

Emergent universe from the Hořava-Lifshitz gravityPuxun Wu^{1,2,3} and Hongwei Yu^{2,1,3,*}¹*Center of Nonlinear Science and Department of Physics, Ningbo University, No.818 Fenghua Road, Ningbo, Zhejiang, 315211 China*²*Department of Physics and Institute of Physics, Hunan Normal University, Changsha, Hunan 410081, China*³*Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha, Hunan 410081, China*

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We study the stability of the Einstein static universe in the Hořava-Lifshitz (HL) gravity and a generalized version of it formulated by Sotiriou, Visser, and Weifurtner. We find that, for the HL cosmology, there exists a stable Einstein static state if the cosmological constant Λ is negative. The universe can stay at this stable state eternally and thus the big bang singularity can be avoided. However, in this case, the Universe can not exit to an inflationary era. For the Sotiriou, Visser, and Weifurtner HL cosmology, if the cosmic scale factor satisfies certain conditions initially, the Universe can stay at the stable state past eternally and may undergo a series of infinite, nonsingular oscillations. Once the parameter of the equation of state w approaches a critical value, the stable critical point coincides with the unstable one, and the Universe enters an inflationary era. Therefore, the big bang singularity can be avoided and a subsequent inflation can occur naturally.

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I. INTRODUCTION

In the standard cosmological model, the existence of a big bang singularity in the early Universe is still an open problem. In order to resolve this problem, Ellis *et al.* proposed, in the context of general relativity, a scenario, called an emergent universe [1,2]. In this scenario, the space curvature is positive and the Universe stays, past eternally, in an Einstein static state and then evolves to a subsequent inflationary phase. So, the Universe originates from an Einstein static state, rather than from a big bang singularity. It is also worth noting that an Einstein static state as the initial state of the Universe is also favored by the entropy considerations [3]. However, the Einstein static universe in the classical general relativity is unstable, which means that it is extremely difficult for the Universe to remain in such an initial static state in a long time due to the existence of perturbations, such as the quantum fluctuations. Therefore, the original emergent model does not seem to resolve the big bang singularity problem successfully as expected.

Since in the early epoch, the Universe is presumably under extreme physical conditions, new effects, such as those stemming from quantization of gravity, or a modification of general relativity or even other new physics, may become important. As a result, the stability of the Einstein static state has been examined in various cases [4–16]. For example, the emergent scenario within the frameworks of loop quantum gravity and braneworld cosmology have been discussed in Refs. [5–8], where it was found that a successful model can be obtained, while the stability in the presence of vacuum energy corresponding to conformally

invariant fields has been studied and a nonsingular emergent cosmological model was reconstructed [4]. In $f(R)$ gravity, it was found that the Einstein static state is stable under homogeneous perturbations [9], but this stability is broken by adding inhomogeneous perturbations [10]. In $f(G)$ gravity, the stability of the Einstein static state against homogeneous perturbations has been analyzed in Ref. [11], where G is the Gauss-Bonnet term. In addition, Barrow *et al.* [12] found, with the covariant techniques, that the Einstein static state is stable for small inhomogeneous vector and tensor perturbations, as well as for adiabatic scalar density inhomogeneities with $c_s^2 > 0.2$.

Recently, motivated by the Lifshitz theory in solid state physics, Hořava proposed a power-counting renormalizable theory of gravity, called Hořava-Lifshitz (HL) gravity [17]. In the ultraviolet (UV) limit, HL has a Lifshitz-like anisotropic scaling between space and time characterized by the dynamical critical exponent $z = 3$ and thus breaks the Lorentz invariance, while in the infrared (IR), it flows to $z = 1$. So, it is expected to reduce to the classical general relativity gravity theory in the low energy limit. Applying the HL gravity to cosmology, it has been found, in a nonflat universe, that the Friedmann equation is modified by a $\frac{1}{a^4}$ term [18–20], where a is the scale factor. The cosmological perturbations with the HL gravity were studied in Refs. [21–30], and the results showed that a scale invariant superhorizon curvature perturbation could be produced without inflation. In the original HL gravity, Hořava assumed two conditions: detailed balance and projectability. More recently, Sotiriou, Visser, and Weifurtner (SVW) [31] suggested to build a general HL theory with projectability but without detailed-balance conditions. For a spatially curved Friedmann-Robertson-Walker universe, the SVW generalization gives an extra $\frac{1}{6}$ correction term and

*Corresponding author.

modifies the coefficient of the $\frac{1}{a^4}$ term in the Friedmann equation as compared to the HL theory. Let us note here that some other issues in HL gravity have been dealt with in Refs. [32–35].

In this paper, we will discuss the stability of the Einstein static universe in the contexts of HL gravity and SVW HL theory, respectively. In the following section, we briefly review the HL and SVW HL cosmology. In Sec. III, we analyze the Einstein static solutions and discuss the stability of these solutions. Finally, in Sec. IV, we present our main conclusions.

II. THE HORÍAVA-LIFSHITZ COSMOLOGY

In HL gravity, it is convenient to use the Arnowitt-Deser-Misner decomposition of the metric

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1)$$

where g_{ij} is the 3-dimensional spatial metric, N is the lapse function, N^i is the shift vector, and the coordinates scale as $t \rightarrow \ell^2 t$, $x^i \rightarrow \ell x^i$. In this paper, we only consider the $z = 3$ case. Next, we turn our attention to the implications of HL gravity in cosmology.

A. The HL cosmology

The action of HL gravity consists of kinetic and potential terms. The former is given by

$$S_k = \frac{2}{\kappa^2} \int dt d^3x \sqrt{g} N (K_{ij} K^{ij} - \lambda K^2), \quad (2)$$

where $K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$ is the extrinsic curvature, $K = g^{ij} K_{ij}$, $K^{ij} = g^{ik} g^{jl} K_{kl}$, and λ is a dimensionless parameter. When $\lambda = 1$, one recovers the kinetic part of the 4-dimensional Einstein-Hilbert action. With the detailed-balance condition, the potential term has the form

$$S_V = \int dt d^3x \sqrt{g} N \left(\beta C_{ij} C^{ij} + \gamma \frac{\epsilon^{ijk}}{\sqrt{g}} R_{il} \nabla_j R_k^l + \zeta R_{ij} R^{ij} + \eta R^2 + \xi R + \sigma \right). \quad (3)$$

Here R_{ij} is the three-dimensional spatial curvature tensor, $R = g^{ij} R_{ij}$, ϵ^{ijk} is the antisymmetric tensor with $\epsilon^{123} = 1$, and $C^{ij} = \frac{\epsilon^{ik}}{\sqrt{g}} \nabla_k (R_l^j - \frac{1}{4} \delta_l^j R)$ is the Cotton tensor. The constants β , γ , ζ , η , ξ , and σ are defined, respectively, as

$$\begin{aligned} \beta &= -\frac{\kappa^2}{2\omega^4}, & \gamma &= \frac{\kappa^2 \mu}{2\omega^2}, & \zeta &= -\frac{\kappa^2 \mu^2}{8}, \\ \eta &= \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)}, & \xi &= \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \Lambda, \\ \sigma &= -\frac{3\kappa^2 \mu^2}{8(1 - 3\lambda)} \Lambda^2, \end{aligned} \quad (4)$$

where Λ is the cosmological constant, and μ and ω are two coupling constants. In this case, the emergent speed of light

becomes

$$c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda}{1 - 3\lambda}}. \quad (5)$$

So, for the case where $3\lambda - 1 > 0$, a negative Λ is required in order to guarantee that the speed of light is real. Let us note that a positive Λ can be obtained by making an analytical continuation for parameters μ and ω^2 by $\mu \rightarrow i\mu$ and $\omega^2 \rightarrow -i\omega^2$ [18]. In addition, it was found in Ref. [36] that a negative cosmological constant may disappear in the different geometries, plane symmetric spacetimes, for example.

For a homogeneous and isotropic universe described by the metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d^2\Omega \right), \quad (6)$$

the Friedmann equation in HL gravity can be expressed as

$$\frac{6}{\kappa^2} (3\lambda - 1) H^2 = \rho - \sigma - \frac{6k\xi}{a^2} - \frac{12k^2(\zeta + 3\eta)}{a^4}. \quad (7)$$

Here $k = 0, \pm 1$, and ρ is the energy density of a perfect fluid in the Universe, which satisfies the conservation equation

$$\dot{\rho} + 3H\rho(1 + w) = 0, \quad (8)$$

where $w = p/\rho$ is the equation of state. In the present paper, a constant w is considered, which is a good approximation, since, as shown in Refs. [1,2], a plateau potential is required in the past-asymptotic limit in the emergent scenario. It is easy to see that, for a spatially flat universe, this Friedmann equation is the same as that in general relativity.

Now, we define two new constants

$$\beta = \frac{\kappa^2}{6(3\lambda - 1)}, \quad \alpha = \frac{\mu^2 \kappa^4}{48(3\lambda - 1)(1 - 3\lambda)}. \quad (9)$$

Apparently, β is positive if $3\lambda - 1 > 0$ and negative if $3\lambda - 1 < 0$, whereas α is always negative if $\mu^2 > 0$. Using these newly defined constants, the above Friedmann equation for a closed universe can be reexpressed as

$$H^2 = \beta\rho + 3\alpha\Lambda^2 - \frac{6\alpha\Lambda}{a^2} + \frac{3\alpha}{a^4}, \quad (10)$$

which be further written as

$$H^2 = \beta\rho + 3\alpha\Lambda^2 \left[1 - \frac{2}{\Lambda a^2} + \frac{1}{\Lambda^2 a^4} \right]. \quad (11)$$

Differentiating this equation with time and using the energy conservation equation, we have

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{1 + 3w}{2} H^2 + \alpha\Lambda^2 \left[\frac{9}{2} (1 + w) - 3(1 + 3w) \frac{1}{\Lambda a^2} \right. \\ &\quad \left. - \frac{3(1 - 3w)}{2} \frac{1}{\Lambda^2 a^4} \right]. \end{aligned} \quad (12)$$

B. The SVW HL cosmology

Sotiriou, Visser, and Weifurtner generalized the original HL theory by keeping the projectability but abandoning the detailed-balance conditions [31]. In this case, the modified Friedmann equation has the following form:

$$\left(1 - \frac{3}{2}\lambda\right)H^2 = \rho + \frac{\Lambda}{3} - \frac{1}{a^2} + \frac{2\beta_1}{a^4} + \frac{4\beta_2}{a^6}, \quad (13)$$

where λ , β_1 , and β_2 are coupling constants. Comparing the above equation with that in HL theory, one can see that the SVW generalization not only gives an extra correction term, but also modifies the coefficients of other terms. It is interesting to note that the $1/a^6$ correction term may also result from a radiation fluid within the UV regime [37]. So, the influence of such a radiation fluid within the UV regime on the stability of an Einstein static state can be regarded as a specific case of the analysis to be carried out next. Defining two dimensionless constants, $\bar{\beta}_1 = \beta_1\Lambda$ and $\bar{\beta}_2 = \beta_2\Lambda^2$, we find that the modified Friedmann equation in SVW HL theory becomes

$$\left(1 - \frac{3}{2}\lambda\right)H^2 = \rho + \Lambda\left[\frac{1}{3} - \frac{1}{\Lambda a^2} + \frac{2\bar{\beta}_1}{\Lambda^2 a^4} + \frac{4\bar{\beta}_2}{\Lambda^3 a^6}\right]. \quad (14)$$

Then, differentiating Eq. (14) with time, one has

$$2\left(1 - \frac{3}{2}\lambda\right)\frac{\dot{a}}{a} = -\left(1 - \frac{3}{2}\lambda\right)(1 + 3w)H^2 + \Lambda\left[(1 + w) - \frac{1 + 3w}{\Lambda a^2} + \frac{6w - 2}{\Lambda^2 a^4}\bar{\beta}_1 + \frac{12w - 12}{\Lambda^3 a^6}\bar{\beta}_2\right]. \quad (15)$$

III. THE EINSTEIN STATIC SOLUTION

The Einstein static solution is given by the conditions $\dot{a} = 0$ and $\ddot{a} = 0$, which imply

$$a = a_{Es}, \quad H(a_{Es}) = 0. \quad (16)$$

A. The HL cosmology

From Eq. (12), it is easy to see that the Einstein static solution satisfies the following equation:

$$\frac{9}{2}(1 + w) - 3(1 + 3w)\frac{1}{\Lambda a^2} - \frac{3(1 - 3w)}{2}\frac{1}{\Lambda^2 a^4} = 0. \quad (17)$$

Solving this equation, one obtains two critical points

$$\text{Point A: } \frac{1}{a_{Es}^2} = \Lambda, \quad (18)$$

and

$$\text{Point B: } \frac{1}{a_{Es}^2} = -\frac{3(1 + w)}{1 - 3w}\Lambda. \quad (19)$$

Substituting these critical points into Eq. (10) reveals that Point A corresponds to

$$\rho_A = 0, \quad (20)$$

and Point B to

$$\rho_B = -\frac{48\alpha}{\beta(1 - 3w)^2}\Lambda^2. \quad (21)$$

If Λ is negative, Point A is physically meaningless since $a_{Es}^2 = \frac{1}{\Lambda}$ is negative. The existence condition for Point B is $-\frac{3(1+w)}{1-3w} < 0$, which leads to $-1 < w < \frac{1}{3}$. For a positive Λ obtained from an analytical continuation for parameters μ and ω^2 by $\mu \rightarrow i\mu$ and $\omega^2 \rightarrow -i\omega^2$, which yields a positive α since μ^2 becomes negative after the analytical continuation [refer to Eq. (9)], it seems that Point A exists and so does Point B if w satisfies the condition $w < -1$ or $w > \frac{1}{3}$. However, in this case, the energy density corresponding to Point B, ρ_B , becomes a negative since α is positive, which is meaningless. So, in the case of a positive Λ , only point A exists physically.

In order to study the stability of these critical points, we introduce two variables

$$x_1 = a, \quad x_2 = \dot{a}. \quad (22)$$

They obey the following equations:

$$\dot{x}_1 = x_2, \quad (23)$$

$$\dot{x}_2 = -\frac{1 + 3w}{2}\frac{x_2^2}{x_1} + \frac{9}{2}\alpha\Lambda^2(1 + w)x_1 - 3\alpha\Lambda(1 + 3w)\frac{1}{x_1} - \frac{3\alpha(1 - 3w)}{2}\frac{1}{x_1^3}. \quad (24)$$

Linearizing the system described by the above two equations near critical points, one can obtain the eigenvalues of the coefficient matrix, which determine the stability of these critical points. After some calculations, we get the eigenvalue ϑ^2 :

$$\text{Point A: } \vartheta^2 = 12\alpha\Lambda^2, \quad (25)$$

$$\text{Point B: } \vartheta^2 = -12\alpha\Lambda\frac{1}{a_{Es}^2}. \quad (26)$$

If $\vartheta^2 < 0$, the corresponding equilibrium point is a center point, otherwise it is a saddle one. In order to analyze the stability of the critical points in detail, we now divide our discussions into two cases, i.e., $\Lambda < 0$ and $\Lambda > 0$.

I. $\Lambda < 0$

In this case, $\alpha < 0$ and Point A is physically meaningless since $a_{Es}^2 < 0$. Therefore, we only discuss the stability of

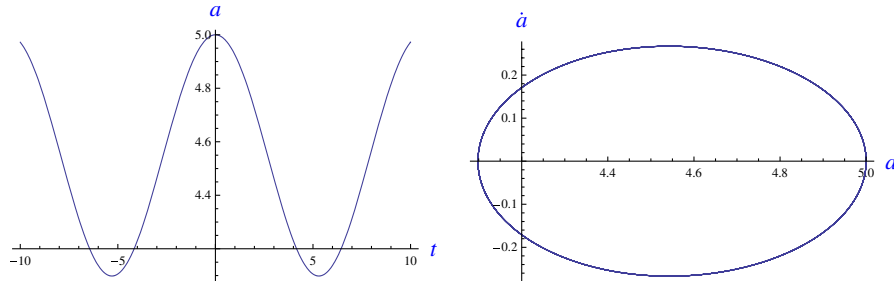


FIG. 1 (color online). The evolutionary curve of the scale factor with time (left) and the phase diagram in space (a, \dot{a}) (right) for the case $\Lambda < 0$ in a Planck unit and with $w = -0.90$, $\Lambda = -0.6$, $\alpha = -1$.

Point B , which exists under the condition $-1 < w < \frac{1}{3}$. From Eq. (26), it is easy to see that $\vartheta^2 < 0$ since $\alpha\Lambda > 0$. This means that point B is stable. In Fig. 1, we plot the evolution of the scale factor with time and the phase diagram in space (a, \dot{a}) . This figure shows that the Universe can stay at the stable state eternally and may undergo a series of infinite, nonsingular oscillations about this point. Thus, the initial big bang singularity can be avoided. By numerical calculation, however, one finds that, when w is larger than $\frac{1}{3}$ or less than -1 , the Universe may undergo an accelerating expansion. It therefore appears that the Universe may enter an inflationary phase from this stable point if the condition $-1 < w < \frac{1}{3}$ is violated. However, from Eq. (19), we find that, once w evolves through -1 or $\frac{1}{3}$, the scale factor a becomes ∞ or 0 . Therefore, in this case, the Universe is essentially stuck at the stable static state unless the scale factor becomes singular.

2. $\Lambda > 0$

A positive Λ can be obtained by making an analytical continuation of the parameters μ and ω^2 by $\mu \rightarrow i\mu$ and $\omega^2 \rightarrow i\omega^2$ [18]. The analytical continuation changes the sign of α and makes it a positive constant ($\alpha > 0$), since it contains a μ^2 factor. Now, critical point A can exist, but it is a saddle point since $\vartheta^2 > 0$. The critical point B is physically meaningless due to $\rho_B < 0$ as we have pointed out. Therefore, in this case, there is no stable Einstein static universe.

A summary of the existence and stability of Points A and B is given in Table I.

B. The SVW HL cosmology

For the SVW HL cosmology, we only consider the case of a positive cosmological constant ($\Lambda > 0$).¹ From Eq. (15), one can see that the critical points are determined by the following cubic equation:

¹The negative cosmological constant case can be treated similarly.

$$(1 + w) - \frac{1 + 3w}{\Lambda a^2} + \frac{6w - 2}{\Lambda^2 a^4} \bar{\beta}_1 + \frac{12w - 12}{\Lambda^3 a^6} \bar{\beta}_2 = 0. \quad (27)$$

When $\beta_2 = 0$, the above equation simplifies to a quadratic one, which is similar with that in the HL theory, but not identical, since coefficients are different. Thus, now we separately discuss two cases, $\beta_2 = 0$ and $\beta_2 \neq 0$.

I. $\beta_2 = 0$

In this case, Eq. (27) reduces to

$$(1 + w) - \frac{1 + 3w}{\Lambda a^2} + \frac{6w - 2}{\Lambda^2 a^4} \bar{\beta}_1 = 0. \quad (28)$$

Solving this equation, one can obtain two critical points:

$$\begin{aligned} \text{Point C: } \frac{1}{\Lambda a_{Es}^2} &= \frac{1}{4(3w - 1)\bar{\beta}_1} [1 + 3w \\ &\quad - \sqrt{(1 + 3w)^2 - 8\bar{\beta}_1(3w^2 + 2w - 1)}], \end{aligned} \quad (29)$$

$$\begin{aligned} \text{Point D: } \frac{1}{\Lambda a_{Es}^2} &= \frac{1}{4(3w - 1)\bar{\beta}_1} [1 + 3w \\ &\quad + \sqrt{(1 + 3w)^2 - 8\bar{\beta}_1(3w^2 + 2w - 1)}]. \end{aligned} \quad (30)$$

When $\beta_1 = \frac{3}{8}$, critical points C and D reduce to $\frac{1}{a_{Es}^2} = \frac{2}{3}\Lambda$ and $\frac{1}{a_{Es}^2} = \frac{2(1+w)}{3w-1}\Lambda$, respectively, which are the same as that in the HL cosmology [given in Eqs. (18) and (19)] after a redefinition of the cosmological constant as $\frac{2}{3}\Lambda$. It follows, from Eqs. (29) and (30), that when

TABLE I. Summary of the critical points and their stability in the case of the HL cosmology.

	$\Lambda < 0$		$\Lambda > 0$	
	Existence	Stability	Existence	Stability
Point A	meaningless		$\forall w$	unstable
Point B	$-1 < w < \frac{1}{3}$	stable	meaningless	

$$\bar{\beta}_1 = \frac{(1+3w)^2}{8(3w-1)(w+1)}, \quad (31)$$

two critical points, C and D , coincide, and thus, in this case, there is only one critical point,

$$\frac{1}{a_{Es}^2} = \frac{w+1}{4(3w+1)} \Lambda. \quad (32)$$

The energy density ρ , at critical points C and D , is

$$\rho(a_{Es}) = \left(-\frac{1}{3} + \frac{1}{\Lambda a_{Es}^2} - 2\bar{\beta}_1 \frac{1}{\Lambda^2 a_{Es}^4} \right) \Lambda, \quad (33)$$

where $\frac{1}{a_{Es}^2}$ is given by (29) or (30). In order to ensure these critical points exist with a physical meaning, it is required that $a_{Es}^2 > 0$ and $\rho(a_{Es}) \geq 0$. This yields a region of existence in the $(w, \bar{\beta}_1)$ parameter space,

(i) For Point C :

$$\begin{aligned} w \leq -\frac{1}{3}, \quad 0 \leq \bar{\beta}_1 \leq \frac{3}{8}, \quad -\frac{1}{3} \leq w \leq \frac{1}{3}, \\ \frac{(1+3w)^2}{8(3w^2+2w-1)} \leq \bar{\beta}_1 \leq \frac{3}{8}, \quad w > \frac{1}{3}, \quad \bar{\beta}_1 \leq \frac{3}{8}. \end{aligned} \quad (34)$$

(ii) For Point D :

$$-\frac{1}{3} \leq w < \frac{1}{3}, \quad \frac{(1+3w)^2}{8(3w^2+2w-1)} < \bar{\beta}_1 < 0. \quad (35)$$

In Fig. 2, we show the regions of existence in the $(w, \bar{\beta}_1)$ parameter space for both critical points C and D .

With the same method as that used in the HL theory, we find the eigenvalue of critical points C and D , which can be expressed as

$$\vartheta^2 = (1+w) + \frac{1+3w}{\Lambda a_{Es}^2} - \frac{3(6w-2)\bar{\beta}_1}{\Lambda^2 a_{Es}^4}. \quad (36)$$

There is no point in the existence region in the parameter

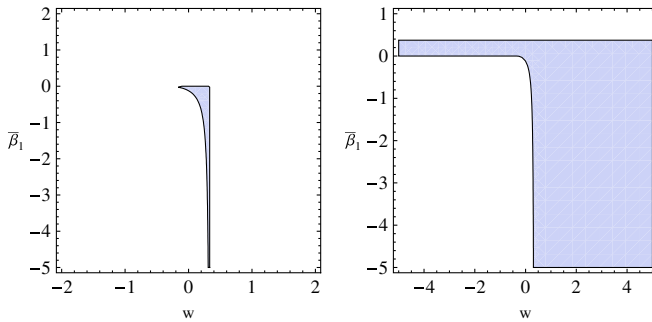


FIG. 2 (color online). Regions of existence in the $(w, \bar{\beta}_1)$ parameter space. The left panel shows the existence region for Point D , while the right panel for Point C .

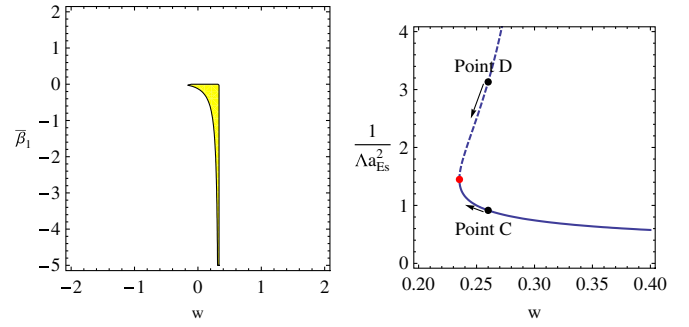


FIG. 3 (color online). Regions of stability in the $(w, \bar{\beta}_1)$ parameter space for Point D (left panel), and the evolution of Points C and D with the decreasing of w (right panel). In the right panel, $\Lambda = 0.6$ and $\bar{\beta}_1 = -1$.

$(w, \bar{\beta}_1)$ space for Point C which gives rise to a negative ϑ^2 . Hence Point C is always unstable. For critical Point D , the region of stability and existence is

$$-\frac{1}{3} \leq w < \frac{1}{3}, \quad \frac{(1+3w)^2}{8(3w^2+2w-1)} < \bar{\beta}_1 < 0, \quad (37)$$

which means that, if Point D exists, it is always stable. The left panel of Fig. 3 shows the region of parameters $(w, \bar{\beta}_1)$ corresponding to Point D . We summarize the existence and stability of points C and D in Table II.

Thus, if the cosmic scale factor satisfies Eq. (30) initially, and w and $\bar{\beta}_1$ lie in the region given in (37), the Universe can stay at a stable state past-eternally and undergo an infinite oscillation. If w evolves in such a way that w and $\bar{\beta}_1$ satisfy Eq. (31), then the stable critical point D coincides with the unstable one (Point C) and becomes unstable. As a result, the Universe goes out of the stable state and enters an inflationary phase naturally. A particular case which realizes a phase transition from a stable state to an inflation era is shown in Fig. 4. So the big rip singularity may be avoided successfully in this case.

2. $\beta_2 \neq 0$

Now, the Einstein static points satisfy Eq. (27), which is a cubic equation of a_{Es}^2 . The solution of Eq. (27) is determined by the following expression:

$$\Delta = B^2 - 4AC, \quad (38)$$

where $A = b^2 - 3ac$, $B = bc - 9ad$, and $C = c^2 - 3bd$

TABLE II. Summary of the critical points and their stability in the SVW HL cosmology with $\beta_2 = 0$.

	Existence	Stability
Point C	$w \leq -\frac{1}{3}, 0 \leq \bar{\beta}_1 \leq \frac{3}{8}$ $-\frac{1}{3} \leq w \leq \frac{1}{3}, \frac{(1+3w)^2}{8(3w^2+2w-1)} \leq \bar{\beta}_1 \leq \frac{3}{8}$ $w > \frac{1}{3}, \bar{\beta}_1 \leq \frac{3}{8}$	unstable
Point D	$-\frac{1}{3} \leq w < \frac{1}{3}, \frac{(1+3w)^2}{8(3w^2+2w-1)} < \bar{\beta}_1 < 0$	stable

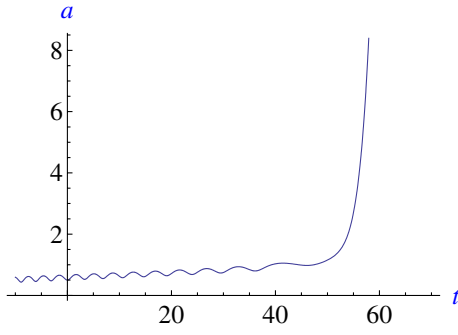


FIG. 4 (color online). The phase transition from a stable state to an inflation by assuming a slowly decreasing equation of state ($w(t) = 0.280 - 0.001t$) for the HL cosmology with the initial conditions $a(0) = 0.5$ and $\dot{a}(0) = 0$. The parameters are set as $\Lambda = 0.6$ and $\bar{\beta}_1 = -1$.

with $a \equiv 12(w - 1)\bar{\beta}_2$, $b \equiv 2(3w - 1)\bar{\beta}_1$, $c \equiv -(1 + 3w)$, and $d \equiv (1 + w)$.

- (i) $\Delta > 0$: there is only one real solution, which corresponds to only one critical point:

$$\text{Point } E: \frac{1}{\Lambda a_{Es}^2} = -\frac{1}{3a} [b + Y_1^{1/3} + Y_2^{1/3}], \quad (39)$$

where $Y_{1,2} = Ab + \frac{3a}{2}(-B \pm \sqrt{\Delta})$.

- (ii) $\Delta < 0$: there are three different real solutions. Thus, in this case, there are three critical points:

$$\text{Point } F: \frac{1}{\Lambda a_{Es}^2} = -\frac{1}{3a} [b + 2\sqrt{A} \cos(\theta/3)], \quad (40)$$

$$\text{Point } G: \frac{1}{\Lambda a_{Es}^2} = -\frac{1}{3a} (b - \sqrt{A} [\cos(\theta/3) + \sqrt{3} \sin(\theta/3)]), \quad (41)$$

$$\text{Point } H: \frac{1}{\Lambda a_{Es}^2} = -\frac{1}{3a} (b - \sqrt{A} [\cos(\theta/3) - \sqrt{3} \sin(\theta/3)]), \quad (42)$$

where $\theta = \arccos(T)$ and $T = \frac{1}{2A^{3/2}}(2Ab - 3aB)$.

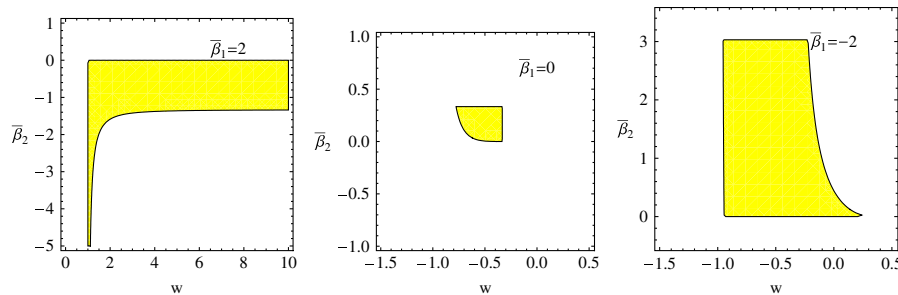


FIG. 5 (color online). Regions of existence for Point E in the $(w, \bar{\beta}_2)$ parameter space. The left, middle, and right panels show the case with $\bar{\beta}_1 = 2, 0$, and -2 , respectively.

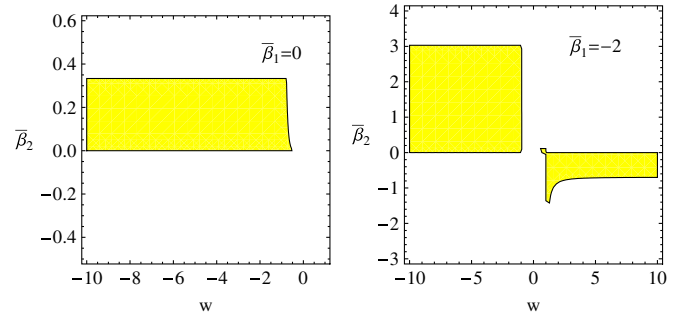


FIG. 6 (color online). Regions of existence for Point F in the $(w, \bar{\beta}_2)$ parameter space. The left and right panels show the case with $\bar{\beta}_1 = 0$ and -2 , respectively.

- (iii) $\Delta = 0$: Points G and H coincide since $\theta = 0$ and thus there are two critical points (Points F and G with $\theta = 0$).

At these critical points, the corresponding energy density ρ has the form

$$\rho(a_{Es}) = -\frac{1}{3} + \frac{1}{\Lambda a_{Es}^2} - 2\bar{\beta}_1 \frac{1}{\Lambda^2 a_{Es}^4} - 4\bar{\beta}_2 \frac{1}{\Lambda^3 a_{Es}^6}, \quad (43)$$

with $\frac{1}{a_{Es}^2}$ given in Eq. (39)–(41), or (42). The conditions for these points to be physically meaningful are that $\rho(a_{Es}) \geq 0$ and $a_{Es}^2 > 0$. Since it is not an easy task to obtain analytic solutions to the existence conditions, we resort to numerical calculations and find that in order to satisfy the existence conditions, for Points F , G , and H , $\bar{\beta}_1$ is required to be less than about $\frac{3}{4}$, while for Point E , there is no constraint on $\bar{\beta}_1$.

In order to show the regions of existence for Points E , F , G , and H in the $(w, \bar{\beta}_2)$ parameter space in detail, we chose $-2, 0$, and 2 as three typical values for $\bar{\beta}_1$. The results are shown graphically in Figs. 5–8. Figure 5 shows the regions of existence for Point E with $\bar{\beta}_1 = 2, 0$ and -2 , Fig. 6, the region of existence for Point F with $\bar{\beta}_1 = 0$ and -2 , and Figs. 7 and 8 the regions of existence for Points G and H with $\bar{\beta}_1 = 0$ and -2 .

In order to discuss the stability of these critical points, we need to calculate the eigenvalues, which can be expressed as

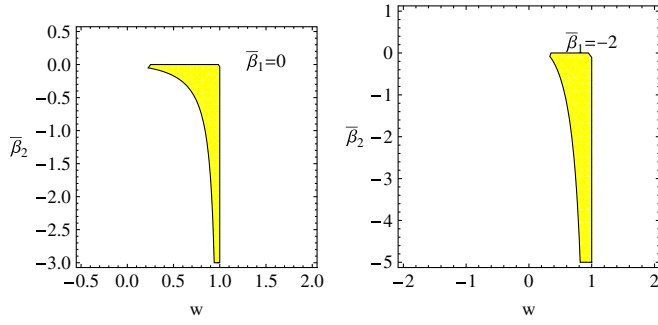


FIG. 7 (color online). Regions of existence for Point G in the $(w, \bar{\beta}_2)$ parameter space. The left and right panels show the case with $\bar{\beta}_1 = 0$ and -2 , respectively.

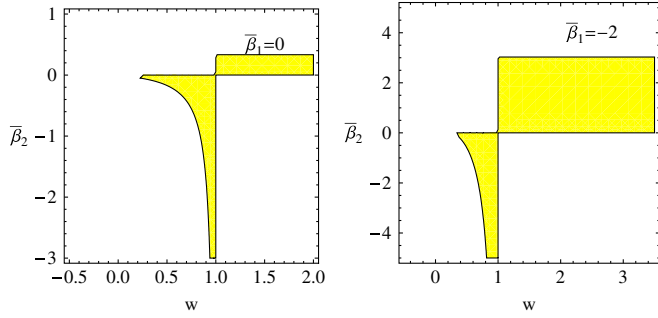


FIG. 8 (color online). Regions of existence for Point H in the $(w, \bar{\beta}_2)$ parameter space. The left and right panels show the case with $\bar{\beta}_1 = 0$ and -2 , respectively.

TABLE III. Summary of the critical points and their stability in the SVW HL cosmology with $\beta_2 \neq 0$.

	Critical point	Stability
$\Delta > 0$	Point E	stable if it exists
$\Delta < 0$	Point F	unstable
	Point G	stable if it exists
	Point H	unstable
$\Delta = 0$	Point G or H with $\theta = 0$	unstable
	Point F with $\theta = 0$	unstable

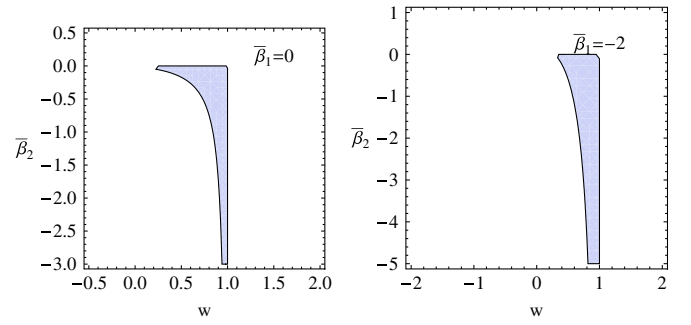


FIG. 10 (color online). Regions of stability for Point G in the $(w, \bar{\beta}_2)$ parameter space. The left and right panels show the case with $\bar{\beta}_1 = 0$ and -2 , respectively.

$$\vartheta^2 = -\frac{1}{3} + \frac{1}{\Lambda a_{Es}^2} - 2\bar{\beta}_1 \frac{1}{\Lambda^2 a_{Es}^4} - 4\bar{\beta}_2 \frac{1}{\Lambda^3 a_{Es}^6}, \quad (44)$$

at each point. We find that the critical points F and H are always unstable, while critical points E and G and, are always stable as long as they exist. A summary of these critical points and their stability is shown in Table III. Therefore, if the initial condition is such that the cosmic scale factor satisfies Eq. (39) or (41), the big bang singularity can be avoided. To illustrate the stability of Points E and G visually, we plot the regions of stability in $(w, \bar{\beta}_2)$ parameter space in Figs. 9 and 10, respectively.

Let us note that Points G and H move closer and closer as Δ changes from $\Delta < 0$ to 0, and coincide once $\Delta = 0$, which means that the stable Point G becomes an unstable one. Hence if the cosmic scale factor satisfies the condition given in Eq. (41) initially, the Universe can evolve slowly from a stable region to an unstable one with the decrease of w , as shown in the Fig. 11. Because of the fact that this unstable state will lead the Universe to enter an inflationary phase, therefore, in this case, the big bang singularity can be avoided and a subsequent inflation can appear naturally. Note, however, that if the cosmic scale factor satisfies the condition given in Eq. (39) initially, the Universe can stay at the Einstein static state eternally and thus avoid the big bang singularity, but it cannot evolve to an inflationary phase with the evolution of w .

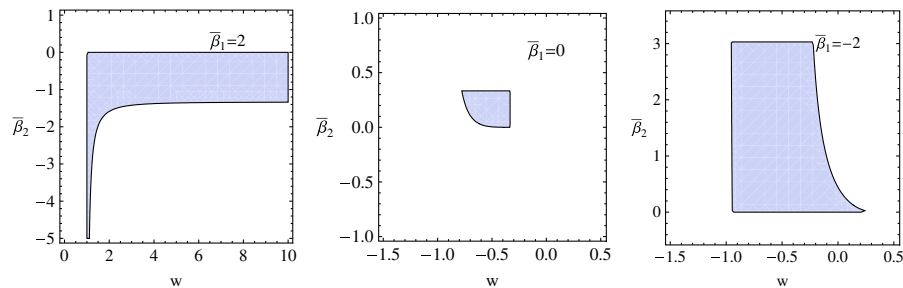


FIG. 9 (color online). Regions of stability for Point E in the $(w, \bar{\beta}_2)$ parameter space. The left, middle, and right panels show the case with $\bar{\beta}_1 = 2, 0$, and -2 , respectively.

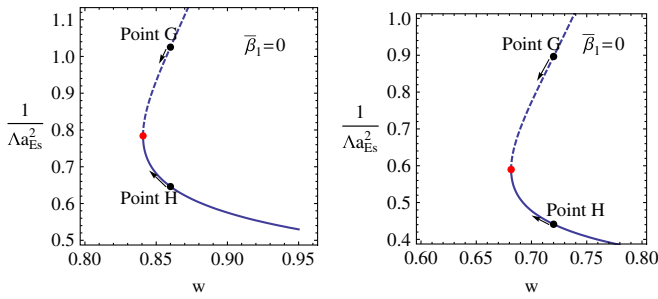


FIG. 11 (color online). The evolutions of the stable Point G and unstable one H with the decreasing of w .

IV. CONCLUSION

The Hořava-Lifshitz gravity is a power-counting renormalizable theory, which has an anisotropic scaling between space and time in the UV limit, and thus breaks the Lorentz invariance. Applying this theory to cosmology, one finds that the Friedmann equation for a nonflat universe is modified by a $\frac{1}{a^4}$ term. The SVW HL theory is a generalization of the original HL gravity by keeping the projectability condition but abandoning the detailed-balance one. This generalization introduces an extra $\frac{1}{a^6}$ correction term to the Friedmann equation and modifies the coefficient for the $\frac{1}{a^4}$ term as compared with the HL theory. In the present paper, we study the influence of these correction terms on the Einstein static state. In the case of HL cosmology, if the cosmological constant Λ is negative, we find that there exists a stable Einstein static state. The Universe can stay at this stable state eternally and thus the big bang singularity can be avoided. However, in this case, the Universe cannot exit to an inflationary era. So the big bang singularity problem cannot be solved successfully. By making an analytical continuation of the model parameters, a positive Λ can be obtained [18]. But, in this case, there is no stable Einstein static state.

For the SVW HL cosmology, when $\beta_2 = 0$, we find that there exists a stable critical point and an unstable one. If the cosmic scale satisfies the condition given in Eq. (30) initially, the Universe can stay at the Einstein static state past eternally. With the decrease of w , the stable point and the saddle one move closer and closer. Once w reaches a critical value, this stable critical point coincides with the saddle one and there is only one critical point, which is unstable. Thus the Universe can go out of the stable state and then enter an inflationary era. Therefore, the big bang singularity can be avoided and a subsequent inflation can occur naturally.

When $\beta_2 \neq 0$, our results show that if the cosmic scale factor satisfies the condition given in Eq. (41) and the equation of state w is larger than a critical value initially, the Universe can evolve from a stable region to an unstable one with the decrease of w . Therefore, in this case, the big bang singularity can also be avoided and an inflation can appear naturally. However, if the cosmic scale factor satisfies the condition given in Eq. (39) initially, although the big bang singularity can also be avoided, the Universe cannot evolve to an inflationary phase with the evolution of w .

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Note added.—While we are in the stage of revising the manuscript, the stability of Einstein static universe in a IR modified HL gravity is discussed in [38].

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- [1] G. F. R. Ellis and R. Maartens, *Classical Quantum Gravity* **21**, 223 (2004).
 - [2] G. F. R. Ellis, J. Murugan, and C. G. Tsagas, *Classical Quantum Gravity* **21**, 233 (2004).
 - [3] G. W. Gibbons, *Nucl. Phys.* **B292**, 784 (1987); **B310**, 636 (1988).
 - [4] S. Carneiro and R. Tavakol, *Phys. Rev. D* **80**, 043528 (2009).
 - [5] D. J. Mulryne, R. Tavakol, J. E. Lidsey, and G. F. R. Ellis, *Phys. Rev. D* **71**, 123512 (2005).
 - [6] L. Parisi, M. Bruni, R. Maartens, and K. Vandersloot, *Classical Quantum Gravity* **24**, 6243 (2007).
 - [7] P. Wu and H. Yu, *J. Cosmol. Astropart. Phys.* **05** (2009) 007.
 - [8] J. E. Lidsey and D. J. Mulryne, *Phys. Rev. D* **73**, 083508 (2006).
 - [9] C. G. Boehmer, L. Hollenstein, and F. S. N. Lobo, *Phys. Rev. D* **76**, 084005 (2007); N. Goheer, R. Goswami, and P. K. S. Dunsby, *Classical Quantum Gravity* **26**, 105003 (2009); S. del Campo, R. Herrera, and P. Labrana, *J. Cosmol. Astropart. Phys.* **11** (2007) 030; R. Goswami, N. Goheer, and P. K. S. Dunsby, *Phys. Rev. D* **78**, 044011 (2008); U. Debnath, *Classical Quantum Gravity* **25**, 205019 (2008); B. C. Paul and S. Ghose, *Gen. Relativ. Gravit.* **42**, 795 (2010).
 - [10] S. S. Seahra and C. G. Bohmer, *Phys. Rev. D* **79**, 064009 (2009).
 - [11] C. G. Boehmer and F. S. N. Lobo, *Phys. Rev. D* **79**, 067504 (2009).

- (2009).
- [12] J.D. Barrow, G. Ellis, R. Maartens, and C. Tsagas, *Classical Quantum Gravity* **20**, L155 (2003).
- [13] T. Clifton and J.D. Barrow, *Phys. Rev. D* **72**, 123003 (2005).
- [14] J.D. Barrow and C.G. Tsagas, *Classical Quantum Gravity* **26**, 195003 (2009).
- [15] C.G. Boehmer, L. Hollenstein, F.S.N. Lobo, and S.S. Seahra, [arXiv:1001.1266](https://arxiv.org/abs/1001.1266).
- [16] A. Odrzywolek, *Phys. Rev. D* **80**, 103515 (2009).
- [17] P. Horava, *J. High Energy Phys.* 03 (2009) 020; *Phys. Rev. D* **79**, 084008 (2009); *Phys. Rev. Lett.* **102**, 161301 (2009).
- [18] H. Lu, J. Mei, and C.N. Pope, *Phys. Rev. Lett.* **103**, 091301 (2009).
- [19] E. Kiritsis and G. Kofinas, *Nucl. Phys.* **B821**, 467 (2009).
- [20] G. Calcagni, *J. High Energy Phys.* 09 (2009) 112.
- [21] S. Mukohyama, *J. Cosmol. Astropart. Phys.* 06 (2009) 001.
- [22] X. Gao, [arXiv:0904.4187](https://arxiv.org/abs/0904.4187).
- [23] R.G. Cai, B. Hu, and H.B. Zhang, *Phys. Rev. D* **80**, 041501(R) (2009).
- [24] B. Chen, S. Pi, and J. Tang, *J. Cosmol. Astropart. Phys.* 08 (2009) 007.
- [25] X. Gao, Y. Wang, R. Brandenberger, and A. Riotto, *Phys. Rev. D* **81**, 083508 (2010).
- [26] K. Yamamoto, T. Kobayashi, and G. Nakamura, *Phys. Rev. D* **80**, 063514 (2009).
- [27] A. Wang and R. Maartens, *Phys. Rev. D* **81**, 024009 (2010).
- [28] T. Kobayashi, Y. Urakawa, and M. Yamaguchi, *J. Cosmol. Astropart. Phys.* 11 (2009) 015.
- [29] Y. Piao, *Phys. Lett. B* **681**, 1 (2009).
- [30] Y.F. Cai and X. Zhang, *Phys. Rev. D* **80**, 043520 (2009).
- [31] T. Sotiriou, M. Visser, and S. Weinfurtner, *Phys. Rev. Lett.* **102**, 251601 (2009); *J. High Energy Phys.* 10 (2009) 033.
- [32] U.H. Danielsson and L. Thorlacius, *J. High Energy Phys.* 03 (2009) 070; R.G. Cai, L.M. Cao, and N. Ohta, *Phys. Rev. D* **80**, 024003 (2009); R.G. Cai, Y. Liu, and Y.W. Sun, *J. High Energy Phys.* 06 (2009) 010; R.G. Cai, L.M. Cao, and N. Ohta, *Phys. Lett. B* **679**, 504 (2009); R.G. Cai and H.Q. Zhang, *Phys. Rev. D* **81**, 066003 (2010); R.G. Cai and N. Ohta, *Phys. Rev. D* **81**, 084061 (2010); R.G. Cai and A. Wang, [arXiv:1001.0155](https://arxiv.org/abs/1001.0155); E. Colgain and H. Yavartanoo, *J. High Energy Phys.* 08 (2009) 021; Y.S. Myung and Y.-W. Kim, *Eur. Phys. J. C* (to be published); Y.S. Myung, *Phys. Lett. B* **684**, 158 (2010); **684**, 1 (2010); *Phys. Lett. B* **679**, 491 (2009); **681**, 81 (2009); [arXiv:0912.3305](https://arxiv.org/abs/0912.3305); *Phys. Lett. B* **685**, 318 (2010); Y.S. Myung, Y.W. Kim, and Y.J. Park, [arXiv:0910.4428](https://arxiv.org/abs/0910.4428); Y.S. Myung, Y.W. Kim, W.S. Son, and Y.J. Park, [arXiv:0911.2525](https://arxiv.org/abs/0911.2525); A. Kehagias and K. Sfetsos, *Phys. Lett. B* **678**, 123 (2009); R.B. Mann, *J. High Energy Phys.* 06 (2009) 075; S. Chen and J. Jing, *Phys. Lett. B* **687**, 124 (2010); *Phys. Rev. D* **80**, 024036 (2009); D.W. Pang, [arXiv:0905.2678](https://arxiv.org/abs/0905.2678); G. Bertoldi, B.A. Burrington, and A. Peet, *Phys. Rev. D* **80**, 126003 (2009); M.i. Park, *J. High Energy Phys.* 09 (2009) 123; A. Castillo and A. Larranaga, [arXiv:0906.4380](https://arxiv.org/abs/0906.4380); J.J. Peng and S.Q. Wu, *Eur. Phys. J. C* **66**, 325 (2010); H.W. Lee, Y.W. Kim, and Y.S. Myung, [arXiv:0907.3568](https://arxiv.org/abs/0907.3568); S.S. Kim, T. Kim, and Y. Kim, *Phys. Rev. D* **80**, 124002 (2009); C. Ding, S. Chen, and J. Jing, *Phys. Rev. D* **81**, 024028 (2010); J.Z. Tang and B. Chen, *Phys. Rev. D* **81**, 043515 (2010); N. Varghese and V.C. Kuriakose, [arXiv:0909.4944](https://arxiv.org/abs/0909.4944); D.Y. Chen, H. Yang, and X.T. Zu, *Phys. Lett. B*, **681**, 463 (2009); T. Harada, U. Miyamoto, and N. Tsukamoto, [arXiv:0911.1187](https://arxiv.org/abs/0911.1187); D. Capasso and A.P. Polychronakos, *Phys. Rev. D* **81**, 084009 (2010); B.R. Majhi, *Phys. Lett. B* **686**, 49 (2010); J.Z. Tang, [arXiv:0911.3849](https://arxiv.org/abs/0911.3849).
- [33] M. Wang, J. Jing, C. Ding, and S. Chen, *Phys. Rev. D* **81**, 083006 (2010); S. Mukohyama, K. Nakayama, F. Takahashi, and S. Yokoyama, *Phys. Lett. B* **679**, 6 (2009); S. Mukohyama, *J. Cosmol. Astropart. Phys.* 09 (2009) 005; S.K. Rama, *Phys. Rev. D* **79**, 124031 (2009); N. Saridakis, [arXiv:0905.3532](https://arxiv.org/abs/0905.3532); M.i. Park, *J. Cosmol. Astropart. Phys.* 01 (2010) 001; S. Koh, [arXiv:0907.0850](https://arxiv.org/abs/0907.0850); C. Appignani, R. Casadio, and S. Shankaranarayanan, *J. Cosmol. Astropart. Phys.* 04 (2010) 006; M.R. Setare, [arXiv:0909.0456](https://arxiv.org/abs/0909.0456); S. Maeda, S. Mukohyama, and T. Shiromizu, *Phys. Rev. D* **80**, 123538 (2009); S. Carloni, E. Elizalde, and P.J. Silva, *Classical Quantum Gravity* **27**, 045004 (2010); G. Leon, and E.N. Saridakis, *J. Cosmol. Astropart. Phys.* 11 (2009) 006; B. Chen, S. Pi, and J.Z. Tang, [arXiv:0910.0338](https://arxiv.org/abs/0910.0338); B. Chen and Q.G. Huang, *Phys. Lett. B* **683**, 108 (2010); S. Dutta and E.N. Saridakis, *J. Cosmol. Astropart. Phys.* 01 (2010) 013; I. Bakas, F. Bourliot, D. Lust, and M. Petropoulos, *Classical Quantum Gravity* **27**, 045013 (2010); E. Czuchry, [arXiv:0911.3891](https://arxiv.org/abs/0911.3891); M. Visser, *Phys. Rev. D* **80**, 025011 (2009); [arXiv:0912.4757](https://arxiv.org/abs/0912.4757); P.R. Carvalho and M. Leite, *Ann. Phys. (N.Y.)* **325**, 151 (2010); A. Volovich and C. Wen, *J. High Energy Phys.* 05 (2009) 087; A. Jenkins, *Int. J. Mod. Phys. D* **18**, 2249 (2009); H. Nikolic, *Mod. Phys. Lett. A* (to be published); H. Nastase, [arXiv:0904.3604](https://arxiv.org/abs/0904.3604); G.E. Volovik, *JETP Lett.* **89**, 525 (2009); D. Orlando and S. Reffert, *Classical Quantum Gravity* **26**, 155021 (2009);
- [34] C. Gao, *Phys. Lett. B* **684**, 85 (2010); T. Nishioka, *Classical Quantum Gravity* **26**, 242001 (2009); A. Ghodsi, [arXiv:0905.0836](https://arxiv.org/abs/0905.0836); J.B. Jimenez and A.L. Maroto, *Phys. Rev. D* **80**, 063512 (2009); Y.W. Kim, H.W. Lee, and Y.S. Myung, *Phys. Lett. B* **682**, 246 (2009); M. Sakamoto, *Phys. Rev. D* **79**, 124038 (2009); M. Botta-Cantcheff, N. Grandi, and M. Sturla, [arXiv:0906.0582](https://arxiv.org/abs/0906.0582); Y.S. Myung, *Phys. Rev. D* **81**, 064006 (2010); A. Ghodsi and E. Hatefi, *Phys. Rev. D* **81**, 044016 (2010); A. Kobakhidze, [arXiv:0906.5401](https://arxiv.org/abs/0906.5401); T. Harko, Z. Kovacs, and F.S.N. Lobo, *Phys. Rev. D* **80**, 044021 (2009); I. Adam, I.V. Melnikov, and S. Theisen, *J. High Energy Phys.* 09 (2009) 130; N. Afshordi, *Phys. Rev. D* **80**, 081502 (2009); I. Cho and G. Kang, [arXiv:0909.3065](https://arxiv.org/abs/0909.3065); T. Suyama, *J. High Energy Phys.* 01 (2010) 93; D. Capasso and A.P. Polychronakos, *J. High Energy Phys.* 02 (2010) 68; S.K. Rama, [arXiv:0910.0411](https://arxiv.org/abs/0910.0411); D. Momeni, [arXiv:0910.0594](https://arxiv.org/abs/0910.0594); M. Park, [arXiv:0910.1917](https://arxiv.org/abs/0910.1917); [arXiv:0910.5117](https://arxiv.org/abs/0910.5117); D. Benedetti and J. Henson, *Phys. Rev. D* **80**, 124036 (2009); Q. Exirifard, [arXiv:0911.4343](https://arxiv.org/abs/0911.4343); W. Chao, [arXiv:0911.4709](https://arxiv.org/abs/0911.4709); R. Garattini, [arXiv:0912.0136](https://arxiv.org/abs/0912.0136); G. Calcagni, *Phys. Rev. D* **81**, 044006 (2010); C. Bogdanos, and E.N. Saridakis, *Classical Quantum Gravity* **27**, 075005 (2010); J. Kluson, [arXiv:0904.1314](https://arxiv.org/abs/0904.1314); [arXiv:0910.5852](https://arxiv.org/abs/0910.5852); J. Chen, and

- Y. Wang, *Int. J. Mod. Phys. A* **25**, 1439 (2010); S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **81**, 043001 (2010); C. Germani, A. Kehagias, and K. Sfetsos, [arXiv:0906.1201](#); D. Blas, O. Pujolas, and S. Sibiryakov, [arXiv:0909.3525](#); [arXiv:0912.0550](#); T. Takahashi and J. Soda, *Phys. Rev. Lett.* **102**, 231301 (2009); R. Brandenberger, [arXiv:0904.2835](#); Y. Cai and E. N. Saridakis, *J. Cosmol. Astropart. Phys.* **10** (2009) 020; S. Mukohyama, [arXiv:0905.3563](#); A. Papazoglou and T. P. Sotiriou, *Phys. Lett. B* **685**, 197 (2010).
- [35] K. Izumi and S. Mukohyama, *Phys. Rev. D* **81**, 044008 (2010); R. Iengo, J. G. Russo, and M. Serone, *J. High Energy Phys.* **11** (2009) 020; R. A. Konoplya, *Phys. Lett. B* **679**, 499 (2009); T. Harko, Z. Kovacs, and F. S. N. Lobo, [arXiv:0908.2874](#); L. Iorio and M. L. Ruggiero, [arXiv:0909.2562](#); [arXiv:0909.5355](#); G. Amelino-Camelia, L. Gualtieri, and F. Mercati, *Phys. Lett. B* **686**, 283 (2010); A. Wang and Y. Wu, *J. Cosmol. Astropart. Phys.*, **07** (2009) 012; J. Greenwald, A. Papazoglou, and A. Wang, [arXiv:0912.0011](#); M. R. Setare and M. Jamil, [arXiv:1001.1251](#); J. Jing, L. Wang, and S. Chen, [arXiv:1001.1472](#); Q. Cao, Y. Chen, and K. Shao, [arXiv:1001.2597](#); F. S. N. Lobo, T. Harko, and Z. Kovacs, [arXiv:1001.3517](#); M. R. Setare and D. Momeni, [arXiv:1001.3767](#); Y. Myung, Y. Kim, W. Son, and Y. Park, [arXiv:1001.3921](#); M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen, and A. Tureanu, [arXiv:1001.4102](#); T. Jacobson, [arXiv:1001.4823](#).
- [36] M. R. Setare and D. Momeni, [arXiv:0911.1877](#).
- [37] M. Minamitsuji, *Phys. Lett. B* **684**, 194 (2010);
- [38] C. G. Boehmer and F. S. N. Lobo, [arXiv:0909.3986](#).