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# Computation of the coefficients for $p^6$ order anomalous chiral Lagrangian

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We present the results of computing the order $p^6$ low energy constants in the anomalous part of the chiral Lagrangian for both two and three flavor pseudoscalar mesons. This is a generalization of our previous work on calculating the order $p^6$ coefficients for the normal part of the chiral Lagrangian in terms of the quark self-energy $\Sigma(p^2)$. We show that most of our results are consistent with those we have found in the literature.

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## I. INTRODUCTION AND BACKGROUND

It is well known that the chiral symmetry in quantum chromodynamics (QCD) suffers anomalies due to the noninvariance of the path integral measure of the quark fields under the chiral symmetry transformation. The anomaly reflects the fact that the classical chiral symmetry may be violated by quantum corrections. At the level of the effective chiral Lagrangian for the pseudoscalar meson field $U$, anomaly no longer comes from the path integral measure. Instead, it is due to the noninvariance of the effective chiral Lagrangian. If we denote by $\Gamma_{\rm eff}[U, J]$ the effective action for the pseudoscalar meson field $U$ and the external source $J$, then this noninvariance can be expressed as

$$\Gamma_{\rm eff}[U, J] - \Gamma_{\rm eff}[U_\Omega, J_\Omega] = \Gamma[\Omega, J], \tag{1}$$

where $U_\Omega \equiv \Omega^\dagger U \Omega^\dagger$ and $J_\Omega \equiv [\Omega P_R + \Omega^\dagger P_L] \times [J + \not\partial][\Omega P_R + \Omega^\dagger P_L]$. $\Gamma[\Omega, J]$ is the anomaly from the light quark path integral measure $\mathcal{D}\bar\psi_\Omega \mathcal{D}\psi_\Omega = \mathcal{D}\bar\psi \mathcal{D}\psi e^\Gamma$ or the well-known Wess-Zumino-Witten term. We can formally express it as

$$\begin{aligned}\Gamma[\Omega, J] &= -\ln\mathrm{Det}[(\Omega P_R + \Omega^\dagger P_L)(\Omega P_R + \Omega^\dagger P_L)] \\ &= -\mathrm{Tr}\ln[\not\partial + J_\Omega] + \mathrm{Tr}\ln[\not\partial + J].\end{aligned} \tag{2}$$

Because for $N_f$ light quarks, each generator of the chiral symmetry $SU(N_f)_L \otimes SU(N_f)_R / SU(N_f)_V$ corresponds to a Goldstone boson, which is treated phenomenologically as the physical pseudoscalar meson field, the phase angle of the chiral rotation group element $\Omega$ can be treated as the pseudoscalar meson field, i.e., $U = \Omega^2$. Then comparing (1) and (2), we can rewrite the effective action $\Gamma_{\rm eff}$ as

$$\begin{aligned}&\Gamma_{\rm eff}[U, J] = -\mathrm{Tr}\ln[\not\partial + J_\Omega] + \mathrm{Tr}\ln[\not\partial + J] + F[U, J], \\ &F[U, J] = F[U_\Omega, J_\Omega].\end{aligned} \tag{3}$$

The $U$ and $J$ dependence for $F[U, J]$ is not fixed by (1), but $F[U, J]$ is invariant on $U \to U_\Omega$ and $J \to J_\Omega$. Hence, $F[U, J]$ represents those chiral invariant terms. In fact, $U_\Omega = \Omega^\dagger \Omega^2 \Omega^\dagger = 1$, and $\Gamma_{\rm eff}[U_\Omega, J_\Omega] = \Gamma_{\rm eff}[1, J_\Omega] = -\mathrm{Tr}\ln[\not\partial + J_\Omega] + \mathrm{Tr}\ln[\not\partial + J_\Omega] + F[U_\Omega, J_\Omega] = F[U, J]$. Note that the effective action is the path integration result for $S_{\rm eff}[U, J]$, the action of the effective chiral Lagrangian for the pseudoscalar meson field $U$ and the external source $J$,

$$\begin{aligned}e^{-\Gamma_{\rm eff}[U_{cl}, J]} &= \int \mathcal{D}U e^{-S_{\rm eff}[U,J]}, \\ U_{cl}(x) &\equiv \int \mathcal{D}U U(x) e^{-S_{\rm eff}[U,J]},\end{aligned} \tag{4}$$

where the second equation gives the definition of $U_{cl}$, which fixes $U_{cl}$ as the functional of the external source $J$. With (3), (4) becomes

$$e^{\mathrm{Tr}\ln[\not\partial + J_\Omega] - \mathrm{Tr}\ln[\not\partial + J] - F(U_{cl}, J)} = \int \mathcal{D}U e^{-S_{\rm eff}[U,J]}. \tag{5}$$

References [1–3] choose as an approximation

$$S_{\rm eff,0}[U, J] = -\mathrm{Tr}\ln[\not\partial + J_\Omega] + \mathrm{Tr}\ln[\not\partial + J], \tag{6}$$

where subscript 0 is used to denote the approximation. From (1), (3), and (5), we find that under the chiral symmetry transformation, $S_{\rm eff,0}[U, J]$, defined in (6), is not invariant. Substituting (6) back into (5) and using standard loop expansion as developed in Ref. [4], we find $F[U_{cl}, J]$ is the pure loop correction from the action $S_{\rm eff,0}[U, J]$. From the action (6), one can calculate various low energy constants (LECs) of the effective chiral Lagrangian for pseudoscalar mesons. In Ref. [5], we call (6) the anomaly approach. In our previous paper [6], we have shown that the finite order $p^4$ LECs of the normal part of $S_{\rm eff,0}[U, J]$ are exactly canceled by the summation of all the $p^6$ and higher order terms. Equation (2) further shows that even for the anomalous part, $S_{\rm eff,0}[U, J]$ only contributes the Wess-Zumino-Witten term; it cannot produce the $p^6$ and higher order anomaly terms. This absence of the normal part and the $p^6$ and more higher order anomalous part reflect the fact that the choice of (6) is not correct, although it offers

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the correct Wess-Zumino-Witten term. Further, (6) is independent of the strong interaction dynamics, i.e., even we switch off the quark-gluon interaction by deleting the strong interaction coupling constant, (6) is not changed. These facts imply that we need to add some strong dynamics dependent correction term $\Delta S_{\rm eff}[U,J]$ to $S_{\rm eff,0}[U,J]$ as given in (6),

$$S_{\rm eff}[U,J] = S_{\rm eff,0}[U,J] + \Delta S_{\rm eff}[U,J]. \qquad (7)$$

From (5) and (6), we find that $\Delta S_{\rm eff}[U,J]$, introduced in (7), must be invariant under chiral symmetry transformations. In Refs. [7,8], $\Delta S_{\rm eff}[U,J]$ is taken to be

$$\Delta S_{\rm eff}[U,J] = {\rm Tr}\ln[\not\partial + J_\Omega + \Sigma(-\bar\nabla^2)], \qquad (8)$$

with $\Sigma$ being the quark self-energy satisfying the Schwinger-Dyson equation (SDE), and $\bar\nabla^\mu$ is defined as $\bar\nabla^\mu \equiv \partial^\mu - iv^\mu_\Omega$. This expression for $\Delta S_{\rm eff}[U,J]$ encodes the dynamics of the underlying QCD through quark self-energy $\Sigma$, and in Ref. [9], we have shown that (8) does not produce the Wess-Zumino-Witten term ensuring the correctness of (1).

In Ref. [7], we have calculated the orders $p^2$ and $p^4$ normal part LECs in terms of the action (7) and (8). The importance of knowledge of LECs of the chiral Lagrangian, especially for order $p^6$ LECs was emphasized in Ref. [10]. Recently, in Ref. [6], we improved the computation procedure and generalized the calculations up to the order $p^6$ normal part LECs. In Ref. [9], we have calculated the $p^4$ order anomalous part and shown that the $\Sigma$ dependent coefficient generates the correct coefficient $N_c$ for the Wess-Zumino-Witten term. It is the purpose of this paper to calculate all order $p^6$ LECs for the anomalous part of the chiral Lagrangian (7). In fact, the general structure of the $p^6$ order anomalous part chiral Lagrangian was first given by Refs. [11,12] and later clarified by Refs. [13,14]. Reference [15] estimates the values of several of the order $p^6$ LECs for the anomalous part of the chiral Lagrangian. Although order $p^6$ LECs for the normal part of the chiral Lagrangian seem to attract more attention in the literature (see references given in [6]), they are the next to next to leading order terms. The order $p^6$ LECs for the anomalous part of the chiral Lagrangian belong to the next leading order terms.

This paper is organized as follows: in Sec. II, we review the calculation of the order $p^4$ anomalous part of the chiral Lagrangian in terms of the action (7). With the method used in Sec. II, in Sec. III, we compute the order $p^6$ LECs for the anomalous part of the chiral Lagrangian, and obtain the analytical expression for the LECs in terms of quark self-energy $\Sigma$. We further compute the numerical values for these LECs. We compare our results with those obtained in the literature. Section IV is the summary and future direction of our work. We list some necessary tables and formulae in the appendices.

## II. REVIEW OF THE ORDER $p^4$ ANOMALOUS PART OF THE CHIRAL LAGRANGIAN

For the anomalous part of the chiral Lagrangian, the leading nontrivial order is $p^4$, and it is the well-known Wess-Zumino-Witten term. In Ref. [9], we have calculated the action (7) by several different methods and all obtain the same Wess-Zumino-Witten term. If we naively apply these methods to the next to leading order $p^6$ computations, we will find that they are too complex to be achieved even with the help of the computer. In this section, we build a method that is suitable to be generalized to the order $p^6$ calculations. The order $p^4$ of the anomalous chiral Lagrangian here is only to be used to explain our method. Reference [9] only expresses the Wess-Zumino-Witten term in terms of a parameter integration. In this section, we will explicitly finish this parameter integration and show that it does recover the Wess-Zumino-Witten term.

Since we are only interested in the $U$ field dependent part of the anomalous part of the chiral Lagrangian, we can drop out the pure source terms. Then our choice of $\Delta S_{\rm eff}[U,J]$ in (8) gives the result that only $\Sigma$ dependent terms in $\Delta S_{\rm eff}[U,J]$ contribute to the chiral Lagrangian, while the $\Sigma$ independent terms in $\Delta S_{\rm eff}[U,J]$ are completely canceled by the term $-{\rm Tr}\ln[\not\partial + J_\Omega]$ in $S_{\rm eff,0}[U,J]$, leaving a pure $U$ field independent term ${\rm Tr}\ln[\not\partial + J]$. So what we need to compute is

$$S_{\rm eff}[U,J] = [{\rm Tr}\ln[\not\partial + J_\Omega + \Sigma(-\bar\nabla^2)] - {\rm Tr}\ln[\not\partial + J + \Sigma(-\nabla^2)]]_{\Sigma{\rm dependent}}, \qquad (9)$$

in which we have added in $S_{\rm eff}[U,J]$ an extra pure source term $-{\rm Tr}\ln[\not\partial + J + \Sigma(-\nabla^2)]|_{\Sigma{\rm dependent}}$ for later use, and we define $\nabla^\mu \equiv \partial^\mu - iv^\mu$. Now we write $\Omega$ as $\Omega = e^{-i\beta}$ and further introduce a parameter $t$ dependent rotation element $\Omega(t) = e^{-it\beta}$. With the help of the relation $\Omega(1) = \Omega$ and $\Omega(0) = 1$, (9) becomes

$$S_{\rm eff}[U,J] = {\rm Tr}\ln[\not\partial + J_{\Omega(t)} + \Sigma(-\nabla_t^2)]|^{t=1}_{t=0,\Sigma\text{ dependent}}, \quad \nabla^\mu_t = \bar\nabla^\mu|_{\Omega\to\Omega(t)}, \qquad (10)$$

with $\nabla^\mu_t = \partial^\mu - iv^\mu_{\Omega(t)}$. $J_{\Omega(t)}$ is $J_\Omega$ with $\Omega$ replaced by $\Omega(t)$. We decompose $J_\Omega$ as $J_\Omega = -i\not v_\Omega - i\not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5$, so we can also decompose $J_{\Omega(t)}$ as $J_{\Omega(t)} = -i\not v_t - i\not a_t\gamma_5 - s_t + ip_t\gamma_5$. Result (10) implies that our chiral Lagrangian can be expressed as the difference of Tr ln(...) at $t$ dependent chiral rotation between $t = 1$ and $t = 0$. Since the $t$ dependent rotated source $J_{\Omega(t)}$ satisfies

$$\frac{\partial J_{\Omega(t)}}{\partial t} = \frac{1}{2}\bigg[\frac{\partial U_t}{\partial t}U_t^\dagger\gamma_5, \not\partial + J_{\Omega(t)}\bigg]_+ \qquad U_t = \Omega^2(t), \qquad (11)$$

we can further proceed to express the chiral Lagrangian in terms of integration over the parameter $t$:

$$\begin{aligned}S_{\rm eff}[U,J] &= \int_0^1 dt \frac{d}{dt} \,\mathrm{Tr}\ln[i\not\partial + J_{\Omega(t)} + \Sigma(-\nabla_t^2)]|_{\Sigma \text{ dependent}} \\ &= \int_0^1 dt\, \mathrm{Tr}\bigg[\bigg[\frac{\partial J_{\Omega(t)}}{\partial t} + \frac{\partial \Sigma(-\nabla_t^2)}{\partial t}\bigg][i\not\partial + J_{\Omega(t)} + \Sigma(-\nabla_t^2)]^{-1}\bigg]_{\Sigma \text{ dependent}} \\ &= \int_0^1 dt\, \mathrm{Tr}\bigg[\bigg(\frac{1}{2}\bigg[\frac{\partial U_t}{\partial t}U_t^\dagger \gamma_5, \not\partial + J_{\Omega(t)}\bigg]_+ + \frac{\partial \Sigma(-\nabla_t^2)}{\partial t}\bigg)[i\not\partial + J_{\Omega(t)} + \Sigma(-\nabla_t^2)]^{-1}\bigg]_{\Sigma \text{ dependent}}. \end{aligned} \tag{12}$$

Equation (12) is the main formula we rely on to calculate LECs. Reference [9] explicitly calculates the order $p^4$ anomalous part of the right-hand side (r.h.s.) of (12) and finds the result

$$\begin{aligned}S_{\rm eff}[U,J]|_{\text{anomalous } p^4} = &-2N_c\epsilon_{\mu\nu\alpha\beta}\int d^4x \int_0^1 dt \int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}_f\bigg[\frac{\partial U_t}{\partial t}U_t^\dagger\bigg(\frac{\Sigma(k^2)[\Sigma^2(k^2)-k^2][\Sigma(k^2)-2k^2\Sigma'(k^2)]}{[\Sigma^2(k^2)+k^2]^4} \\ &\times (2\nabla_t^\mu\nabla_t^\nu\nabla_t^\alpha\nabla_t^\beta + 2a_t^\mu a_t^\nu \nabla_t^\alpha\nabla_t^\beta - 2\nabla_t^\mu a_t^\nu \nabla_t^\alpha a_t^\beta + 2\nabla_t^\mu a_t^\nu a_t^\alpha \nabla_t^\beta + 2a^\mu\nabla_t^\nu\nabla_t^\alpha a_t^\beta - 2a_t^\mu\nabla_t^\nu a_t^\alpha\nabla_t^\beta \\ &+ 2\nabla_t^\mu\nabla_t^\nu a_t^\alpha a_t^\beta + 2a_t^\mu a_t^\nu a_t^\alpha a_t^\beta) + \frac{k^2\Sigma(k^2)[\Sigma(k^2)-2k^2\Sigma'(k^2)]}{[\Sigma^2(k^2)+k^2]^4}(4\nabla_t^\mu\nabla_t^\nu\nabla_t^\alpha\nabla_t^\beta + 2a_t^\mu a_t^\nu\nabla_t^\alpha\nabla_t^\beta \\ &- 2\nabla_t^\mu a_t^\nu\nabla_t^\alpha a_t^\beta + 4a_t^\mu\nabla_t^\nu\nabla_t^\alpha a_t^\beta - 2a_t^\mu\nabla_t^\nu a_t^\alpha\nabla_t^\beta + 2\nabla_t^\mu\nabla_t^\nu a_t^\alpha a_t^\beta)\bigg)\bigg]. \end{aligned} \tag{13}$$

The momentum integration can be calculated analytically, because the integrand is a total derivative. The result is

$$\begin{aligned}S_{\rm eff}[U,J]|_{\text{anomalous } p^4} = &\frac{1}{32\pi^2}\epsilon_{\mu\nu\alpha\beta}\int d^4x\int_0^1 dt\,\mathrm{tr}_f\bigg[\frac{\partial U_t}{\partial t}U_t^\dagger\bigg(V_t^{\mu\nu}V_t^{\alpha\beta} + \frac{2i}{3}[a_t^\mu a_t^\nu, V_t^{\alpha\beta}]_+ + \frac{4}{3}d_t^\mu a_t^\nu d_t^\alpha a_t^\beta + \frac{8i}{3}a_t^\mu V_t^{\nu\alpha}a_t^\beta \\ &+ \frac{4}{3}a_t^\mu a_t^\nu a_t^\alpha a_t^\beta\bigg)\bigg], \end{aligned} \tag{14}$$

where $V_t^{\mu\nu} = \partial^\mu v_t^\nu - \partial^\nu v_t^\mu - i[v_t^\mu, v_t^\nu]$ and $d_t^\mu a_t^\nu = \partial^\mu a_t^\nu - i[v_t^\mu, a_t^\nu]$. Reference [9] only gives the above result (14) without finishing the integration over parameter $t$. References [16,17] obtained the Wess-Zumino-Witten term by integrating out the Bardeen anomaly [18]. Now we use a similar method to achieve this integration, with the help of following relations:

$$\begin{aligned} &\frac{\partial a_t^\mu}{\partial t} = \frac{i}{2}\bigg[\nabla_t^\mu\frac{\partial U_t}{\partial t}U_t^\dagger - \frac{\partial U_t}{\partial t}U_t^\dagger\nabla_t^\mu\bigg], \qquad \frac{\partial v_t^\mu}{\partial t} = \frac{1}{2}\bigg[a_t^\mu\frac{\partial U_t}{\partial t}U_t^\dagger - \frac{\partial U_t}{\partial t}U_t^\dagger a_t^\mu\bigg], \\ &\frac{\partial s_t}{\partial t} = -\frac{i}{2}\bigg[p_t\frac{\partial U_t}{\partial t}U_t^\dagger + \frac{\partial U_t}{\partial t}U_t^\dagger p_t\bigg], \qquad \frac{\partial p_t}{\partial t} = \frac{i}{2}\bigg[s_t\frac{\partial U_t}{\partial t}U_t^\dagger + \frac{\partial U_t}{\partial t}U_t^\dagger s_t\bigg], \\ &\frac{\partial d^\mu a_t^\nu}{\partial t} = \frac{i}{2}\bigg[(\nabla_t^\mu\nabla_t^\nu + a_t^\nu a_t^\mu)\frac{\partial U_t}{\partial t}U_t^\dagger - \nabla_t^\nu\frac{\partial U_t}{\partial t}U_t^\dagger\nabla_t^\mu - \nabla_t^\mu\frac{\partial U_t}{\partial t}U_t^\dagger\nabla_t^\nu - a_t^\nu\frac{\partial U_t}{\partial t}U_t^\dagger a_t^\mu - a_t^\mu\frac{\partial U_t}{\partial t}U_t^\dagger a_t^\nu \\ &\qquad + \frac{\partial U_t}{\partial t}U_t^\dagger(\nabla_t^\nu\nabla_t^\mu + a_t^\mu a_t^\nu)\bigg], \\ &\frac{\partial V_t^{\mu\nu}}{\partial t} = \frac{1}{2}\bigg[-\nabla_t^\mu\frac{\partial U_t}{\partial t}U_t^\dagger a_t^\nu + \frac{\partial U_t}{\partial t}U_t^\dagger\nabla_t^\mu a_t^\nu - \frac{\partial U_t}{\partial t}U_t^\dagger d_t^\mu a_t^\nu + d_t^\mu a_t^\nu\frac{\partial U_t}{\partial t}U_t^\dagger + a_t^\nu\nabla_t^\mu\frac{\partial U_t}{\partial t}U_t^\dagger - a_t^\nu\frac{\partial U_t}{\partial t}U_t^\dagger\nabla_t^\mu \\ &\qquad + \nabla_t^\nu\frac{\partial U_t}{\partial t}U_t^\dagger a_t^\mu - \frac{\partial U_t}{\partial t}U_t^\dagger\nabla_t^\nu a_t^\mu + \frac{\partial U_t}{\partial t}U_t^\dagger d_t^\nu a_t^\mu - d_t^\nu a_t^\mu\frac{\partial U_t}{\partial t}U_t^\dagger - a_t^\mu\nabla_t^\nu\frac{\partial U_t}{\partial t}U_t^\dagger + a_t^\mu\frac{\partial U_t}{\partial t}U_t^\dagger\nabla_t^\nu\bigg], \end{aligned} \tag{15}$$

and by lengthy calculations, we can rewrite (14) as

$$S_{\rm eff}[U,J]|_{\text{anomalous } p^4} = -\frac{N_c}{48\pi^2}\int d^4x\int_0^1 dt\epsilon_{\mu\nu\lambda\rho}\,\mathrm{tr}_f\bigg[\frac{\partial U_t}{\partial t}U_t^\dagger R_t^\mu R_t^\nu R_t^\lambda R_t^\rho + \frac{d}{dt}W^{\mu\nu\lambda\rho}(U_t, l, r)\bigg], \tag{16}$$

with $l^\mu = v^\mu - a^\mu$, $r^\mu = v^\mu + a^\mu$, $R_t^\mu = U_t^\dagger\partial^\mu U_t$, $L_t^\mu = (\partial^\mu U_t)U_t^\dagger$, and

$$W^{\mu\nu\lambda\rho}(U_t, l, r) = iR_t^\mu R_t^\nu R_t^\lambda l^\rho + l^\mu \partial^\nu l^\lambda R_t^\rho + \partial^\mu l^\nu l^\lambda R_t^\rho - \tfrac{1}{2}R_t^\mu l^\nu R_t^\lambda l^\rho + r^\mu U_t l^\nu R_t^\lambda R^\rho U_t^\dagger + iR_t^\mu l^\nu l^\lambda l^\rho + iU_t^\dagger r^\mu U_t \partial^\nu l^\lambda l^\rho + iU_t^\dagger r^\mu \partial^\nu r^\lambda U_t l^\rho - il^\mu U_t^\dagger r^\nu U_t l^\lambda R_t^\rho - R_t^\mu U_t^\dagger \partial^\nu r^\lambda U_t l^\rho + U_t^\dagger r^\mu U_t l^\nu l^\lambda l^\rho + \tfrac{1}{4}U_t^\dagger r^\mu U_t l^\nu U_t^\dagger r^\lambda U_t l^\rho - (U_t \leftrightarrow U_t^\dagger, l^\mu \leftrightarrow r^\mu, L_t^\mu \leftrightarrow -R_t^\mu). \quad (17)$$

In Ref. [9], we already showed that the first term of the r.h.s. of Eq. (16) is just the Wess-Zumino-Witten term of the form defined on a four-dimensional boundary disc $Q$ in five-dimensional space-time

$$-\frac{N_c}{48\pi^2}\int d^4x \int_0^1 dt \epsilon_{\mu\nu\lambda\rho} \operatorname{tr}_f\left[\frac{\partial U_t}{\partial t} U_t^\dagger R_t^\mu R_t^\nu R_t^\lambda R_t^\rho\right] = -\frac{N_c}{240\pi^2}\int_Q d\Sigma_{ijklm} \operatorname{tr}_f[R^i R^j R^k R^l R^m] \quad (18)$$

with $R^i \equiv U^\dagger \partial^i U$. For the second term of the r.h.s. of Eq. (16), the integration over parameter $t$ can be calculated explicitly,

$$-\frac{N_c}{48\pi^2}\int d^4x \int_0^1 dt \epsilon_{\mu\nu\lambda\rho} \operatorname{tr}_f\left[\frac{d}{dt} W^{\mu\nu\lambda\rho}(U_t, l, r)\right] = -\frac{N_c}{48\pi^2}\int d^4x \epsilon_{\mu\nu\lambda\rho} \operatorname{tr}_f[W^{\mu\nu\lambda\rho}(U, l, r) - W^{\mu\nu\lambda\rho}(1, l, r)], \quad (19)$$

which is just the gauge part of the Wess-Zumino-Witten term given by Refs. [13,19]. This finishes the explicit calculation of the order $p^4$ anomalous part of the chiral Lagrangian starting from formula (12). We leave the order $p^6$ part to the next section.

## III. CALCULATION OF THE ORDER $p^6$ ANOMALOUS PART OF THE CHIRAL LAGRANGIAN

In this section, we start from Eq. (12) to calculate its order $p^6$ anomalous part of the chiral Lagrangian. For convenience, we change to the Minkowski space to perform our calculations. Direct computation gives the result

$$S_{\mathrm{eff}}[U, J]|_{\mathrm{anomalous}\ p^6} = \sum_{m=1}^{210}\int d^4x \bar{K}_m^W \int_0^1 dt \operatorname{tr}_f[\bar{O}_m^W(x, t)], \quad (20)$$

where $\bar{K}_m^W$ is the coefficient in front of the operator $\bar{O}_m^W(x, t)$, which depends on quark self-energy $\Sigma(k^2)$. The 210 parameter $t$ dependent operators $\bar{O}_m^W(x, t)$ all have the structure of $\bar{O}_m^W(x, t) = \epsilon_{\mu\nu\lambda\rho}\frac{\partial U_t}{\partial t}U_t^\dagger \bar{O}_m^{\mu\nu\lambda\rho}(x, t)$ and $\bar{O}_m^{\mu\nu\lambda\rho}(x, t)$ are order $p^6$ operators consisting of multiplications of various compositions of $a_t^\mu$, $\nabla_t^\nu$, $s_t$ and $p_t$. In Appendix A we list all of these operators in Table V. In obtaining (20), we have applied the Schouten identity, which reduces the original total 294 operators to the present 210 operators. In the literature, the general $p^6$ order anomalous part of the chiral Lagrangian given in Ref. [13] has only 24 independent operators. For $N_f = 3$, 2 this number reduces to 23 and 5, respectively. Especially for the case of $N_f = 2$, to incorporate the electromagnetic field into the external source $v^\mu$, the original traceless property of $v^\mu$ must be dropped. This changes the original five independent $p^6$ order anomalous operators into 13. If we denote the independent operators by $O_n^W(x)$ [$o_n^W(x)$ for $N_f = 2$] and corresponding coefficients in front of the operators by $C_n^W$ ($c_n^W(x)$ for $N_f = 2$), respectively, then (20) becomes

$$S_{\mathrm{eff}}[U, J]|_{\mathrm{anomalous}\ p^6} = \sum_{n=1}^{24}\int d^4x C_n^W O_n^W(x) \overset{N_f=2}{=\!=\!=\!=\!=} \sum_{n=1}^{13}\int d^4x c_n^W o_n^W(x). \quad (21)$$

Note that our starting chiral Lagrangian (7) only involves one trace for flavor indices. If we further apply the equation of motion to (21), there will appear some operators with two flavor traces. Our result prohibits the appearance of three operators $O_3^W$, $O_{18}^W$, $O_{24}^W$, leaving 21 independent operators. This implies that our formulation gives $C_3^W = C_{18}^W = C_{24}^W = 0$. If we do not apply the equation of motion, there will be more independent operators, and now this number is 23. To make our calculation more convenient, we denote these operators before applying the equation of motion by $\tilde{O}_n^W(x)$ and the corresponding coefficients in front of the operators by $\tilde{K}_n^W$. We list all possible $\tilde{O}_n^W(x)$ in Table VI of Appendix A. With these operators, (21) can also be written as

$$S_{\mathrm{eff}}[U, J]|_{\mathrm{anomalous}\ p^6} = \sum_{n=1}^{23}\int d^4x \tilde{K}_n^W \tilde{O}_n^W(x). \quad (22)$$

Through using the equation of motion, we can obtain the relations among the two sets of operators $\tilde{O}_n^W(x)$ and $O_n^W(x)$ as follows:

$$\tilde{O}_1^W = O_1^W/B_0, \quad \tilde{O}_2^W = O_2^W/B_0, \quad \tilde{O}_3^W = O_4^W/B_0, \quad \tilde{O}_4^W = O_5^W/B_0, \quad \tilde{O}_5^W = O_7^W/B_0, \quad \tilde{O}_6^W = O_9^W/B_0,$$
$$\tilde{O}_7^W = O_{11}^W/B_0, \quad \tilde{O}_8^W = O_{12}^W, \quad \tilde{O}_9^W = O_1^W, \quad \tilde{O}_{10}^W = O_{16}^W, \quad \tilde{O}_{11}^W = O_{17}^W, \quad \tilde{O}_{12}^W = O_{13}^W,$$
$$\tilde{O}_{13}^W = O_{14}^W, \quad \tilde{O}_{14}^W = O_{15}^W, \quad \tilde{O}_{15}^W = -O_4^W + \frac{2}{N_f}O_6^W, \quad \tilde{O}_{16}^W = -O_5^W - \frac{1}{N_f}O_6^W, \quad \tilde{O}_{17}^W = O_{19}^W, \quad \tilde{O}_{18}^W = O_{20}^W,$$
$$\tilde{O}_{19}^W = O_{21}^W, \quad \tilde{O}_{20}^W = O_{22}^W, \quad \tilde{O}_{21}^W = O_{23}^W, \quad \tilde{O}_{22}^W = O_7^W - \frac{1}{N_f}O_8^W, \quad \tilde{O}_{23}^W = O_9^W - \frac{1}{N_f}O_{10}^W, \tag{23}$$

where $B_0$ is the order $p^2$ LEC in the normal part of the chiral Lagrangian. Here, we divide $O_1^W, \cdots, O_7^W$ by $B_0$, making the matrices $A_{mn}$ introduced later in Eq. (27) independent of $B_0$. For $N_f = 2$, (23) is changed to

$$\tilde{O}_1^W = 0, \quad \tilde{O}_2^W = o_1^W/B_0, \quad \tilde{O}_3^W = o_2^W/B_0, \quad \tilde{O}_4^W = -o_2^W/(2B_0) + o_6^W/B_0, \quad \tilde{O}_5^W = o_3^W/B_0,$$
$$\tilde{O}_6^W = o_4^W/B_0, \quad \tilde{O}_7^W = o_5^W/B_0, \quad \tilde{O}_8^W = \tilde{O}_9^W = \tilde{O}_{10}^W = \tilde{O}_{11}^W = 0, \quad \tilde{O}_{12}^W = -o_9^W,$$
$$\tilde{O}_{13}^W = \tilde{O}_{14}^W = -\tfrac{1}{2}o_6^W + o_9^W, \quad \tilde{O}_{15}^W = -o_6^W, \quad \tilde{O}_{16}^W = -\tfrac{1}{2}o_6^W \quad \tilde{O}_{17}^W = o_{10}^W, \quad \tilde{O}_{18}^W = \tilde{O}_{19}^W = -o_{10}^W,$$
$$\tilde{O}_{20}^W = \tfrac{1}{4}o_7^W - \tfrac{1}{8}o_8^W - o_{10}^W + o_{11}^W - 2o_{13}^W, \quad \tilde{O}_{21}^W = 0, \quad \tilde{O}_{22}^W = o_7^W - \tfrac{1}{2}o_8^W, \quad \tilde{O}_{23}^W = 0. \tag{24}$$

Direct comparison between (20) and (22) is difficult, since in (20) we have an extra integration over parameter $t$, and the number of operators in (20) is much larger than it is in (22). Instead of finishing the integration over parameter $t$ in (20), we introduce an integration of parameter $t$ in (22). Since we are only interested in the $U$ dependent part of the chiral Lagrangian, adding some $U$ field independent pure source terms in (22) will not change our result; therefore, we can rewrite (22) as

$$\begin{aligned} S_{\rm eff}[U,J]|_{\text{anomalous } p^6} &= \sum_{n=1}^{23}\int d^4x \tilde{K}_n^W[\tilde{O}_n^W(x) - \tilde{O}_n^W(x)|_{U=1}] = \sum_{n=1}^{23}\int d^4x \tilde{K}_n^W[\tilde{O}_n^W(x)|_{U\to U_{t=1}} - \tilde{O}_n^W(x)|_{U\to U_{t=0}}] \\ &= \sum_{n=1}^{23}\int d^4x \tilde{K}_n^W[\tilde{O}_n^W(x)|_{U\to U_t}]|_{t=0}^{t=1} = \sum_{n=1}^{23}\int d^4x \tilde{K}_n^W \int_0^1 dt \frac{d}{dt}[\tilde{O}_n^W(x)|_{U\to U_t}]. \end{aligned} \tag{25}$$

In expression (25), integration of parameter $t$ is already present in the formula of that in (25), and there are only 23 independent terms acted on by the differential of $t$, while in (20) there are 210 terms. Comparing (25) and (20), we obtain

$$\sum_{n=1}^{23} \tilde{K}_n^W \frac{d}{dt}[\tilde{O}_n^W(x)|_{U\to U_t}] = \sum_{m=1}^{210} \bar{K}_m^W \bar{O}_m^W(x,t). \tag{26}$$

Note that with the help of relation (15), $\frac{d}{dt}[\tilde{O}_n^W(x)|_{U\to U_t}]$ appearing in the above equation can be reduced to the linear composition of $\bar{O}_m^W(x,t)$, i.e.,

$$\frac{d}{dt}[\tilde{O}_n^W(x)|_{U\to U_t}] = \sum_{m=1}^{210} A_{nm}\bar{O}_m^W(x,t), \tag{27}$$

with the $23\times 210$ matrix $A_{nm}$ given by Table VII in Appendix B, Then we rearrange (27) by multiplying both sides of the equation by some $23\times 23$ matrix elements $C_{n'n}$,

$$\begin{aligned} \sum_{n=1}^{23} C_{n'n}\frac{d}{dt}[\tilde{O}_n^W(x)|_{U\to U_t}] &= \sum_{n=1}^{23}\sum_{m=1}^{210} C_{n'n}A_{nm}\bar{O}_m^W(x,t) \\ &= \sum_{m=1}^{210} R_{n'm}\bar{O}_m^W(x,t) \\ R_{n'm} &\equiv \sum_{n=1}^{23} C_{n'n}A_{nm} \end{aligned} \tag{28}$$

and tune $C_{n'n}$ in such a way that a $23\times 23$ submatrix $R'$ is a unit matrix, i.e., $R'_{n'm'} = \delta_{n'm'}$ with $n', m' = 1, 3, 4, 5, 6, 7, 20, 43, 44, 49, 50, 51, 52, 54, 57, 59, 62, 63, 64, 127, 128, 133, 134$. The $C$ matrix is found to be of the form

$$\begin{pmatrix} \bar{C}_{7\times 7} & 0_{7\times 15} \\ 0_{15\times 7} & \tilde{C}_{15\times 15} \end{pmatrix},$$

where $\bar{C}$ and $\tilde{C}$ are $7\times 7$ and $15\times 15$ matrices, respectively. The off diagonal parts are two matrices with null matrix elements and the dimensions are $7\times 15$ and $15\times 7$. We label the dimension of the submatrices as their subscripts. $\bar{C}$ and $\tilde{C}$ matrices are given in Tables VIII and IX in Appendix B. We call the remaining part of $R_{n'm}$ the matrix $R_{n'm''}$, $m'' \neq m'$. Then (28) is changed to

$$\sum_{n=1}^{23} C_{m'n}\frac{d}{dt}[\tilde{O}_n^W(x)|_{U\to U_t}] = \bar{O}_{m'}^W(x,t) + \sum_{m''} R_{m'm''}\bar{O}_{m''}^W(x,t). \quad (29)$$

Multiplying both sides of the above equation by $\bar{K}_{m'}^W$,

$$\sum_{m'}\sum_{n=1}^{23} \bar{K}_{m'}^W C_{m'n}\frac{d}{dt}[\tilde{O}_n^W(x)|_{U\to U_t}] = \sum_{m'}\bar{K}_{m'}^W\bar{O}_{m'}^W(x,t) + \sum_{m'}\sum_{m''}\bar{K}_{m'}^W R_{m'm''}\bar{O}_{m''}^W(x,t). \quad (30)$$

Comparing (30) and (26), to make these two equations consistent with each other, we must have conditions

$$\tilde{K}_n^W = \sum_{m'}\bar{K}_{m'}^W C_{m'n}, \qquad \bar{K}_{m''}^W = \sum_{m'}\bar{K}_{m'}^W R_{m'm''}, \quad (31)$$

in which the second equation is a consistency check for the coefficients $\bar{K}_{m''}^W$ of the dependent operators $\bar{O}_{m''}^W(x,t)$. We have checked analytically that these constraints are all automatically satisfied, and this can be seen as a consistency check of our formulation. The first equation gives $\tilde{K}_n^W$ in terms of $\bar{K}_{m'}^W$ and $C_{m'n}$. Substituting it in the expressions obtained for $\bar{K}_{m'}^W$ and $C_{m'n}$, we finally obtain the 23 order $p^6$ LECs for the three and more flavors anomalous part of the chiral Lagrangian.

The resulting analytical expressions for $\tilde{K}_n^W$ as functions of quark self-energy $\Sigma$ are given in Appendix C. With $\tilde{K}_n^W$ given in Appendix C, we can choose a suitable running coupling constant $\alpha_s(p^2)$, solve the Schwinger-Dyson equation numerically, obtaining the quark self-energy $\Sigma$, and then calculate the numerical values of all order $p^6$ anomalous LECs. To obtain the final numerical result, we have assumed $F_0 = 87$ MeV as input to fix the dimensional parameter $\Lambda_{\rm QCD}$ appearing in the running coupling constant $\alpha_s(p^2)$ and taken momentum cutoff $\Lambda = 1.00^{+0.10}_{-0.10}$ GeV. Because of the appearance of the divergent order $p^2$ LEC $B_0$ in Eqs. (23) and (24), we need a momentum cutoff $\Lambda$ to make $B_0$ finite as we did previously in Ref. [6]. Unlike the case of the normal part, where $\Lambda$ enters into the final expression of LECs either through coefficient $B_0$ or the lower bound of the proper time integration which plays the role of suppressing the ultraviolet momentum contributions to the momentum integration, now $\Lambda$ dependence for anomalous LECs is only from $B_0$. This is due to the fact that all momentum integrations for LECs given in Appendix C are finite which do not need momentum cutoff. In Table I, we give the numerical values for all 21 nonzero LECs for three flavors ($C_3^W = C_{18}^W = 0$ in our formulation). Combined with our numerical result, we also list the numerical estimates for some of the LECs from five different models and different processes given in Refs. [15,21–24]. In Ref. [15], models I and III are all from direct chiral perturbation (ChPT) computations, except that model I is the full ChPT result, while in model III, low energy experiment data are extrapolated to the high energy region; model II is the vector meson dominance model (VMD); models IV and V are the chiral constituent quark model (CQM), with some extrapolations included in model V. For a fixed model, different processes may give different results. For example, in model I for $C_7^W$ and models I and IV for $C_{22}^W$, we obtain two results from two different processes. Further, Refs. [15,23,24] also give estimations on some combinations or ratios of LECs. We list our and their results in Table II. For $N_f = 2$, in Table III, we give the numerical values of all 12 nonzero LECs ($c_{12}^W = 0$ in our formulation), which are actually of the very same structure as that given by [13]. We see that most of our results are consistent with those that we have found in the literature.

To compensate for the missing information in Tables I and III, which only gives the result of LECs at $\Lambda = 1.00^{+0.10}_{-0.10}$ GeV, in the following, we discuss their dependence on $\Lambda$. As we mentioned previously, $\Lambda$'s dependence on LECs is from the coefficient $B_0$. $B_0F_0^2 = -\langle\bar{\psi}\psi\rangle/N_f$ depends on $\Lambda$ going through the light quark condensate $\langle\bar{\psi}\psi\rangle$, which is divergent if we take $\Lambda$ to infinity. $\langle\bar{\psi}\psi\rangle$ and $F_0^2$ both rely on the quark self-energy $\Sigma(p^2)$, which can be obtained from the SDE. We denote the LEC at $\Lambda$ as $C_n^W(\Lambda) \equiv C_n^W|_\Lambda$ for the three flavor $n = 1, 2, \ldots, 23$ and $c_n^W(\Lambda) \equiv c_n^W|_\Lambda$ for the two flavor $n = 1, 2, \ldots, 13$, taking their values at $\Lambda = 1$ GeV (which are our central values given in Table. I and III); as a reference, see plots $C_n^W(\Lambda)/C_n^W(1\text{ GeV})$ and $c_n^W(\Lambda)/c_n^W(1\text{ GeV})$ in Fig. 1.

In Fig. 1, the $\Lambda$ dependence is drawn from 0.75 GeV to 2 GeV. The reason that the lowest $\Lambda$ is only taking 0.75 GeV is that below this value, the SDE cannot give the correct solution $\Sigma(p^2)$ for the fixed $F_0 = 87$ MeV. Further, we find in Fig. 1 that several $C_n^W(\Lambda)/C_n^W$ (1 GeV)s for $n = 2$, 11 or $n = 6$, 8, 12 $-$17, 19 $-$23 and $c_n^W(\Lambda)/c_n^W$ (1 GeV)s for $n = 1 - 5$ or $n = 7 - 11$, 13 shrink to the same curves. This is due to the fact that these LECs have the same $B_0$-dependence behaviors.

It should be emphasized that our calculation is under the large $N_c$ limit. Within this approximation, standard chiral Lagrangian LECs are all finite since all pseudoscalar meson loop effects that cause ultraviolet divergences of LECs belong to the $1/N_c$ order corrections. The reason that our LECs display ultraviolet divergences is due to the fact that our calculation is not exact large $N_c$ limit, i.e., we have made some further approximations within the large $N_c$ limit. And it is these approximations which caused the unexpected ultraviolet divergences. Because of this, we use the difference of LECs at different $\Lambda$s, such as $C_n^W|_{\Lambda=1.1\text{ GeV}} - C_n^W|_{\Lambda=1\text{ GeV}}$, to estimate the errors of our computations.

Further, the suppression of the pseudoscalar meson loop effects from $1/N_c$ implies that the conventional running effects of LECs are also suppressed and then our LECs will not run, i.e., we cannot calculate these running effects

TABLE I. The nonzero values of the order $p^6$ anomalous LECs $C_1^W, C_2^W, C_4^W, \ldots, C_{17}^W, C_{19}^W, \ldots, C_{23}^W$ for three flavors. The LECs are in units of $10^{-3}$ $\mathrm{GeV}^{-2}$. The second column is our resultant LECs with the values at $\Lambda = 1$ GeV with superscript the difference caused at $\Lambda = 1.1$ GeV (i.e., $C_i^W|_{\Lambda=1.1\ \mathrm{GeV}} - C_i^W|_{\Lambda=1\ \mathrm{GeV}}$) and subscript the difference caused at $\Lambda = 0.9$ GeV (i.e., $C_i^W|_{\Lambda=0.9\ \mathrm{GeV}} - C_i^W|_{\Lambda=1\ \mathrm{GeV}}$). The third column to the seventh column contain the results given in Ref. [15]: (I)-ChPT, (II)-VMD, (III)-ChPT (extrapolation), (IV)-CQM, (V)-CQM (extrapolation). The eighth column shows the results from Refs. [21–24].

| $n$ | $C_n^w$ ours | [15] (I) | [15] (II) | [15] (III) | [15] (IV) | [15] (V) | [21–24] |
|---|---|---|---|---|---|---|---|
| 1 | $4.97^{+0.55}_{-0.79}$ | | | | | | |
| 2 | $-1.43^{+0.10}_{-0.12}$ | $-0.32 \pm 10.4$ | | $0.78 \pm 12.7$ | $4.96 \pm 9.70$ | $-0.074 \pm 13.3$ | |
| 4 | $-0.96^{+0.22}_{-0.29}$ | $0.28 \pm 9.19$ | | $0.67 \pm 10.9$ | $6.32 \pm 6.09$ | $-0.55 \pm 9.05$ | |
| 5 | $3.26^{+0.34}_{-0.49}$ | $28.50 \pm 28.83$ | | $9.38 \pm 152.2$ | $33.05 \pm 28.66$ | $34.51 \pm 41.13$ | |
| 6 | $0.91^{+0.03}_{-0.04}$ | | | | | | |
| 7 | $1.68^{-0.24}_{+0.31}$ | $0.013 \pm 1.17$ | | | $0.51 \pm 0.06$ | | $0.1 \pm 1.2$ |
| | | $20.3 \pm 18.7$ | | | | | 0.1[a] |
| 8 | $0.41^{+0.01}_{-0.02}$ | $0.76 \pm 0.18$ | | | | | $0.58 \pm 0.20$ |
| | | | | | | | 0.5[a] |
| 9 | $1.15^{-0.03}_{+0.03}$ | | | | | | |
| 10 | $-0.18^{-0.01}_{+0.01}$ | | | | | | |
| 11 | $-1.15^{+0.08}_{-0.10}$ | $-6.37 \pm 4.54$ | | | $-0.00143 \pm 0.03$ | | $0.68 \pm 0.21$ |
| 12 | $-5.13^{-0.15}_{+0.25}$ | | | | | | |
| 13 | $-6.37^{-0.18}_{+0.31}$ | $-74.09 \pm 55.89$ | $-20.00$ | $-8.44 \pm 69.9$ | $14.15 \pm 15.22$ | $-7.46 \pm 19.62$ | |
| 14 | $-2.00^{-0.06}_{+0.10}$ | $29.99 \pm 11.14$ | $-6.01$ | $0.72 \pm 15.3$ | $10.23 \pm 7.56$ | $-0.58 \pm 9.77$ | |
| 15 | $4.17^{+0.12}_{-0.20}$ | $-25.30 \pm 23.93$ | 2.00 | $-3.10 \pm 28.6$ | $19.70 \pm 7.49$ | $8.89 \pm 9.72$ | |
| 16 | $3.58^{+0.10}_{-0.17}$ | | | | | | |
| 17 | $1.98^{+0.06}_{-0.10}$ | | | | | | |
| 19 | $0.29^{+0.01}_{-0.01}$ | | | | | | |
| 20 | $1.83^{+0.05}_{-0.09}$ | | | | | | |
| 21 | $2.48^{+0.07}_{-0.12}$ | | | | | | |
| 22 | $5.01^{+0.14}_{-0.24}$ | $6.52 \pm 0.78$ | 8.01 | | $3.94 \pm 0.43$ | | $5.4 \pm 0.8$ |
| | | $5.07 \pm 0.71$ | | | $3.94 \pm 0.43$ | | |
| 23 | $2.74^{+0.08}_{-0.13}$ | | | | | | |

[a]This result is just the absolute value given in Ref. [23].

TABLE II. Some combinations or ratios of LECs in units of $10^{-3}$ $\mathrm{GeV}^{-2}$. The second column is our resultant LECs with the values at $\Lambda = 1$ GeV, and with the superscript as the difference caused at $\Lambda = 1.1$ GeV and the subscript as the difference caused at $\Lambda = 0.9$ GeV. The third column to the fifth column contain the results given in Ref. [15]: (I)-ChPT, (II)-VMD, (III)-ChPT (extrapolation), (IV)-CQM, (V)-CQM (extrapolation). The sixth and seventh columns contain the results given in Refs. [23,24], respectively.

| | Ours | [15] | [15] | [15] | [23] | [24] |
|---|---|---|---|---|---|---|
| $C_3^W - C_6^W$ | $-0.91^{-0.03}_{+0.04}$ | $21.67 \pm 17.41$ (I) | $5.07 \pm 5.07$ (IV) | $-2.14 \pm 6.54$ (V) | | |
| $2C_{15}^W - 4C_{14}^W + C_{13}^W$ | $9.95^{+0.29}_{-0.48}$ | $-244.7 \pm 148.4$ (I) | $\approx 8.0$ (II) | $-17.52 \pm 188.3$ (III) | | |
| $2C_{14}^W - C_{13}^W$ | $2.38^{+0.07}_{-0.12}$ | $134.1 \pm 78.17$ (I) | $\approx 8.0$ (II) | $9.88 \pm 100.5$ (III) | | |
| $\lvert C_7^W \rvert / \lvert C_8^W \rvert$ | $4.12^{-0.69}_{+1.01}$ | | | | 0.2 | $<0.1$ |

TABLE III. The nonzero values of the $p^6$ order anomalous LECs $c_1^W, \ldots, c_{11}^W, c_{13}^W$ for the two flavors in units of $10^{-3}$ $\mathrm{GeV}^{-2}$.

| $c_1^W$ | $c_2^W$ | $c_3^W$ | $c_4^W$ | $c_5^W$ | $c_6^W$ | $c_7^W$ | $c_8^W$ | $c_9^W$ | $c_{10}^W$ | $c_{11}^W$ | $c_{13}^W$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $-1.46^{+0.10}_{-0.12}$ | $-1.25^{+0.09}_{-0.11}$ | $2.96^{-0.20}_{+0.25}$ | $0.63^{-0.04}_{+0.05}$ | $-1.17^{+0.08}_{-0.10}$ | $0.77^{+0.26}_{-0.36}$ | $-0.04^{-0.00}_{+0.00}$ | $0.02^{+0.00}_{-0.00}$ | $8.19^{+0.23}_{-0.38}$ | $-8.73^{-0.24}_{+0.41}$ | $4.85^{+0.13}_{-0.23}$ | $-9.70^{-0.27}_{+0.45}$ |

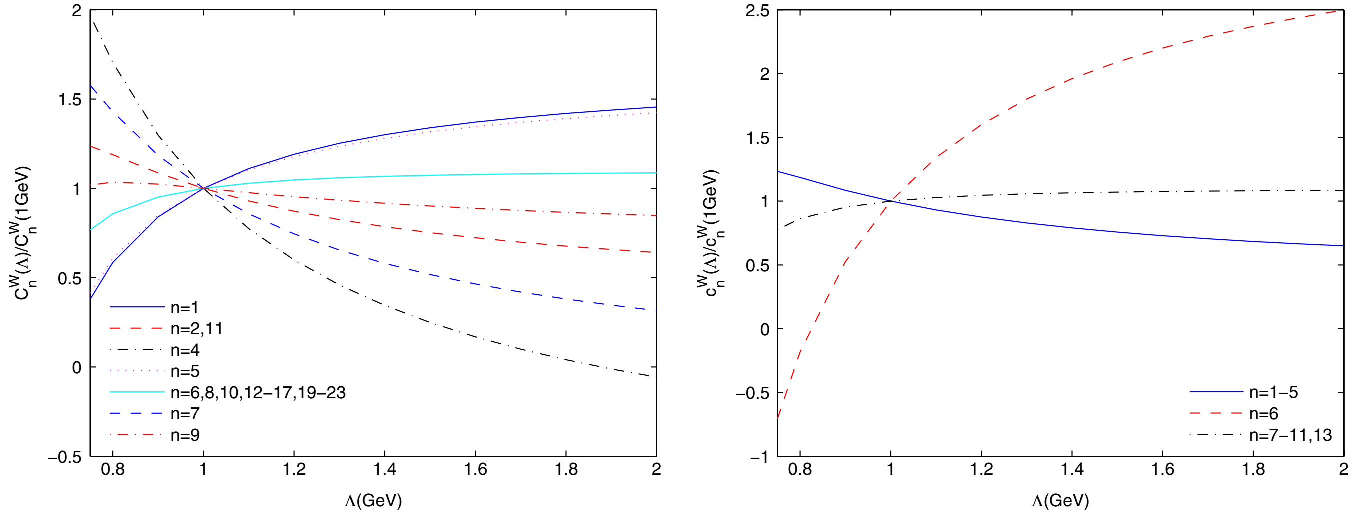


FIG. 1 (color online). The $\Lambda$ dependence of $C_n^W(\Lambda)$ and $c_n^W(\Lambda)$.

caused by pseudoscalar meson loop corrections from our first principle computation at present. Therefore, the $\Lambda$ dependence given in Fig. 1 has nothing to do with the conventional running behavior predicted by the chiral Lagrangian. As a comparison, in Fig. 2, we plot the standard running behavior of LECs given by Ref. [13], where we denote the LEC at scale $\mu$ as $C_n^W(\mu)$ for the three flavor $n = 1, 2, \ldots, 23$ and $c_n^W(\mu)$ for the two flavor $n = 1, 2, \ldots, 13$, taking their values at the $\mu = 1$ GeV to be our central values given in Tables I and III (the same as those used in Fig. 1) as reference.

We see that Figs. 1 and 2 give two completely different behaviors on the scale dependence of LECs. Predicting the conventional running effects of LECs from underlying QCD is a challenging problem that is already beyond the ability of our present formalism. We will investigated it in the future.

At last, as a phenomenological check for the two flavor anomalous LECs, we discuss the $\pi^0 \rightarrow \gamma\gamma$ process. Reference [24] gives the amplitude of this process by

$$T_{\rm LO+NLO} = \frac{1}{F}\bigg\{\frac{1}{4\pi^2} + \frac{16}{3}m_\pi^2(-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr}) + \frac{64}{9}B(m_d - m_u)(5c_3^{Wr} + c_r^{Wr} + 2c_8^{Wr})\bigg\}. \tag{32}$$

In our calculation, we choose the center value $B(m_d - m_u) = 0.32m_{\pi^0}^2$ given in Ref. [24]. Experimentally, the $\pi^0 \rightarrow \gamma\gamma$ process dominates the lifetime of $\pi^0$ to 98.79%, and if we ignore that small fraction from other processes, then the lifetime of $\pi^0$ can be expressed in terms of amplitude $T$ as $1/\tau = \pi\alpha m_\pi^3 T^2/4$. In Table IV we give our result for $\tau_{\rm LO}$ up to the leading order $p^4$, which corresponds to the first term of the r.h.s of Eq. (32), and $\tau_{\rm NLO}$ up to the next leading order $p^6$ of the low energy expansion. Experimental results from the Particle Data Group [25] are also included in the table for comparison.

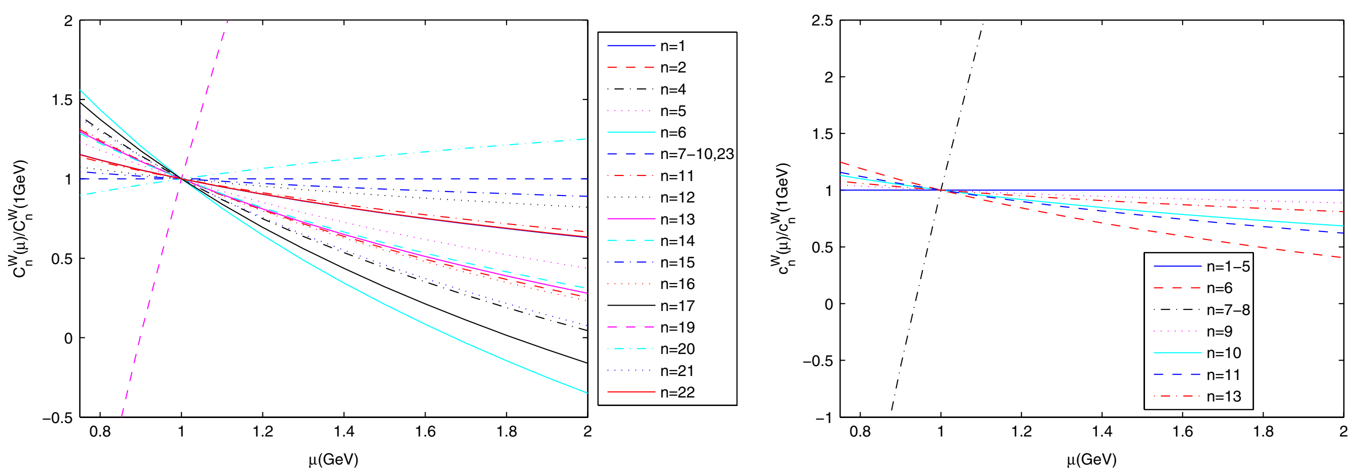


FIG. 2 (color online). The running behavior of $C_n^W(\mu)$ and $c_n^W(\mu)$.

TABLE IV. $\pi^0$ lifetime in units of $10^{-17}$ s.

| | $\tau_{\text{LO}}$ | $\tau_{\text{NLO}}$ |
|---|---|---|
| $F = 87$ MeV | 7.56 | $7.59^{-0.03}_{+0.04}$ |
| $F = 93$ MeV | 8.63 | $8.67^{-0.03}_{+0.04}$ |
| Exp. [25] | $8.4 \pm 0.6$ | |

Our result roughly matches the experimental value and we see that the order $p^6$ results have less effect on the lifetime of $\pi^0$.

## IV. SUMMARY AND FUTURE WORK

In this work, we review the general anomaly structure of the effective chiral Lagrangian and then generalize our order $p^6$ calculation in Ref. [6] from the normal part to the anomalous part of the chiral Lagrangian for pseudoscalar mesons. The result is obtained by computing the imaginary $\Sigma$ dependent part of $\mathrm{Tr}\ln[\not{\partial} + J_\Omega + \Sigma(-\bar{\nabla}^2)]$. To match the calculation of the order $p^4$ anomalous part, in practice we calculate the integration of parameter $t$ over $\frac{d}{dt}\mathrm{Tr}\ln[i\not{\partial} + J_{\Omega(t)} + (-\nabla_t^2)]$. The conventional chiral Lagrangian is also reformulated to an integration of $t$ and through comparison of it with our result, we read out all order $p^6$ anomalous LECs expressed in terms of quark self-energy $\Sigma$. Inputting the SDE solution of $\Sigma(k^2)$, we obtain numerical values and compare them with those we could find in the literature. Some of them are consistent, some are not. We leave those inconsistent results to future investigations. Combined with the previous result on the order $p^6$ normal LECs given in Ref. [6], we have now completed all the order $p^6$ LECs computations. Based on them, one direction of further research is to apply the order $p^6$ chiral Lagrangian to various pseudoscalar meson processes and discuss the corresponding physics. Another direction is to improve the precision of (7) and our ladder approximation of SDE. With these improvements, we expect a more precise estimation on all LECs in the future.

## ACKNOWLEDGMENTS

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## APPENDIX A: LIST OF ALL OPERATORS $\bar{O}_n^{\mu\nu\lambda\rho}$ AND $\bar{O}_n^W$

In this appendix, we first explicitly write down all 210 $\bar{O}_n^{\mu\nu\lambda\rho}$ operators in Table V, which is cited in the sentence after Eq. (20) in Sec. III of the text. In Table V, to save space, we use some simplified symbols to represent the original symbols in the text. Our $\bar{O}_n^{\mu\nu\lambda\rho}$s are constructed in such a way that they are invariant under charge conjugation transformation. This causes the result that most of $\bar{O}_n^{\mu\nu\lambda\rho}$s consist of two terms that are charge conjugates to each other.

TABLE V. $\mu \equiv \nabla_t^\mu$, $\nu \equiv \nabla_t^\nu$, $\lambda \equiv \nabla_t^\lambda$, $\rho \equiv \nabla_t^\rho$, $\bar{\mu} \equiv a_t^\mu$, $\bar{\nu} \equiv a_t^\nu$, $\bar{\lambda} \equiv a_t^\lambda$, $\bar{\rho} \equiv a_t^\rho$, $s \equiv s_t$, $p \equiv p_t \cdots \sigma \cdots \sigma \equiv \cdots \nabla_t^\sigma \cdots \nabla_{t,\sigma}$, $\cdots \bar{\sigma} \cdots \bar{\sigma} \equiv \cdots a_t^\sigma \cdots a_{t,\sigma}$, $\cdots \sigma \cdots \bar{\sigma} \equiv \cdots \nabla_t^\sigma \cdots a_{t,\sigma}$, $\cdots \bar{\sigma} \cdots \sigma \equiv \cdots a_t^\sigma \cdots \nabla_{t,\sigma}$.

| $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ | $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ | $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ | $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ | $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | $s\mu\nu\lambda\rho + \mu\nu\lambda\rho s$ | 43 | $\mu\nu\lambda\rho\sigma\sigma + \sigma\sigma\mu\nu\lambda\rho$ | 85 | $\mu\sigma\bar{\nu}\lambda\rho\bar{\sigma} + \bar{\sigma}\mu\nu\bar{\lambda}\sigma\rho$ | 127 | $\mu\nu\bar{\lambda}\bar{\rho}\bar{\sigma}\bar{\sigma} + \bar{\sigma}\bar{\sigma}\bar{\mu}\bar{\nu}\lambda\rho$ | 169 | $\mu\bar{\nu}\bar{\lambda}\bar{\sigma}\bar{\sigma}\rho + \mu\bar{\sigma}\bar{\sigma}\bar{\nu}\bar{\lambda}\rho$ |
| 2 | $\mu s\nu\lambda\rho + \mu\nu\lambda s\rho$ | 44 | $\mu\nu\lambda\sigma\rho\sigma + \sigma\mu\sigma\nu\lambda\rho$ | 86 | $\mu\sigma\bar{\nu}\lambda\sigma\bar{\rho} + \bar{\mu}\sigma\nu\bar{\lambda}\sigma\rho$ | 128 | $\mu\nu\bar{\lambda}\bar{\sigma}\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\bar{\sigma}\bar{\nu}\lambda\rho$ | 170 | $\mu\bar{\nu}\bar{\sigma}\bar{\lambda}\bar{\rho}\sigma + \sigma\bar{\mu}\bar{\nu}\bar{\sigma}\bar{\lambda}\rho$ |
| 3 | $\mu\nu s\lambda\rho$ | 45 | $\mu\nu\lambda\sigma\sigma\rho + \mu\sigma\sigma\nu\lambda\rho$ | 87 | $\mu\sigma\bar{\nu}\sigma\lambda\bar{\rho} + \bar{\mu}\nu\sigma\bar{\lambda}\sigma\rho$ | 129 | $\mu\nu\bar{\lambda}\bar{\sigma}\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\bar{\sigma}\bar{\nu}\lambda\rho$ | 171 | $\mu\bar{\nu}\bar{\sigma}\bar{\lambda}\bar{\sigma}\rho + \mu\bar{\sigma}\bar{\nu}\bar{\sigma}\bar{\lambda}\rho$ |
| 4 | $s\mu\nu\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\lambda\rho s$ | 46 | $\mu\nu\sigma\lambda\rho\sigma + \sigma\mu\nu\sigma\lambda\rho$ | 88 | $\mu\sigma\bar{\sigma}\nu\lambda\bar{\rho} + \bar{\mu}\nu\lambda\bar{\sigma}\sigma\rho$ | 130 | $\mu\nu\bar{\sigma}\bar{\lambda}\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\bar{\nu}\bar{\sigma}\lambda\rho$ | 172 | $\mu\bar{\sigma}\bar{\nu}\bar{\lambda}\bar{\rho}\sigma + \sigma\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\sigma}\rho$ |
| 5 | $s\mu\bar{\nu}\lambda\bar{\rho} + \bar{\mu}\nu\bar{\lambda}\rho s$ | 47 | $\mu\nu\sigma\lambda\sigma\rho + \mu\sigma\nu\sigma\lambda\rho$ | 89 | $\mu\nu\bar{\lambda}\rho\bar{\sigma}\sigma + \sigma\bar{\sigma}\mu\bar{\nu}\lambda\rho$ | 131 | $\mu\nu\bar{\sigma}\bar{\lambda}\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\bar{\nu}\bar{\sigma}\lambda\rho$ | 173 | $\bar{\mu}\nu\lambda\bar{\rho}\bar{\sigma}\bar{\sigma} + \bar{\sigma}\bar{\sigma}\bar{\mu}\nu\lambda\bar{\rho}$ |
| 6 | $s\mu\bar{\nu}\bar{\lambda}\rho + \mu\bar{\nu}\bar{\lambda}\rho s$ | 48 | $\mu\sigma\nu\lambda\rho\sigma + \sigma\mu\nu\lambda\sigma\rho$ | 90 | $\mu\nu\bar{\lambda}\sigma\bar{\rho}\sigma + \sigma\bar{\mu}\sigma\bar{\nu}\lambda\rho$ | 132 | $\mu\nu\bar{\sigma}\bar{\sigma}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\sigma}\bar{\sigma}\lambda\rho$ | 174 | $\bar{\mu}\nu\lambda\bar{\sigma}\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\bar{\sigma}\nu\lambda\bar{\rho}$ |
| 7 | $s\bar{\mu}\nu\lambda\bar{\rho} + \bar{\mu}\nu\lambda\bar{\rho}s$ | 49 | $\mu\nu\lambda\rho\bar{\sigma}\bar{\sigma} + \bar{\sigma}\bar{\sigma}\mu\nu\lambda\rho$ | 91 | $\mu\nu\bar{\lambda}\sigma\bar{\sigma}\rho + \mu\bar{\sigma}\sigma\bar{\nu}\lambda\rho$ | 133 | $\mu\sigma\bar{\nu}\bar{\lambda}\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\bar{\nu}\bar{\lambda}\sigma\rho$ | 175 | $\bar{\mu}\nu\lambda\bar{\sigma}\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\bar{\sigma}\nu\lambda\bar{\rho}$ |
| 8 | $s\bar{\mu}\nu\bar{\lambda}\rho + \mu\bar{\nu}\lambda\bar{\rho}s$ | 50 | $\mu\nu\lambda\sigma\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\sigma\nu\lambda\rho$ | 92 | $\mu\nu\bar{\sigma}\lambda\bar{\rho}\sigma + \sigma\bar{\mu}\nu\bar{\sigma}\lambda\rho$ | 134 | $\mu\sigma\bar{\nu}\bar{\lambda}\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\bar{\nu}\bar{\lambda}\sigma\rho$ | 176 | $\bar{\mu}\nu\sigma\bar{\lambda}\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\bar{\nu}\sigma\lambda\bar{\rho}$ |
| 9 | $s\bar{\mu}\bar{\nu}\lambda\rho + \mu\nu\bar{\lambda}\bar{\rho}s$ | 51 | $\mu\nu\lambda\sigma\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\sigma\nu\lambda\rho$ | 93 | $\mu\nu\bar{\sigma}\lambda\bar{\sigma}\rho + \mu\bar{\sigma}\nu\bar{\sigma}\lambda\rho$ | 135 | $\mu\sigma\bar{\nu}\bar{\sigma}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\sigma}\bar{\lambda}\sigma\rho$ | 177 | $\bar{\mu}\nu\sigma\bar{\lambda}\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\bar{\nu}\sigma\lambda\bar{\rho}$ |
| 10 | $\mu s\nu\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\lambda s\rho$ | 52 | $\mu\nu\sigma\lambda\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\nu\sigma\lambda\rho$ | 94 | $\mu\nu\bar{\sigma}\sigma\bar{\lambda}\rho + \mu\bar{\nu}\sigma\bar{\sigma}\lambda\rho$ | 136 | $\mu\sigma\bar{\sigma}\bar{\nu}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\lambda}\bar{\sigma}\sigma\rho$ | 178 | $\bar{\mu}\nu\sigma\bar{\sigma}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\sigma}\sigma\lambda\bar{\rho}$ |
| 11 | $\mu s\bar{\nu}\lambda\bar{\rho} + \bar{\mu}\nu\bar{\lambda}s\rho$ | 53 | $\mu\nu\sigma\lambda\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\nu\sigma\lambda\rho$ | 95 | $\mu\sigma\bar{\nu}\lambda\bar{\rho}\sigma + \sigma\bar{\mu}\nu\bar{\lambda}\sigma\rho$ | 137 | $\mu\bar{\nu}\lambda\bar{\rho}\bar{\sigma}\bar{\sigma} + \bar{\sigma}\bar{\sigma}\bar{\mu}\nu\bar{\lambda}\rho$ | 179 | $\bar{\mu}\sigma\nu\bar{\lambda}\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\bar{\nu}\lambda\sigma\bar{\rho}$ |
| 12 | $\mu s\bar{\nu}\bar{\lambda}\rho + \mu\bar{\nu}\bar{\lambda}s\rho$ | 54 | $\mu\nu\sigma\sigma\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\sigma\sigma\lambda\rho$ | 96 | $\mu\sigma\bar{\nu}\lambda\bar{\sigma}\rho + \mu\bar{\sigma}\nu\bar{\lambda}\sigma\rho$ | 138 | $\mu\bar{\nu}\lambda\bar{\sigma}\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\bar{\sigma}\nu\bar{\lambda}\rho$ | 180 | $\bar{\mu}\sigma\nu\bar{\lambda}\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\bar{\nu}\lambda\sigma\bar{\rho}$ |
| 13 | $\bar{\mu}s\nu\lambda\bar{\rho} + \bar{\mu}\nu\lambda s\bar{\rho}$ | 55 | $\mu\sigma\nu\lambda\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\nu\lambda\sigma\rho$ | 97 | $\mu\sigma\bar{\nu}\sigma\bar{\lambda}\rho + \mu\bar{\nu}\sigma\bar{\lambda}\sigma\rho$ | 139 | $\mu\bar{\nu}\lambda\bar{\sigma}\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\bar{\sigma}\nu\bar{\lambda}\rho$ | 181 | $\bar{\mu}\sigma\nu\bar{\sigma}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\sigma}\lambda\sigma\bar{\rho}$ |
| 14 | $\bar{\mu}s\nu\bar{\lambda}\rho + \mu\bar{\nu}\lambda s\bar{\rho}$ | 56 | $\mu\sigma\nu\lambda\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\nu\lambda\sigma\rho$ | 98 | $\mu\sigma\bar{\sigma}\nu\bar{\lambda}\rho + \mu\bar{\nu}\lambda\bar{\sigma}\sigma\rho$ | 140 | $\mu\bar{\nu}\sigma\bar{\lambda}\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\bar{\nu}\sigma\bar{\lambda}\rho$ | 182 | $\bar{\mu}\sigma\bar{\sigma}\bar{\nu}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\lambda}\sigma\bar{\sigma}\bar{\rho}$ |
| 15 | $\bar{\mu}s\bar{\nu}\lambda\rho + \mu\nu\bar{\lambda}s\bar{\rho}$ | 57 | $\mu\sigma\nu\sigma\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\sigma\lambda\sigma\rho$ | 99 | $\mu\nu\bar{\lambda}\bar{\rho}\sigma\sigma + \sigma\sigma\bar{\mu}\bar{\nu}\lambda\rho$ | 141 | $\mu\bar{\nu}\sigma\bar{\lambda}\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\bar{\nu}\sigma\bar{\lambda}\rho$ | 183 | $\bar{\mu}\nu\bar{\lambda}\rho\bar{\sigma}\bar{\sigma} + \bar{\sigma}\bar{\sigma}\mu\bar{\nu}\lambda\bar{\rho}$ |
| 16 | $\mu\nu s\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}s\lambda\rho$ | 58 | $\mu\sigma\sigma\nu\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\lambda\sigma\sigma\rho$ | 100 | $\mu\nu\bar{\lambda}\bar{\sigma}\rho\sigma + \sigma\mu\bar{\sigma}\bar{\nu}\lambda\rho$ | 142 | $\mu\bar{\nu}\sigma\bar{\sigma}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\sigma}\sigma\bar{\lambda}\rho$ | 184 | $\bar{\mu}\nu\bar{\lambda}\sigma\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\sigma\bar{\nu}\lambda\bar{\rho}$ |
| 17 | $\mu\bar{\nu}s\lambda\bar{\rho} + \bar{\mu}\nu s\bar{\lambda}\rho$ | 59 | $\mu\nu\lambda\bar{\rho}\sigma\bar{\sigma} + \bar{\sigma}\sigma\bar{\mu}\nu\lambda\rho$ | 101 | $\mu\nu\bar{\lambda}\bar{\sigma}\sigma\rho + \mu\sigma\bar{\sigma}\bar{\nu}\lambda\rho$ | 143 | $\mu\bar{\sigma}\nu\bar{\lambda}\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\bar{\nu}\lambda\bar{\sigma}\rho$ | 185 | $\bar{\mu}\nu\bar{\lambda}\sigma\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\sigma\bar{\nu}\lambda\bar{\rho}$ |
| 18 | $\mu\bar{\nu}s\bar{\lambda}\rho$ | 60 | $\mu\nu\lambda\bar{\sigma}\rho\bar{\sigma} + \bar{\sigma}\mu\bar{\sigma}\nu\lambda\rho$ | 102 | $\mu\nu\bar{\sigma}\bar{\lambda}\rho\sigma + \sigma\mu\bar{\nu}\bar{\sigma}\lambda\rho$ | 144 | $\mu\bar{\sigma}\nu\bar{\lambda}\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\bar{\nu}\lambda\bar{\sigma}\rho$ | 186 | $\bar{\mu}\nu\bar{\sigma}\lambda\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\nu\bar{\sigma}\lambda\bar{\rho}$ |
| 19 | $\bar{\mu}\nu s\lambda\bar{\rho}$ | 61 | $\mu\nu\lambda\bar{\sigma}\sigma\bar{\rho} + \bar{\mu}\sigma\bar{\sigma}\nu\lambda\rho$ | 103 | $\mu\nu\bar{\sigma}\bar{\lambda}\sigma\rho + \mu\sigma\bar{\nu}\bar{\sigma}\lambda\rho$ | 145 | $\mu\bar{\sigma}\nu\bar{\sigma}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\sigma}\lambda\bar{\sigma}\rho$ | 187 | $\bar{\mu}\nu\bar{\sigma}\lambda\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\nu\bar{\sigma}\lambda\bar{\rho}$ |
| 20 | $s\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\lambda}\bar{\rho}s$ | 62 | $\mu\nu\sigma\bar{\lambda}\rho\bar{\sigma} + \bar{\sigma}\mu\bar{\nu}\sigma\lambda\rho$ | 104 | $\mu\sigma\bar{\nu}\bar{\lambda}\rho\sigma + \sigma\mu\bar{\nu}\bar{\lambda}\sigma\rho$ | 146 | $\mu\bar{\sigma}\sigma\bar{\nu}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\lambda}\sigma\bar{\sigma}\rho$ | 188 | $\bar{\mu}\nu\bar{\sigma}\sigma\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\sigma\bar{\sigma}\lambda\bar{\rho}$ |
| 21 | $\bar{\mu}s\bar{\nu}\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\bar{\lambda}s\bar{\rho}$ | 63 | $\mu\nu\sigma\bar{\lambda}\sigma\bar{\rho} + \bar{\mu}\sigma\bar{\nu}\sigma\lambda\rho$ | 105 | $\mu\bar{\nu}\lambda\rho\sigma\bar{\sigma} + \bar{\sigma}\sigma\mu\nu\bar{\lambda}\rho$ | 147 | $\mu\bar{\nu}\bar{\lambda}\rho\bar{\sigma}\bar{\sigma} + \bar{\sigma}\bar{\sigma}\mu\bar{\nu}\bar{\lambda}\rho$ | 189 | $\bar{\mu}\sigma\bar{\nu}\lambda\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\nu\bar{\lambda}\sigma\bar{\rho}$ |
| 22 | $\bar{\mu}\bar{\nu}s\bar{\lambda}\bar{\rho}$ | 64 | $\mu\nu\sigma\bar{\sigma}\lambda\bar{\rho} + \bar{\mu}\nu\bar{\sigma}\sigma\lambda\rho$ | 106 | $\mu\bar{\nu}\lambda\sigma\rho\bar{\sigma} + \bar{\sigma}\mu\sigma\nu\bar{\lambda}\rho$ | 148 | $\mu\bar{\nu}\bar{\lambda}\sigma\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\sigma\bar{\nu}\bar{\lambda}\rho$ | 190 | $\bar{\mu}\sigma\bar{\nu}\lambda\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\nu\bar{\lambda}\sigma\bar{\rho}$ |
| 23 | $p\mu\nu\lambda\bar{\rho} - \bar{\mu}\nu\lambda\rho p$ | 65 | $\mu\sigma\nu\bar{\lambda}\rho\bar{\sigma} + \bar{\sigma}\mu\bar{\nu}\lambda\sigma\rho$ | 107 | $\mu\bar{\nu}\lambda\sigma\sigma\bar{\rho} + \bar{\mu}\sigma\sigma\nu\bar{\lambda}\rho$ | 149 | $\mu\bar{\nu}\bar{\lambda}\sigma\bar{\sigma}\bar{\rho} + \bar{\mu}\bar{\sigma}\sigma\bar{\nu}\bar{\lambda}\rho$ | 191 | $\bar{\mu}\sigma\bar{\nu}\sigma\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\sigma\bar{\lambda}\sigma\bar{\rho}$ |
| 24 | $p\mu\nu\bar{\lambda}\rho - \mu\bar{\nu}\lambda\rho p$ | 66 | $\mu\sigma\nu\bar{\lambda}\sigma\bar{\rho} + \bar{\mu}\sigma\bar{\nu}\lambda\sigma\rho$ | 108 | $\mu\bar{\nu}\sigma\lambda\rho\bar{\sigma} + \bar{\sigma}\mu\nu\sigma\bar{\lambda}\rho$ | 150 | $\mu\bar{\nu}\bar{\sigma}\lambda\bar{\rho}\bar{\sigma} + \bar{\sigma}\bar{\mu}\nu\bar{\sigma}\bar{\lambda}\rho$ | 192 | $\bar{\mu}\sigma\bar{\sigma}\nu\bar{\lambda}\bar{\rho} + \bar{\mu}\bar{\nu}\lambda\bar{\sigma}\sigma\bar{\rho}$ |

TABLE V. (*Continued*)

| $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ | $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ | $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ | $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ | $n$ | $\bar{O}_n^{\mu\nu\lambda\rho}$ |
|---|---|---|---|---|---|---|---|---|---|
| 25 | $p\mu\bar\nu\lambda\rho - \mu\nu\bar\lambda\rho p$ | 67 | $\mu\sigma\nu\bar\sigma\lambda\bar\rho + \bar\mu\nu\bar\sigma\lambda\sigma\rho$ | 109 | $\mu\bar\nu\sigma\lambda\sigma\bar\rho + \bar\mu\sigma\nu\sigma\bar\lambda\rho$ | 151 | $\mu\bar\nu\bar\sigma\lambda\bar\sigma\bar\rho + \bar\mu\bar\sigma\nu\bar\sigma\bar\lambda\rho$ | 193 | $\bar\mu\nu\bar\lambda\bar\rho\sigma\bar\sigma + \bar\sigma\sigma\bar\mu\bar\nu\lambda\bar\rho$ |
| 26 | $p\bar\mu\nu\lambda\rho - \mu\nu\lambda\bar\rho p$ | 68 | $\mu\sigma\sigma\bar\nu\lambda\bar\rho + \bar\mu\nu\bar\lambda\sigma\sigma\rho$ | 110 | $\mu\bar\nu\sigma\sigma\lambda\bar\rho + \bar\mu\nu\sigma\sigma\bar\lambda\rho$ | 152 | $\mu\bar\nu\bar\sigma\sigma\bar\lambda\bar\rho + \bar\mu\bar\nu\sigma\bar\sigma\bar\lambda\rho$ | 194 | $\bar\mu\nu\bar\lambda\bar\sigma\rho\bar\sigma + \bar\sigma\mu\bar\sigma\bar\nu\lambda\bar\rho$ |
| 27 | $\mu p\nu\lambda\bar\rho - \bar\mu\nu\lambda p\rho$ | 69 | $\mu\nu\lambda\bar\rho\bar\sigma\sigma + \sigma\bar\sigma\bar\mu\nu\lambda\rho$ | 111 | $\mu\bar\sigma\nu\lambda\rho\bar\sigma + \bar\sigma\mu\nu\lambda\bar\sigma\rho$ | 153 | $\mu\bar\sigma\bar\nu\lambda\bar\rho\bar\sigma + \bar\sigma\bar\mu\nu\bar\lambda\bar\sigma\rho$ | 195 | $\bar\mu\nu\bar\lambda\bar\sigma\sigma\bar\rho + \bar\mu\sigma\bar\sigma\bar\nu\lambda\bar\rho$ |
| 28 | $\mu p\nu\bar\lambda\rho - \mu\bar\nu\lambda p\rho$ | 70 | $\mu\nu\lambda\bar\sigma\bar\rho\sigma + \sigma\bar\mu\bar\sigma\nu\lambda\rho$ | 112 | $\mu\bar\sigma\nu\lambda\sigma\bar\rho + \bar\mu\sigma\nu\lambda\bar\sigma\rho$ | 154 | $\mu\bar\sigma\bar\nu\lambda\bar\sigma\bar\rho + \bar\mu\bar\sigma\nu\bar\lambda\bar\sigma\rho$ | 196 | $\bar\mu\nu\bar\sigma\bar\lambda\rho\bar\sigma + \bar\sigma\mu\bar\nu\bar\sigma\lambda\bar\rho$ |
| 29 | $\mu p\bar\nu\lambda\rho - \mu\nu\bar\lambda p\rho$ | 71 | $\mu\nu\lambda\bar\sigma\bar\sigma\rho + \mu\bar\sigma\bar\sigma\nu\lambda\rho$ | 113 | $\mu\bar\sigma\nu\sigma\lambda\bar\rho + \bar\mu\nu\sigma\lambda\bar\sigma\rho$ | 155 | $\mu\bar\sigma\bar\nu\sigma\bar\lambda\bar\rho + \bar\mu\bar\nu\sigma\bar\lambda\bar\sigma\rho$ | 197 | $\bar\mu\nu\bar\sigma\bar\lambda\sigma\bar\rho + \bar\mu\sigma\bar\nu\bar\sigma\lambda\bar\rho$ |
| 30 | $\bar\mu p\nu\lambda\rho - \mu\nu\lambda p\bar\rho$ | 72 | $\mu\nu\sigma\bar\lambda\bar\rho\sigma + \sigma\bar\mu\bar\nu\sigma\lambda\rho$ | 114 | $\mu\bar\sigma\sigma\nu\lambda\bar\rho + \bar\mu\nu\lambda\sigma\bar\sigma\rho$ | 156 | $\mu\bar\sigma\bar\sigma\nu\bar\lambda\bar\rho + \bar\mu\bar\nu\lambda\bar\sigma\bar\sigma\rho$ | 198 | $\bar\mu\sigma\bar\nu\bar\lambda\rho\bar\sigma + \bar\sigma\mu\bar\nu\bar\lambda\sigma\bar\rho$ |
| 31 | $\mu\nu p\lambda\bar\rho - \bar\mu\nu p\lambda\rho$ | 73 | $\mu\nu\sigma\bar\lambda\bar\sigma\rho + \mu\bar\sigma\bar\nu\sigma\lambda\rho$ | 115 | $\mu\bar\nu\lambda\rho\bar\sigma\sigma + \sigma\bar\sigma\mu\nu\bar\lambda\rho$ | 157 | $\mu\bar\nu\bar\lambda\bar\rho\sigma\bar\sigma + \bar\sigma\sigma\bar\mu\bar\nu\bar\lambda\rho$ | 199 | $\bar\mu\bar\nu\lambda\rho\bar\sigma\bar\sigma + \bar\sigma\bar\sigma\mu\nu\bar\lambda\bar\rho$ |
| 32 | $\mu\nu p\bar\lambda\rho - \mu\bar\nu p\lambda\rho$ | 74 | $\mu\nu\sigma\bar\sigma\bar\lambda\rho + \mu\bar\nu\bar\sigma\sigma\lambda\rho$ | 116 | $\mu\bar\nu\lambda\sigma\bar\rho\sigma + \sigma\bar\mu\sigma\nu\bar\lambda\rho$ | 158 | $\mu\bar\nu\bar\lambda\bar\sigma\rho\bar\sigma + \bar\sigma\mu\bar\sigma\bar\nu\bar\lambda\rho$ | 200 | $\bar\mu\bar\nu\lambda\sigma\bar\rho\bar\sigma + \bar\sigma\bar\mu\sigma\nu\bar\lambda\bar\rho$ |
| 33 | $p\mu\bar\nu\bar\lambda\bar\rho - \bar\mu\bar\nu\bar\lambda\rho p$ | 75 | $\mu\sigma\nu\bar\lambda\bar\rho\sigma + \sigma\bar\mu\bar\nu\lambda\sigma\rho$ | 117 | $\mu\bar\nu\lambda\sigma\bar\sigma\rho + \mu\bar\sigma\sigma\nu\bar\lambda\rho$ | 159 | $\mu\bar\nu\bar\lambda\bar\sigma\sigma\bar\rho + \bar\mu\sigma\bar\sigma\bar\nu\bar\lambda\rho$ | 201 | $\bar\mu\bar\nu\lambda\sigma\bar\sigma\bar\rho + \bar\mu\bar\sigma\sigma\nu\bar\lambda\bar\rho$ |
| 34 | $p\bar\mu\nu\bar\lambda\bar\rho - \bar\mu\bar\nu\lambda\bar\rho p$ | 76 | $\mu\sigma\nu\bar\lambda\bar\sigma\rho + \mu\bar\sigma\bar\nu\lambda\sigma\rho$ | 118 | $\mu\bar\nu\sigma\lambda\bar\rho\sigma + \sigma\bar\mu\nu\sigma\bar\lambda\rho$ | 160 | $\mu\bar\nu\bar\sigma\bar\lambda\rho\bar\sigma + \bar\sigma\mu\bar\nu\bar\sigma\bar\lambda\rho$ | 202 | $\bar\mu\bar\nu\sigma\lambda\bar\rho\bar\sigma + \bar\sigma\bar\mu\nu\sigma\bar\lambda\bar\rho$ |
| 35 | $p\bar\mu\bar\nu\lambda\bar\rho - \bar\mu\nu\bar\lambda\bar\rho p$ | 77 | $\mu\sigma\nu\bar\sigma\bar\lambda\rho + \mu\bar\nu\bar\sigma\lambda\sigma\rho$ | 119 | $\mu\bar\nu\sigma\lambda\bar\sigma\rho + \mu\bar\sigma\nu\sigma\bar\lambda\rho$ | 161 | $\mu\bar\nu\bar\sigma\bar\lambda\sigma\bar\rho + \bar\mu\sigma\bar\nu\bar\sigma\bar\lambda\rho$ | 203 | $\bar\mu\bar\nu\sigma\lambda\bar\sigma\bar\rho + \bar\mu\bar\sigma\nu\sigma\bar\lambda\bar\rho$ |
| 36 | $p\bar\mu\bar\nu\bar\lambda\rho - \mu\bar\nu\bar\lambda\bar\rho p$ | 78 | $\mu\sigma\sigma\bar\nu\bar\lambda\rho + \mu\bar\nu\bar\lambda\sigma\sigma\rho$ | 120 | $\mu\bar\sigma\nu\lambda\bar\rho\sigma + \sigma\bar\mu\nu\lambda\bar\sigma\rho$ | 162 | $\mu\bar\nu\bar\sigma\bar\sigma\lambda\bar\rho + \bar\mu\nu\bar\sigma\bar\sigma\bar\lambda\rho$ | 204 | $\bar\mu\bar\sigma\nu\lambda\bar\rho\bar\sigma + \bar\sigma\bar\mu\nu\lambda\bar\sigma\bar\rho$ |
| 37 | $\mu p\bar\nu\bar\lambda\bar\rho - \bar\mu\bar\nu\bar\lambda p\rho$ | 79 | $\mu\nu\bar\lambda\rho\sigma\bar\sigma + \bar\sigma\sigma\mu\bar\nu\lambda\rho$ | 121 | $\bar\mu\nu\lambda\rho\sigma\bar\sigma + \bar\sigma\sigma\mu\nu\lambda\bar\rho$ | 163 | $\mu\bar\sigma\bar\nu\bar\lambda\rho\bar\sigma + \bar\sigma\mu\bar\nu\bar\lambda\bar\sigma\rho$ | 205 | $\bar\mu\bar\nu\bar\lambda\bar\rho\bar\sigma\bar\sigma + \bar\sigma\bar\sigma\bar\mu\bar\nu\bar\lambda\bar\rho$ |
| 38 | $\bar\mu p\nu\bar\lambda\bar\rho - \bar\mu\bar\nu\lambda p\bar\rho$ | 80 | $\mu\nu\bar\lambda\sigma\rho\bar\sigma + \bar\sigma\mu\sigma\bar\nu\lambda\rho$ | 122 | $\bar\mu\nu\lambda\sigma\rho\bar\sigma + \bar\sigma\mu\sigma\nu\lambda\bar\rho$ | 164 | $\mu\bar\sigma\bar\nu\bar\lambda\sigma\bar\rho + \bar\mu\sigma\bar\nu\bar\lambda\bar\sigma\rho$ | 206 | $\bar\mu\bar\nu\bar\lambda\bar\sigma\bar\rho\bar\sigma + \bar\sigma\bar\mu\bar\sigma\bar\nu\bar\lambda\bar\rho$ |
| 39 | $\bar\mu p\bar\nu\lambda\bar\rho - \bar\mu\nu\bar\lambda p\bar\rho$ | 81 | $\mu\nu\bar\lambda\sigma\sigma\bar\rho + \bar\mu\sigma\sigma\bar\nu\lambda\rho$ | 123 | $\bar\mu\nu\lambda\sigma\sigma\bar\rho + \bar\mu\sigma\sigma\nu\lambda\bar\rho$ | 165 | $\mu\bar\sigma\bar\nu\bar\sigma\lambda\bar\rho + \bar\mu\nu\bar\sigma\bar\lambda\bar\sigma\rho$ | 207 | $\bar\mu\bar\nu\bar\lambda\bar\sigma\bar\sigma\bar\rho + \bar\mu\bar\sigma\bar\sigma\bar\nu\bar\lambda\bar\rho$ |
| 40 | $\bar\mu p\bar\nu\bar\lambda\rho - \mu\bar\nu\bar\lambda p\bar\rho$ | 82 | $\mu\nu\bar\sigma\lambda\rho\bar\sigma + \bar\sigma\mu\nu\bar\sigma\lambda\rho$ | 124 | $\bar\mu\nu\sigma\lambda\rho\bar\sigma + \bar\sigma\mu\nu\sigma\lambda\bar\rho$ | 166 | $\mu\bar\sigma\bar\sigma\bar\nu\lambda\bar\rho + \bar\mu\nu\bar\lambda\bar\sigma\bar\sigma\rho$ | 208 | $\bar\mu\bar\nu\bar\sigma\bar\lambda\bar\rho\bar\sigma + \bar\sigma\bar\mu\bar\nu\bar\sigma\bar\lambda\bar\rho$ |
| 41 | $\mu\bar\nu p\bar\lambda\bar\rho - \bar\mu\bar\nu p\bar\lambda\rho$ | 83 | $\mu\nu\bar\sigma\lambda\sigma\bar\rho + \bar\mu\sigma\nu\bar\sigma\lambda\rho$ | 125 | $\bar\mu\nu\sigma\lambda\sigma\bar\rho + \bar\mu\sigma\nu\sigma\lambda\bar\rho$ | 167 | $\mu\bar\nu\bar\lambda\bar\rho\bar\sigma\sigma + \sigma\bar\sigma\bar\mu\bar\nu\bar\lambda\rho$ | 209 | $\bar\mu\bar\nu\bar\sigma\bar\lambda\bar\sigma\bar\rho + \bar\mu\bar\sigma\bar\nu\bar\sigma\bar\lambda\bar\rho$ |
| 42 | $\bar\mu\nu p\bar\lambda\bar\rho - \bar\mu\bar\nu p\lambda\bar\rho$ | 84 | $\mu\nu\bar\sigma\sigma\lambda\bar\rho + \bar\mu\nu\sigma\bar\sigma\lambda\rho$ | 126 | $\bar\mu\sigma\nu\lambda\rho\bar\sigma + \bar\sigma\mu\nu\lambda\sigma\bar\rho$ | 168 | $\mu\bar\nu\bar\lambda\bar\sigma\bar\rho\sigma + \sigma\bar\mu\bar\sigma\bar\nu\bar\lambda\rho$ | 210 | $\bar\mu\bar\sigma\bar\nu\bar\lambda\bar\rho\bar\sigma + \bar\sigma\bar\mu\bar\nu\bar\lambda\bar\sigma\bar\rho$ |

Next, we list all 23 $\tilde{O}_n^W$ operators in Table VI, which is cited in the sentence before Eq. (22) in Sec. III of the text

TABLE VI. List of $\tilde{O}_n^W$ operators, where we divide $\tilde{O}_1^W, \ldots, \tilde{O}_7^W$ by $B_0$, making the matrices $A_{mn}$ introduced in Eq. (27) independent of $B_0$. The symbols are introduced in Ref. [20]. The comparisons between the symbols introduced in Ref. [20] and ours are given in Table XV of Ref. [6].

| $n$ | $\tilde{O}_n^W$ | $n$ | $\tilde{O}_n^W$ |
|---|---|---|---|
| 1 | $\langle iu^\mu u^\nu u^\lambda u^\rho \chi_-\rangle\epsilon_{\mu\nu\lambda\rho}/B_0$ | 13 | $-i\langle f_+^{\mu\nu}u_\sigma u^\lambda h^{\rho\sigma}\rangle\epsilon_{\mu\nu\lambda\rho} - i\langle f_+^{\mu\nu}h^{\lambda\sigma}u^\rho u_\sigma\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 2 | $\langle u^\mu u^\nu \chi_+ f_-^{\lambda\rho}\rangle\epsilon_{\mu\nu\lambda\rho}/B_0 - \langle u^\mu u^\nu f_-^{\lambda\rho}\chi_+\rangle\epsilon_{\mu\nu\lambda\rho}/B_0$ | 14 | $-i\langle f_+^{\mu\nu}u^\lambda h^{\rho\sigma}u_\sigma\rangle\epsilon_{\mu\nu\lambda\rho} - i\langle f_+^{\mu\nu}u_\sigma h^{\lambda\sigma}u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 3 | $\langle f_+^{\mu\nu}u^\lambda u^\rho \chi_-\rangle\epsilon_{\mu\nu\lambda\rho}/B_0 + \langle f_+^{\mu\nu}\chi_- u^\lambda u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}/B_0$ | 15 | $i\langle f_+^{\mu\nu}u^\lambda u^\rho h^\sigma{}_\sigma\rangle\epsilon_{\mu\nu\lambda\rho} + i\langle f_+^{\mu\nu}h^\sigma{}_\sigma u^\lambda u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 4 | $-\langle f_+^{\mu\nu}u^\lambda \chi_- u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}/B_0$ | 16 | $-i\langle f_+^{\mu\nu}u^\lambda h^\sigma{}_\sigma u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 5 | $i\langle f_+^{\mu\nu}f_+^{\lambda\rho}\chi_-\rangle\epsilon_{\mu\nu\lambda\rho}/B_0$ | 17 | $i\langle f_{+\sigma}^{\mu}u^\nu u^\lambda f_-^{\rho\sigma}\rangle\epsilon_{\mu\nu\lambda\rho} + i\langle f_{+\sigma}^{\mu}f_-^{\nu\sigma}u^\lambda u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 6 | $i\langle f_-^{\mu\nu}f_-^{\lambda\rho}\chi_-\rangle\epsilon_{\mu\nu\lambda\rho}/B_0$ | 18 | $i\langle f_+^{\mu\sigma}u^\nu u_\sigma f_-^{\lambda\rho}\rangle\epsilon_{\mu\nu\lambda\rho} - i\langle f_+^{\mu\sigma}f_-^{\nu\lambda}u_\sigma u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 7 | $i\langle f_+^{\mu\nu}f_-^{\lambda\rho}\chi_+\rangle\epsilon_{\mu\nu\lambda\rho}/B_0 - i\langle f_+^{\mu\nu}\chi_+ f_-^{\lambda\rho}\rangle\epsilon_{\mu\nu\lambda\rho}/B_0$ | 19 | $-i\langle f_+^{\mu\sigma}u^\nu f_-^{\lambda\rho}u_\sigma\rangle\epsilon_{\mu\nu\lambda\rho} + i\langle f_+^{\mu\sigma}u_\sigma f_-^{\nu\lambda}u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 8 | $-\langle u_\sigma u^\mu u^\nu u^\lambda h^{\rho\sigma}\rangle\epsilon_{\mu\nu\lambda\rho} + \langle u^\mu u^\nu u^\lambda u_\sigma h^{\rho\sigma}\rangle\epsilon_{\mu\nu\lambda\rho}$ | 20 | $-\langle f_+^{\mu\nu}u^\lambda \nabla_\sigma f_+^{\rho\sigma}\rangle\epsilon_{\mu\nu\lambda\rho} + \langle f_+^{\mu\nu}\nabla_\sigma f_+^{\lambda\sigma}u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 9 | $\langle u^\mu u^\nu u^\lambda u^\rho h^\sigma{}_\sigma\rangle\epsilon_{\mu\nu\lambda\rho}$ | 21 | $-\langle u^\mu \nabla_\sigma f_-^{\nu\sigma}f_-^{\lambda\rho}\rangle\epsilon_{\mu\nu\lambda\rho} - \langle u^\mu f_-^{\nu\lambda}\nabla_\sigma f_-^{\rho\sigma}\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 10 | $\langle u_\sigma u^\mu u^\nu u^\lambda f_-^{\rho\sigma}\rangle\epsilon_{\mu\nu\lambda\rho} - \langle u^\mu u^\nu u^\lambda u_\sigma f_-^{\rho\sigma}\rangle\epsilon_{\mu\nu\lambda\rho}$ | 22 | $\langle f_+^{\mu\nu}f_+^{\lambda\rho}h^\sigma{}_\sigma\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 11 | $\langle u^2 u^\mu u^\nu f_-^{\lambda\rho}\rangle\epsilon_{\mu\nu\lambda\rho} - \langle u^\mu u^\nu u^2 f_-^{\lambda\rho}\rangle\epsilon_{\mu\nu\lambda\rho}$ | 23 | $\langle h^\sigma{}_\sigma f_-^{\mu\nu}f_-^{\lambda\rho}\rangle\epsilon_{\mu\nu\lambda\rho}$ |
| 12 | $i\langle f_+^{\mu\sigma}u^\nu u^\lambda h^\rho{}_\sigma\rangle\epsilon_{\mu\nu\lambda\rho} + i\langle f_+^{\mu\sigma}h^\nu{}_\sigma u^\lambda u^\rho\rangle\epsilon_{\mu\nu\lambda\rho}$ | | |

## APPENDIX B: *A* AND *C* MATRICES

In this appendix, we give matrices $A_{nm}$ and $C_{m'n}$. We first give $A_{nm}$ in Table VII, which is cited in the sentence before Eq. (27) in Sec. III of the text. In Table VII, for convenience, in practice, we do not write the $A$ matrix, but its transverse $A^T$ multiplied by $-i$.

TABLE VII. $-i(A^T)_{mn}$ matrix.

| $m, n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | −64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | −32 | 0 | −32 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | −32 | 0 | 0 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | −32 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | −32 | 0 | 0 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 32 | −32 | 0 | −32 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 0 | −32 | 0 | 0 | 0 | 0 | 64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 17 | 0 | 32 | 0 | 0 | 0 | 0 | −64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 18 | 0 | 64 | 0 | 0 | 0 | 0 | −64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | −64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | −32 | −32 | 64 | 32 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22 | 0 | 64 | 0 | 0 | 0 | 0 | −64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 23 | 0 | 0 | 0 | 0 | −32 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 24 | 0 | 0 | −32 | 0 | −32 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 25 | 0 | 0 | 0 | 0 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 26 | 0 | 0 | 0 | −32 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 27 | 0 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 28 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 29 | 0 | 0 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 30 | 0 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 31 | 0 | 0 | 0 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 32 | 0 | 0 | −32 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 33 | 0 | 32 | 0 | 0 | 0 | 32 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 34 | 0 | 32 | 0 | 0 | 0 | 32 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 35 | 0 | −32 | 32 | 0 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | −32 | −32 | 64 | 32 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | −32 | 0 | 32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 32 | 0 | −32 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 32 | 0 | −64 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | −32 | 0 | −32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | −64 | 0 |
| 44 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −24 | 0 | 32 | 0 |
| 45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | −32 | 0 |
| 47 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 |
| 49 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | −32 | 0 |
| 50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −48 | 0 | −32 | 0 | −8 | 0 | 0 | −8 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 8 | 8 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 32 | 0 | 0 | 0 | 16 | 0 | 0 | 0 | −16 | 0 | 0 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | 0 | 0 | 0 | −8 | 0 | 0 | −8 | 16 | 0 | 0 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 0 | 16 | 0 | 0 | 0 | −32 | 0 |

TABLE VII. (*Continued*)

| $m, n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 56 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −48 | 0 | −32 | 0 | −8 | 0 | 0 | 0 | 0 | 32 | 0 |
| 57 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24 | −16 | 0 | 32 | 0 | 8 | 0 | 0 | 16 | 0 | −32 | 0 |
| 58 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | 0 | −32 | 0 | 0 | 0 | 0 | −8 | 0 | 32 | 0 |
| 59 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | −32 |
| 60 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 |
| 61 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | −8 | 8 | 0 | 0 |
| 62 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | 0 | −16 | −16 | −24 | 16 | 0 | 0 |
| 63 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | 16 | 0 | 0 | 8 | 0 | 16 | 24 | −32 | 0 | 0 |
| 64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | 0 | −8 | 0 | 0 | −16 | 24 | 32 | −32 |
| 65 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | 0 | 0 | 16 | 16 | 16 | 0 | 0 | −32 |
| 66 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | −16 | 0 | 0 | −8 | 0 | −16 | −16 | 0 | 0 | 32 |
| 67 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | 0 | 0 | 0 | 8 | 0 | 0 | 8 | 8 | 0 | −32 |
| 68 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 32 |
| 69 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | −64 | −32 | 0 | −16 | 0 | −8 | −8 | 64 | 32 |
| 70 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 80 | 0 | 64 | 0 | 8 | 0 | 0 | 16 | 8 | −64 | 0 |
| 71 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −32 | 0 | 0 | 0 | −8 | 16 | 0 | −8 | 0 | 0 | 0 |
| 72 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −40 | 16 | −16 | −64 | 0 | −8 | 0 | −16 | −40 | 0 | 64 | 0 |
| 73 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24 | 0 | −16 | 0 | 0 | 8 | −16 | 16 | 48 | −8 | 0 | 0 |
| 74 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 32 | 0 | −32 | 0 | 0 | 0 | 0 | −40 | 8 | 32 | −32 |
| 75 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 16 | 64 | 0 | 0 | 0 | 16 | 24 | 0 | −64 | −32 |
| 76 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | 16 | 0 | 0 | 0 | 0 | −16 | −24 | 0 | 0 | 32 |
| 77 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 8 | 0 | −32 |
| 78 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 32 |
| 79 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | 0 | 0 | 16 | 16 | 16 | 0 | 0 | −32 |
| 80 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | 0 | −16 | −16 | −24 | 16 | 0 | 0 |
| 81 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −8 | 0 | 0 |
| 82 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 |
| 83 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | −16 | 0 | 0 | −8 | 16 | 16 | 0 | −8 | 0 | 0 |
| 84 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 16 | 0 | 0 | 8 | −16 | −16 | 0 | −8 | −32 | 32 |
| 85 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | −32 |
| 86 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | −32 | 8 | −16 | 0 | 0 | 0 | 0 | 32 |
| 87 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 32 | −8 | 16 | 0 | 0 | 16 | 0 | −32 |
| 88 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | −32 | −32 | 0 | 0 | 0 | 0 | −8 | 0 | 32 |
| 89 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | 0 | −16 | −16 | 0 | −32 | 0 | 96 |
| 90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | 16 | 0 | 0 | 8 | 0 | 16 | −8 | 48 | 0 | −64 |
| 91 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 32 | −8 | 16 | 0 | 8 | −16 | 0 | 0 |
| 92 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | −24 | 0 | 64 |
| 93 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 16 | 0 | 0 | 8 | −16 | −16 | 0 | 8 | 0 | 0 |
| 94 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | 32 | 0 | 0 | 0 | 16 | 0 | −16 | −32 | 32 |
| 95 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | −96 |
| 96 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 32 |
| 97 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24 | 0 | −32 |
| 98 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 32 |
| 99 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | −32 | 0 | 0 | 0 | −16 | 0 | −24 | 32 | 32 |
| 100 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 24 | 0 | −32 |
| 101 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | −16 | 0 | −32 | 8 | 0 | 16 | −8 | 0 | 0 | 0 |
| 102 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | 16 | 0 | 0 | 8 | −16 | −16 | 0 | −8 | 0 | 32 |
| 103 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | −32 | 0 | 0 | −8 | 16 | 0 | 8 | 0 | 0 | 0 |
| 104 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 |
| 105 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 0 | 16 | 0 | 0 | 0 | −32 | 0 |
| 106 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 8 | 8 | 0 | 0 |
| 107 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | −8 | 0 | 0 |
| 108 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | 0 | 0 | 0 |
| 109 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | 0 | 0 | 0 | −16 | 32 | 0 | −16 | 0 | 0 | 0 |
| 110 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | 8 | −16 | 0 | 16 | −8 | 0 | 0 |

TABLE VII. (*Continued*)

| $m, n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 111 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | −32 | 0 |
| 112 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | 0 | 0 | 0 | 8 | −16 | 0 | −8 | 0 | 32 | 0 |
| 113 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 8 | −32 | 0 |
| 114 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 0 | 16 | 0 | 0 | 0 | 32 | 0 |
| 115 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | −32 | −32 | −32 | 0 | 0 | 0 | −8 | −8 | 32 | 32 |
| 116 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 32 | −8 | 16 | 0 | 8 | 24 | 0 | −32 |
| 117 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | −16 | 0 | 0 |
| 118 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | −32 | 8 | −16 | 0 | −8 | 8 | 0 | 32 |
| 119 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 0 | 0 | 0 |
| 120 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | −32 |
| 121 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | −64 | 0 |
| 122 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −24 | 0 | 32 | 0 |
| 123 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24 | 0 | 0 | 0 |
| 124 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | −32 | 0 |
| 125 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 |
| 126 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 |
| 127 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | 0 | −16 | −16 | 16 | −32 | 0 | 0 | −16 | −16 | −24 | 0 | 32 | 0 |
| 128 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | −32 | 8 | 16 | −16 | 0 | 0 | 8 | 16 | 16 | 24 | −8 | 0 | 0 |
| 129 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 8 | 0 | 0 | 0 | 0 | −8 | 16 | 0 | 0 | 8 | 0 | 0 |
| 130 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | 32 | −8 | 32 | −16 | 0 | 0 | −8 | 0 | 16 | −16 | 16 | 0 | 0 |
| 131 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 32 | 0 | −16 | 0 | 32 | 0 | 0 | 16 | −32 | −32 | 0 | −16 | 0 | 0 |
| 132 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | 32 | 8 | 16 | −16 | 0 | 0 | −8 | 0 | 16 | 0 | 8 | 0 | 0 |
| 133 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −48 | −32 | 16 | −32 | −16 | 16 | −48 | −64 | −32 | 0 | 0 | −16 | 0 | −16 | 32 | 0 |
| 134 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 32 | −16 | 0 | 8 | 32 | 16 | 64 | 32 | −8 | 16 | 16 | 0 | 16 | −32 | 0 |
| 135 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | −8 | −32 | 0 | −64 | −32 | 8 | −16 | 0 | −8 | −8 | 32 | 0 |
| 136 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 64 | 64 | 0 | 0 | 16 | 0 | 64 | 96 | 64 | 0 | 0 | 0 | 8 | 0 | −32 | 0 |
| 137 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 64 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | −8 | 0 | 32 |
| 138 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | −8 | 0 | 0 |
| 139 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | −32 | 0 | 16 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | 16 | 0 | 0 |
| 140 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | −16 | 32 | 0 | −16 | 16 | 0 | 0 | 0 | 0 | 16 | −16 | 24 | 0 | 0 |
| 141 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 32 | 0 | 8 | 16 | −16 | 0 | 0 | 8 | −32 | −16 | 24 | −32 | 0 | 0 |
| 142 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | −24 | −32 | 0 | −64 | 0 | −8 | 16 | 0 | −24 | 16 | 32 | −32 |
| 143 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | 0 | 0 | −16 | 16 | 0 | 0 | 0 | −16 | −16 | 0 | −24 | 0 | 32 |
| 144 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | −32 | 8 | 0 | −16 | 0 | 0 | −8 | 32 | 16 | 0 | 24 | 0 | −32 |
| 145 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | 32 | −8 | 16 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | −8 | 0 | 32 |
| 146 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −48 | −32 | 16 − 32 | −32 | 0 | 16 | −32 | −32 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | −32 |
| 147 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | 0 | 0 | −16 | 16 | 0 | 0 | 0 | −16 | −16 | 0 | −24 | 0 | 32 |
| 148 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | −16 | 32 | 0 | −16 | 16 | 0 | 0 | 0 | 0 | 16 | −16 | 24 | 0 | 0 |
| 149 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 32 | −16 | 0 | 0 | 16 | 0 | 32 | 32 | 0 | 16 | 0 | 16 | 0 | 0 | 0 |
| 150 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | −8 | 0 | 0 |
| 151 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 32 | 0 | 8 | −32 | 16 | 0 | 0 | −8 | −16 | −16 | 0 | −8 | 0 | 0 |
| 152 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 32 | −16 | 0 | 24 | 16 | 16 | 64 | 0 | 8 | 0 | 16 | 8 | 0 | −32 | 32 |
| 153 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 64 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | −8 | 0 | 32 |
| 154 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | −8 | 16 | 0 | 0 | 0 | 8 | 16 | 0 | 0 | 8 | 0 | −32 |
| 155 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | −16 | 32 | −8 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | −8 | 0 | 0 | 32 |
| 156 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | 0 | 0 | −16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 |
| 157 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −48 | −32 | 16 | −32 | −16 | 16 | −48 | −64 | −32 | 0 | 0 | −16 | 0 | −16 | 32 | 0 |
| 158 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | 32 | −8 | 32 | −16 | 0 | 0 | −8 | 0 | 16 | −16 | 16 | 0 | 0 |
| 159 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | 8 | −48 | 0 | −32 | −32 | 8 | 0 | 0 | 8 | 0 | 0 | 0 |
| 160 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | −32 | 8 | 16 | −16 | 0 | 0 | 8 | 16 | 16 | 24 | −8 | 0 | 0 |
| 161 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 32 | 0 | −16 | 32 | −32 | 0 | 0 | −16 | 0 | −32 | −16 | 0 | 0 | 0 |
| 162 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | −32 | −8 | 32 | −16 | 0 | 0 | 8 | 16 | 16 | 8 | 0 | 0 | 0 |
| 163 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | 0 | −16 | −16 | 16 | −32 | 0 | 0 | −16 | −16 | −24 | 0 | 32 | 0 |
| 164 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | −16 | 32 | 24 | −32 | 16 | 32 | 0 | 8 | 0 | 16 | 24 | 0 | −32 | 0 |
| 165 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 8 | −48 | 0 | −32 | 0 | −8 | 16 | 0 | −8 | 0 | 32 | 0 |

TABLE VII. (*Continued*)

| *m*, *n* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 166 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 64 | 0 | 0 | 0 | 32 | 0 | 0 | −16 | 0 | 0 | 0 | −32 | 0 |
| 167 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 48 | 64 | −16 | 32 | 32 | 16 | 48 | 128 | 64 | 0 | 0 | 16 | 32 | 0 | −64 | 0 |
| 168 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 32 | −16 | −32 | 0 | −64 | −32 | 0 | −16 | 0 | −32 | 0 | 32 | 0 |
| 169 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | −32 | 0 | −16 | 16 | 0 | 0 | 0 | 0 | −16 | 0 | 0 | 0 | 0 |
| 170 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 32 | −16 | 32 | 16 | 16 | 16 | 64 | 32 | 0 | 0 | 16 | 16 | 0 | −32 | 0 |
| 171 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | 0 | 0 | 0 |
| 172 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −64 | −32 | 0 | 0 | −16 | 32 | −64 | −64 | −32 | 0 | 0 | 0 | 0 | 0 | 32 | 0 |
| 173 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | −64 | −32 | 0 | −16 | 0 | −8 | −8 | 64 | 32 |
| 174 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 80 | 0 | 64 | 0 | 8 | 0 | 0 | 16 | 8 | −64 | 0 |
| 175 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −32 | 0 | 0 | 0 | −8 | 16 | 0 | −8 | 0 | 0 | 0 |
| 176 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −40 | 16 | −16 | −64 | 0 | −8 | 0 | −16 | −40 | 0 | 64 | 0 |
| 177 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24 | 0 | −16 | 0 | 0 | 8 | −16 | 16 | 48 | −8 | 0 | 0 |
| 178 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 32 | 0 | −32 | 0 | 0 | 0 | 0 | −40 | 8 | 32 | −32 |
| 179 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 32 | 0 | 16 | 64 | 0 | 0 | 0 | 16 | 24 | 0 | −64 | −32 |
| 180 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | 16 | 0 | 0 | 0 | 0 | −16 | −24 | 0 | 0 | 32 |
| 181 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 8 | 0 | −32 |
| 182 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 32 |
| 183 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | 0 | −16 | −16 | 0 | −32 | 0 | 96 |
| 184 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | 16 | 0 | 0 | 8 | 0 | 16 | −8 | 48 | 0 | −64 |
| 185 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 32 | −8 | 16 | 0 | 8 | −16 | 0 | 0 |
| 186 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | −24 | 0 | 64 |
| 187 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 16 | 0 | 0 | 8 | −16 | −16 | 0 | 8 | 0 | 0 |
| 188 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | 32 | 0 | 0 | 0 | 16 | 0 | −16 | −32 | 32 |
| 189 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | −96 |
| 190 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 32 |
| 191 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24 | 0 | −32 |
| 192 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 32 |
| 193 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | 0 | −16 | −32 | 0 | 0 | 0 | −16 | 0 | −24 | 32 | 32 |
| 194 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 24 | 0 | −32 |
| 195 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | −16 | 0 | −32 | 8 | 0 | 16 | −8 | 0 | 0 | 0 |
| 196 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | 16 | 0 | 0 | 8 | −16 | −16 | 0 | −8 | 0 | 32 |
| 197 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | −32 | 0 | 0 | −8 | 16 | 0 | 8 | 0 | 0 | 0 |
| 198 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 |
| 199 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −16 | −32 | −32 | −32 | 0 | 0 | 0 | −8 | −8 | 32 | 32 |
| 200 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 32 | −8 | 16 | 0 | 8 | 24 | 0 | −32 |
| 201 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −8 | 16 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | −16 | 0 | 0 |
| 202 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | −32 | 8 | −16 | 0 | −8 | 8 | 0 | 32 |
| 203 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 0 | 0 | 0 |
| 204 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | −32 |
| 205 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 48 | 64 | −16 | 32 | 32 | 16 | 48 | 128 | 64 | 0 | 0 | 16 | 32 | 0 | −64 | 0 |
| 206 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 32 | −16 | −32 | 0 | −64 | −32 | 0 | −16 | 0 | −32 | 0 | 32 | 0 |
| 207 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | −32 | 0 | −16 | 16 | 0 | 0 | 0 | 0 | −16 | 0 | 0 | 0 | 0 |
| 208 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 32 | −16 | 32 | 16 | 16 | 16 | 64 | 32 | 0 | 0 | 16 | 16 | 0 | −32 | 0 |
| 209 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | 0 | 0 | 0 |
| 210 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −64 | −32 | 0 | 0 | −16 | 32 | −64 | −64 | −32 | 0 | 0 | 0 | 0 | 0 | 32 | 0 |
| 202 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | −32 | 8 | −16 | 0 | −8 | 8 | 0 | 32 |
| 203 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | 0 | 0 | 0 | −8 | 0 | 0 | 0 | 0 | 0 | 0 |
| 204 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 32 | 0 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | −32 |
| 205 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 48 | 64 | −16 | 32 | 32 | 16 | 48 | 128 | 64 | 0 | 0 | 16 | 32 | 0 | −64 | 0 |
| 206 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 32 | −16 | −32 | 0 | −64 | −32 | 0 | −16 | 0 | −32 | 0 | 32 | 0 |
| 207 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 16 | −32 | 0 | −16 | 16 | 0 | 0 | 0 | 0 | −16 | 0 | 0 | 0 | 0 |
| 208 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 32 | −16 | 32 | 16 | 16 | 16 | 64 | 32 | 0 | 0 | 16 | 16 | 0 | −32 | 0 |
| 209 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −32 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 | 0 | 0 | 0 | 0 |
| 210 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −64 | −32 | 0 | 0 | −16 | 32 | −64 | −64 | −32 | 0 | 0 | 0 | 0 | 0 | 32 | 0 |

TABLE VIII. $\bar{C}_{n'n}$ matrix.

| $n'$, $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | $-\frac{i}{32}$ | 0 | $-\frac{i}{32}$ | 0 | $\frac{i}{32}$ | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | $\frac{i}{64}$ | $-\frac{i}{64}$ | $-\frac{i}{64}$ |
| 4 | $-\frac{i}{16}$ | 0 | $-\frac{i}{32}$ | 0 | 0 | 0 | 0 |
| 5 | 0 | $\frac{i}{32}$ | 0 | $\frac{i}{32}$ | 0 | $-\frac{i}{32}$ | 0 |
| 6 | $\frac{i}{32}$ | $-\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 |
| 7 | $-\frac{i}{32}$ | 0 | 0 | $-\frac{i}{32}$ | 0 | 0 | 0 |
| 20 | $-\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 |

From (28), the $C$ matrix consists of two submatrices, $\bar{C}$ and $\tilde{C}$. We list the $\bar{C}$ matrix in Table VIII, and the $\tilde{C}$ matrix is given in Table IX. These two tables are cited in the sentence before Eq. (29) in Sec. III of the text.

TABLE IX. $\tilde{C}_{n'n}$ matrix

| $n'$ $n$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 43 | $\frac{19i}{160}$ | $-\frac{11i}{80}$ | $-\frac{19i}{160}$ | $-\frac{13i}{160}$ | $-\frac{3i}{80}$ | $-\frac{i}{80}$ | $-\frac{7i}{160}$ | $\frac{9i}{160}$ | $-\frac{i}{160}$ | $-\frac{13i}{80}$ | $-\frac{3i}{40}$ | $-\frac{i}{32}$ | $-\frac{i}{40}$ | $-\frac{i}{10}$ | $-\frac{3i}{160}$ | $-\frac{7i}{160}$ |
| 44 | $\frac{31i}{320}$ | $-\frac{3i}{20}$ | $-\frac{21i}{320}$ | $-\frac{17i}{320}$ | $\frac{3i}{160}$ | $\frac{i}{160}$ | $-\frac{13i}{320}$ | $\frac{11i}{320}$ | $\frac{11i}{320}$ | $-\frac{17i}{160}$ | $-\frac{9i}{160}$ | $\frac{i}{64}$ | $-\frac{i}{20}$ | $-\frac{3i}{40}$ | $-\frac{i}{160}$ | $-\frac{3i}{320}$ |
| 49 | $-\frac{i}{16}$ | $\frac{3i}{32}$ | $\frac{3i}{32}$ | $\frac{i}{16}$ | $\frac{i}{16}$ | $\frac{i}{32}$ | $\frac{i}{32}$ | $-\frac{i}{16}$ | 0 | $\frac{i}{8}$ | $\frac{i}{16}$ | $\frac{i}{32}$ | 0 | $\frac{i}{16}$ | 0 | $\frac{i}{64}$ |
| 50 | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{i}{32}$ | 0 | $\frac{i}{32}$ | 0 | $-\frac{i}{16}$ | $\frac{i}{32}$ | 0 | 0 | 0 | 0 |
| 51 | $-\frac{5i}{64}$ | $\frac{3i}{32}$ | $\frac{5i}{64}$ | $\frac{3i}{64}$ | $\frac{i}{32}$ | 0 | $\frac{i}{64}$ | $-\frac{3i}{64}$ | $\frac{i}{64}$ | $\frac{3i}{32}$ | $\frac{i}{32}$ | $\frac{3i}{64}$ | 0 | $\frac{i}{16}$ | 0 | $\frac{i}{32}$ |
| 52 | $\frac{5i}{32}$ | $-\frac{3i}{16}$ | $-\frac{5i}{32}$ | $-\frac{3i}{32}$ | 0 | 0 | $-\frac{i}{16}$ | $\frac{i}{16}$ | $\frac{i}{32}$ | $-\frac{i}{4}$ | $-\frac{i}{8}$ | $-\frac{i}{16}$ | 0 | $-\frac{i}{8}$ | 0 | $-\frac{i}{32}$ |
| 54 | $-\frac{3i}{32}$ | $\frac{i}{32}$ | $\frac{5i}{32}$ | $\frac{i}{32}$ | 0 | 0 | $\frac{i}{16}$ | $-\frac{i}{32}$ | 0 | $\frac{i}{8}$ | $\frac{i}{16}$ | $\frac{i}{8}$ | 0 | $\frac{i}{8}$ | 0 | $\frac{i}{16}$ |
| 57 | $-\frac{3i}{64}$ | $\frac{i}{32}$ | $\frac{7i}{64}$ | $\frac{i}{64}$ | $\frac{i}{32}$ | 0 | $\frac{i}{64}$ | $-\frac{i}{64}$ | $\frac{i}{64}$ | $\frac{3i}{32}$ | $\frac{i}{32}$ | $\frac{3i}{64}$ | 0 | $\frac{i}{16}$ | 0 | $\frac{i}{32}$ |
| 59 | 0 | $-\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 62 | $-\frac{i}{32}$ | $\frac{i}{32}$ | $-\frac{i}{32}$ | 0 | 0 | 0 | $\frac{i}{32}$ | 0 | $-\frac{i}{32}$ | 0 | 0 | $-\frac{i}{32}$ | 0 | 0 | 0 | 0 |
| 63 | $-\frac{i}{32}$ | $\frac{i}{32}$ | $\frac{i}{32}$ | 0 | 0 | 0 | $\frac{i}{32}$ | 0 | $-\frac{i}{32}$ | 0 | 0 | $\frac{i}{32}$ | 0 | 0 | 0 | 0 |
| 64 | 0 | $\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{i}{32}$ |
| 127 | $\frac{i}{32}$ | 0 | $\frac{i}{32}$ | $-\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 128 | $\frac{i}{32}$ | $-\frac{i}{32}$ | $-\frac{i}{32}$ | $-\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 133 | $-\frac{i}{32}$ | $\frac{i}{32}$ | $\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 134 | $-\frac{i}{32}$ | $\frac{i}{16}$ | $\frac{i}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

## APPENDIX C: FINAL ANALYTICAL RESULT ON $\tilde{K}_n^W$

In this appendix, we list our analytical result on 23 LECs for the $p^6$ order anomalous part of the chiral Lagrangian,

$$\tilde{K}_1^W = \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{2}k^2\Sigma_k^3X^5 - \frac{1}{2}\Sigma_k^5X^5 + \frac{3}{4}k^4\Sigma_k^2\Sigma_k'X^5 + \frac{3}{4}k^2\Sigma_k^4\Sigma_k'X^5\bigg]$$

$$\tilde{K}_2^W = \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{4}\Sigma_k^5X^5 + \frac{1}{8}k^4\Sigma_k^2\Sigma_k'X^5 + \frac{5}{8}k^2\Sigma_k^4\Sigma_k'X^5\bigg]$$

$$\tilde{K}_3^W = \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{4}k^2\Sigma_k^3X^5 - \frac{1}{4}\Sigma_k^5X^5 + \frac{1}{2}k^4\Sigma_k^2\Sigma_k'X^5 + \frac{1}{2}k^2\Sigma_k^4\Sigma_k'X^5\bigg]$$

$$\tilde{K}_4^W = \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{4}k^2\Sigma_k^3X^5 - \frac{1}{4}\Sigma_k^5X^5 + \frac{1}{2}k^4\Sigma_k^2\Sigma_k'X^5 + \frac{1}{2}k^2\Sigma_k^4\Sigma_k'X^5\bigg]$$

$$\tilde{K}_5^W = \int \frac{d^4k}{(2\pi)^4}\bigg[\frac{3}{16}k^2\Sigma_k^3X^5 + \frac{3}{16}\Sigma_k^5X^5 - \frac{1}{16}k^6\Sigma_k'X^5 - \frac{1}{2}k^4\Sigma_k^2\Sigma_k'X^5 - \frac{7}{16}k^2\Sigma_k^4\Sigma_k'X^5\bigg]$$

$$\tilde{K}_6^W = \int \frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{16}k^2\Sigma_k^3X^5 + \frac{1}{16}\Sigma_k^5X^5 - \frac{1}{8}k^4\Sigma_k^2\Sigma_k'X^5 - \frac{1}{8}k^2\Sigma_k^4\Sigma_k'X^5\bigg]$$

$$\tilde{K}_7^W = \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{16}k^2\Sigma_k^3X^5 - \frac{1}{16}\Sigma_k^5X^5 + \frac{1}{32}k^6\Sigma_k'X^5 + \frac{3}{16}k^4\Sigma_k^2\Sigma_k'X^5 + \frac{5}{32}k^2\Sigma_k^4\Sigma_k'X^5\bigg]$$

$$\begin{aligned}\tilde{K}_8^W = \int \frac{d^4k}{(2\pi)^4}\bigg[&\bigg(\frac{9}{40}k^2\Sigma_k'' - \frac{1}{40}\Sigma_k^2\Sigma_k'' + \frac{3}{40}k^4\Sigma_k''' + \frac{1}{180}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_kX^4 + \bigg(-\frac{29}{80}k^4\Sigma_k\Sigma_k' + \frac{17}{80}k^2\Sigma_k^3\Sigma_k' + \frac{3}{40}\Sigma_k^5\Sigma_k'\\ &+ \frac{7}{16}k^6\Sigma_k'^2 - \frac{3}{4}k^4\Sigma_k^2\Sigma_k'^2 - \frac{3}{16}k^2\Sigma_k^4\Sigma_k'^2 - \frac{67}{240}k^6\Sigma_k\Sigma_k'' + \frac{31}{120}k^4\Sigma_k^3\Sigma_k'' + \frac{3}{80}k^2\Sigma_k^5\Sigma_k'' + \frac{27}{80}k^8\Sigma_k'\Sigma_k'' - \frac{1}{2}k^6\Sigma_k^2\Sigma_k'\Sigma_k''\\ &+ \frac{13}{80}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(-\frac{17}{80}k^4\Sigma_k^2 - \frac{1}{8}k^2\Sigma_k^4 - \frac{3}{80}\Sigma_k^6 + \frac{151}{240}k^6\Sigma_k\Sigma_k' - \frac{5}{8}k^4\Sigma_k^3\Sigma_k' + \frac{13}{80}k^2\Sigma_k^5\Sigma_k' - \frac{59}{120}k^8\Sigma_k'^2\\ &+ \frac{893}{480}k^6\Sigma_k^2\Sigma_k'^2 - \frac{217}{240}k^4\Sigma_k^4\Sigma_k'^2 + \frac{39}{160}k^2\Sigma_k^6\Sigma_k'^2 - \frac{293}{240}k^8\Sigma_k\Sigma_k'^3 + \frac{5}{8}k^6\Sigma_k^3\Sigma_k'^3 - \frac{39}{80}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]\end{aligned}$$

$$\begin{aligned}\tilde{K}_9^W = \int \frac{d^4k}{(2\pi)^4}\bigg[&\bigg(-\frac{7}{40}k^2\Sigma_k'' - \frac{3}{40}\Sigma_k^2\Sigma_k'' - \frac{7}{120}k^4\Sigma_k''' - \frac{1}{360}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_kX^4 + \bigg(\frac{37}{80}k^4\Sigma_k\Sigma_k' - \frac{1}{80}k^2\Sigma_k^3\Sigma_k' + \frac{1}{40}\Sigma_k^5\Sigma_k'\\ &- \frac{11}{16}k^6\Sigma_k'^2 + \frac{1}{4}k^4\Sigma_k^2\Sigma_k'^2 - \frac{1}{16}k^2\Sigma_k^4\Sigma_k'^2 + \frac{27}{80}k^6\Sigma_k\Sigma_k'' - \frac{1}{15}k^4\Sigma_k^3\Sigma_k'' + \frac{1}{80}k^2\Sigma_k^5\Sigma_k'' - \frac{8}{15}k^8\Sigma_k'\Sigma_k'' - \frac{1}{6}k^6\Sigma_k^2\Sigma_k'\Sigma_k''\\ &- \frac{7}{15}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(\frac{8}{15}k^4\Sigma_k^2 + \frac{1}{4}k^2\Sigma_k^4 + \frac{3}{10}\Sigma_k^6 - \frac{139}{120}k^6\Sigma_k\Sigma_k' + \frac{7}{12}k^4\Sigma_k^3\Sigma_k' - \frac{17}{40}k^2\Sigma_k^5\Sigma_k' + \frac{17}{20}k^8\Sigma_k'^2\\ &- \frac{317}{240}k^6\Sigma_k^2\Sigma_k'^2 + \frac{23}{120}k^4\Sigma_k^4\Sigma_k'^2 - \frac{51}{80}k^2\Sigma_k^6\Sigma_k'^2 + \frac{197}{120}k^8\Sigma_k\Sigma_k'^3 + \frac{11}{12}k^6\Sigma_k^3\Sigma_k'^3 - \frac{51}{40}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]\end{aligned}$$

$$\begin{aligned}\tilde{K}_{10}^W = \int \frac{d^4k}{(2\pi)^4}\bigg[&\bigg(-\frac{33}{80}k^2\Sigma_k'' + \frac{27}{80}\Sigma_k^2\Sigma_k'' - \frac{11}{80}k^4\Sigma_k''' + \frac{71}{720}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_kX^4 + \bigg(\frac{17}{40}k^4\Sigma_k\Sigma_k' - \frac{31}{40}k^2\Sigma_k^3\Sigma_k' + \frac{3}{10}\Sigma_k^5\Sigma_k'\\ &- \frac{1}{4}k^6\Sigma_k'^2 + 2k^4\Sigma_k^2\Sigma_k'^2 - \frac{3}{4}k^2\Sigma_k^4\Sigma_k'^2 + \frac{107}{240}k^6\Sigma_k\Sigma_k'' - \frac{217}{240}k^4\Sigma_k^3\Sigma_k'' + \frac{3}{20}k^2\Sigma_k^5\Sigma_k'' - \frac{7}{30}k^8\Sigma_k'\Sigma_k'' + \frac{23}{12}k^6\Sigma_k^2\Sigma_k'\Sigma_k''\\ &- \frac{17}{20}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(\frac{11}{240}k^4\Sigma_k^2 - \frac{7}{8}k^2\Sigma_k^4 + \frac{43}{80}\Sigma_k^6 - \frac{37}{80}k^6\Sigma_k\Sigma_k' + \frac{71}{24}k^4\Sigma_k^3\Sigma_k' - \frac{213}{80}k^2\Sigma_k^5\Sigma_k' + \frac{29}{120}k^8\Sigma_k'^2\\ &- \frac{641}{160}k^6\Sigma_k^2\Sigma_k'^2 + \frac{1127}{240}k^4\Sigma_k^4\Sigma_k'^2 - \frac{89}{160}k^2\Sigma_k^6\Sigma_k'^2 + \frac{283}{240}k^8\Sigma_k\Sigma_k'^3 - \frac{97}{24}k^6\Sigma_k^3\Sigma_k'^3 + \frac{89}{80}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]\end{aligned}$$

$$
\begin{aligned}
\tilde{K}^W_{11} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(&-\frac{17}{160}k^2\Sigma_k'' + \frac{3}{160}\Sigma_k^2\Sigma_k'' - \frac{7}{180}k^4\Sigma_k''' + \frac{1}{360}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(\frac{9}{40}k^4\Sigma_k\Sigma_k' - \frac{1}{20}k^2\Sigma_k^3\Sigma_k' \\
&- \frac{1}{40}\Sigma_k^5\Sigma_k' - \frac{5}{16}k^6\Sigma_k'^2 + \frac{1}{4}k^4\Sigma_k^2\Sigma_k'^2 + \frac{1}{16}k^2\Sigma_k^4\Sigma_k'^2 + \frac{49}{240}k^6\Sigma_k\Sigma_k'' - \frac{7}{120}k^4\Sigma_k^3\Sigma_k'' - \frac{1}{80}k^2\Sigma_k^5\Sigma_k'' - \frac{31}{120}k^8\Sigma_k'\Sigma_k'' \\
&+ \frac{1}{8}k^6\Sigma_k^2\Sigma_k'\Sigma_k'' - \frac{7}{60}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(\frac{9}{80}k^4\Sigma_k^2 + \frac{1}{80}\Sigma_k^6 - \frac{3}{10}k^6\Sigma_k\Sigma_k' + \frac{1}{2}k^4\Sigma_k^3\Sigma_k' + \frac{1}{20}k^2\Sigma_k^5\Sigma_k' + \frac{2}{5}k^8\Sigma_k'^2 \\
&- \frac{541}{480}k^6\Sigma_k^2\Sigma_k'^2 - \frac{41}{240}k^4\Sigma_k^4\Sigma_k'^2 - \frac{23}{160}k^2\Sigma_k^6\Sigma_k'^2 + \frac{221}{240}k^8\Sigma_k\Sigma_k'^3 + \frac{5}{24}k^6\Sigma_k^3\Sigma_k'^3 + \frac{23}{80}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{K}^W_{12} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(&-\frac{1}{40}k^2\Sigma_k'' - \frac{1}{40}\Sigma_k^2\Sigma_k'' - \frac{1}{120}k^4\Sigma_k''' + \frac{7}{360}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(-\frac{1}{20}k^4\Sigma_k\Sigma_k' - \frac{1}{10}k^2\Sigma_k^3\Sigma_k' \\
&- \frac{1}{20}\Sigma_k^5\Sigma_k' + \frac{1}{8}k^6\Sigma_k'^2X^5 + \frac{1}{4}k^4\Sigma_k^2\Sigma_k'^2 + \frac{1}{8}k^2\Sigma_k^4\Sigma_k'^2 - \frac{1}{120}k^6\Sigma_k\Sigma_k'' - \frac{1}{5}k^4\Sigma_k^3\Sigma_k'' - \frac{1}{40}k^2\Sigma_k^5\Sigma_k'' + \frac{7}{80}k^8\Sigma_k'\Sigma_k'' \\
&+ \frac{7}{24}k^6\Sigma_k^2\Sigma_k'\Sigma_k'' - \frac{31}{240}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(-\frac{53}{120}k^4\Sigma_k^2 - \frac{1}{2}k^2\Sigma_k^4 + \frac{1}{40}\Sigma_k^6 + \frac{11}{15}k^6\Sigma_k\Sigma_k' + \frac{5}{6}k^4\Sigma_k^3\Sigma_k' - \frac{2}{5}k^2\Sigma_k^5\Sigma_k' \\
&- \frac{7}{60}k^8\Sigma_k'^2 - \frac{37}{80}k^6\Sigma_k^2\Sigma_k'^2 + \frac{59}{120}k^4\Sigma_k^4\Sigma_k'^2 - \frac{13}{80}k^2\Sigma_k^6\Sigma_k'^2 - \frac{3}{40}k^8\Sigma_k\Sigma_k'^3 - \frac{5}{12}k^6\Sigma_k^3\Sigma_k'^3 + \frac{13}{40}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{K}^W_{13} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(&\frac{1}{80}k^2\Sigma_k'' + \frac{1}{80}\Sigma_k^2\Sigma_k'' + \frac{1}{240}k^4\Sigma_k''' + \frac{1}{240}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(-\frac{1}{80}k^2\Sigma_k^3\Sigma_k' - \frac{3}{80}k^4\Sigma_k\Sigma_k' + \frac{1}{40}\Sigma_k^5\Sigma_k' \\
&+ \frac{1}{16}k^6\Sigma_k'^2 - \frac{1}{16}k^2\Sigma_k^4\Sigma_k'^2 - \frac{1}{60}k^6\Sigma_k\Sigma_k'' - \frac{1}{240}k^4\Sigma_k^3\Sigma_k'' + \frac{1}{80}k^2\Sigma_k^5\Sigma_k'' + \frac{7}{240}k^8\Sigma_k'\Sigma_k'' - \frac{7}{240}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 \\
&+ \bigg(-\frac{7}{40}k^4\Sigma_k^2 - \frac{1}{8}k^2\Sigma_k^4 + \frac{1}{20}\Sigma_k^6 + \frac{41}{120}k^6\Sigma_k\Sigma_k' + \frac{1}{6}k^4\Sigma_k^3\Sigma_k' - \frac{7}{40}k^2\Sigma_k^5\Sigma_k' - \frac{1}{15}k^8\Sigma_k'^2 + \frac{13}{240}k^6\Sigma_k^2\Sigma_k'^2 \\
&+ \frac{13}{120}k^4\Sigma_k^4\Sigma_k'^2 - \frac{1}{80}k^2\Sigma_k^6\Sigma_k'^2 - \frac{13}{120}k^8\Sigma_k\Sigma_k'^3 - \frac{1}{12}k^6\Sigma_k^3\Sigma_k'^3 + \frac{1}{40}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{K}^W_{14} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(&-\frac{9}{80}k^2\Sigma_k'' + \frac{11}{80}\Sigma_k^2\Sigma_k'' - \frac{2}{45}k^4\Sigma_k''' + \frac{1}{40}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(\frac{7}{80}k^4\Sigma_k\Sigma_k' - \frac{21}{80}k^2\Sigma_k^3\Sigma_k' + \frac{3}{20}\Sigma_k^5\Sigma_k' \\
&+ \frac{5}{8}k^4\Sigma_k^2\Sigma_k'^2 - \frac{3}{8}k^2\Sigma_k^4\Sigma_k'^2 + \frac{13}{120}k^6\Sigma_k\Sigma_k'' - \frac{19}{60}k^4\Sigma_k^3\Sigma_k'' + \frac{3}{40}k^2\Sigma_k^5\Sigma_k'' - \frac{1}{30}k^8\Sigma_k'\Sigma_k'' + \frac{13}{24}k^6\Sigma_k^2\Sigma_k'\Sigma_k'' \\
&- \frac{17}{40}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(\frac{13}{240}k^4\Sigma_k^2 + \frac{19}{80}\Sigma_k^6 + \frac{1}{120}k^6\Sigma_k\Sigma_k' + \frac{7}{12}k^4\Sigma_k^3\Sigma_k' - \frac{47}{40}k^2\Sigma_k^5\Sigma_k' - \frac{1}{40}k^8\Sigma_k'^2 \\
&- \frac{113}{160}k^6\Sigma_k^2\Sigma_k'^2 + \frac{197}{80}k^4\Sigma_k^4\Sigma_k'^2 - \frac{57}{160}k^2\Sigma_k^6\Sigma_k'^2 - \frac{7}{80}k^8\Sigma_k\Sigma_k'^3 - \frac{41}{24}k^6\Sigma_k^3\Sigma_k'^3 + \frac{57}{80}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{K}^W_{15} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(&\frac{11}{160}k^2\Sigma_k'' - \frac{9}{160}\Sigma_k^2\Sigma_k'' + \frac{19}{720}k^4\Sigma_k''' - \frac{11}{720}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(-\frac{9}{80}k^4\Sigma_k\Sigma_k' + \frac{7}{80}k^2\Sigma_k^3\Sigma_k' \\
&- \frac{1}{20}\Sigma_k^5\Sigma_k' + \frac{1}{8}k^6\Sigma_k'^2 - \frac{1}{4}k^4\Sigma_k^2\Sigma_k'^2 + \frac{1}{8}k^2\Sigma_k^4\Sigma_k'^2 - \frac{2}{15}k^6\Sigma_k\Sigma_k'' + \frac{11}{120}k^4\Sigma_k^3\Sigma_k'' - \frac{1}{40}k^2\Sigma_k^5\Sigma_k'' + \frac{19}{160}k^8\Sigma_k'\Sigma_k'' \\
&- \frac{3}{16}k^6\Sigma_k^2\Sigma_k'\Sigma_k'' + \frac{31}{160}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(\frac{3}{80}k^4\Sigma_k^2 - \frac{13}{80}\Sigma_k^6 + \frac{1}{40}k^6\Sigma_k\Sigma_k' - \frac{1}{4}k^4\Sigma_k^3\Sigma_k' + \frac{19}{40}k^2\Sigma_k^5\Sigma_k' - \frac{1}{5}k^8\Sigma_k'^2 \\
&+ \frac{343}{480}k^6\Sigma_k^2\Sigma_k'^2 - \frac{97}{240}k^4\Sigma_k^4\Sigma_k'^2 + \frac{29}{160}k^2\Sigma_k^6\Sigma_k'^2 - \frac{103}{240}k^8\Sigma_k\Sigma_k'^3 + \frac{5}{24}k^6\Sigma_k^3\Sigma_k'^3 - \frac{29}{80}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]
\end{aligned}
$$

$$\tilde{K}_{16}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(-\frac{7}{80}k^2\Sigma_k'' - \frac{7}{80}\Sigma_k^2\Sigma_k'' - \frac{1}{45}k^4\Sigma_k''' - \frac{1}{120}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(\frac{3}{40}k^4\Sigma_k\Sigma_k' - \frac{1}{10}k^2\Sigma_k^3\Sigma_k' - \frac{7}{40}\Sigma_k^5\Sigma_k' - \frac{1}{16}k^6\Sigma_k'^2 + \frac{3}{8}k^4\Sigma_k^2\Sigma_k'^2 + \frac{7}{16}k^2\Sigma_k^4\Sigma_k'^2 + \frac{23}{240}k^6\Sigma_k\Sigma_k'' + \frac{1}{120}k^4\Sigma_k^3\Sigma_k'' - \frac{7}{80}k^2\Sigma_k^5\Sigma_k'' - \frac{3}{80}k^8\Sigma_k'\Sigma_k'' + \frac{5}{12}k^6\Sigma_k^2\Sigma_k'\Sigma_k'' + \frac{109}{240}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(-\frac{61}{240}k^4\Sigma_k^2 - \frac{3}{8}k^2\Sigma_k^4 - \frac{13}{80}\Sigma_k^6 + \frac{23}{120}k^6\Sigma_k\Sigma_k' + \frac{2}{3}k^4\Sigma_k^3\Sigma_k' + \frac{29}{40}k^2\Sigma_k^5\Sigma_k' + \frac{1}{20}k^8\Sigma_k'^2 - \frac{209}{160}k^6\Sigma_k^2\Sigma_k'^2 - \frac{109}{80}k^4\Sigma_k^4\Sigma_k'^2 + \frac{79}{160}k^2\Sigma_k^6\Sigma_k'^2 + \frac{49}{80}k^8\Sigma_k\Sigma_k'^3 - \frac{1}{24}k^6\Sigma_k^3\Sigma_k'^3 - \frac{79}{80}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]$$

$$\tilde{K}_{17}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(-\frac{2}{5}k^2\Sigma_k'' + \frac{1}{10}\Sigma_k^2\Sigma_k'' - \frac{2}{15}k^4\Sigma_k''' + \frac{11}{180}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(\frac{23}{40}k^4\Sigma_k\Sigma_k' - \frac{19}{40}k^2\Sigma_k^3\Sigma_k' - \frac{1}{20}\Sigma_k^5\Sigma_k' - \frac{5}{8}k^6\Sigma_k'^2 + \frac{3}{2}k^4\Sigma_k^2\Sigma_k'^2 + \frac{1}{8}k^2\Sigma_k^4\Sigma_k'^2 + \frac{8}{15}k^6\Sigma_k\Sigma_k'' - \frac{79}{120}k^4\Sigma_k^3\Sigma_k'' - \frac{1}{40}k^2\Sigma_k^5\Sigma_k'' - \frac{119}{240}k^8\Sigma_k'\Sigma_k'' + \frac{29}{24}k^6\Sigma_k^2\Sigma_k'\Sigma_k'' - \frac{151}{240}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(\frac{37}{120}k^4\Sigma_k^2 - \frac{1}{2}k^2\Sigma_k^4 - \frac{9}{40}\Sigma_k^6 - \frac{14}{15}k^6\Sigma_k\Sigma_k' + \frac{8}{3}k^4\Sigma_k^3\Sigma_k' + \frac{1}{10}k^2\Sigma_k^5\Sigma_k' + \frac{43}{60}k^8\Sigma_k'^2 - \frac{1031}{240}k^6\Sigma_k^2\Sigma_k'^2 + \frac{53}{40}k^4\Sigma_k^4\Sigma_k'^2 - \frac{53}{80}k^2\Sigma_k^6\Sigma_k'^2 + \frac{271}{120}k^8\Sigma_k\Sigma_k'^3 - \frac{13}{12}k^6\Sigma_k^3\Sigma_k'^3 + \frac{53}{40}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]$$

$$\tilde{K}_{18}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(-\frac{9}{80}k^2\Sigma_k'' + \frac{11}{80}\Sigma_k^2\Sigma_k'' - \frac{2}{45}k^4\Sigma_k''' + \frac{7}{180}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(\frac{3}{20}k^4\Sigma_k\Sigma_k' - \frac{1}{5}k^2\Sigma_k^3\Sigma_k' + \frac{3}{20}\Sigma_k^5\Sigma_k' - \frac{1}{8}k^6\Sigma_k'^2 + \frac{1}{2}k^4\Sigma_k^2\Sigma_k'^2 - \frac{3}{8}k^2\Sigma_k^4\Sigma_k'^2 + \frac{13}{120}k^6\Sigma_k\Sigma_k'' - \frac{19}{60}k^4\Sigma_k^3\Sigma_k'' + \frac{3}{40}k^2\Sigma_k^5\Sigma_k'' - \frac{7}{60}k^8\Sigma_k'\Sigma_k'' + \frac{1}{4}k^6\Sigma_k^2\Sigma_k'\Sigma_k'' - \frac{19}{30}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(-\frac{1}{20}k^4\Sigma_k^2 - \frac{1}{8}k^2\Sigma_k^4 + \frac{7}{40}\Sigma_k^6 + \frac{2}{15}k^6\Sigma_k\Sigma_k' + \frac{7}{12}k^4\Sigma_k^3\Sigma_k' - \frac{21}{10}k^2\Sigma_k^3\Sigma_k' + \frac{1}{10}k^8\Sigma_k'^2 - \frac{97}{240}k^6\Sigma_k^2\Sigma_k'^2 + \frac{223}{120}k^4\Sigma_k^4\Sigma_k'^2 - \frac{51}{80}k^2\Sigma_k^6\Sigma_k'^2 + \frac{17}{120}k^8\Sigma_k\Sigma_k'^3 - \frac{7}{12}k^6\Sigma_k^3\Sigma_k'^3 + \frac{51}{40}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]$$

$$\tilde{K}_{19}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(-\frac{1}{4}k^2\Sigma_k'' + \frac{1}{4}\Sigma_k^2\Sigma_k'' - \frac{1}{12}k^4\Sigma_k''' + \frac{5}{72}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(\frac{3}{16}k^4\Sigma_k\Sigma_k' - \frac{9}{16}k^2\Sigma_k^3\Sigma_k' + \frac{1}{4}\Sigma_k^5\Sigma_k' + \frac{11}{8}k^4\Sigma_k^2\Sigma_k'^2 - \frac{5}{8}k^2\Sigma_k^4\Sigma_k'^2 + \frac{1}{4}k^6\Sigma_k\Sigma_k'' - \frac{5}{8}k^4\Sigma_k^3\Sigma_k'' + \frac{1}{8}k^2\Sigma_k^5\Sigma_k'' - \frac{1}{24}k^8\Sigma_k'\Sigma_k'' + \frac{37}{24}k^6\Sigma_k^2\Sigma_k'\Sigma_k'' - \frac{5}{12}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(-\frac{1}{48}k^4\Sigma_k^2 - \frac{1}{4}k^2\Sigma_k^4 + \frac{5}{16}\Sigma_k^6 - \frac{1}{6}k^6\Sigma_k\Sigma_k' + \frac{4}{3}k^4\Sigma_k^3\Sigma_k' - \frac{7}{4}k^2\Sigma_k^5\Sigma_k' - \frac{1}{24}k^8\Sigma_k'^2 - \frac{91}{32}k^6\Sigma_k^2\Sigma_k'^2 + \frac{173}{48}k^4\Sigma_k^4\Sigma_k'^2 - \frac{3}{32}k^2\Sigma_k^6\Sigma_k'^2 + \frac{25}{48}k^8\Sigma_k\Sigma_k'^3 - \frac{29}{8}k^6\Sigma_k^3\Sigma_k'^3 + \frac{3}{16}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]$$

$$\tilde{K}_{20}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\bigg(\frac{1}{40}k^2\Sigma_k'' + \frac{1}{40}\Sigma_k^2\Sigma_k'' - \frac{1}{180}k^4\Sigma_k''' - \frac{1}{180}k^2\Sigma_k^2\Sigma_k'''\bigg)k^2\Sigma_k X^4 + \bigg(\frac{1}{20}k^4\Sigma_k\Sigma_k' + \frac{1}{10}k^2\Sigma_k^3\Sigma_k' + \frac{1}{20}\Sigma_k^5\Sigma_k' - \frac{1}{8}k^6\Sigma_k'^2 - \frac{1}{4}k^4\Sigma_k^2\Sigma_k'^2 - \frac{1}{8}k^2\Sigma_k^4\Sigma_k'^2 + \frac{11}{120}k^6\Sigma_k\Sigma_k'' + \frac{7}{60}k^4\Sigma_k^3\Sigma_k'' + \frac{1}{40}k^2\Sigma_k^5\Sigma_k'' - \frac{13}{120}k^8\Sigma_k'\Sigma_k'' - \frac{1}{12}k^6\Sigma_k^2\Sigma_k'\Sigma_k'' + \frac{1}{40}k^4\Sigma_k^4\Sigma_k'\Sigma_k''\bigg)X^5 + \bigg(\frac{3}{20}k^4\Sigma_k^2 + \frac{1}{4}k^2\Sigma_k^4 + \frac{1}{10}\Sigma_k^6 - \frac{2}{5}k^6\Sigma_k\Sigma_k' - \frac{1}{2}k^4\Sigma_k^3\Sigma_k' - \frac{1}{10}k^2\Sigma_k^5\Sigma_k' + \frac{1}{5}k^8\Sigma_k'^2 - \frac{1}{10}k^6\Sigma_k^2\Sigma_k'^2 - \frac{1}{5}k^4\Sigma_k^4\Sigma_k'^2 + \frac{1}{10}k^2\Sigma_k^6\Sigma_k'^2 + \frac{1}{5}k^8\Sigma_k\Sigma_k'^3 - \frac{1}{5}k^4\Sigma_k^5\Sigma_k'^3\bigg)X^6\bigg]$$

$$\begin{aligned}\tilde{K}^W_{21} = \int \frac{d^4k}{(2\pi)^4}\bigg[&\bigg(-\frac{11}{40}k^2\Sigma''_k + \frac{9}{40}\Sigma_k^2\Sigma''_k - \frac{11}{120}k^4\Sigma'''_k + \frac{3}{40}k^2\Sigma_k^2\Sigma'''_k\bigg)k^2\Sigma_k X^4 + \bigg(\frac{13}{40}k^4\Sigma_k\Sigma'_k - \frac{19}{40}k^2\Sigma_k^3\Sigma'_k + \frac{1}{5}\Sigma_k^5\Sigma'_k \\
&- \frac{1}{4}k^6\Sigma'^2_k + \frac{5}{4}k^4\Sigma_k^2\Sigma'^2_k - \frac{1}{2}k^2\Sigma_k^4\Sigma'^2_k + \frac{13}{40}k^6\Sigma_k\Sigma''_k - \frac{23}{40}k^4\Sigma_k^3\Sigma''_k + \frac{1}{10}k^2\Sigma_k^5\Sigma''_k - \frac{11}{60}k^8\Sigma'_k\Sigma''_k + \frac{5}{4}k^6\Sigma_k^2\Sigma'_k\Sigma''_k \\
&- \frac{17}{30}k^4\Sigma_k^4\Sigma'_k\Sigma''_k\bigg)X^5 + \bigg(\frac{1}{10}k^4\Sigma_k^2 - \frac{1}{4}k^2\Sigma_k^4 + \frac{3}{20}\Sigma_k^6 - \frac{31}{60}k^6\Sigma_k\Sigma'_k + \frac{4}{3}k^4\Sigma_k^3\Sigma'_k - \frac{23}{20}k^2\Sigma_k^5\Sigma'_k + \frac{3}{10}k^8\Sigma'^2_k \\
&- \frac{169}{60}k^6\Sigma_k^2\Sigma'^2_k + \frac{38}{15}k^4\Sigma_k^4\Sigma'^2_k - \frac{7}{20}k^2\Sigma_k^6\Sigma'^2_k + \frac{29}{30}k^8\Sigma_k\Sigma'^3_k - \frac{7}{3}k^6\Sigma_k^3\Sigma'^3_k + \frac{7}{10}k^4\Sigma_k^5\Sigma'^3_k\bigg)X^6\bigg]\end{aligned}$$

$$\begin{aligned}\tilde{K}^W_{22} = \int \frac{d^4k}{(2\pi)^4}\bigg[&\bigg(-\frac{1}{80}k^2\Sigma''_k - \frac{1}{80}\Sigma_k^2\Sigma''_k - \frac{1}{240}k^4\Sigma'''_k - \frac{1}{240}k^2\Sigma_k^2\Sigma'''_k\bigg)k^2\Sigma_k X^4 + \bigg(\frac{3}{80}k^4\Sigma_k\Sigma'_k + \frac{1}{80}k^2\Sigma_k^3\Sigma'_k - \frac{1}{40}\Sigma_k^5\Sigma'_k \\
&- \frac{1}{16}k^6\Sigma'^2_k + \frac{1}{16}k^2\Sigma_k^4\Sigma'^2_k + \frac{3}{80}k^6\Sigma_k\Sigma''_k + \frac{1}{40}k^4\Sigma_k^3\Sigma''_k - \frac{1}{80}k^2\Sigma_k^5\Sigma''_k - \frac{19}{480}k^8\Sigma'_k\Sigma''_k + \frac{1}{48}k^6\Sigma_k^2\Sigma'_k\Sigma''_k \\
&+ \frac{29}{480}k^4\Sigma_k^4\Sigma'_k\Sigma''_k\bigg)X^5 + \bigg(-\frac{1}{80}k^4\Sigma_k^2 - \frac{1}{16}k^2\Sigma_k^4 - \frac{1}{20}\Sigma_k^6 + \frac{1}{80}k^6\Sigma_k\Sigma'_k + \frac{1}{4}k^4\Sigma_k^3\Sigma'_k + \frac{19}{80}k^2\Sigma_k^5\Sigma'_k + \frac{1}{40}k^8\Sigma'^2_k \\
&- \frac{13}{40}k^6\Sigma_k^2\Sigma'^2_k - \frac{11}{40}k^4\Sigma_k^4\Sigma'^2_k + \frac{3}{40}k^2\Sigma_k^6\Sigma'^2_k + \frac{3}{20}k^8\Sigma_k\Sigma'^3_k - \frac{3}{20}k^4\Sigma_k^5\Sigma'^3_k\bigg)X^6\bigg]\end{aligned}$$

$$\begin{aligned}\tilde{K}^W_{23} = \int \frac{d^4k}{(2\pi)^4}\bigg[&\bigg(-\frac{23}{160}k^2\Sigma''_k + \frac{17}{160}\Sigma_k^2\Sigma''_k - \frac{23}{480}k^4\Sigma'''_k + \frac{17}{480}k^2\Sigma_k^2\Sigma'''_k\bigg)k^2\Sigma_k X^4 + \bigg(\frac{19}{160}k^4\Sigma_k\Sigma'_k - \frac{47}{160}k^2\Sigma_k^3\Sigma'_k \\
&+ \frac{7}{80}\Sigma_k^5\Sigma'_k - \frac{1}{32}k^6\Sigma'^2_k + \frac{3}{4}k^4\Sigma_k^2\Sigma'^2_k - \frac{7}{32}k^2\Sigma_k^4\Sigma'^2_k + \frac{31}{240}k^6\Sigma_k\Sigma''_k - \frac{157}{480}k^4\Sigma_k^3\Sigma''_k + \frac{7}{160}k^2\Sigma_k^5\Sigma''_k - \frac{11}{480}k^8\Sigma'_k\Sigma''_k \\
&+ \frac{19}{24}k^6\Sigma_k^2\Sigma'_k\Sigma''_k - \frac{89}{480}k^4\Sigma_k^4\Sigma'_k\Sigma''_k\bigg)X^5 + \bigg(-\frac{9}{80}k^4\Sigma_k^2 - \frac{3}{16}k^2\Sigma_k^4 + \frac{7}{40}\Sigma_k^6 + \frac{1}{120}k^6\Sigma_k\Sigma'_k + \frac{7}{12}k^4\Sigma_k^3\Sigma'_k \\
&- \frac{37}{40}k^2\Sigma_k^5\Sigma'_k + \frac{1}{60}k^8\Sigma'^2_k - \frac{223}{160}k^6\Sigma_k^2\Sigma'^2_k + \frac{371}{240}k^4\Sigma_k^4\Sigma'^2_k - \frac{7}{160}k^2\Sigma_k^6\Sigma'^2_k + \frac{23}{80}k^8\Sigma_k\Sigma'^3_k - \frac{13}{8}k^6\Sigma_k^3\Sigma'^3_k \\
&+ \frac{7}{80}k^4\Sigma_k^5\Sigma'^3_k\bigg)X^6\bigg], \qquad \text{(C1)}\end{aligned}$$

where $B_0$ is the LEC appear in $p^2$ order normal part of the chiral Lagrangian. $\Sigma_k \equiv \Sigma(k^2)$ and $X \equiv \frac{1}{k^2+\Sigma_k^2}$.


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