Applicability of the linearly perturbed FRW metric and Newtonian cosmology

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It has been argued that the effect of cosmological structure formation on the average expansion rate is negligible, because the linear approximation to the metric remains applicable in the regime of nonlinear density perturbations. We discuss why the arguments based on the linear theory are not valid. We emphasize the difference between Newtonian gravity and the weak field, small velocity limit of general relativity in the cosmological setting.

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I. INTRODUCTION

The backreaction conjecture and perturbation theory

It has been proposed that the observed increase in the expansion rate and the distance scale of the Universe at late times relative to the matter-dominated homogeneous and isotropic Friedmann-Robertson-Walker (FRW) model could be explained by the breakdown of the homogeneous and isotropic approximation because of the formation of nonlinear structures [1–5]. The effect of clumpiness on the average expansion rate is called backreaction [6–8]; see [9–11] for reviews. The exact Buchert equations for the average expansion rate show that large variance can lead to accelerated expansion as faster regions come to dominate the volume [8]. This effect has been demonstrated with exact toy models [10,12–16]. At late times there are deviations of order unity in the expansion rate between different regions, so this mechanism could also work in the real Universe. The correct order of magnitude and time scale of the change of the expansion rate have been shown to emerge from the physics of structure formation in a semi-realistic model without any free parameters [17,18]. The relation between the average expansion rate and observations of light is also understood, though it should be established more rigorously and details remain to be worked out [19–21]. However, there is no fully realistic calculation yet, and whether backreaction is important in the real Universe remains an open question. The difference between Newtonian gravity and the weak field, a small velocity limit of general relativity [22–29] plays an important part in the problem. Therefore, quantifying the importance of the growth of structures on the average expansion rate requires treating a statistically homogeneous and isotropic but locally complicated nonlinear system in general relativity.

However, it has been argued that the effect of nonlinear structures on the expansion rate can be evaluated in linear perturbation theory around the FRW metric [3,16,30–39], sidestepping subtleties of nonlinear general relativity and the Newtonian limit. The argument is that even though the density perturbation becomes nonlinear when structures form, the corresponding metric perturbation in the longitudinal gauge, calculated from the Poisson equation, remains much smaller than unity, so the effect on the averages is negligible. There are multiple problems with this argument. Evaluating the effect on averages requires going at least to second order, so using first order perturbation theory is inconsistent, observables are not given by the metric alone, but by the metric and its derivatives (which can become large) and, finally, the linear equations do not, in fact, apply once the density perturbation becomes nonlinear. In short, it is not enough to calculate the magnitude of the effect in linear perturbation theory, the applicability of the linear treatment also has to be considered.

Some of these arguments have been addressed before [5,8,10,17,21,40–42]. However, as they are being repeated in the literature, it may be useful to discuss the issue in more detail than in [5,10,17,21], and from a slightly different perspective than in [8,40–42]. In Sec. II we consider perturbation theory around the FRW metric and show why the linear and second order calculations are not sufficient for evaluating backreaction once the density field becomes nonlinear. We then look at the full nonlinear equations for the averages and consider the Newtonian limit. In Sec. III we discuss previous work on this topic, and in Sec. IV we summarize our conclusions and outlook.

II. PERTURBATIONS AND THE AVERAGE EXPANSION RATE

A. The perturbative calculation

1. The Einstein equation and the metric

We assume that matter and geometry are related by the Einstein equation

$$G_{\alpha\beta} = T_{\alpha\beta}, \tag{2.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor and $T_{\alpha\beta}$ is the energy-momentum tensor; we use units in which $8\pi G_N = 1$, where G_N is Newton's constant. We assume that the matter

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can be described as dust,

$$T_{\alpha\beta} = \rho u_\alpha u_\beta, \tag{2.2}$$

where ρ is the energy density and u^α is the velocity of the observers, taken to be comoving with the dust.

The perturbed FRW metric in the longitudinal gauge is (we consider only a spatially flat background and only scalar perturbations)

$$ds^2 = -(1 + 2\Phi(t, \boldsymbol{x}))dt^2 + (1 - 2\Psi(t, \boldsymbol{x}))a(t)^2\delta_{ij}dx^i dx^j. \tag{2.3}$$

The Einstein tensor for the metric (2.3) is

$$\begin{aligned} G^0{}_0 &\simeq -3H^2(1 - 2\Phi) - 2a^{-2}\nabla^2\Psi + 6H\dot{\Psi}, \\ G^k{}_k &\simeq -(2\dot{H} + 3H^2)(1 - 2\Phi) + 2\ddot{\Psi} + 6H\dot{\Psi} + 2H\dot{\Phi} \\ &\quad - a^{-2}\nabla^2(\Psi - \Phi) + a^{-2}\partial_k^2(\Psi - \Phi), \\ G^i{}_j &\simeq a^{-2}\partial_i\partial_j(\Psi - \Phi) \qquad (i \neq j), \\ G_{0i} &\simeq 2\partial_i(\dot{\Psi} + H\Phi), \end{aligned} \tag{2.4}$$

where $\simeq$ denotes dropping terms which are higher than first order (or, later, second order; it should be clear from the context which is meant) in Φ or Ψ, the dot denotes derivative with respect to the background coordinate time t, and no summation is implied in $G^k{}_k$. We also split the velocity into the background and the perturbation, $u^\alpha = \bar{u}^\alpha + \delta u^\alpha$, and assume that δu^α is small. From the normalization condition $g_{\alpha\beta}u^\alpha u^\beta = -1$ it then follows that $u^0 \simeq 1 - \Phi$.

With the energy-momentum tensor (2.2) and the Einstein tensor (2.4), the Einstein equation (2.1) reduces, at first order, to

$$3H^2(1 - 2\Phi) + 2a^{-2}\nabla^2\Psi - 6H\dot{\Psi} \simeq \rho, \tag{2.5}$$

$$2\dot{H} + 3H^2 - 2\ddot{\Psi} - 6H\dot{\Psi} - 2H\dot{\Phi} + a^{-2}\nabla^2(\Psi - \Phi) - a^{-2}\partial_k^2(\Psi - \Phi) = 0, \tag{2.6}$$

$$\partial_i\partial_j(\Psi - \Phi) = 0, \tag{2.7}$$

$$2\partial_i(\dot{\Psi} + H\Phi) \simeq -\rho\delta u_i. \tag{2.8}$$

Note that we have not made any assumptions about the perturbations of ρ. From (2.7) it follows that $\Psi - \Phi = A(t, x^1) + B(t, x^2) + C(t, x^3)$, where A, B, and C are arbitrary functions. We are mostly interested in the situation when the perturbations are statistically homogeneous and isotropic, in which case $A = B = C = 0$, and we assume this from now on. (The condition $\Psi - \Phi = 0$ would also follow from the technical requirement that the Fourier transform of $\Psi - \Phi$ exists.)

2. *The static case*

Let us first consider the static case $H = 0$, and choose $a = 1$. The set of equations (2.5), (2.6), (2.7), and (2.8) reduces to

$$2\nabla^2\Phi \simeq \rho, \tag{2.9}$$

$$\ddot{\Phi} = 0, \tag{2.10}$$

$$2\partial_i\dot{\Phi} \simeq -\rho\delta u_i. \tag{2.11}$$

To be consistent with neglecting terms which are second order in Φ, we should discard the right-hand side of (2.11), because according to (2.9), ρ is of order Φ. We then obtain the result $\Phi = At + B(\boldsymbol{x})$, where A is a constant and $B(\boldsymbol{x})$ is determined by the density via (2.9). It is possible for ρ to have large variations without Φ becoming large or the first order treatment becoming invalid. (This is the case in the solar system, for example.) However, in that case there is a slight inconsistency in the treatment, because we have assumed that $\nabla^2\Phi = \frac{1}{2}\rho$ is small. If ρ is allowed to be large, we should equally treat $\nabla^2\Phi$ as a large term, so products such as $\Phi\nabla^2\Phi$ should not be discarded. However, we should then take into account second order terms in the metric, because they can be of the same order. Let us look at this in more detail in the cosmological situation.

3. *The cosmological case*

With $H \neq 0$, the Einstein equations (2.5), (2.6), (2.7), and (2.8) read

$$3H^2(1 - 2\Phi) + 2a^{-2}\nabla^2\Phi - 6H\dot{\Phi} \simeq \rho, \tag{2.12}$$

$$2\dot{H} + 3H^2 - 2\ddot{\Phi} - 8H\dot{\Phi} \simeq 0, \tag{2.13}$$

$$2\partial_i(\dot{\Phi} + H\Phi) \simeq -\rho\delta u_i. \tag{2.14}$$

As is usual, we assume that the background and first order equations are separately satisfied. This follows if we assume that the average of Φ over the background space vanishes. We split the density into the background value and the perturbation, $\rho = \bar{\rho} + \delta\rho$, but do not assume that $\delta\rho$ is small. We then have

$$3H^2 = \bar{\rho}, \tag{2.15}$$

$$2a^{-2}\nabla^2\Phi - 6H^2\Phi - 6H\dot{\Phi} = \delta\rho, \tag{2.16}$$

$$2\dot{H} + 3H^2 = 0, \tag{2.17}$$

$$\ddot{\Phi} + 4H\dot{\Phi} = 0, \tag{2.18}$$

$$\delta u_i = -\frac{2}{\rho}\partial_i(\dot{\Phi} + H\Phi). \tag{2.19}$$

Equations (2.17) and (2.18), which come from the pressure-free condition (2.13), determine the evolution of a and Φ, regardless of the energy density. They lead to the standard relations $a \propto t^{2/3}$, $\Phi = A(\boldsymbol{x}) + B(\boldsymbol{x})t^{-5/3}$, where A and B are arbitrary functions. According to (2.15) and (2.16), the density contrast $\delta \equiv \delta\rho/\bar{\rho}$ is related to Φ by

$$\delta = \frac{2}{3(aH)^2}\nabla^2\Phi - 2\Phi - \frac{2}{H}\dot{\Phi}. \tag{2.20}$$

We have nowhere required that δ should be small, so one could at first sight think that (2.20) applies even when δ becomes of order unity, as long as Φ remains small, analogously to the static case. (In the static case $\bar{\rho} = 0$, so δ is not defined, but the variation of ρ between different regions of space can be large.) Keeping to the linear theory, this is not true. The time evolution of Φ is determined independently of ρ, and inserting $\Phi = A + Bt^{-5/3}$ into (2.20) shows that, dropping the decaying mode, the density contrast has a constant part and a part which is proportional to a. Therefore δ grows without limit. For an underdense region, δ cannot go below -1, so this evolution is clearly not correct as δ becomes of order unity. The behavior is also wrong for overdense regions, as is well known from the spherical collapse model [43] (see [44] for reviews).

For the volume expansion rate $\theta = \nabla_\alpha u^\alpha$ we have

$$\theta \simeq 3H - 3(\dot{\Phi} + H\Phi) + \partial_i u^i = 3H(1 - \tfrac{5}{3}\Phi - \tfrac{1}{3}\delta), \tag{2.21}$$

where we have dropped the decaying mode of Φ. (For the expression in terms of the proper time measured by the observers, see [5].) It is clear that the expansion rate given by the linear theory is wrong when δ becomes of order unity.

4. The reason for the breakdown

We have assumed that the Einstein tensor and the velocity u^α can be expanded linearly in the metric perturbations. We have found that the observables calculated using this procedure fail to describe the real behavior when $\delta \sim \nabla^2\Phi/(aH)^2$ becomes of order ± 1, even if Φ would seem to remain small, so the linearly perturbed metric would appear to be valid.

The reason is that in neglecting all terms which are second order in Φ, we have implicitly assumed that terms such as $\Phi\nabla^2\Phi/(aH)^2$ are much smaller than Φ, i.e. that $|\nabla^2\Phi/(aH)^2| \sim |\delta| \ll 1$. To extend the calculation into the regime $|\delta| \gtrsim 1$, we would have to expand to second order in Φ. But to do this consistently, we have to include the intrinsic second order terms in addition to the squares of first order terms. Indeed, the distinction between the two is gauge dependent [45], as first order quantities are not invariant under second order gauge transformations. And at second order, the metric cannot be written in the simple diagonal form (2.3) [46]. It may be that the effect of the second order terms is small for a particular quantity of interest, but this has to be determined via a consistent calculation.

Effectively, there are three expansion parameters: Φ, $\partial_i\Phi/(aH)$, and $\nabla^2\Phi/(aH)^2 \sim \delta$. (No higher derivatives appear, because the Einstein equation is second order.) The formal perturbation expansion is defined in powers of the metric perturbation, treated as an infinitesimal quantity [47]. However, in the real Universe, the metric perturbation has a finite amplitude, so the gradients can make the other expansion parameters large even when Φ remains small. The gradient is a dimensional quantity, so a comparison scale must enter. In cosmology, the relevant scale is aH, and since aH decreases in a decelerating FRW universe, gradients become more important with time. We can also view this as follows: for a time-independent Φ, the magnitude of $\nabla^2\Phi$ is fixed in time, while the curvature scale of the Universe, to which it is compared, decreases. This kind of an instability is not present in the static case.

In a situation with multiple expansion parameters, perturbation theory can be expected to remain valid when all parameters are small, and to fail when all of them are large. When some parameters become large while others remain small, the validity of perturbation theory depends on the system, and on the quantity under consideration. In cosmology, the metric (2.3) is simply the first order term in an expansion, and when gradients of the metric perturbation become large, higher order terms can no longer be neglected. This does not necessarily mean that all first order results are wrong: a consistent calculation with higher order terms may show that some linear relations are valid. However, this cannot be determined using the linear theory.

For backreaction, the important quantity is the average expansion rate. (The primary quantities are of course observables defined in terms of measurements of light; for the connection to the average expansion rate, see [19–21].) One might argue that even if the linear theory fails to correctly describe the local quantities when density perturbations are nonlinear, the effect on the averages nevertheless remains small. To address this issue, let us see what happens when we expand to second order.

5. The average expansion rate at second order

Taking the metric (2.3) and calculating $\theta = \nabla_\alpha u^\alpha$ to second order in Φ, we obtain (we take $\dot{\Phi} = 0$; see [5] for the general expression)

$$\theta \simeq 3H_\tau + \frac{118}{45}\frac{H}{(aH)^2}\partial_i\Phi\partial_i\Phi - \frac{2}{3}\frac{H}{(aH)^2}\partial_i\left(\partial_i\Phi + \Phi\partial_i\Phi - \frac{2}{3}\frac{1}{(aH)^2}\partial_i\Phi\nabla^2\Phi\right), \tag{2.22}$$

where $H_\tau \equiv 2/(3\tau)$ is the background expansion rate in terms of the proper time τ of comoving observers. The last term, with four derivatives, is of order δ^2, so there are large local variations in the expansion rate. Averaging (2.22) on the hypersurface of constant proper time, we obtain [5]

$$\langle\theta\rangle \simeq 3H_\tau\bigg(1 - \frac{22}{135}\frac{1}{(aH)^2}\langle\partial_i\Phi\partial_i\Phi\rangle_0 + \frac{22}{27}\frac{1}{(aH)^2} \times\langle\partial_i(\Phi\partial_i\Phi)\rangle_0 + \frac{8}{27}\frac{1}{(aH)^4}\langle\partial_i(\nabla^2\Phi\partial_i\Phi)\rangle_0\bigg), \tag{2.23}$$

where $\langle\,\rangle$ is a proper average with the correct volume element, $\langle\,\rangle_0$ is an average taken on the background hypersurface of constant proper time, without perturbations in the volume element, and we have assumed $\langle\Phi\rangle_0 = 0$. It is noteworthy that the term with four derivatives, which has the largest amplitude locally, is a boundary term. Before discussing this feature, let us note that this calculation is not consistent, because we have used the first order metric to calculate a second order quantity, i.e. we have neglected intrinsic second order terms. To obtain a result which does not depend on the gauge, it is necessary to truncate the metric consistently at second order instead of first order. The result is then [45][1]

$$\langle\theta\rangle \simeq 3H_\tau\bigg(1 - \frac{5}{27}\frac{1}{(aH)^2}\langle\partial_i\Phi\partial_i\Phi\rangle_0 + \frac{10}{27}\frac{1}{(aH)^2}\langle\partial_i(\Phi\partial_i\Phi)\rangle_0 + \frac{16}{189}\frac{1}{(aH)^4}\langle\partial_i(\nabla^2\Phi\partial_i\Phi - \partial_i\partial_j\Phi\partial_j\Phi)\rangle_0\bigg). \tag{2.24}$$

Comparing (2.23) and (2.24) shows that the first order calculation in the longitudinal gauge happens to give qualitatively the right answer, but the coefficients of the terms are wrong. (In first order perturbation theory, doing the calculation in the synchronous comoving gauge, for example, would give a qualitatively different result.) Note that there is nothing in the result of the first order calculation that would indicate that the answer is wrong. An average of a total derivative can be converted into a surface integral of a flux through the boundary. If the distribution is statistically homogeneous and isotropic, there is no preferred direction, so the integral vanishes (up to statistical fluctuations). (In perturbation theory, the technical requirement that Φ can be expanded in Fourier modes would lead to the same conclusion.) With vanishing boundary terms, the correction to the mean is $\sim\langle\partial_i\Phi\partial_i\Phi\rangle_0/(aH)^2$, which is of the order 10^{-5} for a realistic linear theory power spectrum. To see the failure of the linear theory expanded to second order, we have to work with the second order metric. With the metric truncated at first order, it is impossible to determine the magnitude of the higher order terms which are neglected. As the intrinsic second order terms are as large as the first order terms squared, the question arises as to the magnitude of the terms which are even higher order. While calculating the coefficients of the various terms would be an involved task, it is straightforward to write down their general form.

6. The general structure of the corrections

At second order, the possible correction terms are the squares of the three expansion parameters, $\langle\Phi^2\rangle_0$, $\langle\partial_i\Phi\partial_i\Phi\rangle_0/(aH)^2$, and $\langle\nabla^2\Phi\nabla^2\Phi\rangle_0/(aH)^4 \sim \langle\delta^2\rangle_0$. (The quantity $\langle\Phi\nabla^2\Phi\rangle_0$ is equal to $-\langle\partial_i\Phi\partial_i\Phi\rangle_0$ up to a boundary term.) For simplicity, we take $\dot\Phi = 0$. As long as $|\dot\Phi| \sim H|\Phi|$, taking into account time dependence would simply introduce more terms of the same order of magnitude, and would not lead to any qualitative change. When the average expansion rate is expressed in terms of the proper time τ (as opposed to the unphysical coordinate time t), Φ appears in the expansion rate only with derivatives acting on it [48]. This is to be expected, because if Φ depends only on time, it corresponds to using a different time coordinate, not to having a physical degree of freedom. Assuming that the higher order equations are satisfied order by order and the perturbations are Gaussian, all higher order terms from scalar perturbations factorize into products of these three expectation values, since they are sourced by the first order terms. In contrast, vector and tensor perturbations (which necessarily arise at higher orders) have solutions which do not need to be supported by a source, so their contribution cannot be completely expressed in terms of the first order seed fields. (See [46] for the second order case.) However, such terms are expected to be subdominant to the scalar perturbations in the nonlinear regime.

We now return to the feature that at second order, the term with the highest number of derivatives (and therefore locally the largest amplitude) is a boundary term and as such vanishes upon averaging, up to statistical fluctuations. In [5] it was argued that at higher orders there might not be such a cancellation for the leading terms. However, in [49] it was realized that because each factor of $\partial_i/(aH)$ is accompanied by one power of the speed of light c, the terms with the highest number of spatial derivatives are the ones which dominate in the Newtonian limit $c \to \infty$. In Newtonian gravity, the backreaction correction is exactly a boundary term [7]. Thus, in general relativity the term with the highest number of spatial derivatives at each order in

[1]As an aside, in [45] it is assumed, as is usual, that the equations are satisfied separately order by order in perturbation theory. While this procedure is self-consistent, there seems to be no rigorous justification for it beyond first order. At first order, the equations for the background and perturbations decouple, assuming that the average of the perturbations vanishes. Starting at second order, the average of the perturbations does not vanish, so decoupling of the background and perturbations is an extra assumption.

perturbation theory is a boundary term.[2] The general structure of the corrections from scalar perturbations to the average expansion rate is therefore (see also [40])

$$\langle\theta\rangle = 3H_\tau\left(1 + \frac{1}{(aH)^2}\langle\partial_i\Phi\partial_i\Phi\rangle_0 \sum_{n=0}^{\infty}\lambda_n\langle\delta^2\rangle_0^n + \cdots\right), \tag{2.25}$$

where λ_n are constants and ... indicates subleading terms with a smaller number of derivatives, such as $\langle\partial_i\Phi\partial_i\Phi\rangle_0^m \times \langle\delta^2\rangle_0^{n-m}/(aH)^{2m}$, with $n \geq m \geq 1$. In powers of Φ, the λ_n term is of order $2n+2$. The term $\langle\delta^2\rangle_0$ can appear at fourth order in Φ at the earliest, where the leading correction is $\langle\partial_i\Phi\partial_i\Phi\rangle_0/(aH)^2 \times \langle\delta^2\rangle_0$. This term grows without bound with increasing $|\delta|$, so the breakdown of the perturbative expansion is transparent. None of the coefficients λ_n have been calculated. It is possible to determine λ_1 in third order perturbation theory, which is being developed [50], without a full fourth order calculation [51]. However, calculating λ_1 would be inconclusive, because at every order, there are an increasing number of terms that grow even faster as $|\delta|$ becomes of order unity.

In [40] it was argued that the series (2.25) would have only a finite number of gradient terms when Φ is taken as the full metric perturbation and not only the linear part (and $\Psi \neq \Phi$ is included). However, this is not the case: for a metric of the form (2.3), the velocity u^i (and thus also θ) expanded as a series in Φ necessarily contains an infinite (or zero) number of spatial derivatives [5]. And as we have noted, beyond first order, the metric cannot be written in the form (2.3) [46].

From the fact that the series (2.25) would naively seem to diverge at $\langle\delta^2\rangle_0 \sim 1$ we cannot conclude that the sum of the correction terms would be large. However, we can definitely say that the series expansion does not prove that the correction would be small when Φ is small. The magnitude of the effect has to be established with nonperturbative methods, or a resummation of the series. For studies in the spherically symmetric situation where the exact solution is known, see [16,52–55]. In particular, [55] shows that it is possible to have a large effect on the observable distance-redshift relation even when the metric can be written in the form (2.3) (at least on the light cone). These models are not conclusive of the cosmological situation, which is not spherically symmetric.[3] Different resummation schemes have been applied in Newtonian cosmology [56], and it would be interesting if such methods could be extended to general relativity.

We would still expect to recover linear equations for perturbations with wavelengths much larger than the size of the structures, as is usual in statistical physics. These should look similar to perturbation equations around the FRW Universe, with correction terms due to the underlying structure [10]. This is also suggested by the success of FRW perturbation theory in describing observations of large-scale structure. Such equations would be analogous to the Buchert equations, which look like FRW equations with correction terms, though their physical content is different, as they involve only average quantities and not local expansion. The effect of backreaction cannot be described merely as a change in the FRW background [10,17,19,21], unlike argued in [38,57,58]. Even though the average expansion rate will always agree with that of some FRW model, other observables will in general not be the same as in that FRW Universe. In particular, the relationship between the average expansion rate and the luminosity distance is different than in FRW models if backreaction is important [19,21,59,60].

We have discussed perturbation theory as it is most commonly formulated, by adding perturbations on top of a background (and previous perturbations). The alternative is to take the full nonlinear system and linearize it. In cosmology (unlike in the spherically symmetric case) we cannot write down the exact solution to linearize. However, it is at least possible to build perturbation theory by starting from the full exact equations, written in the covariant formalism, and linearize around the FRW solution [61–63]. This has the benefit that all terms are included to begin with, so it is transparent to estimate what is being dropped, unlike in the case when perturbations are added to a background order by order. In addition, the covariant formalism deals only with measurable quantities and the physical spacetime, so there are no gauge artifacts. Instead of a perturbative analysis, we go directly to the exact nonperturbative equations for physical insight into the effect of perturbations becoming large. A comparison of the general relativistic and Newtonian cases is also instructive, given that backreaction vanishes in the latter, for a statistically homogeneous and isotropic distribution.

B. The Newtonian limit

1. The Buchert equations

If the matter is irrotational dust, the exact equations which describe the effect of inhomogeneities on the average expansion rate $\langle\theta\rangle \equiv 3\partial_\tau a/a$ in general relativity are [8] (for the case with nondust matter or rotation, see [21,64–66])

$$3\frac{\partial_\tau^2 a}{a} = -4\pi G_{\rm N}\langle\rho\rangle + \mathcal{Q}, \tag{2.26}$$

$$3\frac{(\partial_\tau a)^2}{a^2} = 8\pi G_{\rm N}\langle\rho\rangle - \frac{1}{2}\langle{}^{(3)}R\rangle - \frac{1}{2}\mathcal{Q}, \tag{2.27}$$

$$\partial_\tau\langle\rho\rangle + 3\frac{\partial_\tau a}{a}\langle\rho\rangle = 0, \tag{2.28}$$

[2] Assuming that the leading order general relativity result reduces to the Newtonian theory at all orders. As we discuss in Sec. II B, this is not necessarily true. If that is not the case, the series (2.25) is even more divergent.

[3] Note that in [54] the average spatial curvature is small, so it is clear that backreaction is not important.

where ${}^{(3)}R$ is the spatial curvature and $\mathcal{Q}$ is the backreaction variable defined as

$$\mathcal{Q} \equiv \tfrac{2}{3}(\langle\theta^2\rangle - \langle\theta\rangle^2) - 2\langle\sigma^2\rangle, \tag{2.29}$$

where σ^2 is the shear scalar. The integrability condition between (2.26) and (2.27) is

$$\partial_\tau\langle{}^{(3)}R\rangle + 2\frac{\partial_\tau a}{a}\langle{}^{(3)}R\rangle = -\partial_\tau\mathcal{Q} - 6\frac{\partial_\tau a}{a}\mathcal{Q}. \tag{2.30}$$

If $\mathcal{Q} = 0$, we have $\langle{}^{(3)}R\rangle \propto a^{-2}$; in particular, this holds for all exactly homogeneous and isotropic universes [67]. The system of equations (2.26), (2.27), and (2.28) closes once we are given $\mathcal{Q}$ or $\langle{}^{(3)}R\rangle$; because of the integrability condition (2.30), the effect of clumpiness can be viewed equivalently in terms of either quantity. The Raychaudhuri equation (2.26) together with (2.29) shows that, apart from a possible $\langle{}^{(3)}R\rangle \propto a^{-2}$ term, deviations from homogeneity and isotropy have a large effect on the average expansion rate only when the variance of the expansion rate is large, and is not canceled by the shear (or the shear is large, and is not canceled by the variance). This shows that $\langle\delta^2\rangle_0 = 1$ is not a sufficient condition for a large effect on the average expansion rate. However, it is necessary that the deviation of the expansion rate from the mean is large in a large fraction of space (assuming that the deviation is at most of the same order of magnitude as the mean, which is true in cosmology).

2. Newtonian gravity and beyond

In Newtonian gravity, the counterparts of the Raychaudhuri equation (2.26) and the conservation equation (2.28) are identical to the relativistic equations, but there is no analog of the Hamiltonian constraint (2.27). The variance of the expansion rate and the shear combine to give a total derivative, so $\mathcal{Q}$ reduces to a boundary term [7]. (If the vorticity is nonzero, it is included in this boundary term.) Thus, if backreaction is important in a statistically homogeneous and isotropic universe, this must be due to non-Newtonian aspects of general relativity [10,11,17,40,41,49,68].

In the expansion (2.25), all correction terms are post-Newtonian. The term $c^2\langle\partial_i\Phi\partial_i\Phi\rangle_0/(aH)^2$, which may be identified as the square of a peculiar velocity, v^2/c^2, suppresses the post-Newtonian terms. However, the terms it multiplies can become very large as $c^4\langle\nabla^2\Phi\nabla^2\Phi\rangle_0/(aH)^4 \sim \langle\delta^2\rangle_0$ grows. This demonstrates that in general relativity, non-Newtonian effects can be important even when velocities are small and fields are weak. An exact example is given by rotating and expanding dust. In general relativity, there are no dust solutions which have nonzero expansion and rotation but zero shear [69]. However, in Newtonian gravity such solutions are exactly known [22,25]. Analysis of the Newtonian theory in this case would be misleading, because the Newtonian solutions betray no sign of the fact that starting from general relativity, they do not exist, even at small velocities and weak fields.[4]

This issue arises due to the indeterminacy of Newtonian cosmology, which is related to the absence of Newtonian analogs of the magnetic component of the Weyl tensor[5] and the evolution equation of the electric component of the Weyl tensor [22,24,26,28,29,71,72]. Newtonian cosmology is only defined up to boundary conditions [24,28,29]. This shows up in the fact that $\mathcal{Q}$ is a boundary term, and the average expansion rate is determined by what happens at the boundary. In general relativity this is not the case, and backreaction is given by integrals over the volume.

If the volume considered has periodic boundary conditions or is statistically homogeneous and isotropic, then in Newtonian gravity $\mathcal{Q}$ vanishes, and the first integral of the Raychaudhuri equation (2.26) leads to an equation which looks like the Hamiltonian constraint (2.27) with $\mathcal{Q} = 0$ and $\langle{}^{(3)}R\rangle = Ea^{-2}$, where E is a constant of motion which may be identified with (being proportional to) the conserved energy of the isolated system. In relativistic cosmology, the conserved energy is replaced by spatial curvature, which has no physical analog in Newtonian gravity, so the interpretation of this term is different in the two theories, even in the FRW case. There is no conservation law for the spatial curvature in general relativity, so $\langle{}^{(3)}R\rangle$ can evolve in a nontrivial manner, unlike the total energy of an isolated Newtonian system [17,68]. The equivalent statement in terms of the backreaction variable $\mathcal{Q}$ is that in Newtonian gravity the variance of the expansion rate always equals 3 times average shear scalar (up to a boundary term), as we see from (2.29), while in general relativity there is no such constraint.

In second order perturbation theory we have, dropping boundary terms and taking $\dot{\Phi} = 0$,

$$\begin{aligned}
\langle\theta^2\rangle - \langle\theta\rangle^2 &\simeq \frac{4}{9}\frac{H^2}{(aH)^4}\langle\nabla^2\Phi\nabla^2\Phi\rangle_0 \approx \langle\delta^2\rangle_0 H^2,\\
\langle\sigma^2\rangle &\simeq \frac{4}{27}\frac{H^2}{(aH)^4}\langle\nabla^2\Phi\nabla^2\Phi\rangle_0 \approx \frac{1}{3}\langle\delta^2\rangle_0 H^2,\\
\langle{}^{(3)}R\rangle &\simeq \frac{50}{9}\frac{1}{a^2}\langle\partial_i\Phi\partial_i\Phi\rangle_0,
\end{aligned} \tag{2.31}$$

where the second approximation holds if Φ is small compared to its gradient. Second order relativistic perturbation theory around a spatially flat FRW background is close to Newtonian gravity in the sense that there is an exact cancellation between the variance and the shear, so $\mathcal{Q} = 0$. The variance and the shear can be calculated using first

[4]This underlines the fact that it is not sufficient to look at the Newtonian limit of the equations of general relativity, but it is necessary to consider the limit of solutions, because in general the operations of taking the limit and solving the equations do not commute [23,25,70].

[5]We can equivalently say that in Newtonian gravity the magnetic component vanishes identically.

order theory, because they vanish for the background [51]. There is only a single non-Newtonian term, the spatial curvature, which is proportional to a^{-2} according to the integrability condition (2.30). To determine the coefficient of this term, it is necessary to go to second order.

Already in first and second order perturbation theory the variance and the shear are large. A large backreaction effect in general relativity does not require the variance of the expansion rate to be larger than expected; it is enough that the cancellation with the shear is not perfect, unlike in Newtonian cosmology. However, the spatial curvature does have to become large [10,17,73]. In a realistic cosmological setting, this is easy to understand. The spatial curvature of the initial overdense and underdense regions averages to zero in the linear regime, but once perturbations become nonlinear, the evolution of overdense and underdense regions is different, and the average will in general deviate from zero. It is to be expected that if the volume of the Universe becomes dominated by underdense voids which expand faster than overdense regions, the average spatial curvature will be negative.

Comparing Newtonian gravity and general relativity in cosmology is different than in the case of isolated, asymptotically flat systems. For isolated systems, both Newtonian gravity and general relativity are well defined. In contrast, while relativistic cosmology is well defined there is no unique Newtonian theory of cosmology, because the Newtonian equations are only defined up to boundary terms which have to be specified at all times [24,28,29]. (We could alternatively say that Newtonian gravity is a theory of isolated systems only, and there are an infinite number of possible generalizations to the cosmological setting.) This shortcoming of the Newtonian theory is often hidden in cosmology by the assumption of periodic boundary conditions (sometimes implicitly through the use of Fourier series). For periodic boundary conditions, Newtonian cosmology does have a well-defined initial value problem [28,74], at least to all orders in perturbation theory, but this situation does not correspond to the real Universe, which is not periodic on the observed scales.

So in the cosmological context, post-Newtonian corrections refer to the difference between a realistic relativistic cosmological model, which is thus far not tractable, and a Newtonian model which is defined only when it does not correspond to the real situation. In particular, it is not possible to estimate backreaction from usual N-body simulations, because the relativistic degrees of freedom are absent. Relativistic cosmological simulations would in principle provide an answer. The initial conditions, the matter model, and the equations of motion are known and well defined, but solving the system in full generality is not computationally feasible. It would be interesting to obtain a reduced system that would retain the relevant relativistic degrees of freedom while being tractable. Because relativistic cosmology is not sensitive to boundary conditions in the same way as Newtonian cosmology, the periodicity required for a numerical implementation would not be a crucial limiting factor.

One check on the correctness of the Newtonian treatment in the nonlinear regime is provided by comparison of N-body simulations with observations of structures. (In the simulations, matter with negative pressure or modified gravity is introduced to change the background expansion rate. Without such an addition, observations of the expansion rate and the distance scale are, of course, already completely discrepant with the Newtonian model.) On large scales significant differences between the simulations and observations have been reported. The observed homogeneity scale is an order of magnitude larger than in N-body simulations [75], and the number of very luminous superclusters is about 5 times larger in observations than in simulations [76]. Whether this reflects a deficiency of the Newtonian treatment, or instead indicates a problem in the way simulations are done or the observational data are analyzed is not clear. The discrepancy could also be due to an incorrect choice of initial conditions, matter content, or theory of gravity.

Relativistic dust models which are Newtonian-like are a very restricted class [77]. The close relation of linearly perturbed relativistic FRW models and Newtonian gravity may be misleading because Newtonian-like models suffer from a linearization instability. In general relativity, the Newtonian constraint that the magnetic component of the Weyl tensor vanishes identically is, in general, not propagated in time. However, in the linear theory, the constraint is trivially satisfied at all times. There are thus linear theory dust solutions which are not the limit of any nonlinear solution. More importantly, this shows that relativistic dust models do not, in general, have Newtonian counterparts and their evolution cannot be described in Newtonian theory. For practical applications in cosmology, the important issue is the quantitative importance of the non-Newtonian features, which depends on the solution under consideration. For addressing this question it would be useful to understand better the relation between the evolution of the electric and magnetic components of the Weyl tensor and spatial curvature in the context of cosmological structure formation.

III. COMPARISON TO PREVIOUS WORK

Arguments in linear theory

There have been various claims that backreaction in the real Universe is negligible [3,16,30–39]. In particular, it has been argued that the relative magnitude of backreaction corrections is given by the square of the peculiar velocity. All these studies, except for [31,38], expand quantities calculated with the first order metric to second order, which, as we have seen, is not in general consistent. Often the physical expansion rate, proper time, and hypersurface of averaging are also not correctly identified. That was first done in [5], while the first consistent second order calculation was done in [45]. Let us discuss some of the other shortcomings of these studies.

A numerical estimate of the correction to the expansion rate from expanding the first order metric to second order was first given in [30]. In [31] the Zel'dovich approximation was used to obtain the second order metric and calculate corrections to the expansion rate. In [3] the correction to the Einstein equation was calculated in the same manner. In [34,36,38] a calculation similar to the one in [5,45] was done, with some variations. (In [36], the perturbation of the volume element was inconsistently neglected.) In [32] a similar calculation for the correction to the 00-component of the Einstein equation was done, and nonlinear scaling relations were used for the density power spectrum, but this cannot compensate for using only the linear metric. As discussed in Sec. II A, such calculations lead to the correction term $\langle\partial_i\Phi\partial_i\Phi\rangle_0/(aH)^2$, which is of the order 10^{-5}. (In [31], the value 10^{-3} was obtained instead.) This result is the origin of the idea that the magnitude of backreaction is given by the square of the peculiar velocity, because at second order we have $\langle\partial_i\Phi\partial_i\Phi\rangle_0/(aH)^2 \sim \langle u_i u^i\rangle_0$. However, beyond second order, the expansion parameter $\langle\delta^2\rangle_0$ is also involved, so the second order calculation is inconclusive, and $\langle\partial_i\Phi\partial_i\Phi\rangle_0/(aH)^2 \ll 1$ is not a sufficient condition for small backreaction.

As an aside, note that u^i is the (spatial component of the) deviation of the physical velocity of observers comoving with the dust fluid from a fictitious background velocity. This is a coordinate-dependent quantity, and we can always set $u^i = 0$ by choosing coordinates which are comoving with the observers. In order to determine a physical peculiar velocity, we have to define another physical velocity field to compare u^α to [78]. (In the longitudinal gauge in the linear theory, $u_i u^i$ does give the physical magnitude of the deviation from uniform motion.)

In [33] it was asserted that the linear metric (2.3) (or the equivalent with a spatially curved background) describes the Universe on all scales, except in the vicinity of black holes and neutron stars. It was then claimed that if the conditions $|\Phi| \ll 1$, $|\dot\Phi|^2 \ll a^{-2}\partial_i\Phi\partial_i\Phi$, $(\partial_i\Phi\partial_i\Phi)^2 \ll \partial_i\partial_j\Phi\partial_i\partial_j\Phi$ hold, nonlinear corrections are negligible. In fact, the metric (2.3) cannot (with a dust source) simultaneously describe the static metric in the solar system and cosmological expansion, as is clear from the expressions for θ and δ in (2.20) and (2.21) together with (2.18). We can either use the metric with $H = 0$ to describe a static structure or take $H \neq 0$ to describe a cosmologically evolving region, but these conditions are obviously mutually incompatible. Apart from the inconsistency of using the linear theory to calculate second order quantities, if we nevertheless took (2.3) and expanded observables in terms of Φ, then at fourth order and higher we would expect to obtain terms involving $\langle\delta^2\rangle_0$, which are not necessarily small. (Of course, these corrections are meaningless without accounting for the intrinsic higher order terms.)

It was also argued in [33] that since the average expansion rate depends on the choice of the averaging hypersurface, accelerated expansion could arise as a gauge artifact. However, we should distinguish three different concepts, namely, gauge dependence, coordinate dependence, and dependence on the averaging hypersurface. Gauge dependence arises due to ambiguity in the mapping between the perturbed physical spacetime and a fictitious background spacetime. When points of the fictitious and real spacetime with the same coordinate values are taken to map to each other, gauge dependence reduces to choice of coordinates, but in general it is a distinct issue. In the covariant formalism with the full nonlinear equations, we deal only with physical quantities and the real spacetime, so there is no gauge issue. As all quantities are defined covariantly, independent of the choice of coordinates, the dependence on the coordinate system appears only in the usual transformation properties of tensors under coordinate changes. In particular, covariantly defined averages of scalar quantities such as the volume expansion rate do not depend on the coordinate system when expressed in terms of a physical observable such as the observer's proper time [45]. They do, however, depend on the choice on the averaging hypersurface [48,79,80]. The reason is that the averaging hypersurface is a physical issue, unlike coordinates or gauge. For irrotational dust, there is a preferred foliation which is orthogonal to the fluid flow, and which coincides with the hypersurface of constant proper time [10,17,41]. However, the hypersurface should be chosen based on analysis of observables, and cannot be determined on abstract mathematical grounds [10,19,21]. Any average quantities are of course useful only insofar they give an approximate description of what is actually measured. (For discussion of gauge invariance in averaging, see also [66,81].)

In [35] the linear metric was again expanded to second order. It was assumed that the average energy density is the same as the background energy density, which is not true beyond first order. Accordingly, one obtains equations which are inconsistent [17]. It was also argued that backreaction vanishes in a $2+1$-dimensional model. This is not surprising, because in $2+1$ dimensions, the integral of the spatial curvature is a topological invariant. Therefore it cannot evolve in time, similarly to the total energy of a Newtonian universe discussed in Sec. II B [17]. Therefore, in $2+1$ dimensions, backreaction can only give $\mathcal{Q} \propto a^{-4}$, and it is not clear whether the coefficient can be nonzero. [This follows from the $2+1$-dimensional analog of (2.30): in $d+1$ dimensions, the last term on the right-hand side is $-2d\frac{\partial_\tau a}{a}\mathcal{Q}$.]

In [16,37] an iterative calculation was done in the macroscopic gravity formalism [82], which is an extension of general relativity. We are interested in what happens in general relativity, but let us note that the method of [16,37] was to take the first order metric, expand to second order to obtain a new background, and then repeat the process. This way, one never moves beyond first order perturbation theory. As we have seen, in general relativity the higher order terms in general do not have the same form as the first and second order terms, so this kind of an analysis would not be correct there.

In [39] quantities were expanded to second order in the linear metric, with the usual result. The correction $\langle\partial_i\Phi\partial_i\Phi\rangle_0/(aH)^2 \times \bar{\rho} \sim v^2\bar{\rho}$ was identified with a pressure term. For clarity we note that the pressure measured by observers comoving with dust is zero by definition. The physical interpretation of the second order correction is spatial curvature, not pressure. To determine the pressure (and anisotropic stress and energy flux) generated in the process of structure formation, it is necessary to go beyond the ideal fluid treatment [83]. For the importance of non-dust terms for backreaction, see [21].

Let us also comment on some studies which claim not a small, but instead a possibly large backreaction effect from perturbation theory.

A series expansion similar to (2.25) was presented in [49]. It was argued that the expansion parameter is not $\langle\delta^2\rangle_0$, but a quantity which becomes of order unity around the present time. However, the expansion in [49] is incorrect, because it does not take into account the factorization of higher order terms into two-point functions and the constraint that a nonzero two-point function must contain an even number of momenta [17]. The only preferred era in the perturbative expansion is $\langle\delta^2\rangle_0 = 1$, signifying the formation of the first generation of gravitationally bound objects. For typical models of supersymmetric dark matter this happens around a redshift of 40–60 [84], considerably earlier than the present day. [As discussed above, the failure of the simple perturbative expansion (2.25) at $\langle\delta^2\rangle_0 = 1$ does not alone indicate that backreaction would be large.]

In [85] first order theory expanded to second order was used to estimate backreaction and compared to observations. Apart from the question of the applicability of first (or second) order perturbation theory, the correction term used is qualitatively wrong, because the momentum scale in the integral is misidentified with the size of the averaging domain (the suppression of the leading correction due to the fact that it is a boundary term is also neglected); see Sec. 5.1 of [17].

IV. CONCLUSION

If we consider cosmological perturbations which are initially small and Gaussian with zero mean, it is necessary to go at least to second order to find their effect on the average expansion of the Universe, called backreaction. Nevertheless, taking the first order metric in the longitudinal gauge and expanding quantities to second order serendipitously gives almost the correct second order result [5,45]. At second order, the perturbations only lead to a spatial curvature term with an amplitude of 10^{-5}. Several papers have thus claimed that backreaction is negligible in the real Universe, based on the argument that the linear metric is a good approximation even when density perturbations are nonlinear.

We have discussed why the linearly perturbed FRW metric does not in general correctly describe the situation once density perturbations become nonlinear. The effect on the average expansion rate vanishes at the linear level by construction, and at second order, the intrinsic second order terms are of the same order as squares of the first order terms (in fact, the division between the two is gauge dependent). At higher orders, generic correction terms become larger than unity as density perturbations become nonlinear. This does not necessarily mean that the effect on the average expansion rate is large, simply that the naive perturbative expansion is no longer valid.

The important question is not in which form the metric can be written, but what happens to measurable quantities. For this purpose it is useful to consider the covariant formalism, which deals only with physical degrees of freedom and is fully nonlinear. The effect of deviations from homogeneity and isotropy is quantified by the Buchert equations, which show that the average expansion rate will significantly deviate from the FRW behavior when the variance of the expansion rate is of order unity and does not cancel against the shear (or vice versa) [8]. Even in the linear (and second order) theory, the variance of the expansion rate becomes of order unity as density perturbations become nonlinear. However, there is no significant backreaction in relativistic second order theory because the variance cancels exactly against the shear apart from a boundary term, a feature shared by nonlinear Newtonian gravity. In exact general relativity, there is no such cancellation, so Newtonian theory is not sufficient for evaluating backreaction.

Determining whether structure formation in the real Universe leads to a large enough variance for backreaction to be important requires dealing with a locally complex nonperturbative system in general relativity. However, details of the local behavior are not needed, statistical information about the distribution of the expansion rate in different regions is enough. A semirealistic statistical calculation found a rise of 10%–30% in the expansion rate relative to the FRW value around a time of 10×10^9 years [17,18], which agrees with the observations within an order of magnitude. The calculation involved several approximations, and a more careful treatment is needed. In particular, the difference between relativistic and Newtonian cosmology should be better understood to isolate the relevant relativistic degrees of freedom, related to spatial curvature and the electric and magnetic components of the Weyl tensor.

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