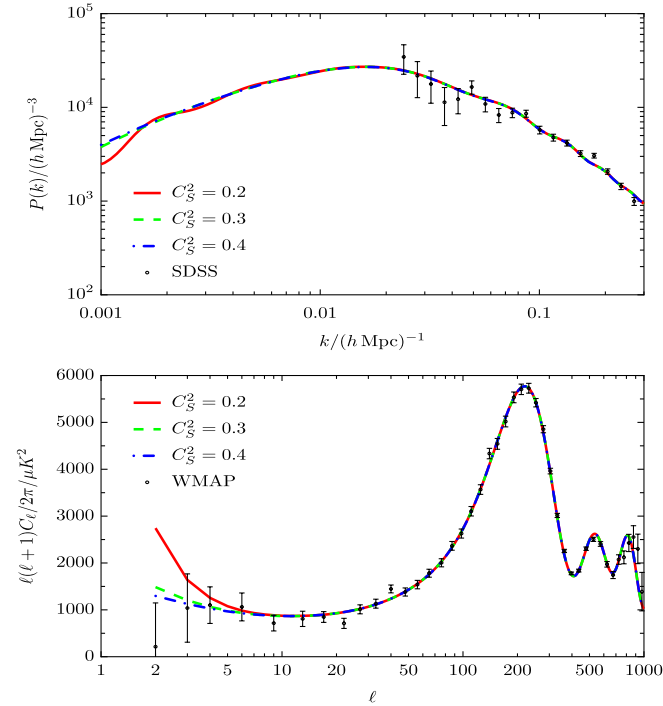


FIG. 11 (color online). Matter power (top) and CMB (bottom) power spectra for the various GEA dark energy models, with WMAP and SDSS constraints. The squared speed of sound of vector field perturbations is varied.

Variation of parameters other than n could also have a significant impact on the success of the models. Given the results of the previous section, it seems unlikely that any influence of the field ξ would tend to improve the models. This indeed seems to be the case. Figure 11 depicts various models where the function is varied C_S^2 for fixed values of the other parameters. In particular, n takes the value 0.1. The function C_S^2 is ultimately a measure of the ability of the field ξ to sustain any homogeneous growing behavior for $(k\tau) > 1$; higher values will tend to limit the effect of the vector field to larger and larger scales. The sets of parameters were chosen so such a growing solution indeed existed on superhorizon scales. The figure indicates that a growing ξ field will indeed have deleterious effects on scales where it is not suppressed. There is a significant ISW effect evident for the red curve CMB; the corresponding model must be considered as being at the edge of acceptability. The corresponding large-scale matter power spectrum exhibits exotic oscillations, entirely unrelated to the baryon-acoustic oscillations which occur on other scales. Their presence in this model is thus reflective of dynamics in the dark energy sector.

VI. CONCLUSIONS

A. Being dark matter is hard

Generalized Einstein-aether can, with different parameter choices, resemble dark matter in some important ways

but never all of them at once. Specifically, the new degrees of freedom introduced by the model may conspire to identically replicate one or more but not all of the following properties of cold dark matter.

1. Background dynamics

To accomplish identical background dynamics to cold dark matter, one must introduce considerable fine-tuning into the function F of the theory. Specifically the function must change form on either side of (dark) matter-radiation equality. The parameter we tune to make this happen is in the action itself, unlike the usual case where we simply alter the abundance Ω_c . Changes to fit cosmological observations can therefore have a larger impact on the small-scale behavior of gravity.

2. The speed of sound

If the speed of sound is too high then structure cannot form on small enough scales. In this model we may reduce the sound speed to be close to zero, at the cost of one of our parameters. When designing new gravity theories this is perhaps the easiest structure formation constraint to investigate, and it should be examined to see if it conflicts with other constraints needed to make the theory useful—for an example, see [48].

3. Growth rate of “overdensity”

Theories of modified gravity designed to replace dark matter must necessarily have growing modes of fluctuation in at least one of the new degrees of freedom they introduce, in order to sufficiently source gravitational collapse and structure formation on scales within their own sound horizon. There remains some flexibility in the perturbation growth rate, since the bias on the galaxy power spectrum measurements is a free factor. As measurements of weak lensing (which samples gravitation directly) and semianalytic models (which predict bias) improve this freedom will be reduced.

4. Absence of anisotropic stress and contribution the cosmological Poisson equation

A sufficiently small anisotropic stress associated with the vector field may be implemented by fine-tuning the parameter c_{13} to be very small [see Eq. (32)]. However, as was discussed in Sec. IV B, even this will tend to come at the expense of other desired behavior of the field. An appreciable time variation of the anisotropic stress over the time from last scattering to today can result in a very poor fit to the low- l CMB C_l . This problem is likely to be common in theories of modified gravity.

As we have seen, it is a combination of time variation of the anisotropic stress and time variation of the field Φ via the vector field’s effect on the cosmological Poisson equation that contribute to the ISW effect. Though both effects

are absent in the cold dark matter case, one may imagine both effects being present in the vector field model but being of equal and opposite sign. As with the case of the isolated anisotropic shear contribution, it seems this is not possible while maintaining the other constraints like the existence of a growing mode.

5. Effective minimal coupling to the gravitational field

Even if the vector field growing mode gives an appropriate (dark matter-like) contribution to the Poisson equation, the link between the overdensity and the corresponding Φ can differ from the cold dark matter case. This difference can come from curvature terms in the vector field stress-energy tensor, and its main consequence is a time-dependent rescaling of Newton's constant G . Thus Φ may gain a time dependence during the matter era even when there is a completely standard dark matter contribution to the Poisson equation. The converse may also be possible: the time dependence of an incorrect Poisson contribution could be counteracted by a time dependence of the effective G .

B. Being dark energy is easy

As a model for dark energy the generalized Einstein-aether theory is more successful: we have obtained constraints on its parameters and found that it generates spectra that fit the data across a reasonable range of its parameter space. It is clear that modified gravity approaches to explaining late-time acceleration are viable and can provide motivated explanations for dark energy (though this theory retains the coincidence problem in the guise of the parameter M).

1. Closeness to Λ

The most interesting constraint on this branch of the theory is on the parameter n_{ae} and is shown in Fig. 6. In some sense this parameter describes how closely the theory mimics Λ (which has $n_{\text{ae}} = 0$). The fact that this parameter is rather free, $n_{\text{ae}} < 0.126$ (95% CL), is consistent with the fact that a wide variety of other theories can also explain dark energy: present structure formation data are not very informative about the nature of dark energy, and deeper require expansion probes like baryon-acoustic oscillation and supernovae.

2. Sound speed

The other notable constraint on Einstein-aether dark energy, which may extend to other modified gravity approaches, is the limits on the sound speed, illustrated in Fig. 7. As shown in Fig. 11, an incorrect sound speed can lead to large-scale oscillations by modifying the other parameters of the theory and permitting a growing mode excitation at late time.

3. Other constraints

The other constraints on the theory (which are easily fulfilled by choosing the c parameters) come from ensuring that no growing mode can disrupt the power spectra, that the acceleration is close to the Λ CDM value, and that the value of effective G remains positive at all times.

C. Future issues for modified gravity and structure formation

Because Λ CDM is such a good fit to current cosmological data, modified gravity will never be favored in a model comparison exercise using only current data about linear structure. It is only in combination with physics on galactic and smaller scales that it can be persuasive. This work highlights a few issues for future model-building in this vein.

The generalized Einstein-aether model is a member of a class of models in which the scale $M \sim H_0$ associated with dark energy is visible to dark matter. Many of the issues raised here will be relevant to any such models which try to use a dark matter scale consistent with small-scale modifications to gravity.

A combination of probes sensitive only to the background (like type 1A supernovae) and to the behavior of cosmological perturbations is needed to fully constrain these theories. For example, a value $n_{\text{ae}} = 0.3$ has $w(z) \sim 1$ at low redshift but is ruled out by our constraints. Similarly there are models with $n_{\text{ae}} \sim 0.75$ that provide reasonable spectra, but they are ruled out by $w(z)$ constraints.

There are, of course, a number of extensions to the theory and features of it that could be changed; we could, for example, allow M to vary dynamically, or add more terms to the kinetic component F in Eq. (19). There are also myriad possibilities in changing more general aspects of the primordial conditions or cosmological parameters: what happens if we add tensors? Can we include an isocurvature mode? Would massive neutrinos help? Or curvature? This leads us to a key caveat that applies to this and all similar work constraining new physics with linear structure: a simple constraint from the data alone is worthless, since any of the numerous other parameters we could change might conspire to counteract whatever problem it solves. We need a physical explanation of a constraint's origin to understand whether it is robust to the cosmologist's tinkering.

ACKNOWLEDGMENTS

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APPENDIX A: THE MOND REGIME $F(K)$ CANNOT YIELD LATE-TIME ACCELERATION

The MOND branch of F applies in the regime $0 \leq K \ll 1$; outside this range MOND does not prescribe its form. Inside that range the MOND value is

$$c_1 F = -2K + \frac{2}{3}K^{3/2} \quad (0 \leq K \ll 1). \quad (\text{A1})$$

The weak field limit of the Einstein-aether theory is

$$K_{\text{WF}}^{\text{ae}} = -c_1 \frac{(\nabla\phi)^2}{M^2}, \quad (\text{A2})$$

where ϕ is the conformal Newtonian potential. The weak field MOND limit is

$$K_{\text{WF}}^{\text{MOND}} = \frac{(\nabla\phi)^2}{a_0^2}. \quad (\text{A3})$$

Clearly we can equate these by setting $M^2 = -c_1 a_0^2$. Using this relation with Eq. (14) gives us this value for the presentday cosmological K :

$$K_{\text{FRW}}(t_0) = -3 \frac{\alpha}{c_1} \frac{H_0^2}{a_0^2}. \quad (\text{A4})$$

There are now two cases: $\alpha/c_1 < 0$ and $\alpha/c_1 > 0$. In either case we need the measured values of H_0 and a_0 ; they are suggestively similar:

$$H_0 \equiv \kappa a_0 \quad \text{where } \kappa \approx 6. \quad (\text{A5})$$

If $\alpha/c_1 < 0$ then $K > 0$ and we are directly in the MOND regime. Then we obtain the modified Friedmann equation

$$H^2 \left(1 + \frac{\alpha}{c_1} + \frac{3}{2} \kappa^2 \left(\frac{H}{H_0} \right) \left(\frac{-3\alpha}{c_1} \right)^{3/2} \right) = 8\pi G\rho. \quad (\text{A6})$$

A self-accelerating solution to this is only possible if the quantity in brackets is positive definite, so that $H \rightarrow \text{const}$ as $\rho \rightarrow 0$. This could only happen if $\alpha/c_1 < -1$, but that would violate our requirements that $K \ll 1$.

In the other case $\alpha/c_1 > 0$ we must extrapolate the MOND form of F in Eq. (A2) to $K < 0$. To make the extension continuous across the $K = 0$, we should set $F(-K) = -F(K)$, so that the MOND form becomes

$$c_1 F = -2K - \frac{2}{3}(-K)^{3/2} \quad (0 \leq -K \ll 1). \quad (\text{A7})$$

The Friedmann equation then becomes

$$H^2 \left(1 + \frac{\alpha}{c_1} - \frac{3}{2} \kappa^2 \left(\frac{H}{H_0} \right) \left(\frac{3\alpha}{c_1} \right)^{3/2} \right) = 8\pi G\rho, \quad (\text{A8})$$

which does have an accelerating solution. Unfortunately,

solving this equation at the present day for reasonable values of Ω_m shows that it requires $K \sim 2$, which again violates our requirement that $|K| \ll 1$. Having exhausted our other options we are forced to require $K \geq 1$, outside the true MOND regime. This is consistent with another separate analysis of the solutions of (16) [49]. It seems likely that if the vector field is responsible for the late-time acceleration then it is a result of behavior of the function away from the MONDian limit. There is still though a role for the near numerical coincidence of a_0 and H_0 —it lessens the fine-tuning of the other parameters in F in order for the acceleration to happen at suitably late times.

APPENDIX B: VECTOR AND TENSOR MODES

In this appendix, we provide the perturbed equations of motion for the vector and tensor modes of the various fields appearing in these models, namely, the metric, the vector field, and the Lagrange multiplier. The latter, being scalar, only have a spin-0 mode. The other two fields, g_{ij} and A^i , are perturbed as

$$\begin{aligned} ds^2 &= -a^2 d\tau^2 + a^2 B_i d\tau dx^i + a^2 (\gamma_{ij} + h_{ij}) dx^i dx^j \\ h_{ij} &= 2\partial_{(i} E_{j)} + 2E_{ij} \quad A^\mu = \left(\frac{1}{a}, \frac{A^i}{a} \right), \\ A_\mu &= (-a, aV_i), \end{aligned} \quad (\text{B1})$$

where the different fields introduced satisfy

$$\partial_i B^i = 0 = \partial^j E_{ij} = \partial_i E^i = E_i^i \quad \text{and} \quad \partial_i A^i = 0. \quad (\text{B2})$$

We also introduce the useful quantity $V^i = A^i + B^i$ and remind that all the latin indices on the perturbed fields are raised and lowered thanks to the Kronecker flat metric δ_{ij} .

In the following we use the unperturbed results

$$\begin{aligned} K &= 3M^{-2} \mathcal{H}^2 \alpha, \quad J^0_0 = 6c_2 \mathcal{H}, \quad J^i_j = 2\mathcal{H} \alpha \delta_j^i, \\ I^{\sigma}_{00} &= 6c_2 \mathcal{H} \delta_0^\sigma, \quad I^{\sigma}_{0i} = 0, \quad I^{\sigma}_{ij} = -2\mathcal{H} \alpha \delta_0^\sigma \delta_{ij}. \end{aligned} \quad (\text{B3})$$

The Einstein equation (5) without matter introduces the stress tensor

$$\begin{aligned} \tilde{T}_{\alpha\beta} &= \frac{1}{2} \nabla_\sigma [I^{\sigma}_{\alpha\beta}] + \hat{T}_{\alpha\beta} \\ \hat{T}_{\alpha\beta} &= -F_K Y_{(\alpha\beta)} + \frac{1}{2} M^2 g_{\alpha\beta} \mathcal{F} + \lambda A_\alpha A_\beta, \end{aligned}$$

where

$$I^{\sigma}_{\alpha\beta} = F_K [J_{(\alpha}{}^\sigma A_{\beta)} - J^\sigma{}_{(\alpha} A_{\beta)} - J_{(\alpha\beta)} A^\sigma]. \quad (\text{B4})$$

As intermediate results we have

$$\delta K^{\alpha\beta}{}_{\gamma\delta} = c_1 (\delta g^{\alpha\beta} g_{\gamma\delta} + g^{\alpha\beta} \delta g_{\gamma\delta}), \quad (\text{B5})$$

$$\delta(\nabla_\alpha A^\gamma) = \delta_i^\gamma \partial_\alpha \left[\frac{A^i}{a} \right] + \Gamma_{\alpha k}^\gamma \frac{A^k}{a} + \frac{\delta \Gamma_{\alpha 0}^\gamma}{a}, \quad (\text{B6})$$

$$\delta Y_{0i} = \delta Y_{i0} = c_1(\mathcal{H}V_i + V'_i), \quad (\text{B7})$$

$$\delta \hat{T}_{ij} = \frac{1}{2}M^2 F a^2 h_{ij}, \quad (\text{B8})$$

$$\delta \hat{T}_{0i} = \frac{1}{2}a^2 B_i M^2 F - \lambda a^2 V_i - F_K \delta Y_{(0i)},$$

$$\delta J^0_i = \frac{2\mathcal{H}c_3 V_i}{a} - \frac{2c_1 V'_i}{a}, \quad (\text{B9})$$

$$\delta J_0^i = -\delta J^0_i + \frac{2\mathcal{H}}{a}(c_1 + c_3)B^i,$$

$$\delta J_i^0 = -\delta J^j_0 \delta_{ij} + \frac{2\mathcal{H}}{a}(c_1 + c_3)B_i, \quad (\text{B10})$$

$$\delta J^i_0 = 2\frac{c_{13}B^i}{a}\mathcal{H} - 2\frac{c_1\mathcal{H}}{a}V^i + \frac{2c_3V'^i}{a}; \quad (\text{B11})$$

and

$$\begin{aligned} \delta J^i_j &= 2c_1 \left[\frac{\partial^i A_j}{a} + \frac{h'_j{}^i}{2a} + \frac{\partial^{[i} B_{j]}}{a} \right] \\ &+ 2c_3 \left[\frac{\partial_j A^i}{a} + \frac{h'_j{}^i}{2a} + \frac{\partial_{[j} B^{i]}}{a} \right]. \end{aligned} \quad (\text{B12})$$

We then obtain

$$\begin{aligned} \delta I_{0i}^\sigma &= F_K [2\mathcal{H}c_3 V_i - 2c_1 V'_i - 2\mathcal{H}\alpha B_i] \delta_0^\sigma \\ &+ 2F_K (c_1 - c_3) \partial^{[k} V_{i]} \end{aligned} \quad (\text{B13})$$

with $\alpha = c_1 + 3c_2 + c_3$, and

$$\begin{aligned} \delta I_{00}^i &= F_K [-2c_1 V'^i - 2c_1 \mathcal{H}V^i + 2\mathcal{H}c_3 V^i \\ &+ 2c_3 V'^i + 6c_2 \mathcal{H}A^i]. \end{aligned} \quad (\text{B14})$$

We also get

$$\delta I_{ij}^\sigma = -\delta_0^\sigma F_K \delta J_{(ij)}. \quad (\text{B15})$$

To obtain the perturbed Einstein equation, we then plug these results into the relations

$$\begin{aligned} \delta \nabla_\sigma I_{0i}^\sigma &= \partial_0(\delta I_{0i}^0) + \partial_k(\delta I_{0i}^k) + 2\mathcal{H}\delta I_{0i}^0 \\ &- (B^j + \mathcal{H}B^j)I_{ji}^0 - 6\mathcal{H}^2 c_2 B_i F_K - \Gamma_{ji}^0 \delta I_{00}^i, \end{aligned}$$

$$\begin{aligned} \delta \nabla_\sigma I_{ij}^\sigma &= \partial_0(\delta I_{ij}^0) + 2\mathcal{H}\delta I_{ij}^0 - I_{kj}^0 \delta \Gamma_{0i}^k \\ &- \mathcal{H}\delta_{ki}\delta I_{0j}^k - \delta \Gamma_{0j}^k I_{ik}^0 - \mathcal{H}\delta_{kj}\delta I_{i0}^k. \end{aligned}$$

We decompose these equations into Fourier components:

$$\begin{aligned} X^i(t, \vec{x}) &= \sum_{\vec{k}} \sum_{m=0,1} X(t, \vec{k}) Y_{\vec{k}}^{i(\pm m)}, \\ T^{ij}(t, \vec{x}) &= \sum_{\vec{k}} \sum_{m=0,1,2} T(t, \vec{k}) Y_{\vec{k}}^{ij(\pm m)}, \end{aligned} \quad (\text{B16})$$

where the orthonormal modes

$$Y^{(0)}, \quad Y^{(\pm 1)}, \quad Y^{(\pm 2)} \quad (\text{B17})$$

are eigenmodes of the Laplace-Beltrami operator: $\Delta Y_{\mathbf{I}}^{(m)} = -k^2 Y_{\mathbf{I}}^{(m)}$, \mathbf{I} being an arbitrary set of Lorentz indices. For more information on these functions, see [17]. The perturbed vector field equation can then be written

$$\begin{aligned} 2\lambda a V &= F_{KK} K'(2\mathcal{H}c_3 V - 2c_1 V') - 2c_1 F_K V'' \\ &- 4F_K c_1 \mathcal{H}V' \\ &+ V \left(2\frac{a''}{a} c_3 F_K + \mathcal{H}^2 (2c_3 + 2c_1) F_K \right) \\ &- F_K (2c_1 k^2 A + (c_1 - c_3) k^2 B) + 3F_K c_2 k^2 E'. \end{aligned} \quad (\text{B18})$$

The spin-2 part of the ij Einstein equations gives

$$E'' + 2\mathcal{H}E' + \frac{k^2 E}{1 + F_K(c_1 + c_3)} + 2F_{KK} K'(c_1 + c_3)E' = 0, \quad (\text{B19})$$

while the spin-1 part of the 0_i Einstein equations gives the same equation as (B18) and the spin-1 component of the ij Einstein equation is, in space conventions,

$$\begin{aligned} 0 &= \partial_{(i} E'_{j)} + \frac{2(1 - 2\alpha)\mathcal{H} + 2c_{13}\frac{1}{a^2}(a^2 F_K)' + 4\alpha\mathcal{H}F_K}{1 + 2F_K c_{13}} \\ &\times \partial_{(i} E'_{j)} - \frac{\partial_{(i} B'_{j)} - c_{13}F_K \partial_{(i} A'_{j)}}{1 + 2F_K c_{13}} \\ &- \frac{2(2\mathcal{H}' + \mathcal{H}^2) + M^2 \mathcal{F} a^2 + 4\alpha(\mathcal{H}F_K)' + 8\mathcal{H}^2 \alpha \mathcal{F}}{1 + 2F_K c_{13}} \\ &\times \partial_{(i} E_{j)} - \frac{2\mathcal{H}\partial_{(i} B_{j)}}{1 + 2F_K c_{13}} + \frac{c_{13}F_{KK}K' + 2c_{13}\mathcal{H}F_K}{1 + 2F_K c_{13}} \partial_{(i} A_{j)}. \end{aligned} \quad (\text{B20})$$

In order to compare with the results of [17] let us consider the gauge $E_j = 0$. The spin-1 part of the ij Einstein equations (B20) implies

$$B_i = c_{13} F_K A_i. \quad (\text{B21})$$

After that we can rewrite (B18) as

$$\begin{aligned} A'' &= - \left(2\mathcal{H} + \frac{F_{KK}}{c_1 F_K} K' \right) A' - \frac{F_K}{2c_1} \\ &\times \frac{2c_1 + F_K(c_1^2 - c_3^2)}{1 + F_K c_{13}} k^2 A + \left[\mathcal{H} \frac{c_3 F_{KK}}{c_1 F_K} K' + \frac{a''}{a} \right. \\ &\times \left. \frac{c_3}{c_1} + \mathcal{H}^2 \frac{c_{13}}{c_1} - \frac{\lambda a}{F_K c_1} \right] A \end{aligned}$$

and obtaining the exact expression of λ in the case $F(K) = K$,

$$\lambda = 3c_{13} \frac{\mathcal{H}^2}{a} - 3c_2 \frac{a''}{a^2} + 6c_2 \frac{\mathcal{H}^2}{a}, \quad (\text{B22})$$

implies in this latter case

$$c_1(2\mathcal{H}A' + A'') = -\frac{1}{2} \left[c_1 - c_3 + \frac{c_{13}}{1 + c_{13}} \right] k^2 A - \left[2\alpha\mathcal{H}^2 - \alpha \frac{a''}{a} + c_1 \frac{a''}{a} \right] A. \quad (\text{B23})$$

It is interesting to find that in the limit where $\mathcal{F}(K) = K$ we recover results from [17] for (B19), (B22), and (B21).

A quick look at (B19) and (B22) implies the existence of two different speeds

$$C_T^2 \equiv \frac{1}{1 + F_K c_{13}}, \quad C_V^2 \equiv \frac{F_K}{2c_1} \frac{2c_1 + F_K(c_1^2 - c_3^2)}{1 + F_K c_{13}}, \quad (\text{B24})$$

being the sound speeds, respectively, of the tensor perturbation E_{ij} and the vector perturbation A_i . These speeds must be positive, so that it implies constraints on the parameters of the theory and the function $F(K)$. In particular, $F_K \neq \text{const}$ will imply nontrivial constraints between the fields of the theory and the parameters c_i :

$$1 + F_K c_{13} > 0 \quad F_K \left[1 + F_K \frac{c_1^2 - c_3^2}{2c_1} \right] > 0. \quad (\text{B25})$$

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