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Gravitational instantons, self-duality, and geometric flows

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We discuss four-dimensional "spatially homogeneous" gravitational instantons. These are self-dual solutions of Euclidean vacuum Einstein equations. They are endowed with a product structure $\mathbb{R}\times\mathcal{M}_3$ leading to a foliation into three-dimensional subspaces evolving in Euclidean time. For a large class of homogeneous subspaces, the dynamics coincides with a geometric flow on the three-dimensional slice, driven by the Ricci tensor plus an $\mathfrak{so}(3)$ gauge connection. The flowing metric is related to the vielbein of the subspace, while the gauge field is inherited from the anti-self-dual component of the four-dimensional Levi-Civita connection.

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The aim of the present paper is to report on an intriguing and potentially important relationship between four-dimensional self-dual gravitational instantons and three-dimensional geometric flows. The framework is that of four-dimensional Euclidean geometry $\mathcal{M}_4$, which is topologically $\mathbb{R}\times\mathcal{M}_3$ with leaves $\mathcal{M}_3$ assumed to be homogeneous spaces of Bianchi type.

Gravitational instantons are classical solutions of general relativity and potential tools to handle quantum transitions; they are also important ingredients of strings and branes, entering in various symmetry-breaking schemes. Many solutions of Einstein's equations were made available in the late 1970's (see e.g. [1–5]); however, no criterion has ever been presented that allows one to foresee, according to an elegant, unified and comprehensive pattern, the expected classes of Bianchi solutions. In this paper we present such a criterion, with possible applications to more general situations.

On the other side, geometric flows of three-dimensional homogeneous spaces are interesting in their own right; they turned out to play a role in Hamilton's program for proving Poincaré's and Thurston's conjectures. A relevant question is to ask whether and how this flow behavior of one-parameter families of three-dimensional spaces is related to the Euclidean-time evolution inside a gravitational instanton, where the homogeneous spaces appear as the leaves of the foliation. This question is motivated by several facts.

First, Ricci-flow equations are renormalization-group equations for two-dimensional sigma models, with $t\propto -\log\mu$ the renormalization-group time [6–9]. Setting a relation between the latter and the Euclidean time of a gravitational instanton would be one more indication in favor of the dynamical generation of time in string theory—similar in spirit to the role of the Liouville field in noncritical strings. Second, it was noticed, but overlooked as a coincidence in [10,11], that some Ricci-flow interpretation may exist for a particular class of Bianchi IX metrics. Geometric-flow first-order equations also emerge as classical equations of motion in modified Einstein gravity [12] as a consequence of a Bogomol'nyi-Prasad-Sommerfield–like condition (the detailed balance). It is thus legitimate to ask whether a self-duality condition could have a similar effect, on more general grounds, under appropriate homogeneity and foliation assumptions. Lastly, the governing of gravitational behavior by a first order equation is reminiscent of holographic situations. Along the lines of [13], we hope that the first order flow equations can ultimately be used to reconstruct the bulk fields from the boundary data.

The achievements we will exhibit are twofold. On the one hand, we show that real, nondegenerate, self-dual solutions exist only for unimodular Bianchi groups or for the one of type III, and are classified in terms of the homomorphisms of $\mathfrak{g}\rightarrow\mathfrak{so}(3)$, where $\mathfrak{g}$ is the real Lie algebra of the Bianchi group under consideration and $SO(3)$ the anti-self-dual factor of the group $SO(4)$. On the other hand, we observe that the self-duality requirement leads to first-order equations, which turn out to describe a geometric flow for three-dimensional Bianchi manifolds, driven by the Ricci tensor combined with a flat $SO(3)$ gauge field (tildes refer to three-dimensional tensors):

$$\frac{d\tilde{g}_{ij}}{dt}=-\tilde{R}_{ij}-\frac{1}{2}\,\mathrm{tr}(\tilde{A}_i\tilde{A}_j). \tag{1}$$

As already stated, we seek Euclidean four-dimensional spaces of the type $\mathcal{M}_4=\mathcal{M}_3\times\mathbb{R}$ with homogeneous spatial sections $\mathcal{M}_3$. The latter are assumed to be of Bianchi type: a three-dimensional group G acts simply transitively on the leaves, which are therefore endowed with the structure of a group manifold (hence we exclude H_3, $H_2\times S^1$ or $S^2\times S^1$). Unimodular groups are referred to as Bianchi A and consist of the Abelian three-dimensional translation group, the Heisenberg group, $E(1,1)$, $E(2)$, $SL(2,\mathbb{R})$, and $SU(2)$. The metric for $\mathcal{M}_4$ can always be of the form

$$ds^2 = N^2 dT^2 + g_{ij}\sigma^i\sigma^j, \tag{2}$$

where $g_{ij}(T)$ are functions to be determined and σ^i are G-invariant forms. It is convenient to introduce an orthonormal frame $\{\theta^a, a = 0, 1, 2, 3 = \{0, \alpha\}\}$,

$$ds^2 = \delta_{ab}\theta^a\theta^b, \tag{3}$$

by setting

$$\theta^0 = NdT, \qquad \theta^\alpha = \Theta^\alpha{}_j\sigma^j \quad \text{with} \quad g_{ij} = \delta_{\alpha\beta}\Theta^\alpha{}_i\Theta^\beta{}_j, \tag{4}$$

where $\alpha, \beta, \ldots$ label orthonormal space indices. We will make the convenient gauge choice $N = \Theta = \sqrt{\det g_{ij}}$, and later use another "time" t defined as $dt = \Theta dT$.

Euclidean solutions to vacuum Einstein equations can be obtained by imposing (anti-)self-duality of the Riemann curvature twoform. This is a well-studied topic, and for reasons that will become clear, we would first like to elaborate. Spin connection and curvature forms belong to the antisymmetric **6** representation of $SO(4)$. In four dimensions, this group of local frame rotations factorizes as $SO(3)_{\rm sd} \otimes SO(3)_{\rm asd}$, and the connection ω_{ab} and curvature $\mathcal{R}_{ab}$ $SO(4)$-valued forms can be reduced with respect to the $SO(3)_{({\rm a})sd}$ as $\mathbf{6} = (\mathbf{3}_{\rm sd}, \mathbf{3}_{\rm asd})$ [14]:

$$\Sigma_\alpha = \frac{1}{2}\left(\omega_{0\alpha} + \frac{\epsilon_{\alpha\beta\gamma}}{2}\omega^{\beta\gamma}\right), \qquad A_\alpha = \frac{1}{2}\left(\omega_{0\alpha} - \frac{\epsilon_{\alpha\beta\gamma}}{2}\omega^{\beta\gamma}\right) \tag{5}$$

for the connection, and similarly for the curvature which now reads

$$\mathcal{S}_\alpha = d\Sigma_\alpha - \epsilon_{\alpha\beta\gamma}\Sigma^\beta \wedge \Sigma^\gamma, \qquad \mathcal{A}_\alpha = dA_\alpha + \epsilon_{\alpha\beta\gamma}A^\beta \wedge A^\gamma. \tag{6}$$

The $\{\mathcal{S}_\alpha, \Sigma_\alpha\}$ are vectors of $SO(3)_{\rm sd}$ and singlets of $SO(3)_{\rm asd}$ and vice versa for $\{\mathcal{A}_\alpha, A_\alpha\}$.

Imposing that $\mathcal{S}_\alpha$ or $\mathcal{A}_\alpha$ be zero is sufficient to solve vacuum Einstein equations. We will focus here on the self-dual solutions; anti-self-dual solutions are obtained by $O(4)$ parity or time-reversal transformations.

First-order equations can be obtained by taking the spin connection A_α in (6) to be

$$A_\alpha = 0. \tag{7}$$

This first integral raises two questions: (i) does $A_\alpha = 0$ lead to *consistent* vacuum solutions, and (ii) is this unique? Concerning the second question, it is known that, barring global issues, if $\mathcal{A}_\alpha = 0$ one can always find an $SO(3)_{\rm asd}$ local transformation (see e.g. [2]) such that (7) holds in the rotated frame, which is no longer invariant though. This property opens a practical issue: if we insist on keeping the original invariant frame, we must allow for flat anti-self-dual parts in the connection. Listing all nonequivalent connections of this type will provide a classification of all possible spatially homogeneous self-dual instantons.

Both questions can be answered accurately. First, we can prove that Eq. (7) admits nondegenerate solutions, except at isolated points where the metric determinant may vanish, for the Bianchi A class and Bianchi III only. Second, we show that there are as many nonequivalent G-invariant connections A with vanishing anti-self-dual curvature $\mathcal{A}$ as homomorphisms of $\mathfrak{g} \to \mathfrak{so}(3)$. We will refer to them as *branches* of solutions, even though this distinction is to some extent bound to our requirement of invariant frame [16]. It will turn out that in every case there are two such branches. We define general $I_{\alpha i}$ such that $A_\alpha = \frac{1}{2}I_{\alpha i}\sigma^i$ and introduce this ansatz in $\mathcal{A}_\alpha = 0$ together with (2) and (4). The equations we obtain are [17]

$$\dot{\Theta}_{\alpha i} = \Theta_{\alpha j}[(n^{j\ell} - a_k\epsilon^{kj\ell})g_{\ell i} - \tfrac{1}{2}\delta^j_i n^{k\ell}g_{k\ell}] - \Theta I_{\alpha i} \tag{8}$$

(where the dot stands for a derivative with respect to T) plus a constraint on the constants of motion $I_{\alpha i}$,

$$I_{\alpha\ell}c^\ell{}_{jk} + \epsilon_{\alpha\beta\gamma}I^\beta{}_j I^\gamma{}_k = 0. \tag{9}$$

This constraint defines a homomorphism of $\mathfrak{g} \to \mathfrak{so}(3)$ and its solutions are classified in terms of these homomorphisms. By using appropriate transformations, one can bring the $I_{\alpha\ell}$ into a diagonal form with entries $\{\lambda_1, \lambda_2, \lambda_3\}$ taking the values 0 or 1.

To make contact with existing literature [5], we note that Eq. (9) can lead to imaginary solutions. These are related to homomorphisms of $\mathfrak{g}$ into real subalgebras of $\mathfrak{su}(2, \mathbb{C})$, which provide more freedom but *are not* genuine instantons. We summarize the various possibilities as follows:

Bianchi Class A.—The rank-zero homomorphism which maps $\mathfrak{g}$ to the null generator of $\mathfrak{so}(3)$ with $\lambda_i = 0$ is always available and leads to consistent solutions. There is always a second homomorphism (unique up to trivial algebra automorphisms), which is rank one in types I, II, VI_{-1}, and VII_0, where it maps one generator of $\mathfrak{g}$ onto one of $\mathfrak{so}(3)$ with a single nonvanishing λ_i; and rank three in type IX, where it is the isomorphism of $\mathfrak{g} \equiv \mathfrak{so}(3)$ to itself with all $\lambda_i = 1$. The case of VIII exhibits a rank-three homomorphism in $\mathbb{C}$—$\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -i$—and corresponds to a real solution in a space with signature $(-, -, +, +)$. The case VI_{-1} similarly admits a rank-one homomorphism in $\mathbb{C}$: $\lambda_1 = i$, $\lambda_2 = \lambda_3 = 0$. These cases turn out to be necessary in setting the advertised relation with the Ricci flow of three-dimensional Bianchi spaces.

Bianchi Class B.—Only rank-zero and rank-one homomorphisms are *a priori* possible. However, they generally lead to singular metrics, except for a special case in Bianchi III, which requires a nondiagonal metric.

We now focus on the Bianchi A class, where we may always assume a diagonal coframe without restriction:

$\Theta_{\alpha i} = \gamma_i \delta_{\alpha i}$. With this simplification, we can turn to the interpretation of the Euclidean-time evolution in the above gravitational instantons as a geometric flow of a family of three-dimensional homogeneous spaces. For concreteness we carry out first a well-studied case, that of Bianchi IX [4]. We take $I_{\alpha i} = (1 - \tilde{\lambda})\delta_{\alpha i}$, with $\tilde{\lambda} = 0$ (isomorphism) corresponding to the Taub-NUT branch and $\tilde{\lambda} = 1$ (trivial homomorphism) to the Eguchi-Hanson branch. The self-duality equations (8) read [(i, j, k) are circular permutations of (1, 2, 3)]

$$2\frac{\dot{\gamma}_i}{\gamma_i} = (\gamma_j - \gamma_k)^2 - \gamma_i^2 + 2\tilde{\lambda}\gamma_j\gamma_k. \tag{10}$$

For the Taub-NUT branch ($\tilde{\lambda} = 0$) the observation (already made in [10,11]) is that Eqs. (10) are *Ricci-flow equations* for three-dimensional Bianchi IX geometries

$$d\tilde{s}^2 = \tilde{g}_{ij}\sigma^i\sigma^j = \delta_{\alpha\beta}\tilde{\theta}^\alpha\tilde{\theta}^\beta, \tag{11}$$

which are also of the diagonal type: $\tilde{g}_{ij}(t) = \delta_{ij}\gamma_i(t)$.

For the Eguchi-Hanson branch ($\tilde{\lambda} = 1$), the flowing three-dimensional geometries are again Bianchi IX with (11). Examining the self-duality equations (10), we now find in addition to the Ricci tensor an $\mathfrak{so}(3)$ gauge field $\tilde{A}$ on the flowing three-spheres. It originates from the Levi-Civita anti-self-dual connection A and reads

$$\tilde{A} = \tilde{A}_i\sigma^i = -\tilde{\lambda}\delta_{\alpha i}T^\alpha\sigma^i, \tag{12}$$

where T^α are the generators of $\mathfrak{so}(3)$ in the adjoint, satisfying $\mathrm{tr}(T^\alpha T^\beta) = -2\delta^{\alpha\beta}$. This $\mathfrak{so}(3)$ gauge field vanishes for the Taub-NUT case but is nonzero for the Eguchi-Hanson case. In both cases, however, its field strength is zero. With this field, Eq. (10) is recast as announced in the beginning:

$$\frac{d\tilde{g}_{ij}}{dt} = -\tilde{R}_{ij} - \frac{1}{2}\mathrm{tr}(\tilde{A}_i\tilde{A}_j). \tag{13}$$

The flow equation (13) follows directly from (8) with an $\mathfrak{so}(3)$ gauge field

$$\tilde{A} = -\tilde{I}_{\alpha i}T^\alpha\sigma^i. \tag{14}$$

As a consequence of (9), the $\tilde{I}_{\alpha i}$'s are subject to

$$\tilde{I}_{\alpha\ell}c^\ell{}_{jk} + \epsilon_{\alpha\beta\gamma}\tilde{I}^\beta{}_j\tilde{I}^\gamma{}_k = 0, \tag{15}$$

which is a flatness condition for $\tilde{A}$: $\tilde{F} = d\tilde{A} + [\tilde{A}, \tilde{A}] \equiv 0$. The gauge field is a *flat background field*: it does not flow ($\dot{\tilde{A}} = 0$) but contributes to the flow of the metric.

The above developments set a correspondence between the time evolution in self-dual gravitational instantons foliated with homogeneous leaves and the flow (parametric in time) evolution of homogeneous Bianchi IX spaces, valid for branches labeled by flat $\mathfrak{so}(3)$ connections over G. We now proceed to show that this correspondence holds for all Bianchi A classes. We illustrate this correspondence explicitly for the remaining Bianchi I, II, VI_{-1}, VII_0, and VIII classes. Hence, the corresponding three-manifolds $\mathcal{M}_3$ are endowed with a metric (11), where

$$\tilde{g}_{ij}(t) = \delta_{ij}\tilde{\gamma}_i(t). \tag{16}$$

Note that the correspondence does not take the three-dimensional part of the four-dimensional metric to be equal to the three-dimensional metric. Similarly, in the diagonal ansatz, we take (14) as an $\mathfrak{so}(3)$ gauge field with

$$\tilde{I}_{\alpha i} = \tilde{\lambda}_i\delta_{\alpha i}, \tag{17}$$

where $\tilde{\lambda}_i$ are subject to the constraints (15), recast as

$$\tilde{\lambda}_i c^i{}_{jk} + \epsilon_{ijk}\tilde{\lambda}_j\tilde{\lambda}_k = 0 \tag{18}$$

(no summation on i, j, k). Consequently, the geometric-flow equations obtained from (13) can be written as

$$\frac{\dot{\tilde{\gamma}}_i}{\tilde{\gamma}_i} = -\sum_{j,k=1}^{3}\frac{1}{4}[(c^i{}_{jk})^2\tilde{\gamma}_i^2 - 2(c^j{}_{ki})^2\tilde{\gamma}_j^2 + 2c^j{}_{ki}c^k{}_{ij}\tilde{\gamma}_j\tilde{\gamma}_k] + \tilde{\lambda}_i^2\frac{\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3}{\tilde{\gamma}_i}, \tag{19}$$

where the dot stands for $\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\ d/dt$. Correspondingly, the self-duality equations (8) read

$$\frac{\dot{\gamma}_i}{\gamma_i} = \sum_{j,k=1}^{3}\frac{\epsilon_{ijk}}{2}\left[-\frac{c^i{}_{jk}}{2}\gamma_i^2 + \frac{1}{2}(c^j{}_{ki}\gamma_j^2 + c^k{}_{ij}\gamma_k^2)\right] + \lambda_i\frac{\gamma_1\gamma_2\gamma_3}{\gamma_i} \tag{20}$$

(as already quoted, here the dot stands for $\gamma_1\gamma_2\gamma_3\ d/dt$).

Each of the above equations has two branches. From the self-dual four-dimensional side, this is determined by each of the two nonequivalent homomorphisms of $\mathfrak{g} \to \mathfrak{so}(3)$, as a consequence of the flatness of the anti-self-dual Levi-Civita connection A. From the three-dimensional viewpoint, this corresponds to the two nonequivalent flat $\mathfrak{so}(3)$—G-invariant—connections $\tilde{A}$ over the group manifold G. This holds over the real numbers for Bianchi I, II, VII_0, and IX, whereas Bianchi VI_{-1} and VIII require us to pass to the complex i.e. change signature. In all Bianchi A spaces, the advertised correspondence holds as one-to-one for each class and each branch. It goes as follows: in cases I, II, VII_0, IX, we must set for the metric $\tilde{\gamma}_i = \gamma_i,\quad \forall\ i$, whereas there is a fine structure for the gauge field: $\tilde{\lambda}_i = \lambda_i\quad \forall\ i$ for I and II, and $\tilde{\lambda}_i = 1 - \lambda_i\quad \forall\ i$ for VII_0 and IX [18].

For VI_{-1} we find the correspondence $\{\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3\} = \{i\gamma_1, i\gamma_2, \gamma_3\}$ for the metric coefficients, while for the connection, the coefficients $\{\lambda_i\}$ and $\{\tilde{\lambda}_i\}$ are interchanged as $\{0, 0, 0\} \leftrightarrow \{i, 0, 0\}$. Similarly, for VIII we find the correspondence $\{\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3\} = \{\gamma_1, -i\gamma_2, -i\gamma_3\}$ with $\{\lambda_i\}$ and $\{\tilde{\lambda}_i\}$ interchanged as $\{0, 0, 0\} \leftrightarrow \{0, -i, -i\}$.

Because of their complex nature, the classes VI_{-1} and VIII are not interesting for gravitational instantons. They are nevertheless useful for setting the correspondence on universal grounds and may play a more physical role in the search of self-dual solutions in a four-dimensional setting with signature $(-, -, ++)$.

The correspondence described here, involving in three dimensions the ''square root'' of the four-dimensional metric, has a genuine intrinsic geometrical meaning, related to a remarkable scaling property of the Riemann tensor in three-dimensional homogeneous spaces [19]. It has, furthermore, some immediate consequences. For example, there is a potential application of the integrability properties of self-duality equations to tackle Ricci or related flows beyond the usual asymptotic analysis [20]. Another, more fundamental result is the appearance of a new and yet unravelled kind of flow, namely, Ricci flow in the presence of a gauge field. As it stands, this flow exhibits two intriguing features: the absence of evolution for $\tilde{A}$ and its flatness, both resulting from the flatness of the anti-self-dual four-dimensional Levi-Civita connection which in turn follows from the self-duality requirement.

More general flows with nonvanishing $\dot{\tilde{A}}$ and $\tilde{F}$ can be obtained by replacing the self-duality condition on the *Riemann* curvature with a milder one, still allowing a first-order time evolution without imposing that the anti-self-dual Levi-Civita connection be a pure gauge. This is possible by allowing for a *cosmological constant* in four dimensions. In this case, self-duality of the Riemann is traded for that of the Weyl tensor. The anti-self-dual part of the connection now explicitly depends on time, and the corresponding curvature is nonzero. This is illustrated in the celebrated solution of Fubini-Study for Bianchi IX. Translated in the three-dimensional side, the equation for the metric flow is still given by (13) but is now accompanied by a flow for $\tilde{A}$ and a constraint for $\tilde{F}$. The gauge field thus carries dynamics, which decouples when the cosmological constant is turned off.

As a conclusion we would like to make some final remarks and stress the role of each ingredient that we have used in setting the gravitational-instanton/geometric-flow correspondence. We worked in four dimensions, where the orthogonal group is factorized into two three-dimensional subgroups and all degrees of freedom are reduced as self-dual plus anti-self-dual. The foliation plus homogeneity assumption further introduces three-dimensional leaves and another three-dimensional group, G related to $SO(3)$ with nontrivial homomorphisms. Finally, the self-duality requirement effectively reduces the system to a three-dimensional one, whose dynamics turns out to be equivalent to a geometric flow on homogeneous three-manifolds endowed with an $\mathfrak{so}(3)$ gauge connection. Although the present scheme seems quite rigid, generalizations are possible in several ways. Self-duality can be directly implemented in seven dimensions and possibly generalized in higher dimensions in the sense of reducing degrees of freedom by trading second-order equations for first-order ones. Understanding the geometrical underpinnings of the correspondence, based on higher-dimensional flows, may give insight into such generalizations. Multi-instantons [15] provide another rich playground for extensions, although the absence of homogeneity may be an obstruction.

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[14] The splitting into self-dual and anti-self-dual parts of the $SO(4)$ group acting on the tangent space spanned by the orthonormal vierbein is adapted to the slicing of $\mathcal{M}_4$ into the homogeneous space and the orthogonal direction. This choice is mandatory for revealing the geometric-flow interpretation (1). Metric representations other than (2) exist, such as the one used in [15], which are convenient for Bianchi IX generalizations like multicenter instantons, but not suitable for our purposes. We thank the referee for an interesting exchange on this issue.
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[16] For Bianchi IX, in the Gibbons-Hawking coordinates [15], the distinction between the Taub-NUT and Eguchi-Hanson solutions is less sharp because of the choice made of self-dual connections.
[17] The structure constants of Bianchi groups are given in terms of n^{ij} and a^i: $c^k{}_{ij} = \epsilon_{ij\ell} n^{\ell k} + \delta^k_j a_i - \delta^k_i a_j$. Here, we focus on unimodular groups, for which $a^i = 0$ and n^{ij} is diagonal: I (0, 0, 0), II (1, 0, 0), VI_{-1} (0, 1, −1), VII_0 (1, 1, 0), VIII (−1, 1, 1), and IX (1, 1, 1).
[18] Whenever $\exists\, i \neq j \neq k$ such that $c^i{}_{jk} c^k{}_{ij} \neq 0$, that is to say for Bianchi VII_0 and IX, as well as VI_{-1} and VIII, the branches are crossed.
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